Aeolian sand transport rates and mid-air collisions

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The Aarhus sand gang

Transport rate

TRANSPORT RATE:

$$Q = \Phi \cdot \text{MEAN JUMP LENGTH} = \Phi \cdot \overline{x(t_i)}$$

Φ = MASS FLUX FROM THE BED INTO THE AIR

t_i time of impact $\quad x(t_i)$ jump length

Grain borne shear stress

Owen (1964)

$$\begin{aligned} \rho U_*^2 &= \text{AIR BORNE SHEAR STRESS + GRAIN BORNE SHEAR STRESS} \\ &= T_a(y) + T_g(y) \end{aligned}$$

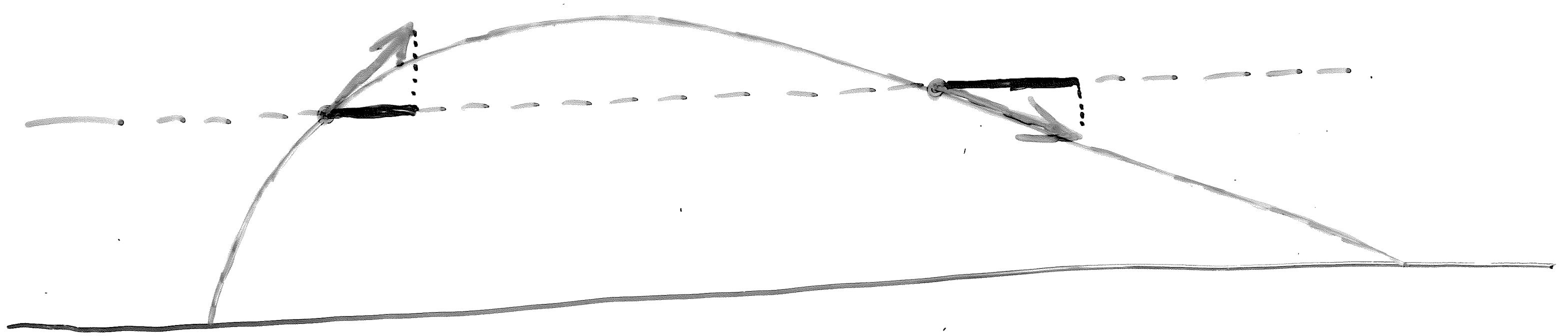
U_* friction speed in the grain free wind, ρ density of air

Grain borne shear stress at height y: $T_g(y) = \Phi v(y)$

$v(y)$ = the average increase of the horizontal velocity component of a saltating grain while it is above the level y

$$v(y) = \overline{[\dot{x}(t_y^2) - \dot{x}(t_y^1)]1_{\{y^{\text{top}} > y\}}}$$

t_y^1 1st time at height y t_y^2 2nd time at height y
y^{top} height of grain trajectory



Grain borne shear stress

AIR BORNE SHEAR STRESS AT THE BED: $T_a(0) = \pi\rho U_*^2 \quad 0 < \pi < 1$

It follows from $T_a(0) + \Phi v(0) = \rho U_*^2$ that

$$\Phi = \rho U_*^2 \left(1 - \pi\right) / v(0),$$

and

$$T_g(y) = \rho U_*^2 \left(1 - \pi\right) a(y)$$

where

$$a(y) = v(y)/v(0).$$

Friction speed

Anderson (1986)

FRICTION SPEED AT HEIGHT y:

$$U_*(y) = \sqrt{U_*^2 - T_g(y)/\rho} = U_* \sqrt{1 - (1 - \pi)a(y)}$$

Sørensen (1991, 2004): $\sqrt{1-x} \approx 1 - x$

$$U_*(y) = U_*(1 - (1 - \pi)a(y))$$

Durán and Herrmann (2006):

$$U_*(0) = \pi U_* \qquad \rho U_*(0)^2 = \pi^2 \rho U_*^2 \neq \pi \rho U_*^2$$

Wind profile

We only need a good approximation to $\sqrt{1-x}$ in the interval $[0, 1-\pi]$:

$$\sqrt{1-x} \;\approx\; 1 - \frac{1-\sqrt{\pi}}{1-\pi}x$$

$$U_*(y) = U_*(1-(1-\sqrt{\pi})a(y))$$

Eddy viscosity/Prandtl turbulence closure ($\kappa =$ von Kármán's constant):

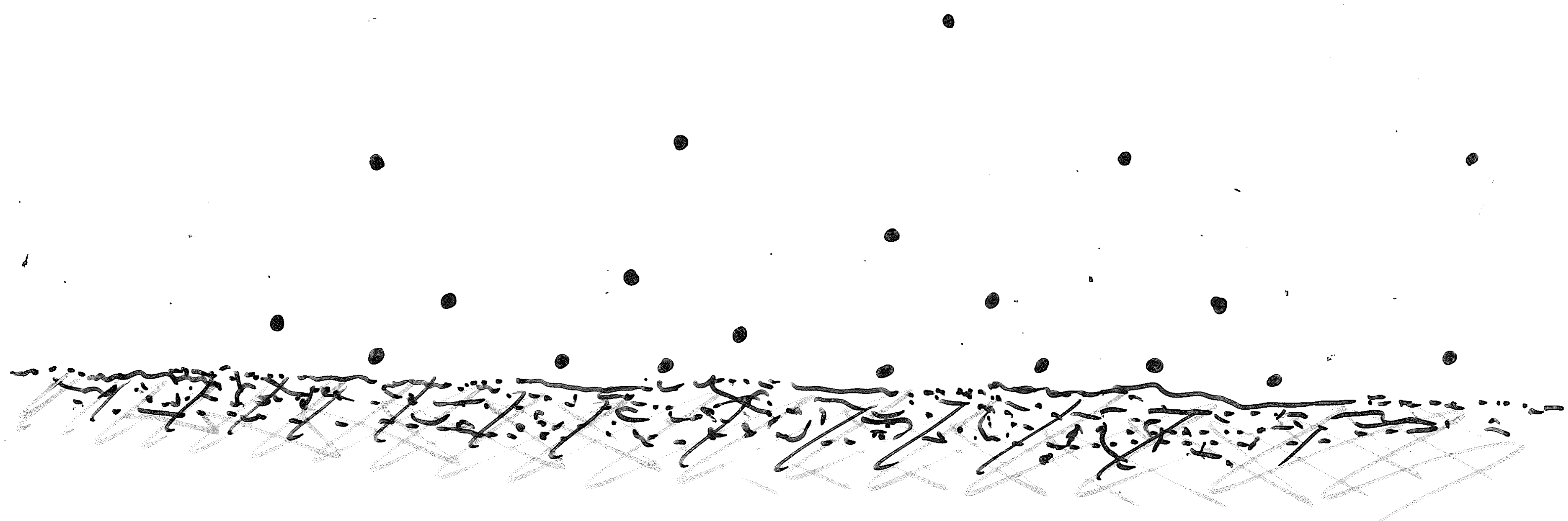
$$\frac{dU}{dy} = \frac{U_*}{\kappa y}(1-\left(1-\sqrt{\pi}\right)a(y))$$

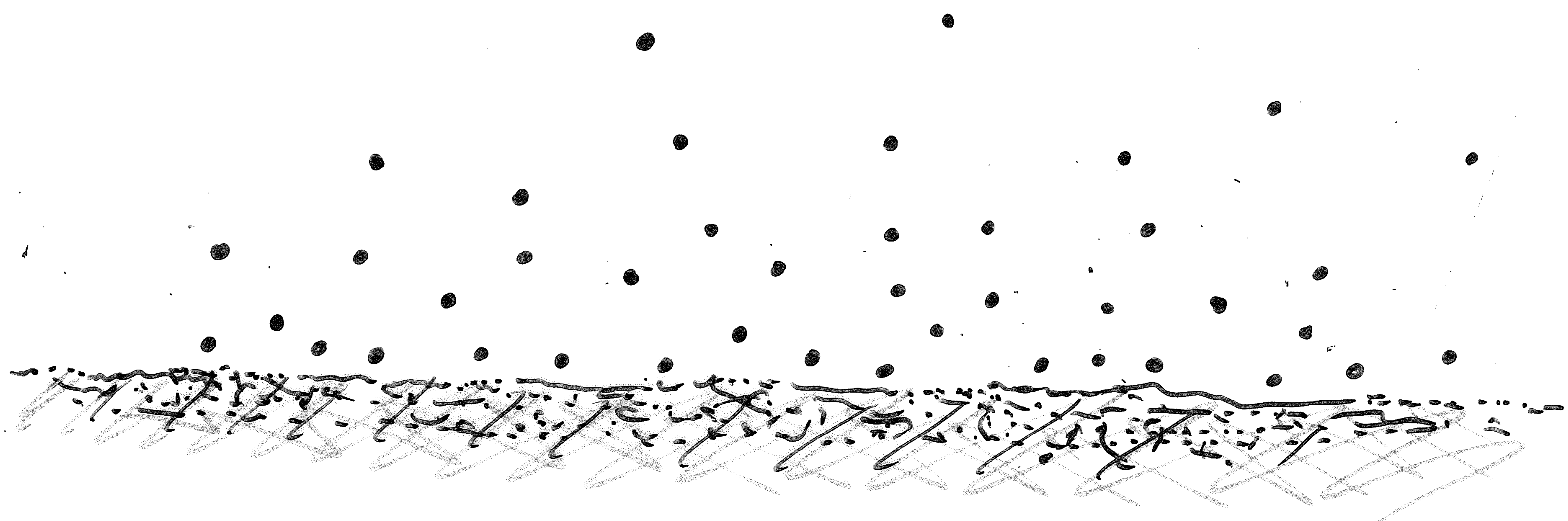
$$U(y) = \kappa^{-1}U_*\left[\ln(y/y_0) - \left(1-\sqrt{\pi}\right)b(y)\right]$$

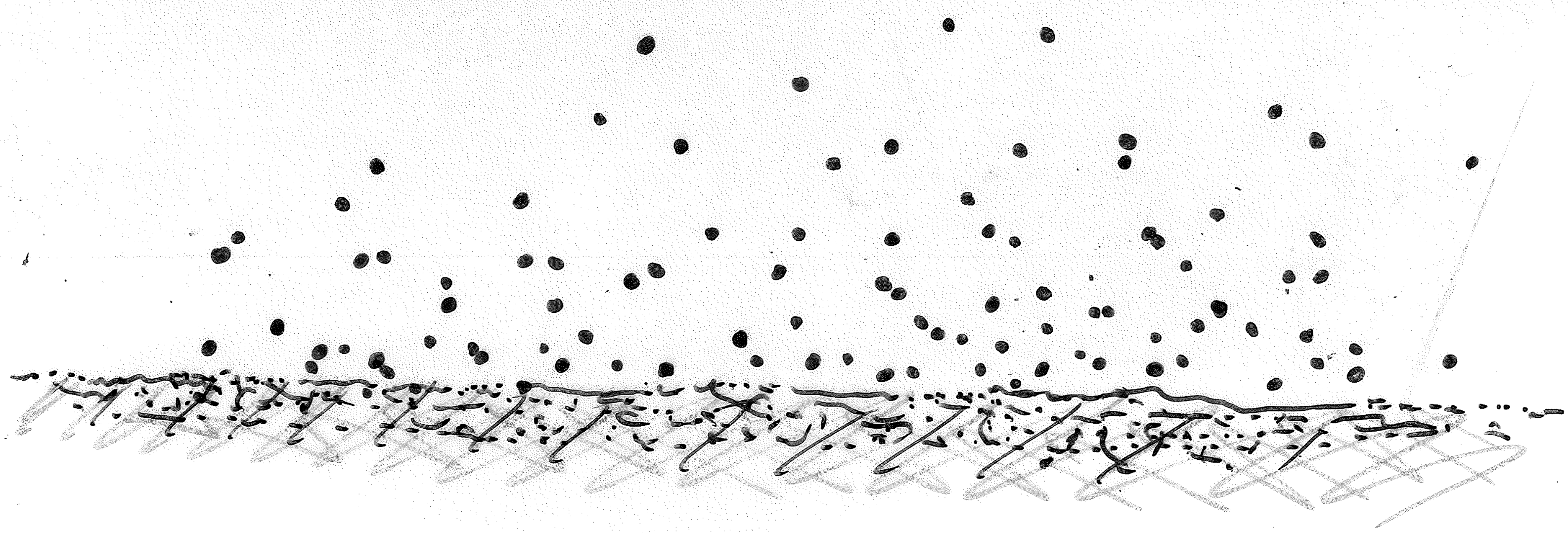
where

$$b(y) = \int_{y_0}^{y} z^{-1}a(z)dz.$$

Assumption: The function b is independent of U_*







A simple saltation model: the grain motion

$$\ddot{x} = H(v)(U(y) - \dot{x})$$

$$\ddot{y} + g + H(v)\dot{y} = 0$$

$(x(0), y(0)) = (0, 0)$ $\quad\quad$ $(\dot{x}(0), \dot{y}(0)) = (v_1^0, v_2^0)$ is random

$H(v) = D(v)/(mv)$

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Owen (1964): $\quad D(v) = \delta v \qquad H(v) = t_*^{-1}$

$t_* = m/\delta$ is the response time of a grain to changes in the wind speed

A simple saltation model: the grain motion

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$$\begin{aligned} x(t) &= \int_0^t \left(1 - e^{-(t-s)/t_*}\right) U(y(s))ds + t_* v_1^0 \left(1 - e^{-t/t_*}\right) \\ y(t) &= t_*(v_f + v_2^0)(1 - e^{-t/t_*}) - v_f t \qquad v_f = g t_* \end{aligned}$$

A simple saltation model: the transport rate

$Q = \Phi \cdot \overline{x(t_i)}$ $\qquad$ t_i impact time

$$\overline{x(t_i)} = \kappa^{-1} U_* \left[\overline{\int_0^{t_i} \left(1 - e^{-(t_i - s)/t_*}\right) \log(y(s)/y_0) ds} - \left(1 - \sqrt{\pi}\right) \overline{\int_0^{t_i} \left(1 - e^{-(t_i - s)/t_*}\right) b(y(s)) ds} \right] + t_* \overline{v_1^0 \left(1 - e^{-t_i/t_*}\right)}.$$

$$\Phi = \rho U_*^2 \left(1 - \pi\right) / v(0)$$

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$$\Phi = \rho U_*^2 \left(1 - \pi\right) / v(0)$$

$$\frac{Qg}{\rho U_*^3} = (1 - \pi) \left[\alpha + \beta\sqrt{\pi} + \gamma / U_*\right]$$

Saturated saltation

Owen (1964):

AIR BORNE SHEAR STRESS AT THE BED IS EQUAL TO ρU_{*c}^2
FOR ALL $U_* \geq U_{*c}$

U_{*c} friction speed at the impact threshold

$$\pi = \frac{U_{*c}^2}{U_*^2} = V^{-2}$$

$$V = \frac{U_*}{U_{*c}} \qquad \text{dimensionless friction speed}$$

Saturated saltation

$$\Phi = \rho U_*^2 \left(1 - V^{-2}\right) / v(0),$$

$$T_g(y) = \rho U_*^2 \left(1 - V^{-2}\right) a(y)$$

$$U(y) = \kappa^{-1} U_* \left[\ln(y/y_0) - \left(1 - V^{-1}\right) b(y)\right]$$

$$\frac{Qg}{\rho U_*^3} = \left(1 - V^{-2}\right)\left[\alpha + \beta V^{-1}\right] = \left(1 - V^{-1}\right)\left(1 + V^{-1}\right)\left[\alpha + \beta V^{-1}\right]$$

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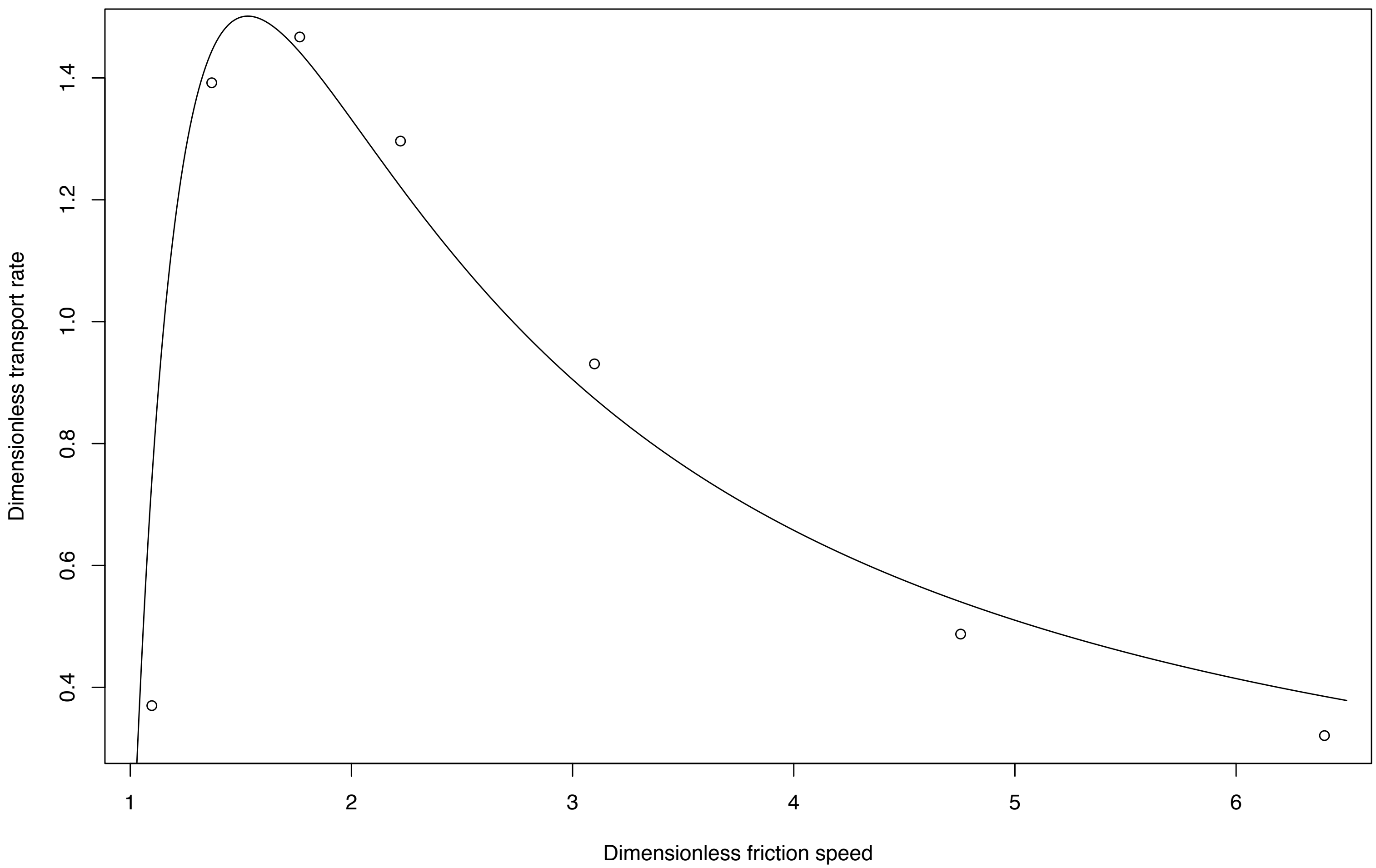
Sørensen (2004): $$\frac{Qg}{\rho U_*^3} = \left(1 - V^{-2}\right)\left[\alpha + \beta V^{-1} + \gamma V^{-2}\right]$$

Sørensen (1991): $$\frac{Qg}{\rho U_*^3} = \left(1 - V^{-1}\right)\left[\alpha + \beta V^{-1}\right]$$

Sørensen (2004) formula

Iversen and Rasmussen (1999): Homogeneous sand of size 170 μm

$\alpha = 0, \beta = 2.1, \gamma = 3.0$

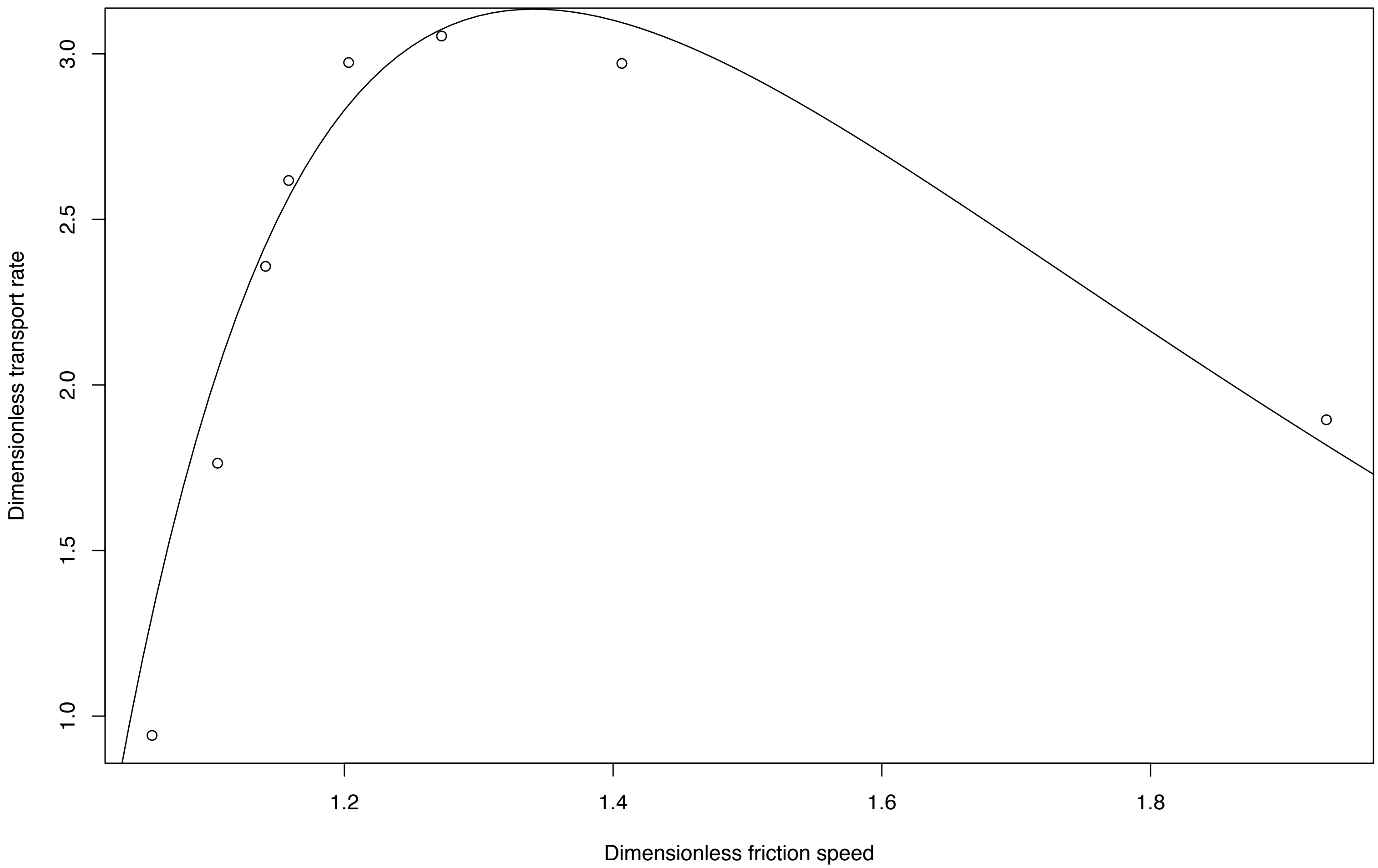


Sørensen (2004) formula

Iversen and Rasmussen (1999): Natural sand of typical size 230 μm

Råbjerg Mile

$\alpha = -1.8, \beta = 0, \gamma = 15.9$



Bagnold's focal point

$$
\begin{aligned}
U(y) &= \kappa^{-1} U_* \left[\ln(y/y_0) - \left(1 - V^{-1}\right) b(y)\right] \\
&= \kappa^{-1} U_* \left[\ln(y/y_0) - b(y)\right] + \kappa^{-1} U_{*c} b(y)
\end{aligned}
$$

Suppose a height $\bar{y}$ exists where $U(\bar{y})$ does not depend on U_*

$$
U(\bar{y}) = \kappa^{-1} U_* \left[\ln(\bar{y}/y_0) - b(\bar{y})\right] + \kappa^{-1} U_{*c} b(\bar{y})
$$

Only possible if $\ln(\bar{y}/y_0) = b(\bar{y})$, but

$$
\ln(\bar{y}/y_0) - b(\bar{y}) = \int_{y_0}^{\bar{y}} z^{-1}(1 - a(z))dz \geq 0 \qquad (a(z) \leq 1)
$$

$\ln(\bar{y}/y_0) = b(\bar{y})$ only possible, if $\bar{y}$ is sufficiently close to the bed that $a(\bar{y}) \sim 1$

In that case $\ln(y/y_0) = b(y)$ for all $y \leq \bar{y}$

Bagnold's focal point

$$\begin{aligned} U(y) &= \kappa^{-1}U_* \left[\ln(y/y_0) - \left(1 - \sqrt{\pi}\right) b(y)\right] \\ &= \kappa^{-1}U_* \left[\ln(y/y_0) - b(y)\right] + \kappa^{-1}U_*(0)b(y) \end{aligned}$$

$\sqrt{\pi}U_* = U_*(0)$

$U_*(0) = f(U_*)$

Suppose $\kappa^{-1}U_* \left[\ln(\bar{y}/y_0) - b(\bar{y})\right] + \kappa^{-1}f(U_*)b(\bar{y})$ does not depend on U_*

Then

$$f'(U_*) = -(\mu - 1)$$

where

$$\mu = \ln(\bar{y}/y_0)/b(\bar{y}) \geq 1$$

Bagnold's focal point

$$f(U_*) = -(\mu - 1)U_* + \kappa$$

$$U_{*c} = f(U_{*c}) = -(\mu - 1)U_{*c} + \kappa$$

So $\kappa = \mu U_{*c}$ and

$$U_*(0) = -(\mu - 1)U_* + \mu U_{*c}$$

Owen (1964): $\mu = 1$

Since necessarily $U_*(0) \geq 0$, this formula can only hold for sufficiently small values of U_*:

$$V \leq \frac{1}{1 - \mu^{-1}}$$

Wind profile

$$\sqrt{\pi} U_* = U_*(0) = -(\mu - 1)U_* + \mu U_{*c}$$

implies

$$\sqrt{\pi} = 1 - \mu(1 - V^{-1})$$

$$\pi = (\mu - 1)^2 - 2\mu(1 - \mu)V^{-1} + \mu^2 V^{-2}$$

$$1 - \pi = \mu(1 - V^{-1})(2 - \mu + \mu V^{-1})$$

$$U(y) = \kappa^{-1} U_* \left[\ln(y/y_0) - \mu\left(1 - V^{-1}\right) b(y)\right]$$

$$U(\bar{y}) = \kappa^{-1} U_* \left[\ln(\bar{y}/y_0) - \mu\left(1 - V^{-1}\right) b(\bar{y})\right] = \kappa^{-1} U_{*c} \ln(\bar{y}/y_0)$$

Transport rate

$$\Phi = \rho U_*^2 \mu (1 - V^{-1})(2 - \mu + \mu V^{-1})/v(0)$$

$$\frac{Qg}{\rho U_*^3} = \mu (1 - V^{-1})(2 - \mu + \mu V^{-1}) \left[\alpha + \beta V^{-1}\right]$$

Durán and Herrmann (2006)

Transport rate

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Durán and Herrmann (2006)

$$\mu = \ln(\bar{y}/y_0)/b(\bar{y})$$

Transport rate

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Durán and Herrmann (2006)

$$\mu = \ln(\bar{y}/y_0)/b(\bar{y})$$

$$U(y) = \kappa^{-1} U_* \left[\ln(y/y_0) - \mu \left(1 - V^{-1}\right) b(T)\right] = \kappa^{-1} U_* \ln(y/\tilde{y}_0) \quad \text{for } y \geq T$$

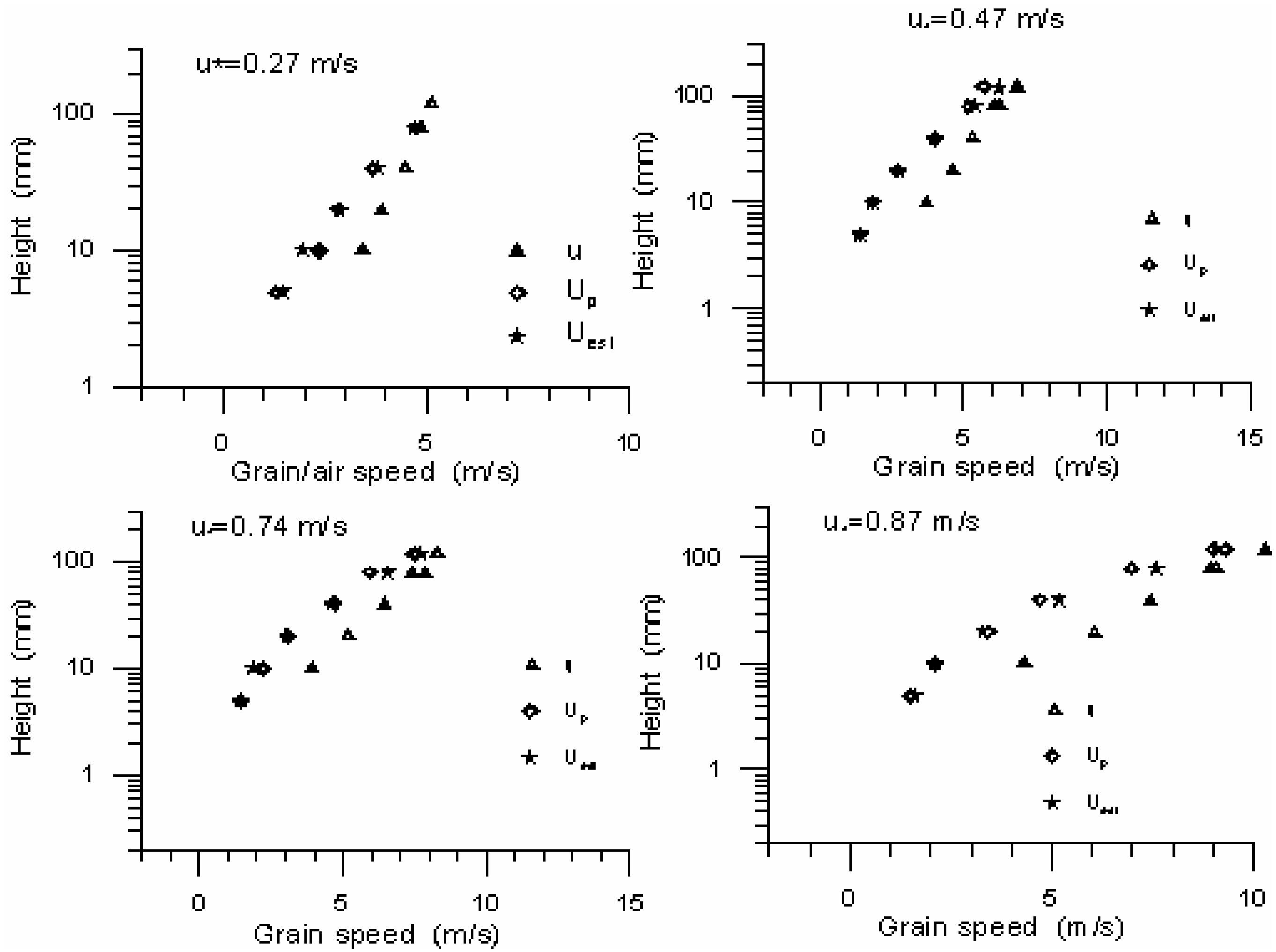
$T =$ top of saltation layer

$$\ln(\tilde{y}_0) = \ln(y_0) + \mu(1 - V^{-1}) b(T)$$

$$b(y) = \int_{y_0}^{y} z^{-1} a(z) dz$$

Grain speeds

Horizontal component of grain speeds measured by laser-Doppler technology, Rasmussen and Sørensen (2005).



The function v

$$b(y) = \int_{y_0}^{y} z^{-1} a(z) dz \qquad a(y) = v(y)/v(0)$$

$$v(y) = \overline{[\dot{x}(t_y^2) - \dot{x}(t_y^1)] 1_{\{y^{\text{top}} > y\}}}$$

By approximations in Rasmussen and Sørensen (2005):

$$\begin{aligned} v(y) \quad &= \quad 2\nu\lambda^{-1}\tilde{y}\left[\frac{e^{-a_y}}{a_y^2+1}K_2\left(2\sqrt{\tilde{y}(a_y^2+1)}\right) - K_2\left(2\sqrt{\tilde{y}}\right)\right] \\ &\quad + 2\sqrt{\tilde{y}}\left(\nu u_2(y) - U\right)\left[\frac{1}{\sqrt{a_y^2+1}}K_1\left(2\sqrt{\tilde{y}(a_y^2+1)}\right) - K_1\left(2\sqrt{\tilde{y}}\right)\right] \end{aligned}$$

The function v

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&\quad + 2\sqrt{\tilde{y}}\,(\nu u_2(y) - U)\left[\frac{1}{\sqrt{a_y^2+1}}K_1\left(2\sqrt{\tilde{y}(a_y^2+1)}\right) - K_1\left(2\sqrt{\tilde{y}}\right)\right]
\end{aligned}
$$

$$U(y) = U$$

$$u_{20} \sim \text{Exponential}(\lambda) \qquad E(u_{10}\,|\,u_{20}) = \nu u_{20}$$

$$y = t_*[u_2(y) - v_f \ln(1 + u_2(y)/v_f)] \qquad v_f = g t_*$$

$$a_y = \frac{1}{\lambda(v_f + u_2(y))} \qquad \tilde{y} = y/(a_y t_* v_f)$$

Other transport rate formulae

$$\frac{Qg}{\rho U_*^3} = \left(1 - V^{-1}\right) f(V)$$

Forms of the function $f(V)$ proposed by other authors

Owen (1964)	$(1 + V^{-1})(\alpha + \beta V^{-1})$
Kind (1976)	$C(1 + V^{-1})$
Owen (1980)	$C(1 + V^{-1})$
Ungar and Haff (1987)	$\beta V^{-1}(1 + V^{-1})$
Sørensen (1991)	$\alpha + \beta V^{-1}$
Sauermann et al. (2001)	$(1 + V^{-1})(\alpha\sqrt{1 + CV^{-2}} + \gamma V^{-1})$
Sørensen (2004)	$(1 + V^{-1})(\alpha + \beta V^{-2} + \gamma V^{-1})$
Durán and Herrmann (2006) and present	$\mu(2 - \mu + \mu V^{-1}) \left[\alpha + \beta V^{-1}\right]$

Mid-air collisions

Sørensen (1988, 1991): Soft-bed hypothesis

Sørensen and McEwan (1996)

Probability that a saltator hits a reptator below height 1 cm

Grains are spheres with radius r

Saltator moves along a straight line with speed u and angle φ

Time of mid-air collision: $T \in [0, 1/u_2]$

$c(y)$ concentration (by number) of grains at height y

$$P(T \in [t, t+dt] \,|\, T > t) = c(t - u_2 t) 4\pi r^2 u dt$$

Probability of mid-air collision

Probability density function of T:

$$f_T(t) = au_2 c(1-u_2 t)\exp\left(-au_2\int_0^t c(1-u_2 s)ds\right), \qquad 0 \le t \le 1/u_2$$

Probability of mid-air collision:

$$1 - P(T = u_2^{-1}) = 1 - \exp\left(-a\int_0^1 c(s)ds\right)$$

Probability density of hight at which collision occurs ($Y = 1 - u_2 T$):

$$f_Y(y) = ac(y)\exp\left(-a\int_y^1 c(s)ds\right) \qquad 0 \le y \le 1$$

Concentration profile

$$c(y) = m^{-1}\Phi\sqrt{\left(\frac{1}{\dot{y}(t_y^1)} - \frac{1}{\dot{y}(t_y^1)}\right)}$$

Fitted an exponential curve to concentration found by McEwan and Willetts (1991) for spherical grains of radius 150 μm

$$c(y) = c_0 \exp(-\nu u)$$

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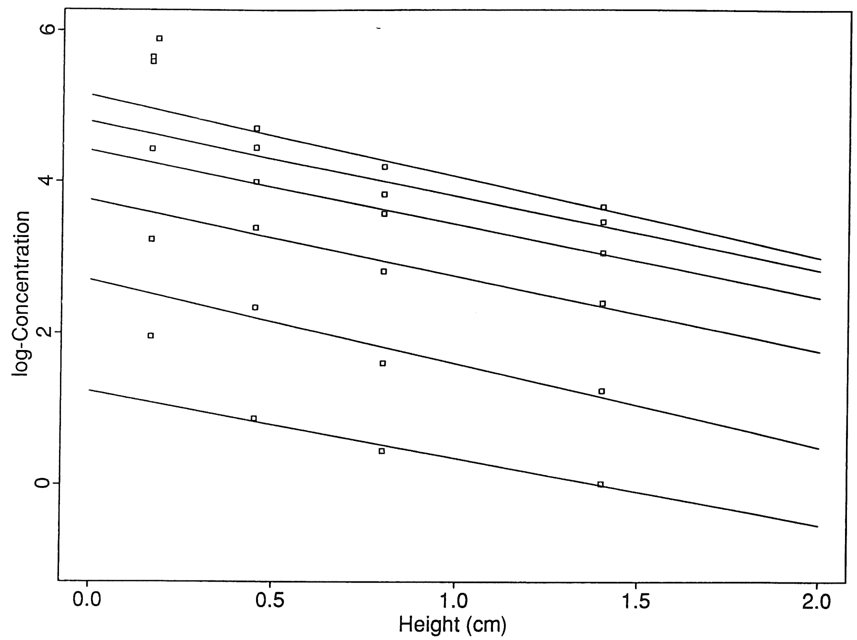

Fig. 1. Concentration profiles (grains cm^{-3}) calculated by the McEwan–Willetts model (□) and the fitted curves of the type given by Eq. (5) (full drawn lines). The grain concentration increases with shear velocity (U_*=28, 35, 49, 66, 79, 90 cm s^{-1}).

Probability of mid-air collisions

Probability of mid-air collisions (p)

U_*	angle	c_0	ν	p
28	12.8	3.4	0.88	0.03
35	11.7	15.0	1.11	0.12
49	10.4	42.9	1.00	0.35
66	9.3	82.3	0.97	0.60
79	8.7	120.3	0.98	0.76
90	8.5	170.7	1.07	0.87

Height of collision

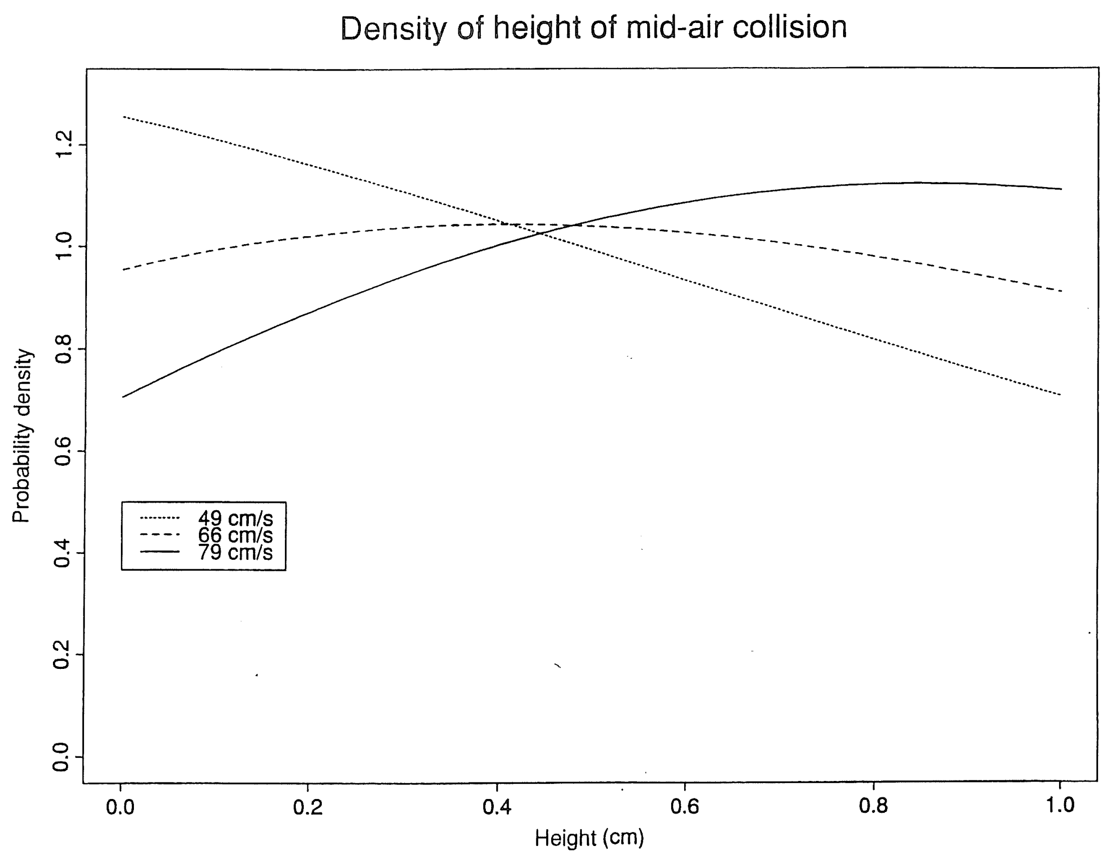

Fig. 2. The probability density function $f_Y{}^*$ of the height (cm) at which a mid-air collision happens as given by Eq. (9) with parameter values from Table 1.

Therefore, it follows from Eq. (11) that the simultaneous cumulative distribution function of v and w is given by

descending saltator, we cannot assume that the grains move along straight lines after the collision. Instead we use the analytical solution

Velocities after the collision

Coefficient of restitution for quartz grains of sand size: 0.7

Anderson and Haff (1991), experiments at University of Aberdeen

Velocity of descending saltator after collision:

$$\vec{v}(u,\alpha,\beta) = u\left(\begin{bmatrix} \cos(\varphi) \\ -\sin(\varphi) \\ 0 \end{bmatrix} - \vec{h}(\alpha,\beta)\right)$$

Velocity of target grain after collision: $u\vec{h}(\alpha,\beta)$

$$\vec{h}(\alpha,\beta) = 0.85\begin{pmatrix} \cos(\varphi)\cos^2(\beta)+\sin(\varphi)\cos(\beta)\sin(\beta)\sin(\alpha) \\ -\sin(\varphi)\cos^2(\beta)+\cos(\varphi)\cos(\beta)\sin(\beta)\sin(\alpha) \\ \cos(\beta)\sin(\beta)\cos(\alpha) \end{pmatrix}, \quad \sin(\beta) = \frac{L}{2r}$$

7-8 % of kinetic energy of saltators lost

Grain trajectories after collision

The impact velocity $\vec{u}_i$ of a grain that at height y_0 has velocity $\vec{v}_0$:

$$u_{i1} = v_{01} + (U - v_{01})(1 - \exp(-t_i/t_*))$$

$$u_{i2} = v_{02} - t_i g + y_0/t_*$$

$$u_{i3} = v_{03} \exp(-t_i/t_*)$$

$$\exp(-t_i/t_*) = 1 - \frac{v_f t_i - y_0}{t_*(v_{02} + v_f)}$$

Previous model with $U(y) = U$

D we put $\mathbf{v}_0 = \mathbf{w}(u, \alpha, \beta)$ given by Eq. (12). In the case of no mid-air collision the mean horizontal displacement of the saltator after a potential collision is given by

$$\int_0^\infty \int_0^1 D(u,y) f_U(u) f_Y^*(y) \mathrm{d}y \, \mathrm{d}u,$$

where D is given by Eq. (24) with $\mathbf{v}_0 = (u \cos \varphi, u \sin \varphi)$. As earlier, φ denotes the impact angle of a saltator before a collision, cf. the section on the probability of mid-air collisions. For the target grain the mean horizontal displacement after a potential collision is

$$\int_0^\infty D(y) f_Y^*(y) \mathrm{d}y,$$

where D is given by Eq (24) with $\mathbf{v}_0 = 0$.

The probability density of the impact speed of the saltator and of the target grain in the collision situation as well as in the no-collision situation is plotted in Fig. 5 for shear velocities of 28, 49 and 79 cm s^{-1}. We see that for the saltator the probability mass has been moved towards smaller values of the impact speed by the collision. On the other hand, the target grain typically hits the surface with much higher speeds in the collision situation than in the case of no collision.

The probability densities of the impact speed for the saltator and for the target grain are remarkably similar in the collision case. Moreover, the impact speeds for both grains are in the collision case considerably larger than the impact speeds for the target grain when there is no mid-air collision. An explanation of both findings is indicated by Table 2 where mean values of the horizontal displacements of the grains after a collision (in the no-collision case after a potential collision) are given.

We see that on average both grains essentially make an extra saltation jump after the collision. Probably what happens is that in most cases one of the grains makes a long extra jump while the other hits the bed immediately after the mid-air collision. The net effect, however, is that in the collision case the grains on average jump further and extract more momentum from the wind without hitting the bed, i.e. without splashing up more grains.

Because of this effect, it is not straightforward to compare the outcome of the grain–bed collision after a mid-air collision to the grain–bed collision when there is no mid-air collision. In the mid-air

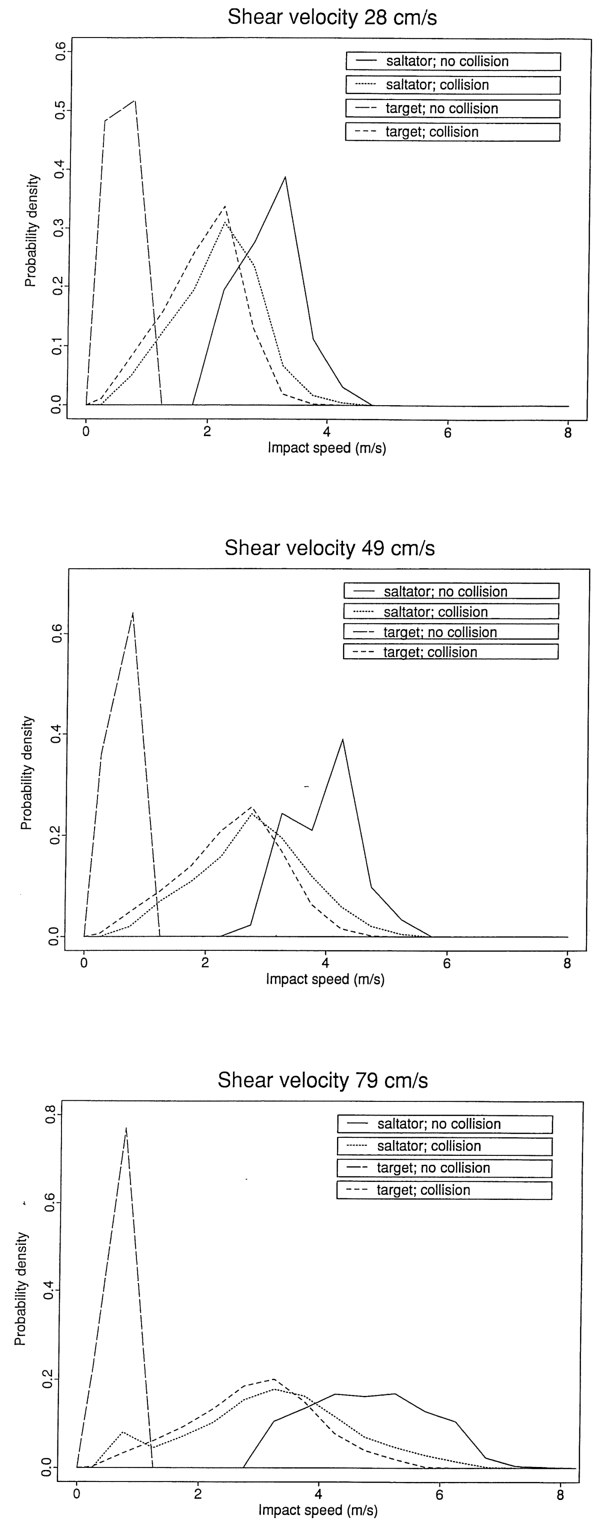

Fig. 5. The probability density functions of the grain–

Mean horizontal displacement

Mean horizontal displacement (cm) after collision (or potential collision)

U_* (cm/s)	No Collision		Collision	
	Saltator	Target	Saltator	Target
28	1.80	0.7	7.47	7.38
35	1.93	0.75	9.56	9.38
49	2.35	0.87	13.20	12.76
66	2.87	0.99	17.73	16.86
79	3.31	1.07	20.67	19.47
90	3.60	0.97	20.79	19.32

Mean number of grains ejected

Mean number of grains ejected when the grains hit the sand surface

Anderson and Haff (1991): $n = 1.75 v_i$

U_* (cm/s)	No Collision		Collision
	Saltator	Both grains	Both grains
28	5.2	6.1	7.1
35	5.8	6.7	8.0
49	6.8	7.8	9.1
66	7.8	8.9	10.4
79	8.5	9.6	11.0
90	8.6	9.6	11.0

Momentum extracted from wind

Mean momentum extracted from the wind by the saltator and the target grain after collision (or potential collision)

U_* (cm/s)	No Collision	Collision
28	42.5	50.5
35	45.7	56.8
49	50.7	61.9
66	54.0	60.4
79	55.4	56.0
90	47.5	38.3