Lecture 4: Theories with Two Supersymmetries

To date we have discussed many generalities about field theories and supersymmetry, but we have only seen one supersymmetric theory in detail—the superparticle. In this lecture we study in detail the classical supersymmetric nonlinear σ-model with two supersymmetries, and at the end of the lecture we briefly discuss supersymmetric gauge theory with two supersymmetries. These models have minimal supersymmetry in $n = 3$ dimensions (see Table 5) and can be dimensionally reduced to $n = 2$ and $n = 1$ dimensions.

At the end of Lecture 3 we introduced the free field theories—both classical and quantum—for scalar fields, spinor fields, and connections. Here we begin with a general discussion of nonfree bosonic classical theories: nonlinear σ-models and nonabelian gauge theories. We introduce some terminology and important basic concepts from physics. With this groundwork we turn to a brief discussion of the superspacetime $M^{3|2}$ on which these models are defined. Since we covered the general theory of superspacetime in Lecture 3, we need only review the specific coordinates and conventions for $M^{3|2}$.

We define the supersymmetric σ-model with target a general Riemannian manifold X directly in superspacetime. Physicists call this the 'superfield formulation' or 'superspace formulation', since the fields in this approach—called *superfields*—are functions on superspacetime. The superfield formulation has the advantage that the supersymmetry is manifest and closes off-shell. However, it is not as immediate to read off the basic physics in these terms, and so we derive the *component field formulation* of the theory. Here the fields are defined on ordinary Minkowski spacetime and the lagrangian can be analyzed according to our general discussion. We also introduce a potential energy term in terms of a function $h\colon X \to \mathbb{R}$ and analyze that theory as well. Upon reduction to $n = 2$ dimensions we meet a new phenomenon—solitons—here derived using the BPS condition. Finally, we briefly discuss the reduction to mechanics, which is a superparticle with two supersymmetries.

In the superfield formulation of pure supersymmetric gauge theory in $M^{3|2}$ we meet *constrained* superfields for the first time. Constraints play a crucial role in theories with more supersymmetry and in supergravity.

σ-models and gauge theories

Throughout we let $\check{M}^n$ denote n dimensional Minkowski spacetime. We fix standard coordinates $x^0 = ct,\ x^1,\ \dots,\ x^{n-1}$ and the metric

$$(dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{4.1}$$

Here c is the speed of light and t is time. In the sequel we work in natural units where $c = 1$, and so then $x^0 = t$. Let

$$|d^n x| = |dx^0 dx^1 \dots dx^{n-1}|$$

be the standard density on $\check{M}^n$.

Fix a Riemannian manifold X. Then the field in a σ-model is a map

$$\phi\colon \check{M}^n \longrightarrow X.$$

The other piece of data is a *potential energy* function

$$V\colon X \longrightarrow \mathbb{R}.$$

The fundamental lagrangian for the model is

$$L = \{\frac{1}{2}|d\phi|^2 - V(\phi)\}\;|d^nx|. \tag{4.2}$$

The classical solutions obey the higher dimensional version of Newton's law:

$$\Box\phi = -\operatorname{grad} V(\phi). \tag{4.3}$$

Unless X is an affine space and V a function at most quadratic, then the space of solutions is a *nonlinear* infinite dimensional symplectic manifold.

Of special interest are solutions of minimal *energy*, which comprise the *moduli space of vacuua* $\mathcal{M}_{\text{vac}}$. This is sometimes termed the *classical moduli space*, though one has to be careful to distinguish from the moduli space of *all* solutions to (4.3). The energy is computed by integrating minus the Noether current for infinitesimal time translation over a spacelike codimension one submanifold of $\check{M}^n$. For simplicity we take that manifold to be $\{x^0 = \text{const}\}$; then the energy density is

$$\mathcal{E} = \left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 + \sum_{i=1}^{n-1}\frac{1}{2}\left(\frac{\partial\phi}{\partial x^i}\right) + V(\phi)\right\}\;|d^{n-1}x|. \tag{4.4}$$

On the space of classical solutions the energy E, which is the integral of (4.4) over the manifold $x^0 =$ *const*, is independent of the constant time, since energy is a conserved Noether charge (for minus infinitesimal time translation). The density in (4.4) is *not* Poincaré-invariant; it is just one piece of the Poincaré-invariant energy-momentum tensor. From (4.4) we see that a scalar field has minimal energy if it is static and constant in space, with constant value at a minimum of V. Assume that V is bounded below and shift the minimum value of V to be 0. (Recall that in supersymmetric theories we necessarily have $V \geq 0$.) Therefore,

$$\mathcal{M}_{\text{vac}} = V^{-1}(0).$$

Notice that any vacuum solution is obviously Poincaré-invariant.

We can consider local minima of V as vacuua also, but in that case we need to adjust V by a constant to make V vanish at the vacuum we are considering. Note that we are free to redefine the energy by a constant, but we must do so in a Poincaré-invariant fashion by modifying the whole energy-momentum tensor.

The group H of global symmetries of the σ-model is the group of isometries of X which preserve V. They act as manifest symmetries of the theory. In particular, they preserve the moduli space of vacuua $\mathcal{M}_{\text{vac}}$. Now a given vacuum ϕ_0 may or may not be fixed by H. Physicists say that the group H is *broken down* to the subgroup $\operatorname{Stab}_H \phi_0$. For example, in any vacuum the Poincaré symmetry is *unbroken*.

We set up perturbation theory by expanding around a given vacuum ϕ_0. We simply write ϕ as a Taylor series around ϕ_0. Then the potential V has no linear term, since we are at a critical point. The quadratic term, which is the Hessian of V at ϕ_0, is the mass matrix of the fluctuating scalar field. (Compare with the lagrangian (3.???) for the free scalar field.) The kernel of the Hessian is comprised of massless fields, and if we are at a nondegenerate point of $\mathcal{M}_{\text{vac}}$ it is equal to the tangent space $T_{\phi_0}\mathcal{M}_{\text{vac}}$. The global symmetry group determines a subspace of $T_{\phi_0}\mathcal{M}_{\text{vac}}$; it is the tangent space to the orbit through ϕ_0. Massless scalars which lie in the tangent to the orbit are called *Goldstone bosons*. Here they are classical fields; a theorem in quantum field theory guarantees that they persist at the quantum level.

For gauge theory we fix a compact Lie group G and a bi-invariant scalar product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$. The field in the theory is a connection A on some principal G bundle. We identify connections which are isomorphic and only write gauge-invariant terms in the lagrangian. (The collection of all connections is a category, but the collection of equivalence classes under gauge transformations is a set.) The fundamental lagrangian in a pure gauge theory is

$$L = -\frac{1}{2}|F_A|^2\,|d^nx|, \tag{4.5}$$

where F_A is the curvature of the connection A. Here the bars denote both the norm in the Lie algebra and the norm on differential forms. The factor in front is determined to have the canonical normalization of the kinetic term (1/2 the time derivative squared), as the reader may easily check in coordinates. The classical equation of motion is the *Yang-Mills equation*

$$d_A^*F_A = 0,$$

and for any connection A we have the Bianchi identity

$$d_AF_A = 0.$$

To write the energy density we split the curvature into the *electric field* 1-form E_A and the magnetic field 2-form B_A:

$$F_A = dx^0 \wedge E_A + B_A.$$

Here the electric and magnetic fields do not involve dx^0. Then the energy density is a generalization of a standard formula from classical electrodynamics:

$$\mathcal{E} = \left\{ \frac{1}{2}|E_A|^2 + \frac{1}{2}|B_A|^2 \right\} |d^{n-1}x|. \tag{4.6}$$

We remark that the energy-momentum tensor is *not* simply computed as minus the canonical Noether current of translations, as it is for a scalar field; there is a (Belanfante) correction term. A complete discussion would take us too far afield. In any case from (4.6) we see that there is a single equivalence class of minimal energy solutions—the trivial connections. In this case the moduli space of vacuua is trivial:

$$\mathcal{M}_{\text{vac}} = \text{pt}\,. \tag{4.7}$$

We expand about the minimum, which simply amounts to replacing the connection A, which lives in an affine space, with a 1-form, which lives in the associated vector space of translations. So the lagrangian is essentially unchanged and there is are $\dim G$ massless 1-form fields.

Notice again that the vacuum solution is Poincaré-invariant. In fact, the equivalence class of trivial connections is the unique field configuration which is Poincaré-invariant. Though we won't discuss it here, the only minimal energy solution for a spinor field ψ with the usual kinetic term is $\psi \equiv 0$; it is also Poincaré-invariant.

Finally, we consider the most general bosonic field theory with scalar fields and gauge fields. It is defined by the following data:

$$\begin{array}{ll} G & \text{compact Lie group} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant scalar product on } G \\ X & \text{Riemannian manifold on which } G \text{ acts by isometries} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential function invariant under } G \end{array} \tag{4.8}$$

There are two fields in the model:

$$\begin{array}{ll} A & \text{connection on some principal } G \text{ bundle } P \longrightarrow \check{M}^n \\ \phi & \text{section of the associated bundle } P \times_G X \longrightarrow \check{M}^n \end{array} \tag{4.9}$$

Of course, a given example might be written in terms of many fields, but we can always consolidate them into the concise form written here. Then the fundamental lagrangian is

$$L = \left\{ -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - V(\phi) \right\} |d^n x|. \tag{4.10}$$

Again we assume that the minimum of the potential occurs at 0. Then the moduli space of vacuua is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{4.11}$$

The quotient by G comes from the gauge symmetry: a vacuum solution consists of a trivial connection A and a (covariant) constant scalar field ϕ. The global gauge transformations fix A and act on the constant scalar ϕ to produce (4.11). (We identify field configurations which differ by a gauge transformation.)

The global symmetry group H of the model is the group of isometries of X which commute with G and preserve V.

As before, we can do perturbation theory about some vacuum $[\phi_0] \in \mathcal{M}_{\text{vac}}$. We use the brackets to indicate the equivalence class of $\phi_0 \in V^{-1}(0)$ under the G action. Then we have

$T_{[\phi_0]}\mathcal{M}_{\text{vac}}$	massless scalar fields (if $[\phi_0]$ nondegenerate)
$\operatorname{Stab}_G \phi_0 \subset G$	massless gauge fields
$\operatorname{Stab}_H[\phi_0] \subset H$	unbroken global symmetry group

The massless scalar fields at a singular point lie in the kernel of the Hessian of V on the quotient of $T_{\phi_0}X$ by the G action. Physicists sometimes say that the gauge group G is *broken* to the stabilizer subgroup $\operatorname{Stab}_G \phi_0$. That reduction is the *Higgs mechanism*. To see how it works let A_0 be a trivial connection, and write $A = A_0 + \alpha$, where A is a 1-form with values in $\mathfrak{g} = \operatorname{Lie}(G)$. Let $\{\zeta_i\}_{i=1}^k$ be a basis of $\mathfrak{g}$, with $\{\zeta_i\}_{i=1}^\ell$ a basis of $\operatorname{Stab}_G \phi_0$. Write $\alpha = \alpha^i\zeta_i$. Now ζ_i induces a vector field $\hat{\zeta}_i$ on X. Then the quadratic terms in α in the expansion of (4.10) about $\phi = \phi_0,\ A = A_0$ are

$$-\frac{1}{2}|d\alpha|^2 + \sum_{i=1}^k \frac{1}{2}|\hat{\zeta}_i(\phi_0)|^2|\alpha^i|^2 = \sum_{i=1}^\ell -\frac{1}{2}|d\alpha^i|^2 + \sum_{j=\ell+1}^k -\frac{1}{2}|d\alpha^j|^2 + \frac{1}{2}|\hat{\zeta}_j(\phi_0)|^2|\alpha^j|^2. \tag{4.12}$$

Thus α^i $(i = 1,\dots,\ell)$ are massless 1-forms and α^j $(j = \ell+1,\dots,k)$ are massive 1-forms with masses $|\hat{\zeta}_j(\phi_0)|$.

This classical picture is usually corrected in the quantum theory. In a supersymmetric theory the corrections are constrained by the supersymmetry. An exciting development over the past few years is the realization that in many cases the constraints are stringent enough to determine the exact quantum picture of the moduli space, massless spectrum, and symmetry structure.

The superspacetime $M^{3|2}$

Recall from Lecture 3 that $M^{3|2}$ is defined from a 3 dimensional real vector space V with Lorentz metric and a 2 dimensional real vector space S^* which is a minimal real representation of the spin group $\mathrm{Spin}(1,2) \cong SL(2,\mathbb{R})$. We fix isomorphisms $V \cong \mathrm{Sym}^2 S^*$ and

$$\epsilon\colon \textstyle\bigwedge^2 S^* \to \mathbb{R}. \tag{4.13}$$

As coordinates on the superspace we fix

$$\begin{aligned} &\theta^1, \theta^2 &&\text{on } \Pi S^*,\\ &y^{11}, y^{12}, y^{22} &&\text{on } V. \end{aligned}$$

The indices on the y are symmetric and the underlying linear basis of V is determined from the underlying linear basis of S^* using the isomorphism $V \cong \mathrm{Sym}^2 S^*$. The y coordinates are related to the standard x coordinate system (4.1) by the formulas

$$\begin{aligned} y^{11} &= \frac{x^0 + x^1}{2},\\ y^{22} &= \frac{x^0 - x^1}{2},\\ y^{12} &= \frac{x^2}{2}. \end{aligned} \tag{4.14}$$

Instead of computing in coordinates, we use the global framing by the bi-invariant vector fields

$$\partial_{\alpha\beta} = \frac{\partial}{\partial y^{\alpha\beta}}$$

and the left invariant vector fields

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \theta^\beta \partial_{\alpha\beta}.$$

These satisfy the bracket relation

$$[D_\alpha, D_\beta] = -2\partial_{\alpha\beta}.$$

Recall that we also use the right invariant vector fields

$$\tau_{Q_\alpha} = \frac{\partial}{\partial \theta^\alpha} + \theta^\beta \partial_{\alpha\beta}.$$

They commute with the D_α.

Our choice of coordinates defines an inclusion

$$i\colon \check{M}^3 \hookrightarrow M^{3|2}.$$

The pullback i^* simply sets $\theta^1 = \theta^2 = 0$.

Finally, we choose the basis of S^* to put the bilinear form (4.13) and its inverse into standard form:

$$\epsilon^{12} = \epsilon_{12} = 1. \tag{4.15}$$

Since ϵ is a *skew* form, there is a sign ambiguity in raising and lowering a single index. Rather than fix some convention, which we might easily forget, we agree to raise and lower only two indices at a time. This is unambiguous if we agree to raise each of the indices using the same sign convention. For example, the bilinear form ϵ canonically determines a bilinear form on the dual space, as indicated in (4.15). We warn the reader that physicists often use a different convention in which the form used on the dual space is not the naturally induced one. *Oy!*

Supersymmetric σ-model in 3 dimensions

Fix a Riemannian manifold X. We first construct the field theory directly on the superspacetime $M^{3|2}$ where the supersymmetry is manifest. The field Φ is a scalar field on superspacetime with values in X:

$$\Phi\colon M^{3|2} \longrightarrow X.$$

Fields on superspacetime are usually called *superfields*. The (left) action of the supersymmetry group $P^{3|2}$ on $M^{3|2}$ induces an action on superfields by pullback. Recall that the infinitesimal action is by *right* invariant vector fields.

Just as we did for the superparticle (see (2.31)) we define *component fields* on Minkowski space $\check{M}^3$ by restricting normal derivatives of the superfield to $\check{M}^3 \subset M^{3|2}$. Since the θ^α are nilpotent, a small number of normal derivatives determines the superfield completely. In other words, the Taylor expansion normal to $\check{M}^3$ is a finite power series in θ^α, and the component fields are essentially the coefficients in that expansion. However, rather than make such an expansion in some coordinate system, we use the global left-invariant framing to define global component fields. That said, we define

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_\alpha &= i^* D_\alpha \Phi \\ F &= i^*\big(-\frac{1}{2}\epsilon^{\alpha\beta} D_\alpha D_\beta\big)\Phi \end{aligned} \tag{4.16}$$

A few remarks are in order:

- The leading bosonic component ϕ is a scalar field on $\check{M}^3$ with values in X, namely

$$\phi\colon \check{M}^3 \to X.$$

  The field ψ_α is odd, since the vector field D_α is odd. It is an odd section of the pullback tangent bundle. Together the two fields ψ_1, ψ_2 transform as a spinor on Minkowski space, so we can combine them into

$$\psi \in C^\infty\big(\check{M}^3, \phi^*\Pi(TX \otimes S)\big).$$

  The field F is even and is also a section of the pullback tangent bundle:

$$F \in C^\infty(\check{M}^3, \phi^*TX).$$

- The outer (leftmost) derivative in the definition of F is a covariant derivative using the Levi-Civita connection, and in subsequent computations we will meet multiple covariant derivatives of Φ. We will use the fact that Levi-Civita is torsionfree when commuting the first two derivatives acting on Φ (see (2.3)); when commuting further derivatives we pick up curvature terms.
- In case $X = \mathbb{R}$ then the fields ϕ, ψ comprise precisely the field content of a massless or massive scalar multiplet in $n = 3$ dimensions. (Recall the discussion of representations of $P^{3|2}$ in Lecture 3.) That is, if we write the free massless or massive lagrangian for these fields, then upon quantization the one particle Hilbert space is that irreducible supersymmetry multiplet. So we are led to inquire: What is F doing here? We will see that F is an *auxiliary field* in the sense that it does not contribute physical degrees of freedom in the quantization. In fact, F enters the lagrangian only algebraically—no derivatives—so the equations of motion determine F algebraically in terms of the other fields. We will eliminate F using this algebraic equation.
- Still, the presence of F allows us to write down supersymmetry transformation laws (see (4.18) below) for the component fields which close *off-shell*. If we substitute for F in the transformation laws, then this is no longer true.
- The component fields $\{\phi, \psi, F\}$ are often referred to as a *multiplet*, in this case a *scalar multiplet*.
- We can do the classical field theory in superspacetime with the superfield Φ or in ordinary spacetime with the component fields ϕ, ψ, F. In the first case we do calculus on $\mathcal{F}_\Phi \times M^{3|2}$; in the second calculus on $\mathcal{F}_{\{\phi,\psi,F\}} \times \check{M}^3$. The supersymmetry is *manifest* in the superfield formulation but not in the component formulation.

Next, we write down the supersymmetry transformation law for the component fields. Recall that in superspacetime the infinitesimal supersymmetry is the induced action on Φ of $-\tau_{Q_\alpha}$, where we use a minus sign to obtain a left action as explained at the end of Lecture 2. Introduce odd parameters η^α and consider the action of the even vector field $-\eta^\alpha\tau_{Q_\alpha}$. This induces a vector field $\hat{\zeta}$ on the space of component fields $\mathcal{F}_{\{\alpha,\psi,F\}}$. Quite generally, a component field f (see (4.16)) is defined by a formula

$$f = i^* D^r \Phi,$$

where 'D^r' denotes a sum of products of D_α. Then a short argument shows that under an infinitesimal supersymmetry transformation f transforms as

$$\hat{\zeta} f = -\eta^\alpha i^* D_\alpha D^r \Phi. \tag{4.17}$$

The right hand side can be rewritten in terms of component fields. This formula works in any model with a superfield formulation, not just in the $3|2$ supersymmetric σ-model. In the case at hand we find

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^\alpha \psi_\alpha \\ \hat{\zeta}\psi_\alpha &= \quad \eta^\beta(\partial_{\alpha\beta}\phi - \epsilon_{\alpha\beta}F) \\ \hat{\zeta}F &= \quad \eta^\alpha\big[(\not{D}\psi)_\alpha + \frac{1}{3}\epsilon^{\beta\gamma}R(\psi_\alpha,\psi_\beta)\psi_\gamma\big], \end{aligned} \tag{4.18}$$

where R is the Riemann curvature tensor of X and the Dirac operator is

$$(\not{D}\psi)_\alpha = -\epsilon^{\beta\gamma}\partial_{\alpha\beta}\psi_\gamma.$$

This computation shows off the advantages of the superspacetime formulation—the supersymmetry transformation laws of the component fields are determined *a priori*.

The lagrangian density for the supersymmetric σ-model is

$$\mathcal{L} = \frac{1}{4}\epsilon^{\alpha\beta}\langle D_\alpha\Phi, D_\beta\Phi\rangle\,|d^3x|\,d^2\theta. \tag{4.19}$$

We have not given a detailed discussion of densities and integration on supermanifolds, and as before we finesse that point. Suffice it to say that $|d^3x|\,d^2\theta$ is a $P^{3|2}$-invariant density on $M^{3|2}$. The lagrangian function ℓ which appears in front of it in the lagrangian density (4.19) is obviously $P^{3|2}$-invariant, since $P^{3|2}$ acts on the left and the D_α are left-invariant. So $\mathcal{L}$ is manifestly supersymmetric—invariant under $P^{3|2}$.

We need to "integrate out the θs" to define a component lagrangian L. Quite generally, if we have a superspacetime model on $M^{n|s}$ the superspacetime lagrangian density is

$$\mathcal{L} = \ell \; |d^n x| \; d^s\theta.$$

There is a standard notion of (Berezin) integration to integrate out the odd variables. Let

$$\pi\colon M^{n|s} \longrightarrow \check{M}^n$$

be the projection given by our construction of superspacetime (see (3.???)) and π_* the corresponding integration. Rather than integrate, our approach is to find in each case a certain sum of products of D_α, which we denote 'D^s', such that

$$\pi_*\mathcal{L} = (i^* D^s \ell + \Delta i^* \ell) \; |d^n x|$$

for some Poincaré invariant differential operator Δ on $\check{M}^n$. Then instead of using $\pi_*\mathcal{L}$ as the component lagrangian, we define the component lagrangian to be

$$L = (i^* D^s \ell) \; |d^n x|. \tag{4.20}$$

A few brief remarks:

- In general this differs from the straight integration of the odd variables, as indicated by the presence of Δ. The particular D^s that we use is chosen so that the component lagrangian density L involves only first derivatives of the fields.
- In this lecture we have $\Delta = 0$. Examples with $\Delta \neq 0$ occur in superspacetimes with $s = 4$ odd directions, as we will see in Lecture 5.
- Formula (4.20) is effective in that we find strings of D_α acting on superfields which we then express in terms of component fields using the commutation relations.
- The component lagrangian is supersymmetric, but the supersymmetry is not manifest. There is an explicit formula to determine the exact term by which it changes under a supersymmetry transformation. Recall from (2.20) that this term appears in the Noether current for the supersymmetry transformation, often called the *supercurrent*.

For $M^{3|2}$ we define the component lagrangian to be

$$L = i^*\big(-\frac{1}{2}\epsilon^{\alpha\beta} D_\alpha D_\beta\big)\ell \; |d^3 x|. \tag{4.21}$$

As remarked above, this agrees with the definition using the Berezinian integral $\int d^2\theta$. For the σ-model lagrangian (4.19) we have

$$L = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}i^* D_\alpha D_\beta \langle D_\gamma \Phi, D_\delta \Phi\rangle. \tag{4.22}$$

To illustrate the manipulations, consider a typical term in (4.22) without the numerical factor:

$$i^*\langle D_1 D_2 D_1 \Phi, D_2\Phi\rangle.$$

We emphasize that the outer two derivatives are *covariant* derivatives. We when we commute them we pick up a curvature term:

$$\langle D_1 D_2 D_1\Phi, D_2\Phi\rangle = -\langle D_2 D_1 D_1 \Phi, D_2\Phi\rangle - 2\langle \partial_{12} D_1\Phi, D_2\Phi\rangle + \langle R(D_1\Phi, D_2\Phi)D_1\Phi, D_2\Phi\rangle.$$

Since $D_1^2 = -\partial_{11}$ and $[\partial_{11}, D_2] = 0$, we have

$$D_2 D_1 D_1 \Phi = -D_2\partial_{11}\Phi = -\partial_{11}D_2\Phi.$$

Restricting to $\check{M}^3$ we obtain

$$i^*\langle D_1 D_2 D_1\Phi, D_2\Phi\rangle = \langle \partial_{11}\psi_2, \psi_2\rangle - 2\langle\partial_{12}\psi_1, \psi_2\rangle + \langle R(\psi_1,\psi_2)\psi_1,\psi_2\rangle.$$

Of course, there are systematic algebraic shortcuts one can develop for these manipulations. In any case, at the end of the day one obtains the component lagrangian for the supersymmetric σ-model:

$$L = \left\{\frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}\psi\rangle + \frac{1}{12}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\langle\psi_\alpha, R(\psi_\beta,\psi_\gamma)\psi_\delta\rangle + \frac{1}{2}|F|^2\right\} |d^3x|. \tag{4.23}$$

Some remarks:

- As promised, F appears only algebraically in the lagrangian. Its equation of motion is simply
$$F = 0.$$
- The *bosonic* lagrangian is obtained by setting the fermion ψ to zero:
$$L_{\text{bos}} = \frac{1}{2}|d\phi|^2\ |d^3x|$$
after eliminating the auxiliary field F. This is the nonlinear σ-model lagrangian (4.2) with zero potential. So we have indeed obtained a supersymmetric extension of the nonlinear σ-model with two supersymmetries in three dimensions.
- There is not much to say about this model: the moduli space of vacuua is X and the global symmetry group is the group of isometries of X.

We need the formula for the *supercurrent* j, which is *minus* the Noether current corresponding to the supersymmetry $\eta^\alpha \tau_{Q_\alpha}$. Here we write the full $|-1|$-form on $\check{M}^3$:

$$j = \Big\{ \eta^\alpha \epsilon^{\beta\gamma} \epsilon^{\delta\epsilon} \langle \partial_{\alpha\delta}\phi, \psi_\epsilon \rangle - \eta^\beta \epsilon^{\alpha\gamma} \langle F, \psi_\alpha \rangle \Big\} \, \iota(\partial_{\beta\gamma}) \, |d^3x|.$$

We are only interested in this on-shell, in which case we could set $F = 0$, but we need the formula later in cases where $F \neq 0$. The *supercharges* are obtained by integrating j over a fixed time slice:

$$\begin{aligned} Q_1 &= \int_{x^0=\mathrm{const}} \Big\{ \langle \partial_{11}\phi, \psi_1 \rangle + \langle \partial_{12}\phi, \psi_2 \rangle \Big\} \, |dx^1 dx^2| \\ Q_2 &= \int_{x^0=\mathrm{const}} \Big\{ \langle \partial_{22}\phi, \psi_2 \rangle + \langle \partial_{12}\phi, \psi_1 \rangle \Big\} \, |dx^1 dx^2| \end{aligned}$$

In these last formulas we did set $F = 0$.

A supersymmetric potential

The theory is more fun when we add a potential term, which of course we must do in a supersymmetric way. This is easy in superspacetime. Let

$$h \colon X \longrightarrow \mathbb{R}$$

be a real-valued function on X. It is *not* the potential function V of the model; we compute V in terms of h below. Simply add the pullback of h by the superfield Φ to the superspacetime lagrangian (4.19):

$$\mathcal{L}' = \Big\{ \frac{1}{4} \epsilon^{\alpha\beta} \langle D_\alpha \Phi, D_\beta \Phi \rangle + \Phi^*(h) \Big\} \, |d^3x| \, d^2\theta.$$

It is easy to work out the contribution of the new term to the component lagrangian:

$$\Big\{ \langle F, \phi^* \operatorname{grad} h \rangle + \frac{1}{2} \epsilon^{\alpha\beta} \phi^*(\operatorname{Hess} h)(\psi_\alpha, \psi_\beta) \Big\} \, |d^3x|. \tag{4.24}$$

So the new component lagrangian is the sum of (4.23) and (4.24). Now the equation of motion of F is more interesting than before:

$$F = -\phi^* \operatorname{grad} h.$$

So upon eliminating F from the component lagrangian we obtain

$$\begin{aligned} L' = \Big\{ \frac{1}{2} |d\phi|^2 + \frac{1}{2} \langle \psi \not{D} \psi \rangle - \frac{1}{2} \phi^* |\operatorname{grad} h|^2 + \frac{1}{2} \epsilon^{\alpha\beta} \phi^*(\operatorname{Hess} h)(\psi_\alpha, \psi_\beta) \\ + \frac{1}{12} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \langle \psi_\alpha, R(\psi_\beta, \psi_\gamma) \psi_\delta \rangle + \frac{1}{2} |F|^2 \Big\} \, |d^3x|. \end{aligned} \tag{4.25}$$

The presence of h leads to two new terms:

- A potential energy
$$V = \frac{1}{2}|\operatorname{grad} h|^2$$
for the scalar field ϕ. Note that the potential is nonnegative, as it must be in a supersymmetric theory.
- A mass term
$$\frac{1}{2}\epsilon^{\alpha\beta}\phi^*(\operatorname{Hess} h)(\psi_\alpha, \psi_\beta)$$
for the fermions.

The bosonic lagrangian in this case is

$$L_{\mathrm{bos}} = \left\{\frac{1}{2}|d\phi|^2 - \frac{1}{2}\phi^*|\operatorname{grad} h|^2\right\} |d^3x|,$$

a nonlinear σ-model with nonzero potential.

We also record the *on-shell* supersymmetry transformation laws for the physical fields ϕ, ψ from (4.18), using the equation of motion for F:

$$\begin{aligned}\hat{\zeta}\phi &= -\eta^\alpha\psi_\alpha \\ \hat{\zeta}\psi_\alpha &= \eta^\beta(\partial_{\alpha\beta}\phi + \epsilon_{\alpha\beta}\phi^* \operatorname{grad} h).\end{aligned} \tag{4.26}$$

The supercharges are now given by the formulas

$$\begin{aligned}Q_1 &= \int_{x^0=\mathrm{const}} \Big\{\langle\partial_{11}\phi, \psi_1\rangle + \langle\partial_{12}\phi, \psi_2\rangle - \langle\phi^*\operatorname{grad} h, \psi_2\rangle\Big\}\, |dx^1dx^2| \\ Q_2 &= \int_{x^0=\mathrm{const}} \Big\{\langle\partial_{22}\phi, \psi_2\rangle + \langle\partial_{12}\phi, \psi_1\rangle + \langle\phi^*\operatorname{grad} h, \psi_1\rangle\Big\}\, |dx^1dx^2|\end{aligned} \tag{4.27}$$

Now we investigate the classical vacuum states—the minimal energy field configurations. As we mentioned before these occur when $\psi = 0$ and $\phi \equiv \phi_0$ is constant. If ϕ_0 is at a minimum of the potential V—in other words, $V(\phi_0) = 0$—then ϕ_0 is a critical point of h. These field configurations are invariant under supersymmetry. In other words, the vector field $\hat{\zeta}$ vanishes at such points of field space, as is obvious from (4.26). Thus we say that *supersymmetry is unbroken* in these vacuua. It is natural to assume that h is a Morse function, or perhaps a generalized Morse function in the sense of Bott. Then we might have critical manifolds, and at vacuua which lie on such manifolds there are massless scalars and massless fermions. The fact that supersymmetry is unbroken means that these come in supersymmetric pairs, i.e., in multiplets. More generally, the masses of the scalar fields and of the fermions agree at such a vacuum. This is easily verified by computing the masses directly from the lagrangian (4.25). (Recall how masses appear in the free lagrangian (3.???) for the scalar field and (3.???) for the spinor field.) For the scalar field ϕ the squares of the masses are the eigenvalues of the hessian of the potential V, which at a critical point of h are the squares of the eigenvalues of $\operatorname{Hess} h$. On the other hand the mass matrix M for the fermions is $\operatorname{Hess} h$, and so the fermion mass squares are again the squares of the eigenvalues of the hessian.

We can also consider a vacuum which is a local minimum of V, but not a global minimum. (A critical point of V which is not a local minimum is unstable.) For such a local minimum ϕ_0 we need to shift the energy by $-V(\phi_0)$ in order to have zero energy for the vacuum. Since

$$dV = \langle \operatorname{grad} h, \nabla \operatorname{grad} h \rangle,$$

such critical points of V occur when $\operatorname{grad} h$ is nonzero but in the kernel of $\operatorname{Hess} h$. Such a point could be isolated and at such a vacuum we might have no massless scalar fields. But there is a massless fermion since the fermion mass matrix $\operatorname{Hess} h$ has a kernel. In fact, this massless fermion can be predicted from the supersymmetry. Notice that this vacuum is *not* supersymmetric—the vector field $\hat{\zeta}$ in (4.26) does not vanish there. So we say that *supersymmetry is broken* in this vacuum. The Poincaré symmetry is always assumed to be unbroken in a vacuum, but as we see this is not true for the *super*Poincaré symmetry. Now just as broken (even) global symmetries lead to massless bosons—Goldstone bosons—so too do broken odd symmetries lead to massless fermions—*Goldstone fermions*.

Dimensional reduction to $n = 2$ dimensions

We dimensionally reduce the model in components, though we could dimensionally reduce the superspacetime formulation to obtain a superspacetime formulation in $n = 2$ dimensions.

Let's begin by collecting the formulas we need. We restrict to fields f which satisfy $\partial_{12} f = 0$. (Recall from (4.14) that y^{12} is a spatial direction.) It is convenient to use *lightcone coordinates* on $\check{M}^2$, or equivalently to use a basis of vector fields ∂_+, ∂_- which are lightlike. (See (3.13).) Recall that these are related to ∂_0, ∂_1 by

$$\begin{aligned} \partial_+ + \partial_- &= 2\partial_0 \\ \partial_+ - \partial_- &= 2\partial_1. \end{aligned}$$

Also for spinors we use '$+$' for the index '1' and '$-$' for the index '2'. In more invariant language, we decompose spinors under the inclusion $\mathrm{Spin}(1,1) \subset \mathrm{Spin}(1,2)$. Then from (4.26) we find the supersymmetry transformations

$$\begin{aligned} \hat{\zeta}\phi &= -(\eta^+\psi_+ + \eta^-\psi_-) \\ \hat{\zeta}\psi_+ &= \eta^+\partial_+\phi + \eta^-\phi^* \operatorname{grad} h \\ \hat{\zeta}\psi_- &= \eta^-\partial_-\phi - \eta^+\phi^* \operatorname{grad} h \end{aligned} \tag{4.28}$$

and from (4.27) the supercharges

$$\begin{aligned} Q_+ &= \int_{x^0=\text{const}} \Big\{ \langle \partial_+\phi, \psi_+\rangle + \langle \phi^* \operatorname{grad} h, \psi_-\rangle \Big\} \, |dx^1| \\ Q_- &= \int_{x^0=\text{const}} \Big\{ \langle \partial_-\phi, \psi_-\rangle + \langle \phi^* \operatorname{grad} h, \psi_+\rangle \Big\} \, |dx^1|. \end{aligned} \tag{4.29}$$

Field theory in $n = 2$ dimensions is special for many reasons, among them the fact that spatial infinity is disconnected. (We fix a splitting of Minkowski spacetime into time cross space, where "space" is an $n-1$ dimensional Euclidean space.) This has the following consequence. Let $\mathcal{F}\mathcal{E}_{\text{bos}}$ denote the space of bosonic field configurations on space of finite energy. Now the energy density of a field $\phi(x^1)$ on space is, according to (4.4),

$$\begin{aligned} \mathcal{E} &= \Big\{ \frac{1}{2}\, |\partial_1\phi|^2 + \frac{1}{2}\, \phi^* |\operatorname{grad} h|^2 \Big\} \, |dx^1| \\ &= \frac{1}{2}\, |\partial_1\phi \pm \phi^* \operatorname{grad} h|^2 \, |dx^1| \;\mp\; d\phi^*(h). \end{aligned} \tag{4.30}$$

This is the typical argument in certain calculus of variations problems which expresses an energy functional as a square plus a topological term. If a field $\phi(x^1)$ has finite energy then the integral of the first term is finite, in which case $\phi(x^1)$ has limits as $x^1 \to \pm\infty$ which are critical points of h. Then the integral over space of the topological term is plus or minus the difference

$$Z = -\{h(\infty) - h(-\infty)\}$$

of the limiting values. So the space $\mathcal{F}\mathcal{E}_{\text{bos}}$ of finite energy field configurations splits into a disjoint union according to the limiting values at plus or minus infinity. The parameter space for the components is the Cartesian square $\pi_0 \operatorname{Crit}(h)^{\times 2}$ of the set of components of the critical point set. The *central charge* Z is constant on each component.

Note that in $n = 3$ spacetime dimensions spatial infinity is connected and a field with finite energy has a single limit at infinity. So the components of $\mathcal{F}\mathcal{E}_{\text{bos}}$ in that case are labeled by $\pi_0 \operatorname{Crit}(h)$.

A frequent example is $X = \mathbb{R}$ with

$$h(\phi) = \frac{\phi^3}{3} - a^2\phi, \qquad a \in \mathbb{R}.$$

Then we have a quartic potential

$$V(\phi) = \frac{1}{2}(\phi^2 - a^2)^2 \tag{4.31}$$

with nondegenerate zeros at $\phi = \pm a^2$. So in the $n = 2$ dimensional theory $\mathcal{FE}_{\mathrm{bos}}$ has four components according to the limiting values as $x^1 \to \pm\infty$ are $\pm a^2$.

Now in a "diagonal" component of $\mathcal{FE}_{\mathrm{bos}}$, where $h(\infty) = h(-\infty)$, there are field configurations of zero energy—the constant fields with values in the critical point set. (For an isolated critical point there is a unique such field.) These are the classical vacuua. But in components where $h(\infty) \neq h(-\infty)$ there are no vacuua. In any component the energy E, which is the spatial integral of the energy density $\mathcal{E}$ (4.30), is bounded below by the absolute value of the central charge:

$$E \geq |Z|. \tag{4.32}$$

From (4.30) we see that a field configuration of minimal energy satisfies the *first order* equation for a flow line:

$$\partial_1\phi \pm \phi^* \operatorname{grad} h = 0. \tag{4.33}$$

Again, this is the typical derivation in geometry of first order equations for fields of minimal energy.

In general, nonvacuum field configurations of locally minimal energy are called *solitons*. In the example (4.31) they are sometimes called *kinks* or *lumps*. We can take the soliton to be constant in time. Physically it represents a stable localized lump sitting still in space. Upon quantization we might expect that it gives a particle, and because of the inequality (4.32) we can expect it to be massive. In more detail: When we quantize the theory we need to quantize the space $\mathcal{M}$ of all finite energy classical solutions to the field equations. This space divides into components as indicated above. The lowest energy state in a diagonal component is a vacuum of zero energy. The small fluctuations around the vacuum give massless and massive particles according to the Hessian of V, at least in perturbation theory. Nondiagonal components have no state of zero mass. Rather, the states of smallest mass saturate the inequality (4.32) (where we replace energy by mass) and form a symplectic manifold on which the Poincaré group acts. The quantization of this symplectic manifold is then a collection of *quantum solitons*. They are massive, and in perturbation theory their mass is very large compared to the masses of the particles constructed from small fluctuations of the vacuum.

Returning to the Morse theory example, in the simplest case there is a single geometric flow line between distinct critical points. However, spatial translations act freely on the space of parametrized flow lines $\phi(x^1)$, and the space of static solitons is one dimensional. We act on these flow lines by Lorentz boosts to obtain moving solitons; the collection of these is two dimensional and is a homogeneous space for the Poincaré group. We can identify this space with the tangent bundle to the space of parametrized flow lines, and upon quantization we obtain a massive scalar particle.

Our discussion so far has ignored the fermion and the supersymmetry. In fact, the central charge Z appears in the supersymmetry algebra as was anticipated in (3.13), where we discussed a central extension of the supersymmetry algebra $\mathfrak{p}^{2|(1,1)}$. Now we realize the central extension classically by computing the Poisson bracket $\{Q_+, Q_-\}$ of the supersymmetry charges. The Poisson bracket of Noether charges can be computed in different ways (see (2.???)), and is in fact the spatial integral of a Poisson bracket of Noether currents. We use the explicit formulas (4.29) and note that the nontrivial contributions come from the brackets $\{\psi_+, \psi_+\}$ and $\{\psi_-, \psi_-\}$. The result is

$$\begin{aligned}\frac{1}{2}\{Q_+, Q_-\} &= \frac{1}{2}\int_{x^1=\mathrm{const}} \Big\{-\langle \partial_+\phi, \phi^* \operatorname{grad} h\rangle + \langle \partial_-\phi, \phi^* \operatorname{grad} h\rangle\Big\}\ |dx^1| \\ &= \int_{x^1=\mathrm{const}} -\langle \partial_1\phi, \phi^* \operatorname{grad} h\rangle\ |dx^1| \\ &= \int_{x^1=\mathrm{const}} -\partial_1(\phi^* h)\ |dx^1| \\ &= Z.\end{aligned}$$

So as promised we obtain a central extension of the supersymmetry algebra. The energy inequality (4.32) is the classical version of (3.16), and it follows from the supersymmetry algebra as in (3.15). Field configurations which saturate this classical *BPS inequality* are annihilated by the supersymmetry $\tau_{Q_+} + \tau_{Q_-}$ or by $\tau_{Q_+} - \tau_{Q_-}$. This provides an alternative derivation—using supersymmetry—of the first order equation (4.33) for minimal energy bosonic field configurations. Namely, we look at the supersymmetry transformation (4.28) and ask that the vector field $\hat{\zeta}$ vanish. Inspecting the variation of the fermions ψ_+, ψ_- we learn that such *BPS field configurations* satisfy

$$\begin{aligned}\partial_+\phi \pm \phi^* \operatorname{grad} h &= 0 \\ \pm\partial_-\phi - \phi^* \operatorname{grad} h &= 0.\end{aligned} \tag{4.34}$$

These equations are equivalent to the equations

$$\begin{aligned}\partial_0\phi &= 0 \\ \partial_1\phi \mp \phi^* \operatorname{grad} h &= 0,\end{aligned}$$

which are the equations for a static flow line. We also obtain an equation on fermions by requiring that the variation of the boson ϕ vanish in (4.28):

$$\psi_+ \pm \psi_- = 0. \tag{4.35}$$

To analyze this we must also bring in the equations of motion for the fermions, which are

$$\begin{aligned}-\partial_+\psi_- &= R(\psi_+, \psi_-)\psi_+ + \phi^*(\nabla \operatorname{grad} h)(\psi_+) \\ \partial_-\psi_+ &= -R(\psi_-, \psi_+)\psi_- + \phi^*(\nabla \operatorname{grad} h)(\psi_-).\end{aligned} \tag{4.36}$$

Combining (4.35) and (4.36) we see that the curvature terms vanish by the Bianchi identity and we are left with equations for a single fermion ψ_+:

$$\begin{aligned} \partial_0 \psi_+ &= 0 \\ \partial_1 \psi_+ \mp \phi^*(\nabla \operatorname{grad} h)(\psi_+) &= 0. \end{aligned} \tag{4.37}$$

Equations (4.37) are the variations of equations (4.34) in the direction ψ_+. Thus the BPS fermion field ψ_+ is an odd tangent vector to the manifold of flow lines.

Recapping: The BPS condition leads to a first order equation on bosons which implies the second order equation of motion. This demonstrates that classical fermionic fields, which are difficult to visualize in a geometric manner, can impact classical bosonic fields.

Finally, we comment on the quantization including the fermions. In this supersymmetric theory we expect to get an action of the supersymmetry group, and indeed what we must quantize is now a symplectic *super*manifold. The bosonic part as before is the tangent bundle TN to a space of parametrized flow lines N. The fermion field, according to (4.37), is an odd tangent vector, so the supermanifold can be described as

$$\pi^* \Pi TN \longrightarrow TN,$$

where $\pi\colon TN \to N$. This is the same supermanifold which entered into our discussion of the superparticle—see (2.14). So we see that the classical system we must quantize is supersymmetric quantum mechanics on the space N of static solitons, here parametrized flow lines. In models where we start with more supersymmetry, we end up with supersymmetric quantum mechanics with more supersymmetry, from which we learn about the geometry of the space of solitons.

Dimensional reduction to $n = 1$ dimensions

We make some informal remarks about the *quantum* mechanical model.

Classically, we reduce to a mechanical model by requiring the fields in the $n = 3$ dimensional model to be invariant under all spatial translations. We obtain a model with supersymmetry group $P^{1|2}$. It has a superspacetime formulation on $M^{1|2}$, which is the dimensional reduction of the superspacetime formulation in $n = 3$ dimensions. This theory has twice the minimal amount of supersymmetry. In particular, it has twice as much supersymmetry as the superparticle model considered in Lecture 2. There we indicated that the quantum Hilbert space of that model is the space of spinor fields on X. With twice as much supersymmetry we obtain instead the space of differential forms on X as the quantum Hilbert space; the $\mathbb{Z}/2$-grading is by the parity of the degree. In the minimal superparticle there is a single supercharge, and its quantization is the Dirac operator. Here there are two supercharges (4.29) and the corresponding quantum operators are, up to factors, the first order differential operators $d + d^*$ and $\sqrt{-1}(d - d^*)$. This is if we have no potential term. In case there is a potential term, then the second term in the supercharges (4.29) gives an additional term which is a combination of exterior and interior multiplication by dh. In this way we obtain the modification of the de Rham complex used by Witten in his study of Morse theory. The hamiltonian in this model is the Hodge laplacian on differential forms modified by lower order terms involving dh. The solitons we discussed in $n = 2$ dimensions are *instantons* in $n = 1$ dimension, and they enter into the Morse theory discussion.

The dimensional reduction from $n = 3$ dimensions to $n = 1$ dimensions has a global $SO(2)$ symmetry from spatial rotations. From the $n = 3$ dimensional point of view these symmetries are part of the Poincaré group; from the $n = 1$ dimensional point of view they are an example of an R-symmetry. In the quantum mechanical theory this global $SO(2)$ symmetry gives a $\mathbb{Z}$-grading on the quantum Hilbert space; it is simply the usual $\mathbb{Z}$-grading on differential forms.

Supersymmetric gauge theory in $n = 3$ dimensions

Fix a compact Lie group G and a bi-invariant inner product $\langle\cdot,\cdot\rangle$ on its Lie algebra. Just as in the bosonic case, the field in the superspacetime formulation of a minimal supersymmetric pure gauge theory is a G connection. However, in the supersymmetric case the connection $\mathcal{A}$ on superspacetime is *constrained* to be flat along the odd distribution spanned by the left invariant vector fields D_α:

$$\iota(D_\alpha)\iota(D_\beta)F_{\mathcal{A}} = 0 \qquad \text{for all } \alpha,\beta. \tag{4.38}$$

Here $F_{\mathcal{A}}$ is the curvature of $\mathcal{A}$. Recall from Table 5 that minimal supersymmetric gauge theories have a superspacetime formulation on $M^{2|(1,0)}$, $M^{3|2}$, $M^{4|4}$, and $M^{6|(8,0)}$.

We specialize to pure supersymmetric gauge theory on $M^{3|2}$. Rewrite the constraint (4.38) in terms of covariant derivatives $\mathcal{D}_\alpha$ and $\nabla_{\alpha\beta}$ along the vector fields D_α, $\partial_{\alpha\beta}$ (acting on the adjoint bundle):

$$[\mathcal{D}_\alpha,\mathcal{D}_\beta] = -2\nabla_{\alpha\beta}. \tag{4.39}$$

In this case it is easy to find an unconstrained field—a partial connection along the odd distribution. In other words, given any $\mathcal{D}_\alpha$ we use (4.39) to define $\nabla_{\alpha\beta}$ unambiguously, since $V \cong \operatorname{Sym}^2 S^*$.

The Bianchi identities together with the constraint (4.39) imply that all components of the curvature can be written in terms of

$$\Lambda_\alpha = -\frac{1}{3}\epsilon^{\beta\gamma}[\nabla_{\alpha\beta},\mathcal{D}_\gamma].$$

This is a particular component of the curvature, and it is a section of the adjoint bundle. We define the component fields corresponding to $\mathcal{A}$ to be

$$\begin{aligned} A &= i^*\mathcal{A}, &&\text{connection on } \check{M}^3;\\ \lambda_\alpha &= i^*\Lambda_\alpha, &&\lambda \in C^\infty\bigl(\check{M}^3,\Pi(\operatorname{ad}P\otimes S)\bigr). \end{aligned}$$

There are no auxiliary fields in this model. The space of constrained connections $\mathcal{A}$ up to equivalence is isomorphic to the space of component fields $\{A, \lambda\}$ up to gauge equivalence. (In fact, the collection of fields in each case is a category, and the categories are equivalent.) Note that in case $G = \mathbb{T}$ is the circle group, then the component fields comprise a *vector multiplet* in $n = 3$ dimensions with $s = 2$ supersymmetries. That is, when we quantize the free theory, as in Lecture 3, we obtain the vector multiplet representation of $P^{3|2}$.

For the curvature of A we write

$$*F_A = f_{\alpha\beta}\, dy^{\alpha\beta}.$$

Then the supersymmetry transformation laws of the component fields, computed from the general formula (4.17), are

$$\begin{aligned}(\hat{\zeta}A)_{\alpha\beta} &= \eta^\gamma(\epsilon_{\alpha\gamma}\lambda_\beta + \epsilon_{\beta\gamma}\lambda_\alpha)\\ \hat{\zeta}\lambda_\alpha &= -\eta^\beta f_{\alpha\beta}.\end{aligned} \tag{4.40}$$

The variation of the connection A is an odd 1-form with values in the adjoint bundle; the coefficient of $dy^{\alpha\beta}$ is written in (4.40).

The superfield lagrangian is

$$\mathcal{L} = -\frac{1}{4}\epsilon^{\alpha\beta}\langle \Lambda_\alpha, \Lambda_\beta\rangle\; |d^3x|\, d^2\theta,$$

from which we derive the component lagrangian

$$L = \left\{-\frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \lambda \not{D}_A \lambda\rangle\right\} |d^3x|$$

using (4.21). Notice that the bosonic lagrangian (set $\lambda = 0$) is the standard Yang-Mills lagrangian (4.5), so this is indeed a minimal supersymmetric extension of bosonic Yang-Mills theory.

Just as for bosonic pure gauge theory, there is not much to say about the classical theory. For example, the moduli space of vacuua is a point (4.7). The coupling of the supersymmetric gauge theory to the supersymmetric σ-model is straightforward in superspace; the construction of the theory is the exact analog of the bosonic construction in (4.8)–(4.10). There are no surprises. We will have more interesting discussions in Lecture 5 when we consider theories with more supersymmetries and their dimensional reduction.