Jamming and the Glass Transition

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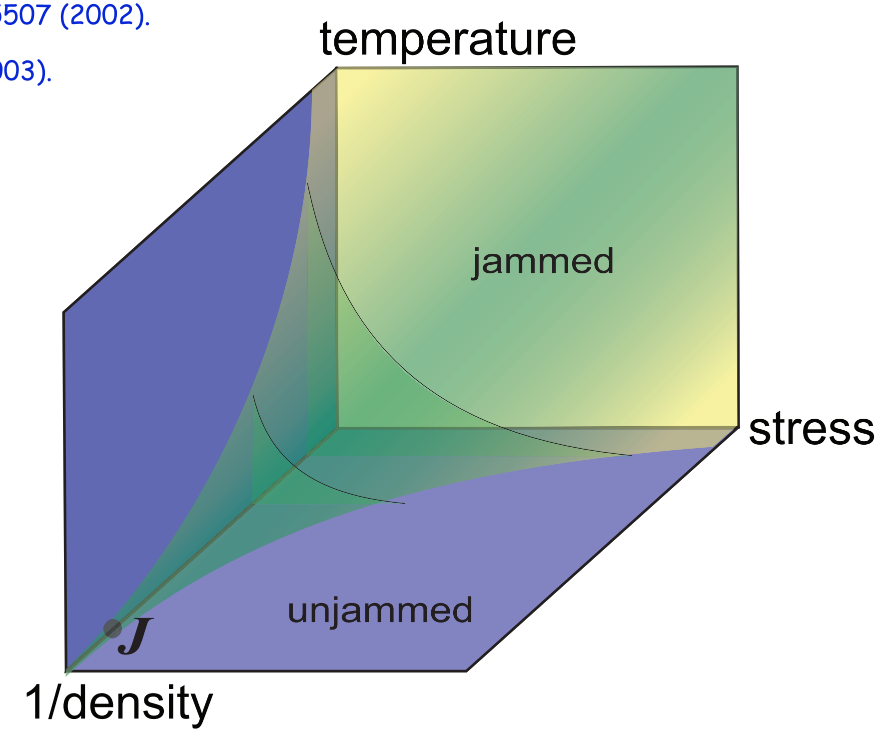
Sidney R. Nagel	UChicago

The Jamming Scenario

C. S. O'Hern, S. A. Langer, A. J. Liu and S. R. Nagel, Phys. Rev. Lett. **88**, 075507 (2002).

C. S. O'Hern, L. E. Silbert, A. J. Liu, S. R. Nagel, Phys. Rev. E **68**, 011306 (2003).



- Focus on foams/emulsions or frictionless granular material
  - soft, repulsive, finite-range spherically-symmetric potentials
- Such systems have T=0 1st/2nd-order phase transition

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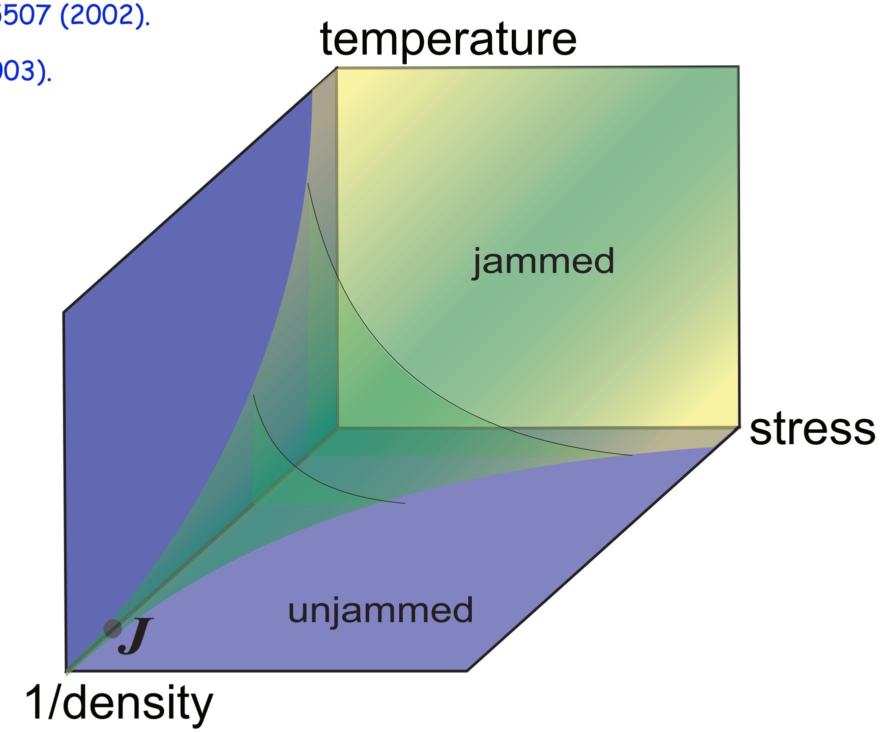

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  - soft, repulsive, finite-range spherically-symmetric potentials
- Such systems have T=0 1st/2nd-order phase transition

How we study Point J

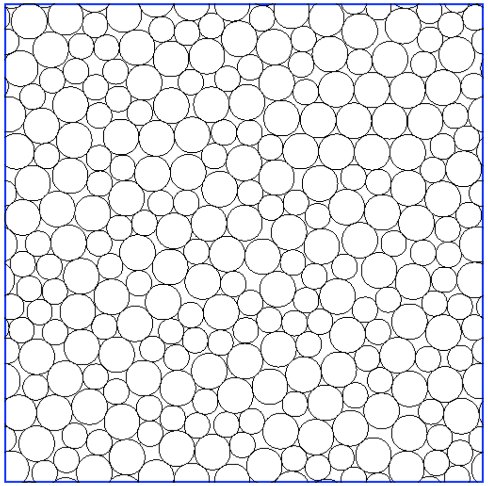
- Generate configurations near J
  - Start with some initial config

$$V(r) = \begin{cases} \dfrac{\varepsilon}{\alpha}\left(1-\dfrac{r}{\sigma}\right)^{\alpha} & r \le \sigma \\ 0 & r > \sigma \end{cases}$$

  - Conjugate gradient energy minimization (inherent structures, Stillinger & Weber)

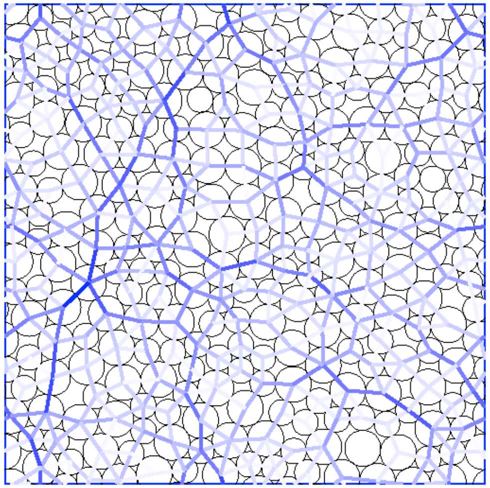
- Classify resulting configurations

non-overlapped
$V=0$
$p=0$



$T_f=0$

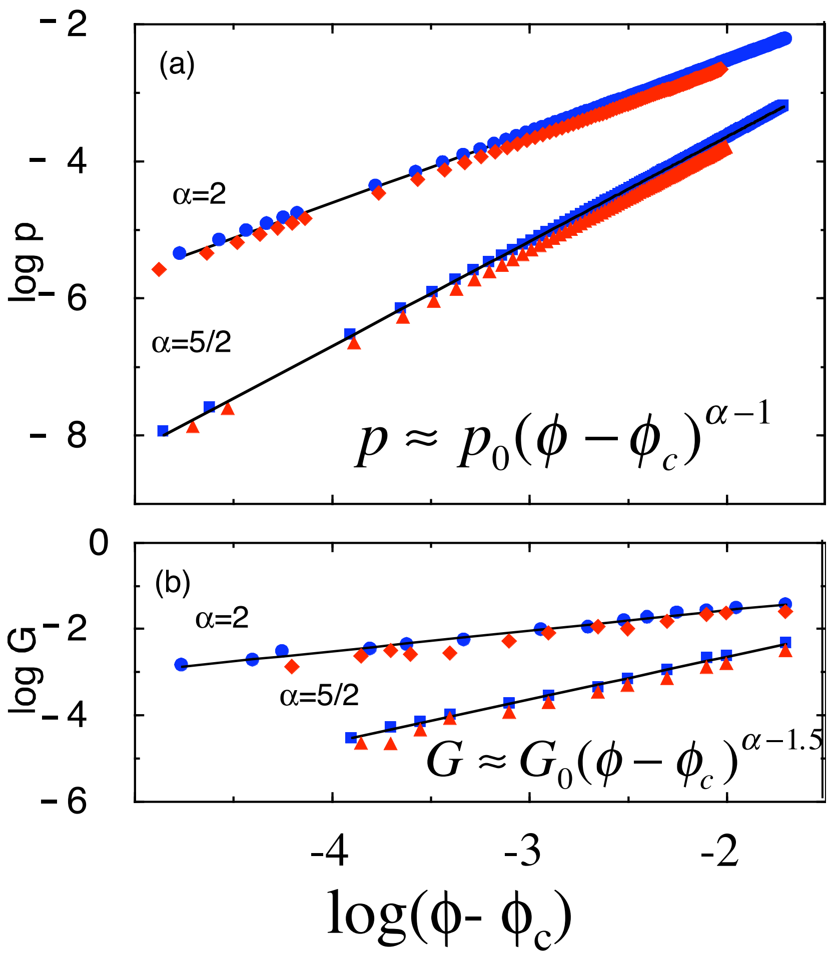
or



$T_f=0$

overlapped
$V>0$
$p>0$

Onset of Overlap is Onset of Jamming



D=2

D=3

- Pressures for different states collapse on a single curve

- Shear & bulk moduli, G & B, vanish at same ϕ_c

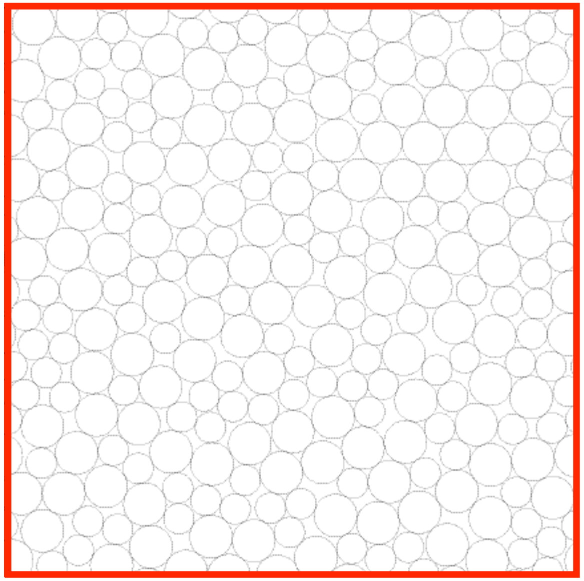
- $G/B \sim (\phi - \phi_c)^{\gamma \approx 1/2}$

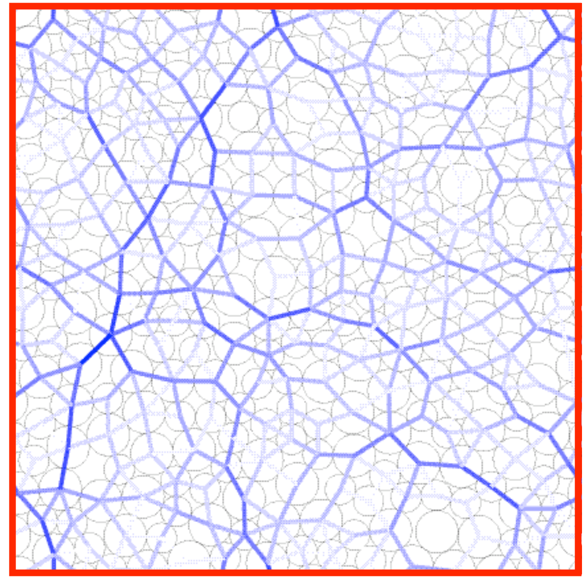
D. J. Durian, PRL **75**, 4780 (1995);

C. S. O'Hern, S. A. Langer, A. J. Liu, S. R. Nagel, PRL **88**, 075507 (2002).

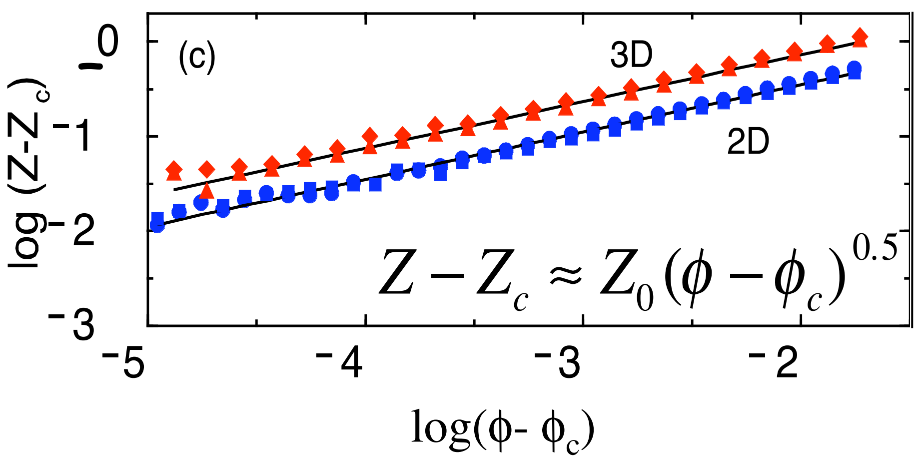
Onset of Overlap has 1st-Order Character

Just below ϕ_c, no particles overlap





Just above ϕ_c there are Z_c overlapping neighbors per particle



$$Z_c = 3.99 \pm 0.01 \quad \text{(2D)}$$

$$Z_c = 5.97 \pm 0.03 \quad \text{(3D)}$$

Durian, PRL **75**, 4780 (1995).
O'Hern, Langer, Liu, Nagel, PRL **88**, 075507 (2002).

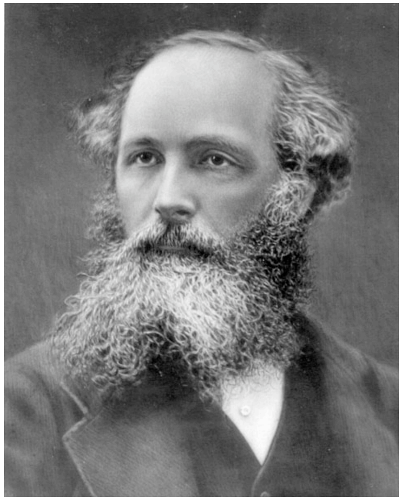
Verified experimentally:

Majmudar, Sperl, Luding, Behringer, PRL **98**, 058001 (2007).

Isostaticity

- What is the minimum number of interparticle contacts needed for mechanical equilibrium?

- No friction, spherical particles, D dimensions
  - Match
    unknowns (number of interparticle normal forces)
    equations (force balance for mechanical stability)
  - Number of unknowns per particle=Z/2
  - Number of equations per particle = D



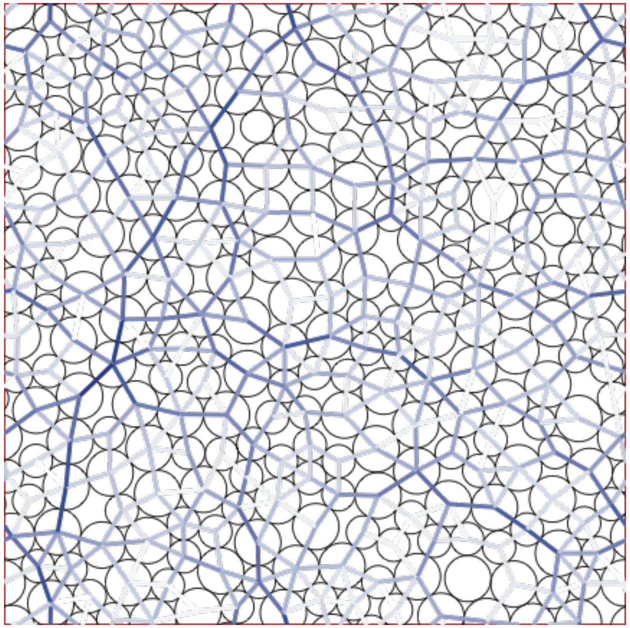
James Clerk Maxwell

$$Z = 2D$$

Phillips, Thorpe, Boolchand, Edwards, Ball, Blumenfield

Isostaticity and Diverging Length Scale

M. Wyart, S.R. Nagel, T.A. Witten, EPL **72**, 486 (05)



- For system at ϕ_c, Z=2d
- Removal of one bond makes entire system unstable by adding one soft mode
- This implies diverging length as $\phi \to \phi_c^{\,+}$

For $\phi > \phi_c$, cut bonds at boundary of circle of size L

Count number of soft modes within circle

$$N_s \approx L^{d-1} - (Z - Z_c) L^d$$

Define length scale at which soft modes just appear

$$\ell \approx \frac{1}{Z - Z_c} \approx (\phi - \phi_c)^{-0.5}$$

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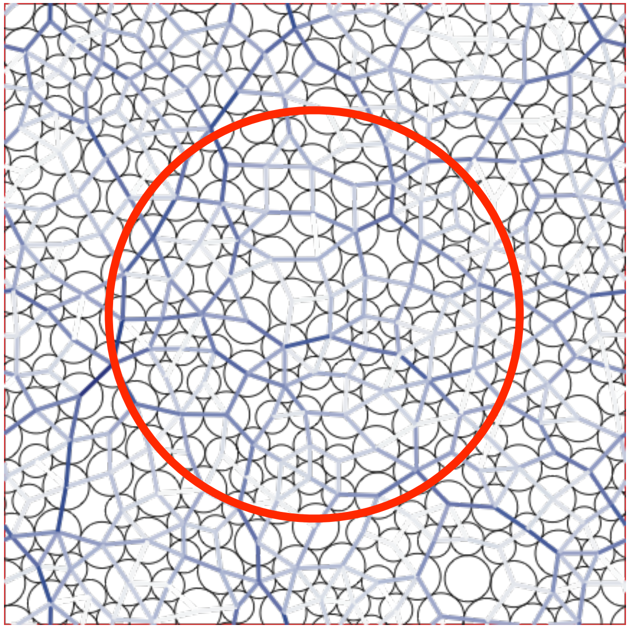

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Consequences of Diverging Length Scale

- Solids are all alike at low T or ω:
  - density of vibrational states $D(\omega) \sim \omega^2$ in d=3
  - vibrational heat capacity $C(T) \sim T^3$
  - thermal conductivity $\kappa(T) \sim Cv\ell \sim T^3$
- Why?

Low-frequency excitations are sound modes. At long length scales all solids look elastic

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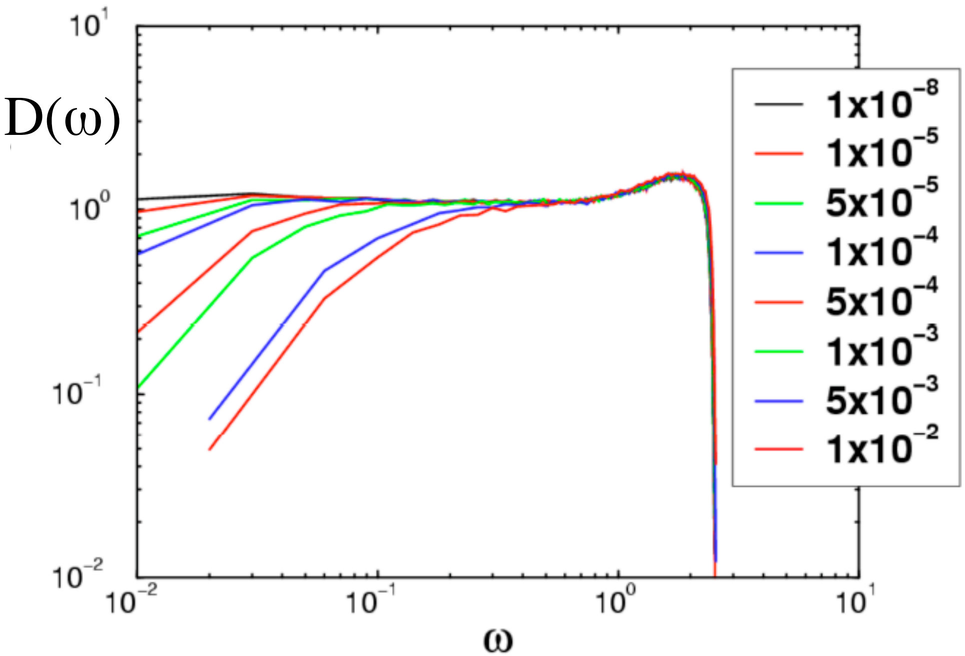
Low-frequency excitations are sound modes. At long length scales all solids look elastic

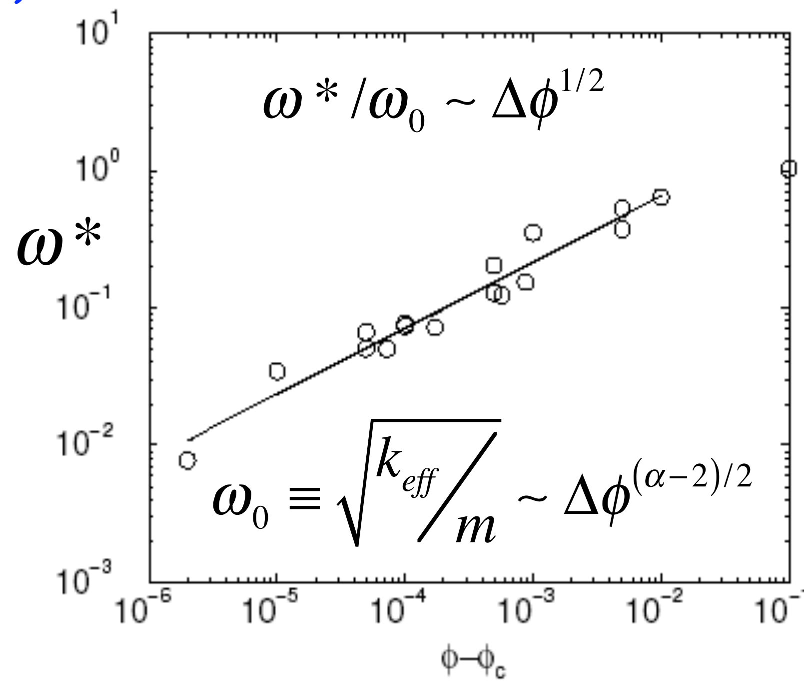
BUT at Point J, there is a diverging length scale ℓ^*

So what happens?

Vibrations in Marginally Jammed Solids

L. E. Silbert, A. J. Liu, S. R. Nagel, PRL **95**, 098301 ('05)





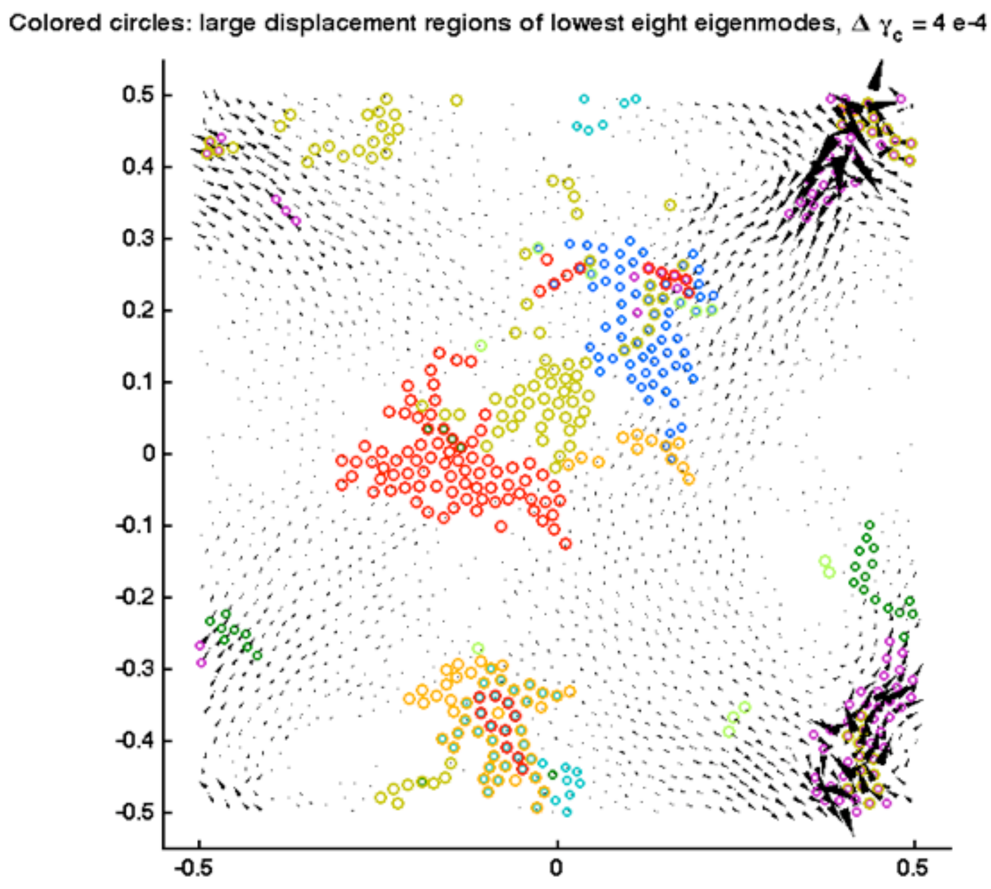
- More and more modes in excess of Debye prediction as $\phi \to \phi_c$ (boson peak)

- New class of excitations distinct from sound modes originates from soft modes at Point J M. Wyart, S.R. Nagel, T.A. Witten, EPL **72**, 486 (05)
- Robust for systems near isostaticity Souslov, Liu, Lubensky, PRL **103**, 205503 (2009); Mao, Xu, Lubensky arXiv: 09092616

Vibrational Modes Predict Soft Spots

- Adams-Gibbs Cooperatively-Rearranging Regions?
- Shear Transformation Zones?

  M. Lisa Manning's talk next week



Summary of Point J

$$V(r) = \begin{cases} \frac{\varepsilon}{\alpha}\left(1 - \frac{r}{\sigma}\right)^{\alpha} & r \le \sigma \\ 0 & r > \sigma \end{cases}$$

- Mixed first-order/second-order transition (RFOT)
- Number of overlapping neighbors per particle

$$Z = \begin{cases} 0 & \phi < \phi_c \\ Z_c + z_0\left(\phi - \phi_c\right)^{\beta \cong 1/2} & \phi \ge \phi_c \end{cases}$$

- Static shear modulus/bulk modulus

$$G / B \sim \left(\phi - \phi_c\right)^{\gamma \cong 1/2}$$

- Two diverging length scales

$$\ell^{*} \sim \left(\phi - \phi_c\right)^{-\nu \cong -1/2}$$

$$\ell^{\dagger} \sim \left(\phi - \phi_c\right)^{-\nu^{\dagger} \cong -1/4}$$

Similarity to Other Models

- This behavior has only been found in a few models, all in mean-field limit
  - 1-RSB p-spin interaction spin glass Kirkpatrick, Thirumalai, Wolynes
  - compressible frustrated Ising antiferromagnet Yin, Chakraborty
  - kinetically-constrained Ising models Sellitto, Toninelli, Biroli, Fisher
  - k-core percolation and variants Schwarz, Liu, Chayes, Toninelli, Biroli, Fisher,Harris, Jeng
  - Mode-coupling approximation of glasses Biroli, Bouchaud
  - 1-RSB solution of hard spheres Zamponi, Parisi

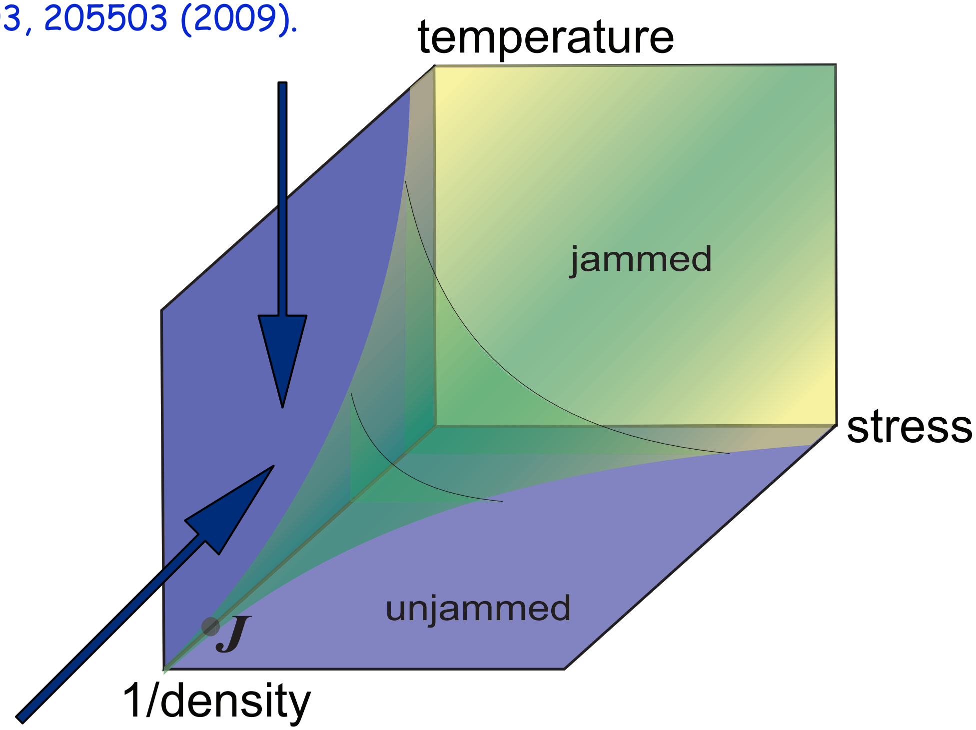
- These other models all exhibit glassy dynamics!!

First hint of **quantitative** connection between sphere packings

and glass transition

Point J and the Glass Transition

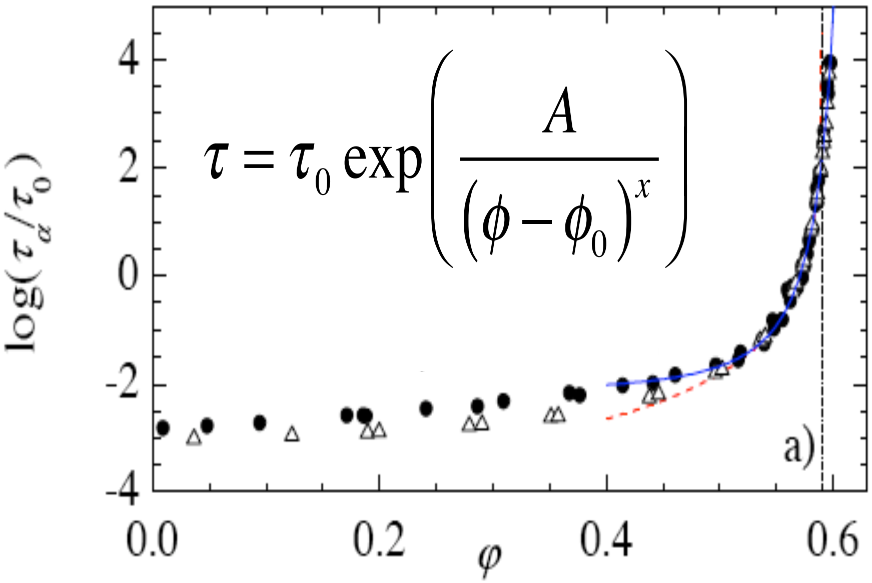
N. Xu, T. K. Haxton, A. J. Liu and S. R. Nagel, PRL 103, 205503 (2009).



- How do ideal spheres behave at nonzero temperature?

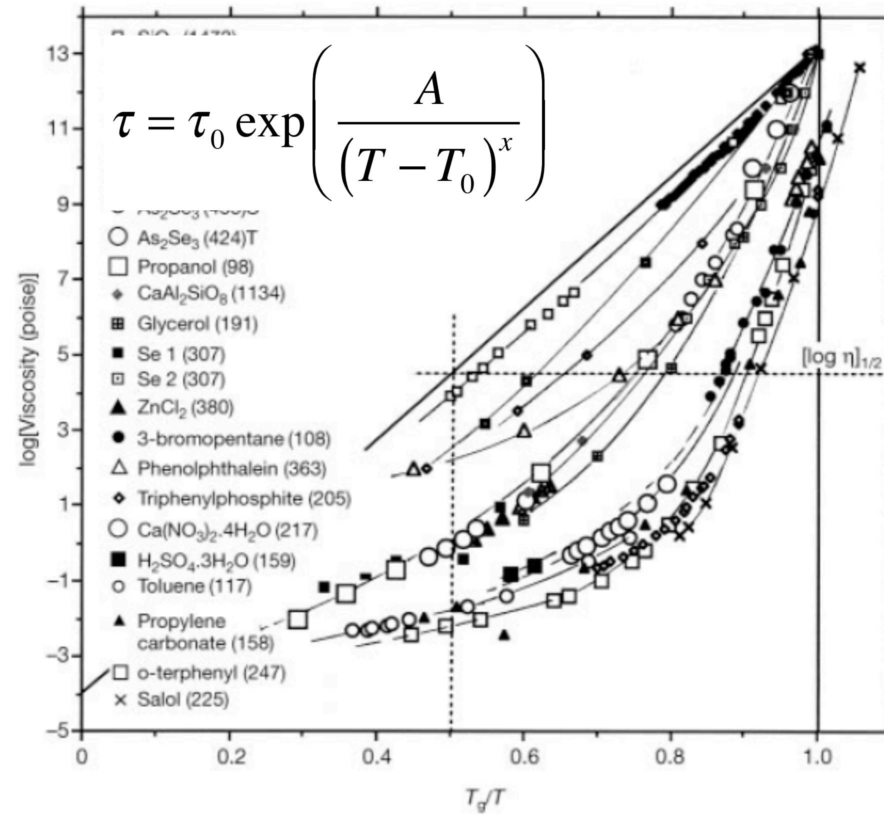
Dramatic Increase of Relaxation Time

Colloidal glass transition



Brambilla, et al., arXiv/0809.3401

Glass transition



L.-M. Martinez and C. A. Angell, Nature **410**, 663 (2001).

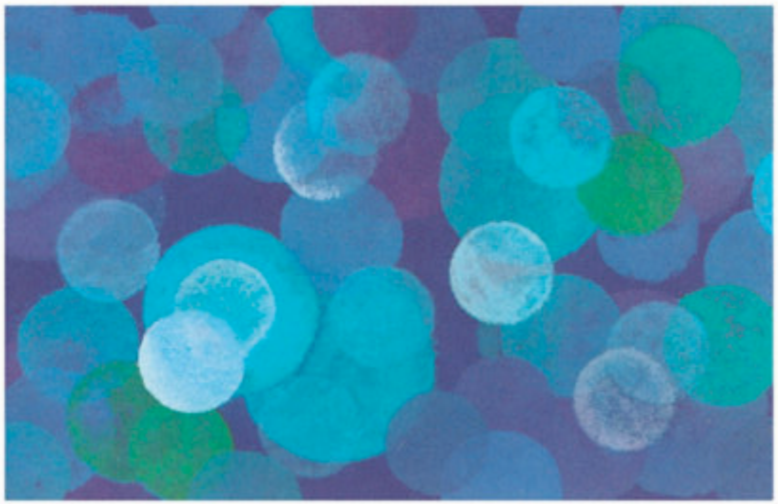
Similar fitting functions.

WHY??

Very Different Underlying Pictures

Colloidal glass transition

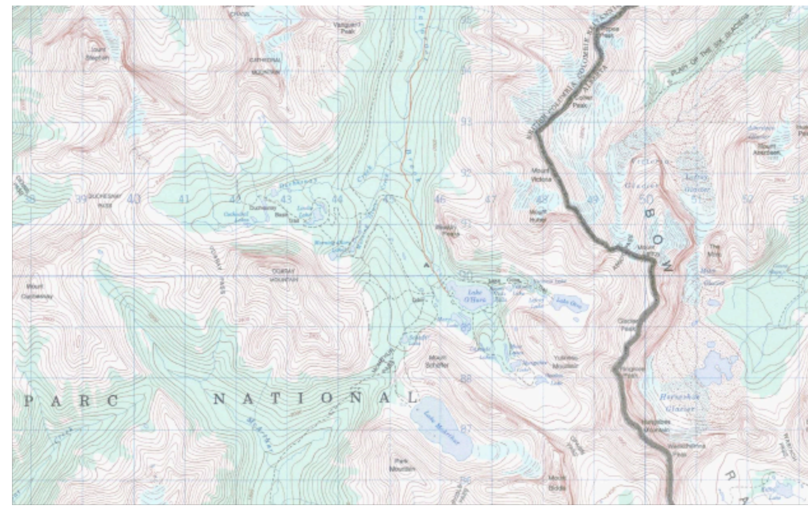
free volume



Pressure is most important
It governs amount of free volume

Glass transition

complex energy landscape



Temperature is most important
It allows system to overcome energy barriers

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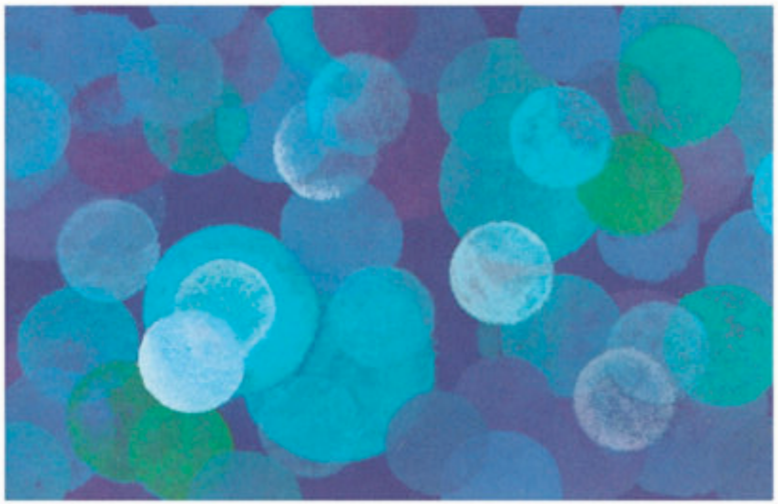

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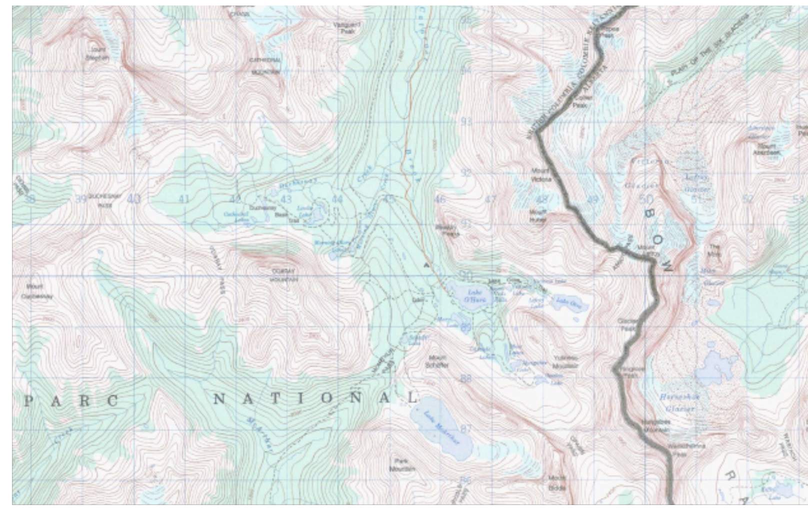
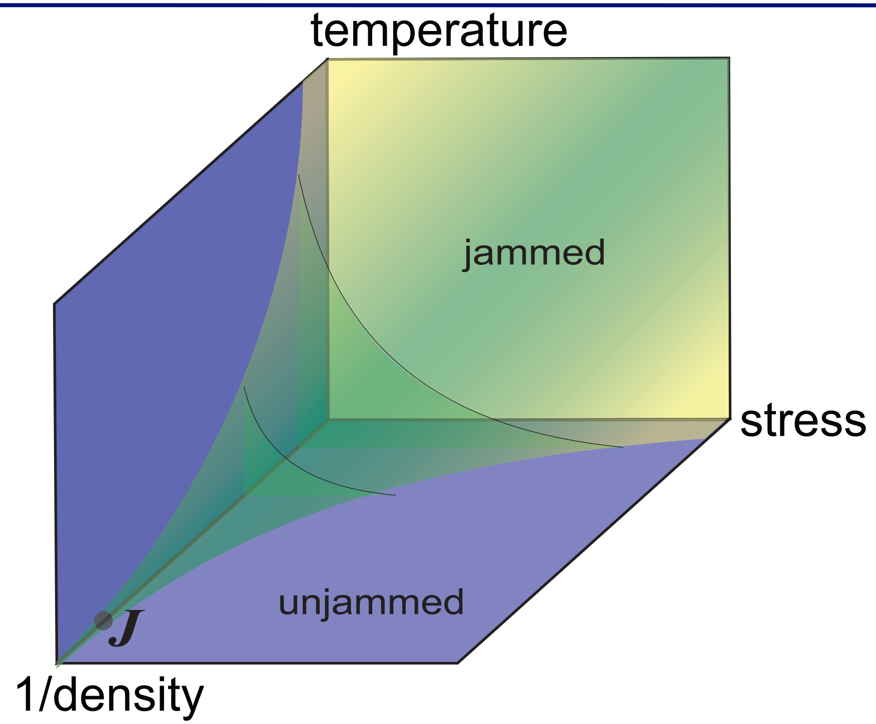


Temperature is most important
It allows system to overcome energy barriers

There are systems for which these two transitions are the same phenomenon

Relaxation Time

- Look at relaxation time along different trajectories
  - Fix p, lower T
  - Fix T, raise p



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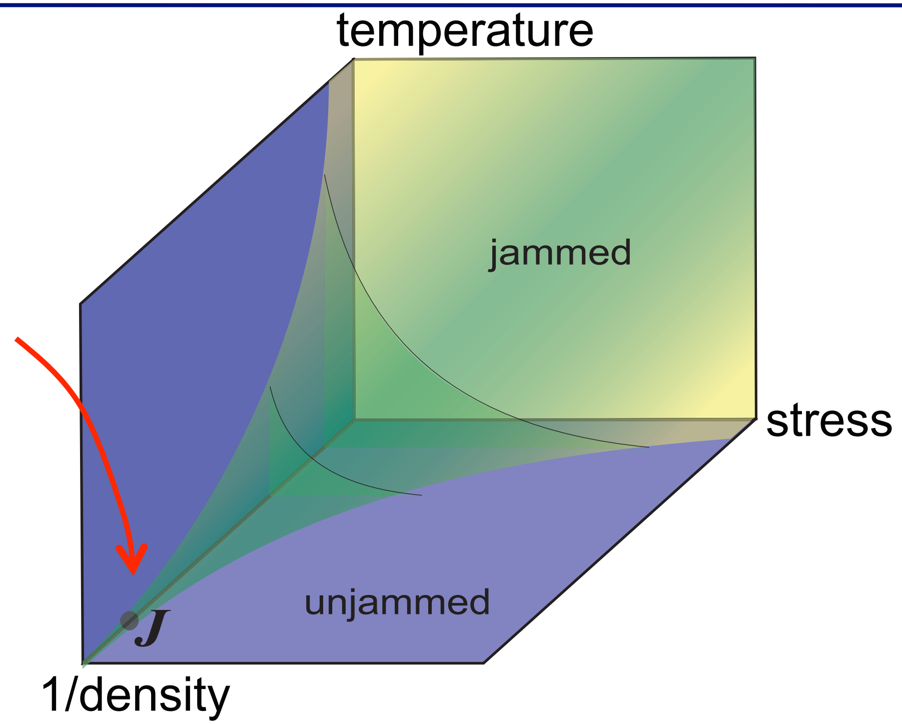

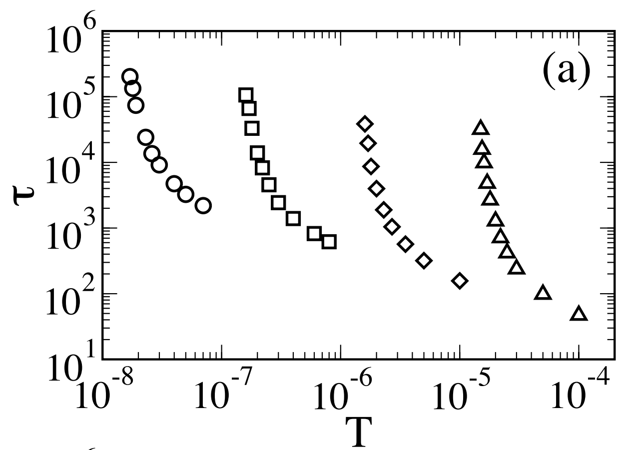

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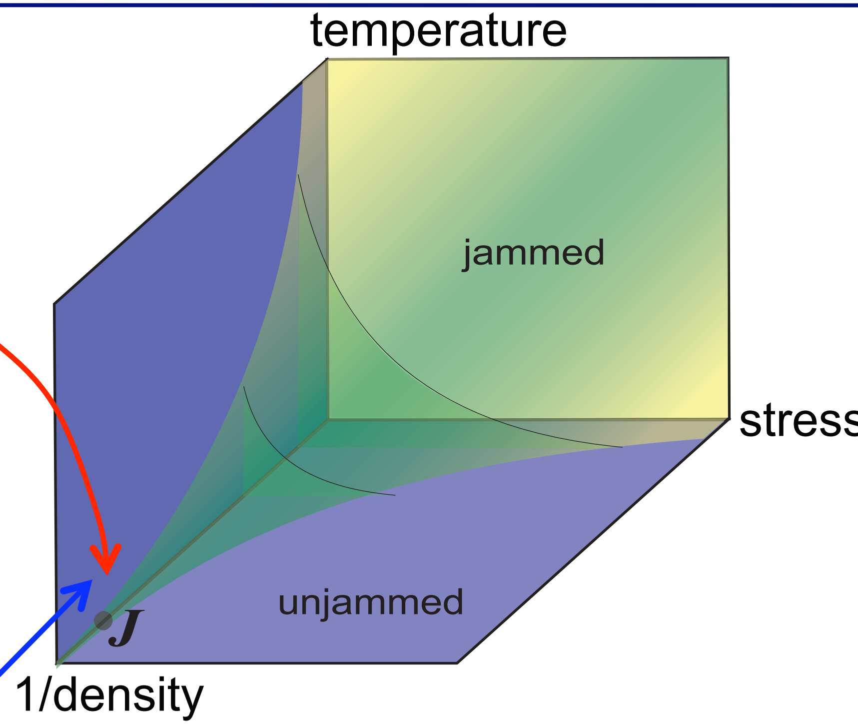
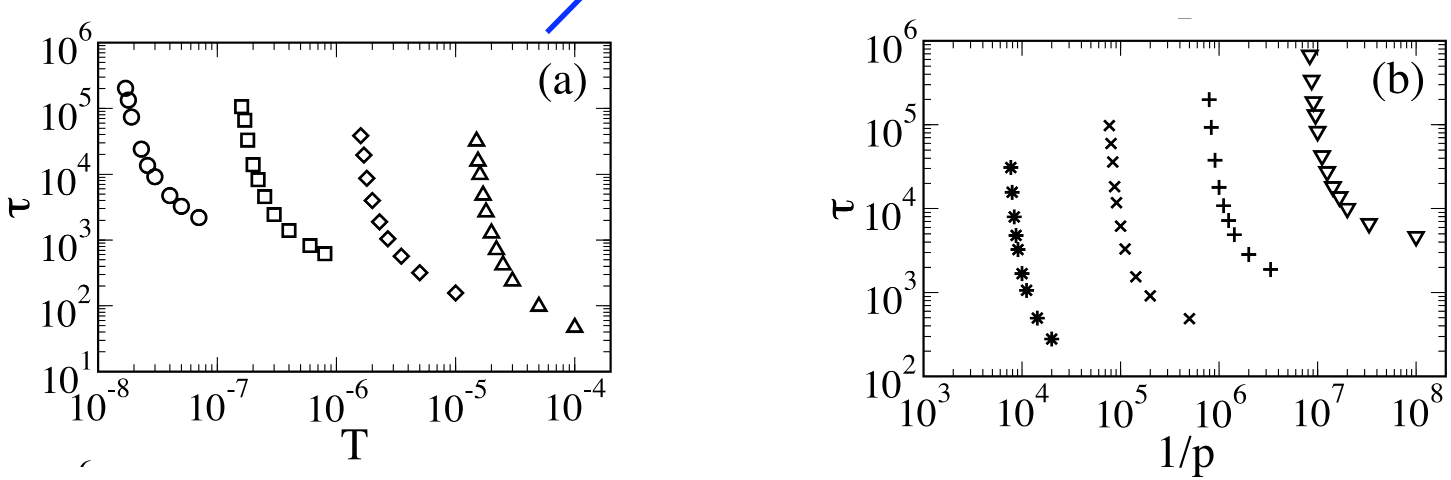
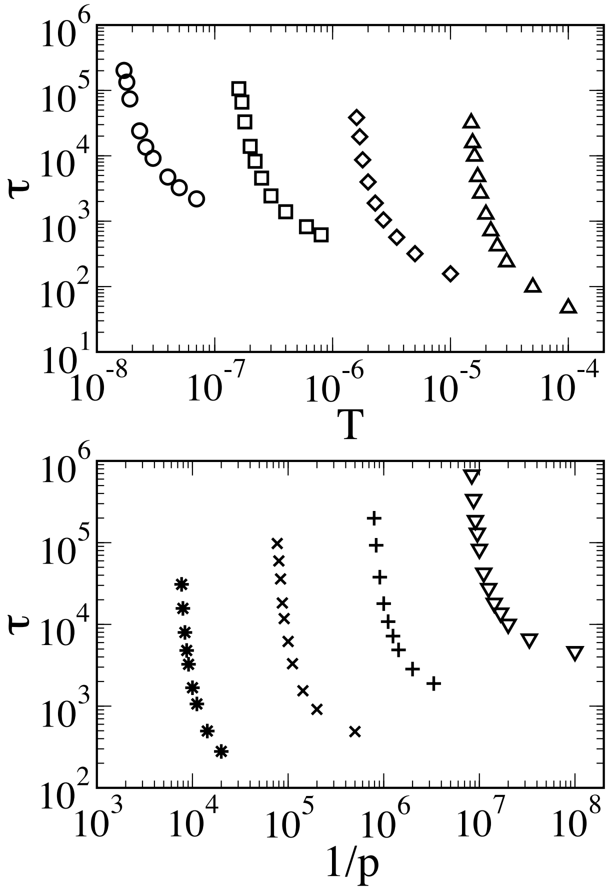
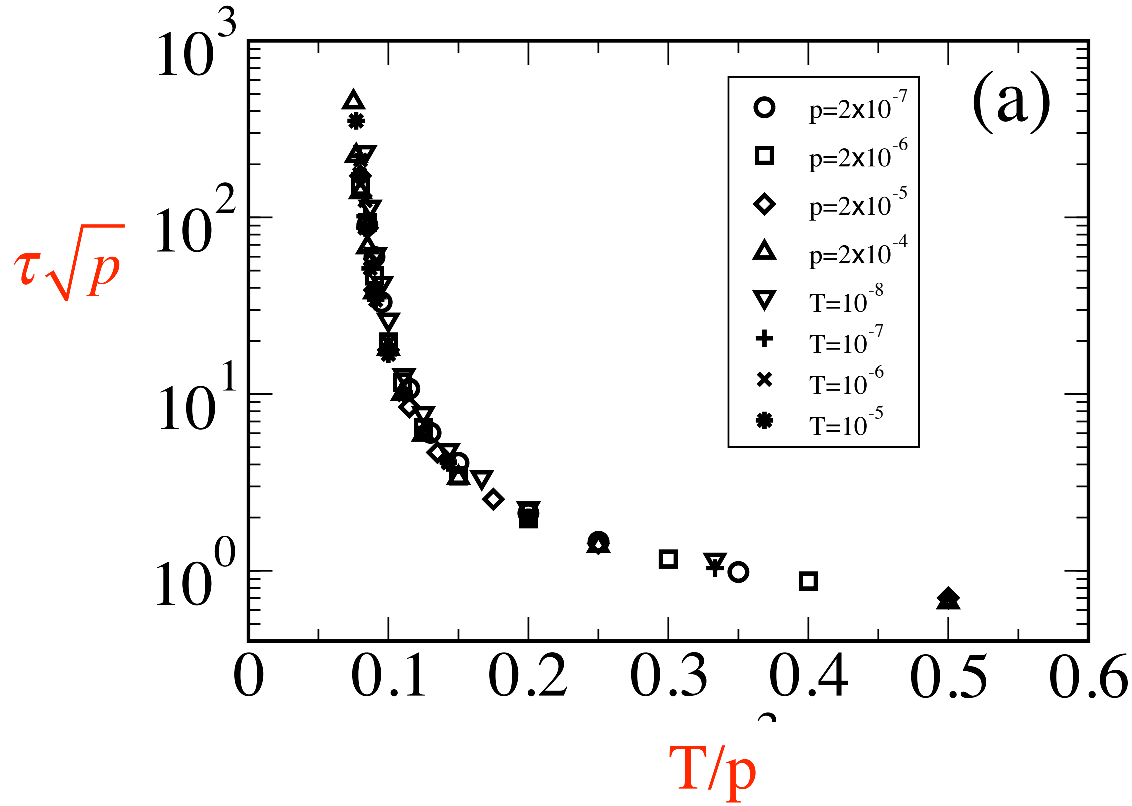




Data Collapse!





Data for different (p, T) collapse on to single scaling curve!

Dimensional Analysis

- Recall interaction potential

$$V(r) = \begin{cases} \frac{\varepsilon}{2}\left(1 - \frac{r}{\sigma}\right)^2 & r \le \sigma \\ 0 & r > \sigma \end{cases}$$

- We have 3 dimensional parameters in model: ε, m, σ

particle mass (m)

interaction energy (ε)

particle diameter (σ)

Relaxation time

$$\tau\sqrt{\frac{\varepsilon}{m\sigma^2}} = h\left(\frac{T}{\varepsilon}, \frac{p\sigma^3}{\varepsilon}\right)$$

or equivalently

$$\tau\sqrt{\frac{p\sigma}{m}} = g\left(\frac{T}{p\sigma^3}, \frac{p\sigma^3}{\varepsilon}\right)$$

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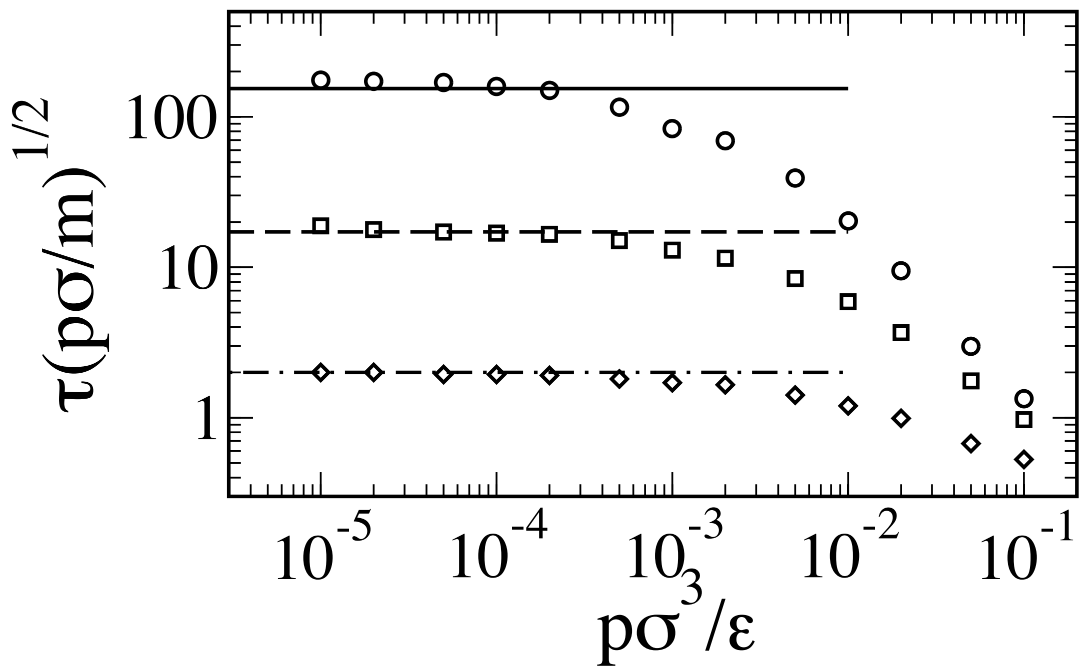
$$\tau\sqrt{\frac{p\sigma}{m}} = g\left(\frac{T}{p\sigma^3}, \frac{p\sigma^3}{\varepsilon} \to 0\right)$$

Dimensional Analysis



$$\lim_{p\sigma^3/\varepsilon\to 0} \tau\sqrt{\frac{p\sigma}{m}} = \lim_{p\sigma^3/\varepsilon\to 0} g\left(\frac{T}{p\sigma^3},\frac{p\sigma^3}{\varepsilon}\right) = f\left(\frac{T}{p\sigma^3}\right)$$

T, p equally important!

- In low p limit, the relaxation time depends only on T/p
- Data collapse for different trajectories

Data Collapse for Different Potentials

- $p\sigma^3/\varepsilon \to 0$ corresponds to low p limit for soft spheres AND to the hard sphere limit
- Should see collapse for any $\alpha \geq 0$ including $\alpha = 0$ (hard spheres)

$$V(r) = \begin{cases} \dfrac{\varepsilon}{\alpha}\left(1 - \dfrac{r}{\sigma}\right)^{\alpha} & r \leq \sigma \\ 0 & r > \sigma \end{cases}$$

Data Collapse for Different Potentials

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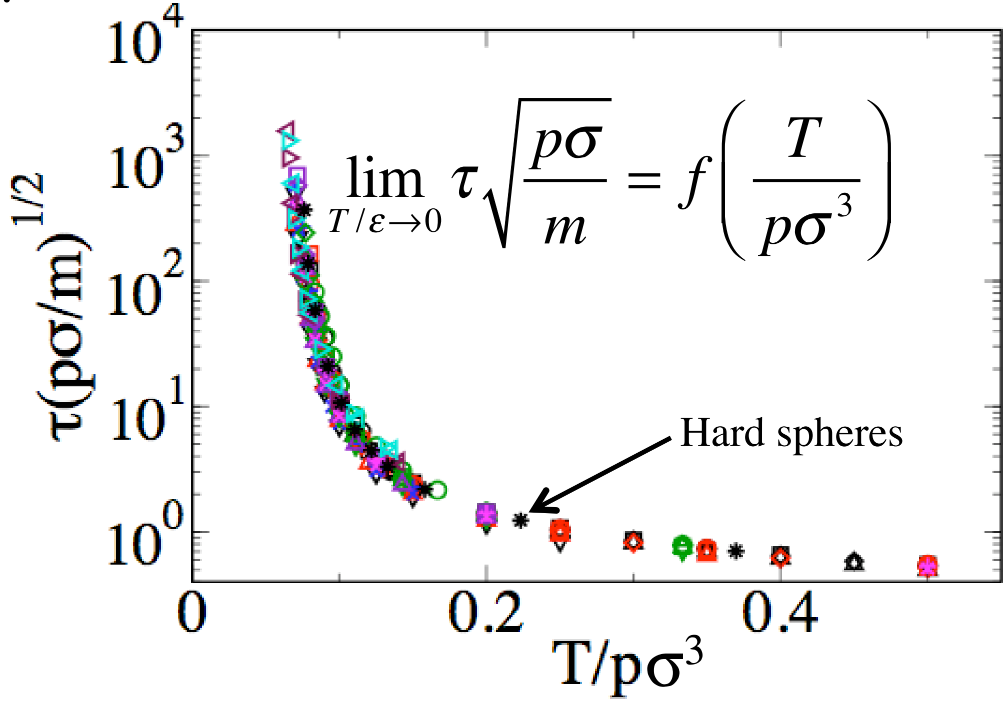
YES!

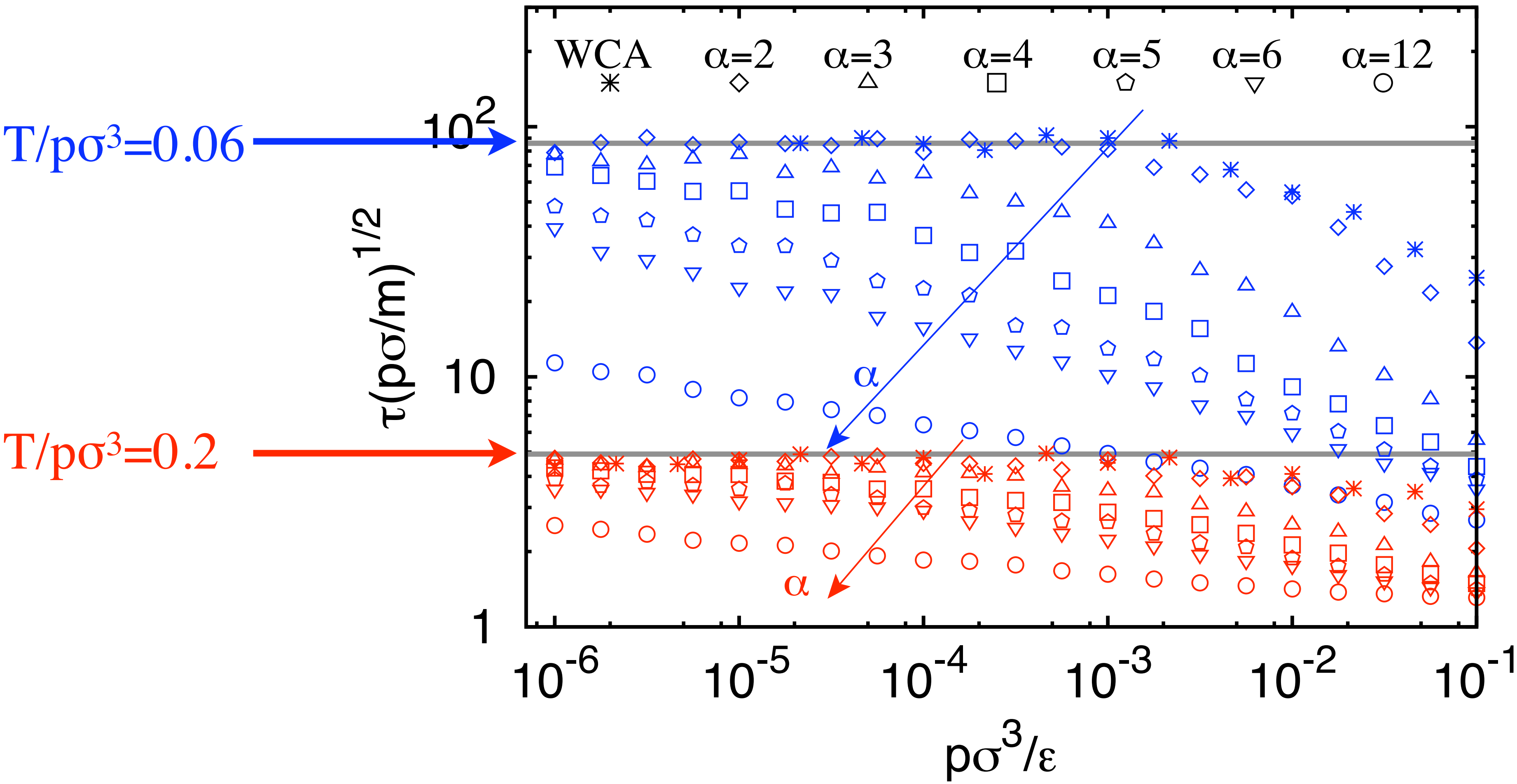


$\Delta V \sim T/p$

T does work against pressure to open up free volume

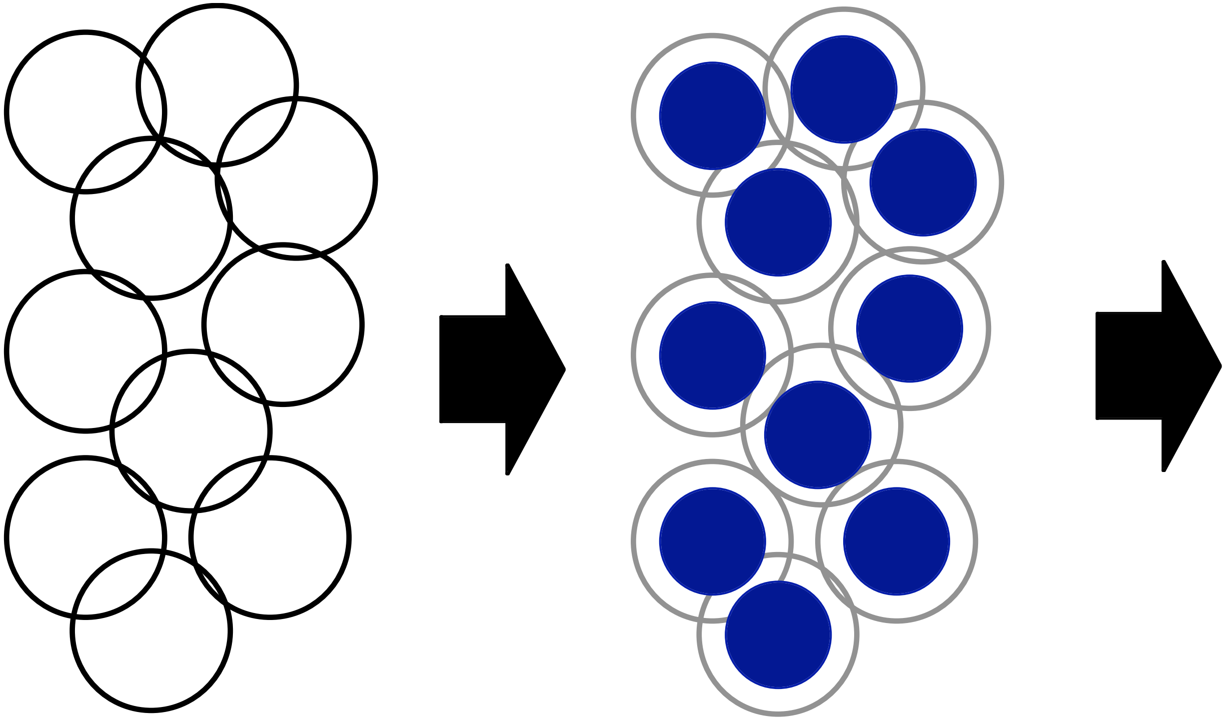
Glass transition **equivalent** to colloidal glass transition as $p\sigma^3/\varepsilon$-> 0

Relaxation times at high pressure



- As spheres soften, relaxation time decreases (Weitz/Reichman)
- New mechanism of relaxation controlled by $p\sigma^3/\varepsilon$ emerges

Soft spheres have smaller equivalent hard sphere diameter



Soft spheres with diameter σ

Hard spheres with reduced diameter

$\sigma_{eff} < \sigma$

smaller effective volume fraction

more free volume

faster relaxation

How to choose σ_{eff}?

Rowlinson, Barker-Henderson, Andersen-Weeks-Chandler

Strategy for choosing σ_{eff}

Andersen, Weeks, Chandler J. Chem. Phys. 54, 5237 (1971)

- Taylor expand free energy around hard-sphere potential
- Choose σ_{eff} so that first-order functional derivative of free energy with respect to exp(-V(r)/T) vanishes

$$\int d\vec{r}\, y_{eff}(r)\left\{\exp\left[-V(r)/T\right]-\exp\left[-V_{eff}(r)/T\right]\right\}=0$$

$y_{eff}(r) = e^{\beta V_{eff}(r)} g_{eff}(r)$

$\exp[-V(r)/T]$: original soft-sphere potential

$V_{eff}(r)$: hard-sphere potential with σ_{eff}

- Can calculate σ_{eff} from the soft-sphere potential and hard-sphere properties alone!
- This approximation reproduces captures static quantities beautifully

Relaxation time

- Start with soft spheres at arbitrary pressure
- Use ACW to calculate effective hard sphere diameter σ_{eff}
- Obtain new packing fraction for effective hard spheres ϕ_{eff}

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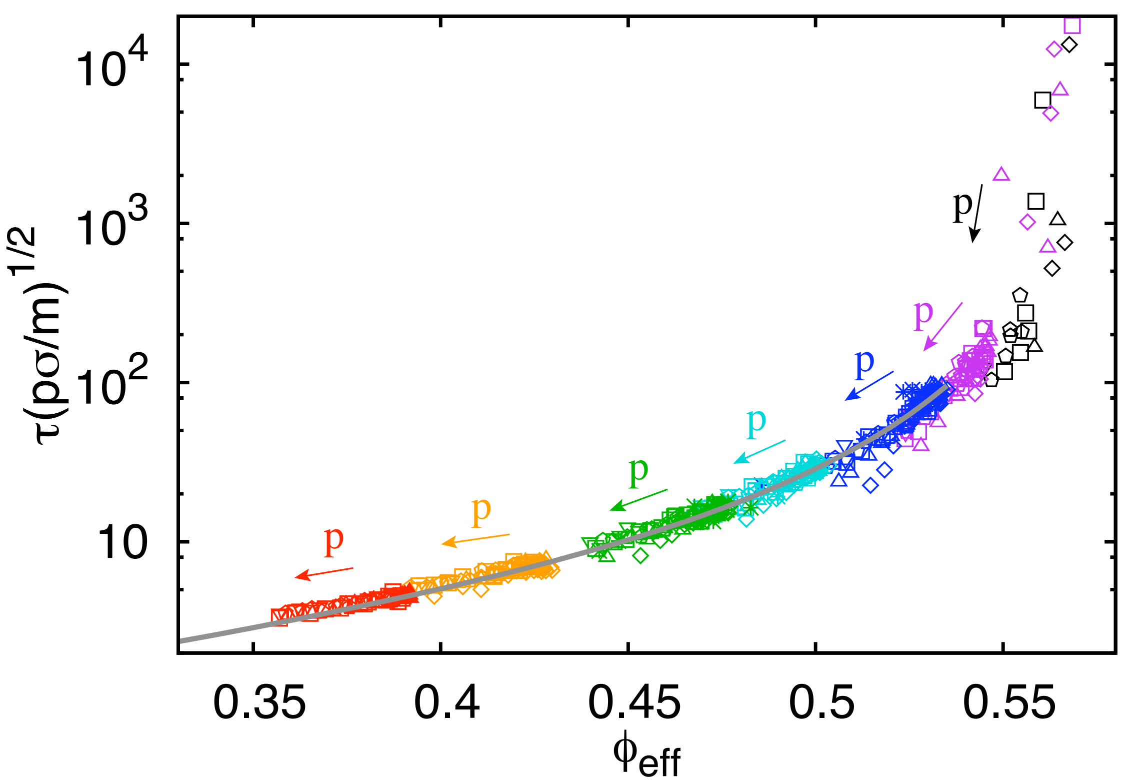


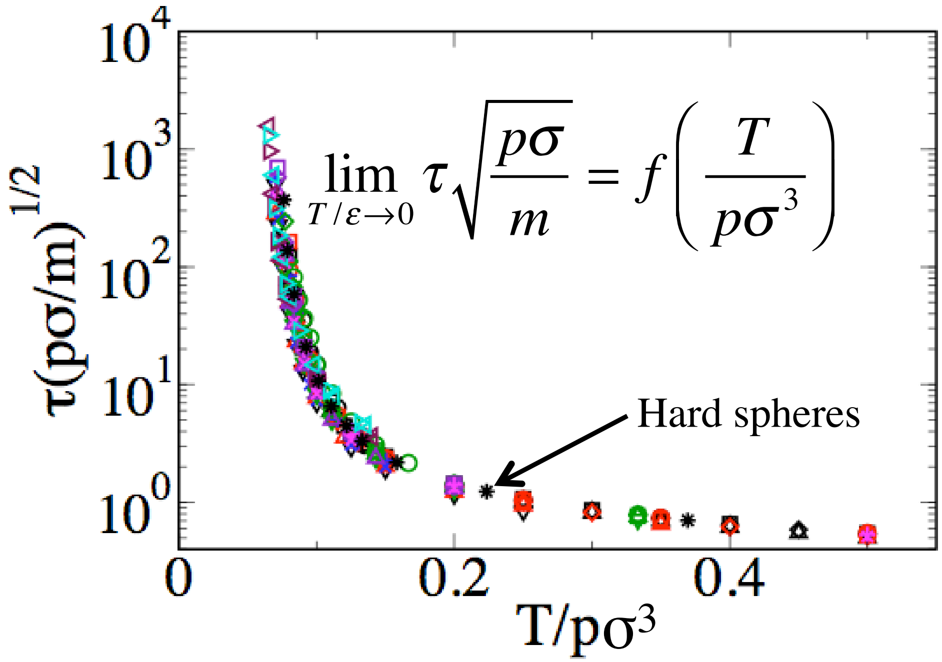
ACW approx rocks!!

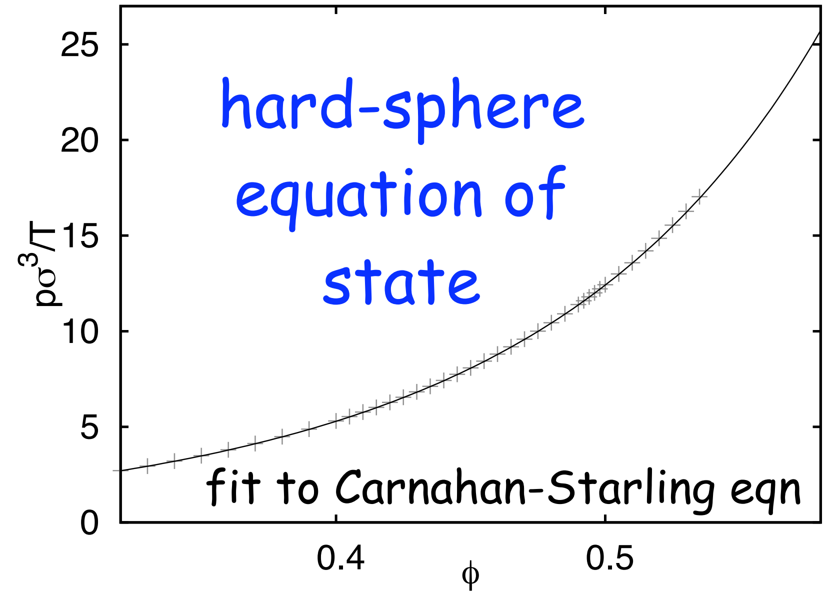
τ collapses onto

hard-sphere curve

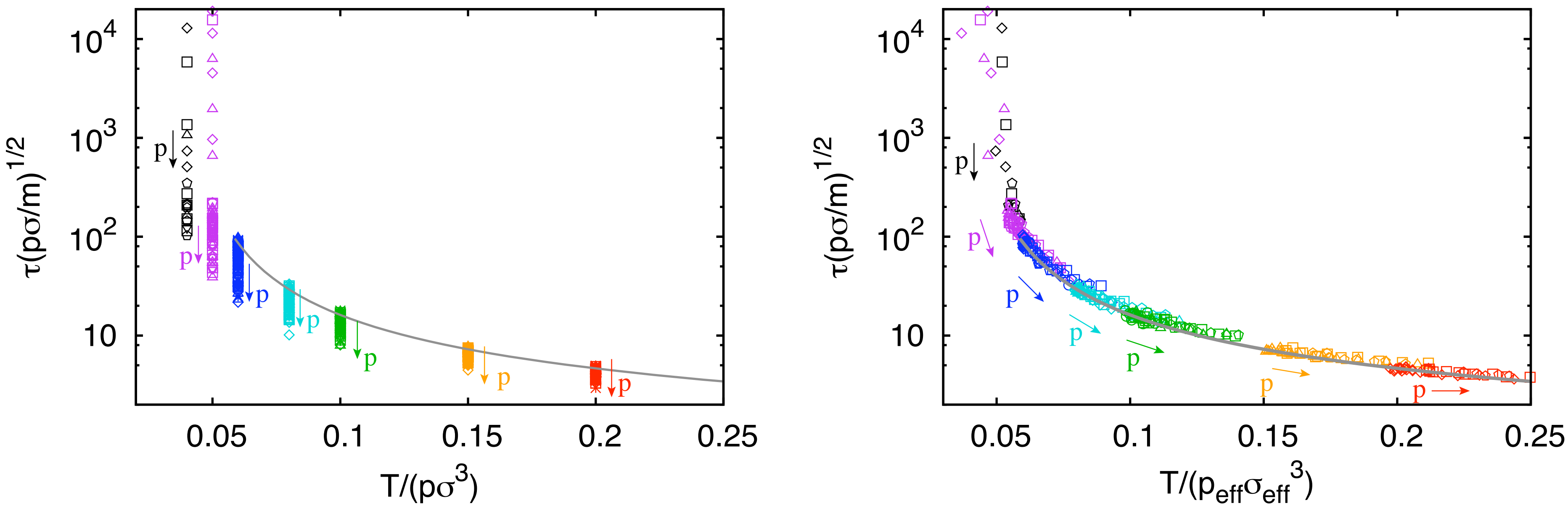
for all finite-ranged repulsive potentials and pressures studied

Recall Hard-Sphere Curve





Universal Hard-Sphere Master Curve



- This implies that $\tau\sqrt{p\sigma/m} = \tau_{eff}\sqrt{p_{eff}\sigma_{eff}/m}$ or $\tau = \tau_{eff}\sqrt{p_{eff}\sigma_{eff}/p\sigma}$
- Given
  - hard-sphere master curve
  - pair interaction potential of soft-sphere system
- can calculate relaxation time of soft-sphere system!
- Can also use soft spheres to extend master curve

Two Mechanisms of Relaxation in Soft Spheres

- Temperature opens up free volume against the pressure

$$T = p\Delta V$$

  - hard spheres are fragile glassformers (super-Arrhenius increase of relaxation time)

- Temperature allows soft spheres to overlap so they behave as hard spheres with smaller diameter (less super-Arrhenius with increasing overlap)

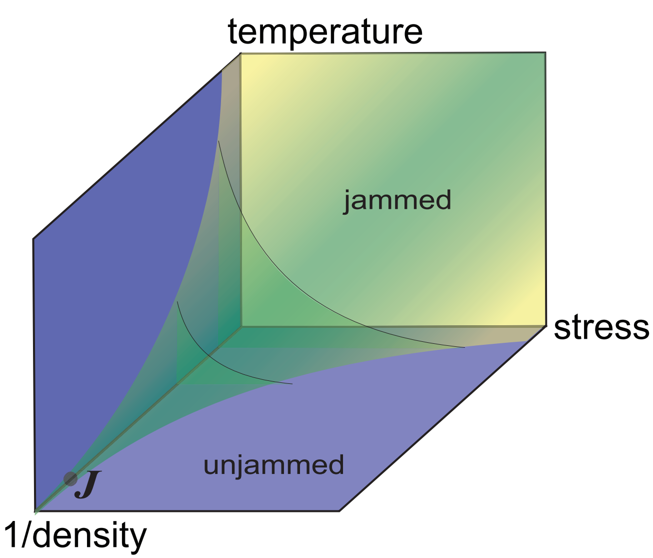
- In energy landscape, canyons do not become deeper but become narrower and more convoluted as $p \uparrow$ or $T \downarrow$

Connection to Point J

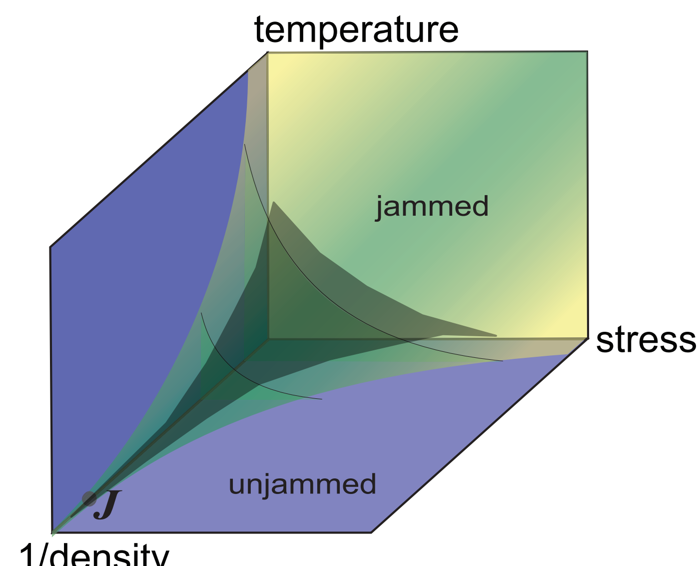
- Point J corresponds to double limit $T/p\sigma^3 \to 0$, $p\sigma^3/\varepsilon \to 0$
- What is

$$\lim_{T/p\sigma^3 \to 0} \lim_{p\sigma^3/\varepsilon \to 0} \tau\sqrt{\frac{p\sigma}{m}} = \lim_{T/p\sigma^3 \to 0} \lim_{p\sigma^3/\varepsilon \to 0} g\left(\frac{T}{p\sigma^3}, \frac{p\sigma^3}{\varepsilon}\right) = \lim_{T/p\sigma^3 \to 0} f\left(\frac{T}{p\sigma^3}\right)$$

- Does τ diverge at Point J? Or does it diverge at $T/p\sigma^3 > 0$?
- Does Pt J control glass transition? Or is there an underlying thermodynamic glass transition?



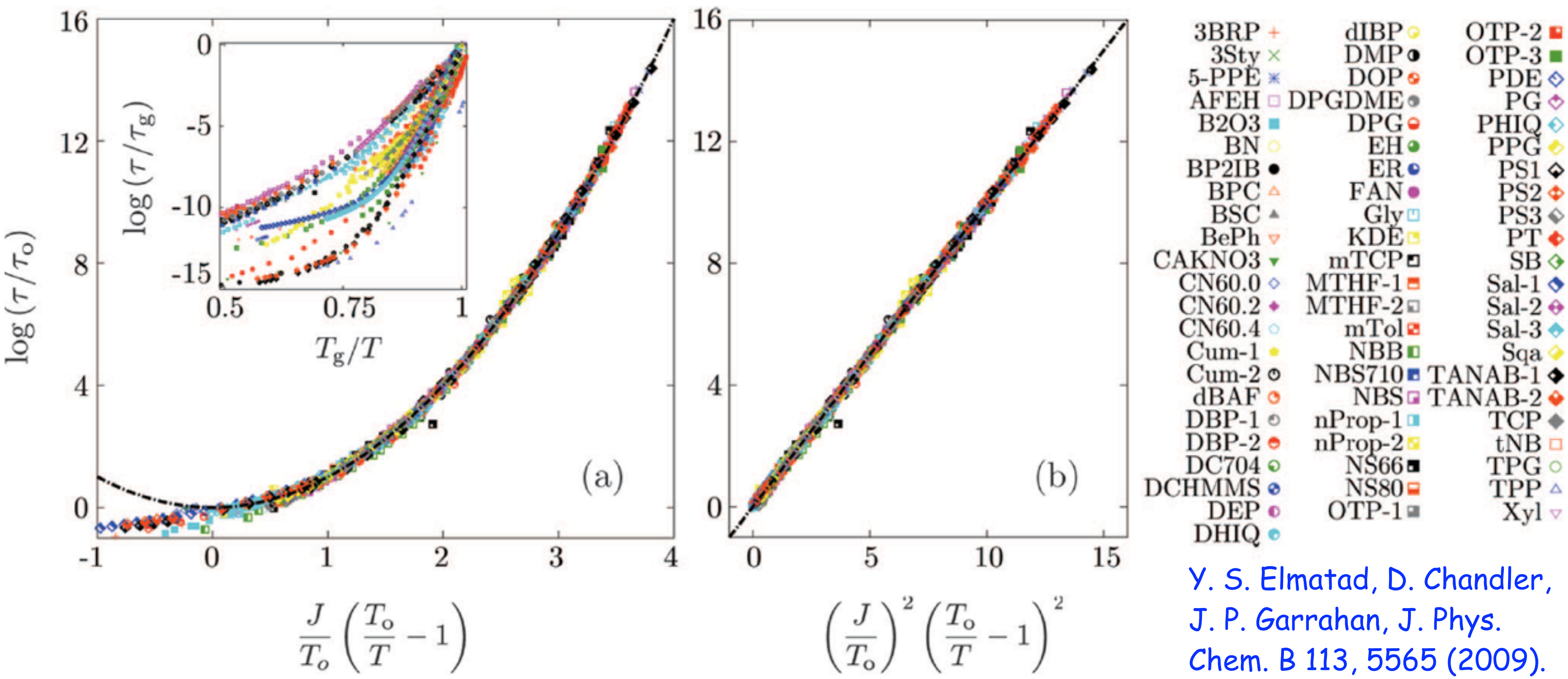
vs.



Fitting Forms

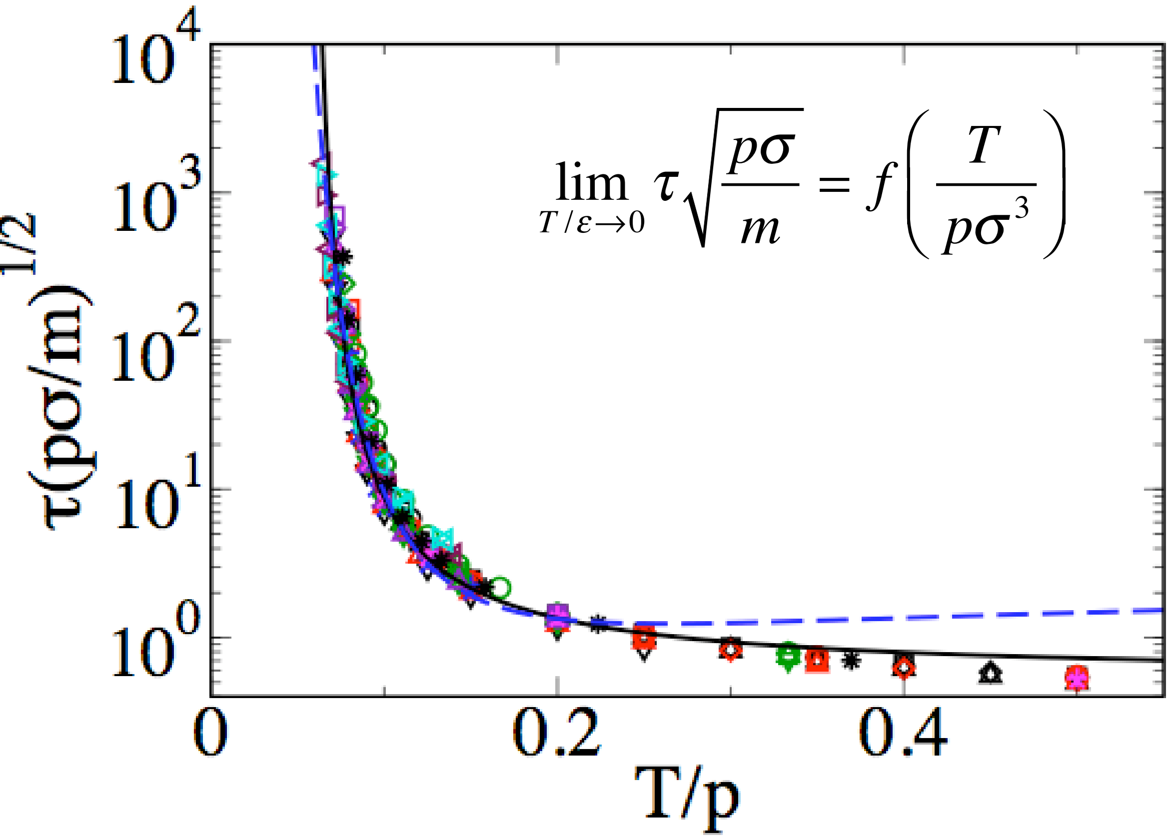
- Vogel-Fulcher form: $$\tau/\tau_0 = \exp\left(\frac{A}{T - T_0}\right)$$

- Elmatad-Chandler-Garrahan form:



Y. S. Elmatad, D. Chandler, J. P. Garrahan, J. Phys. Chem. B 113, 5565 (2009).

Form of Scaling Function



Fits:

$$f(x) = 0.5\exp\left(\frac{0.15}{x - 0.05}\right)$$

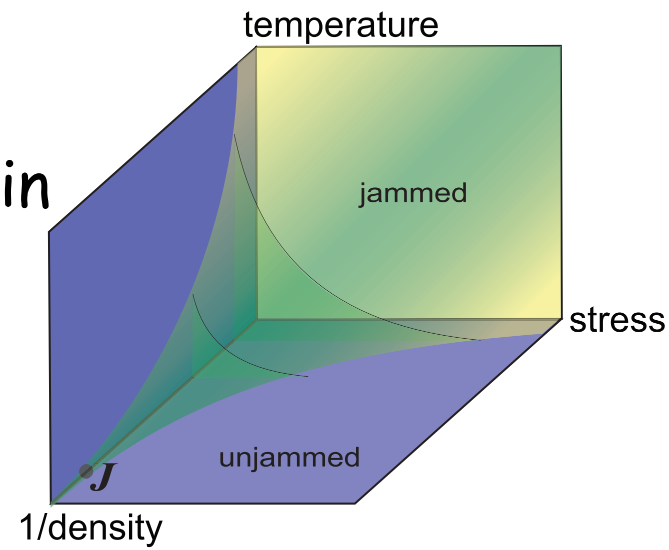
Vogel-Fulcher form

$$f(x) = A\exp\left[B\left(\frac{1}{x} - \frac{1}{x_1}\right)^2\right]$$

Elmatad-Chandler-Garrahan form

- Hard spheres are very fragile!
- Can't distinguish between V-F form and E-C-G form
- Can't tell if there is a thermodynamic glass transition or not
- But if not, then Point J controls dynamical glass transition

Conclusions



- Point J is a special point
- Hint of connection to glass transition in exponents for jamming transition
- Similarity in form of slow down in dynamics due to equivalence of
  - hard sphere glass transition
  - thermal glass transition of soft spheres at low p
- Hard spheres tell us everything about soft spheres
- Point J controls dynamical glass transition of hard spheres if thermodynamic glass transition does not exist
- Still ahead: **attractions**

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Matthieu Wyart Princeton

Anton Souslov UPenn
Tom Lubensky UPenn

Lynn Daniels UPenn
Doug Durian UPenn

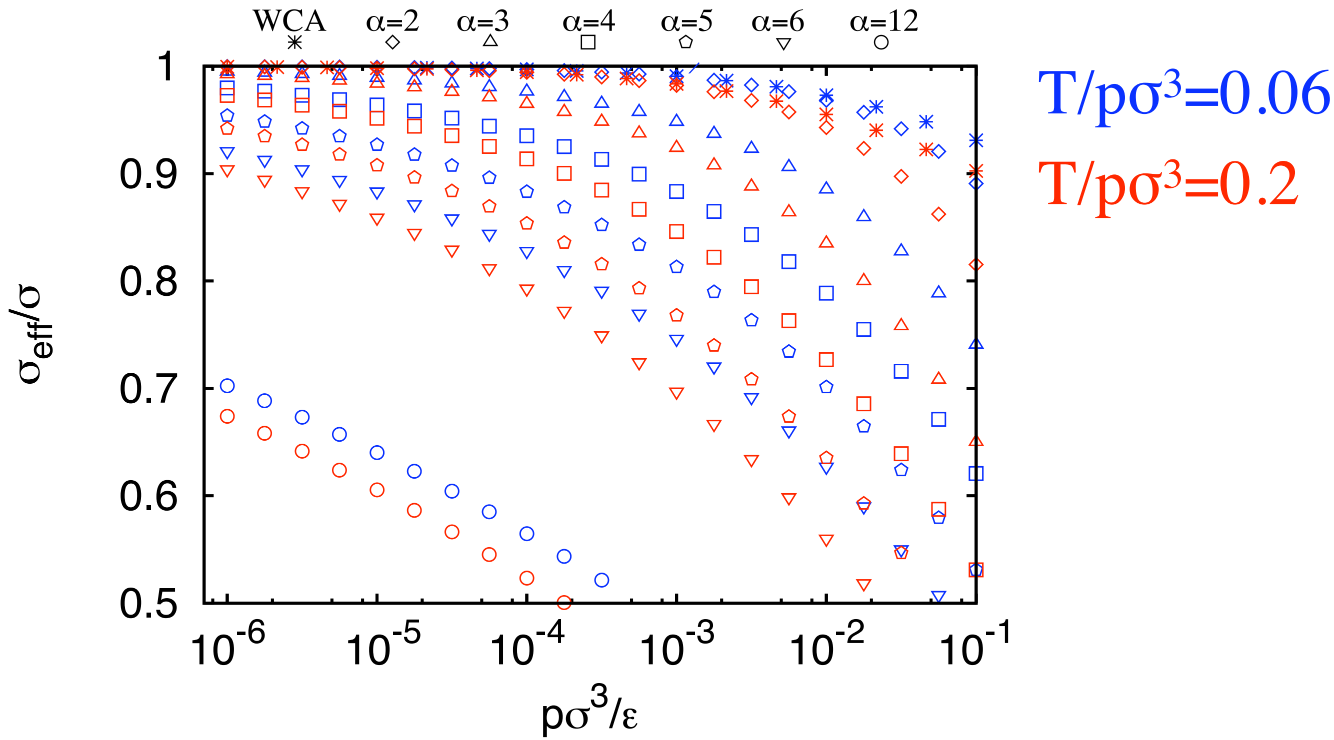
Zexin Zhang UPenn
Ke Chen UPenn
Arjun Yodh UPenn

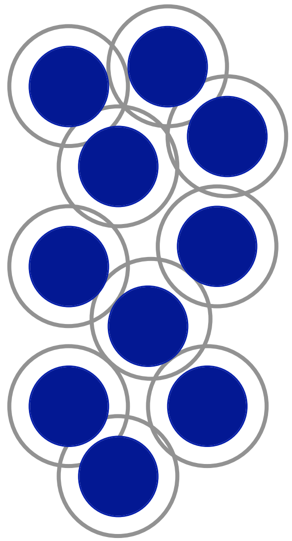
Randy Kamien UPenn

Bread for Jam:

DOE DE-FG02-03ER46087
German Academic Exchange Service (DAAD)

Effective diameter



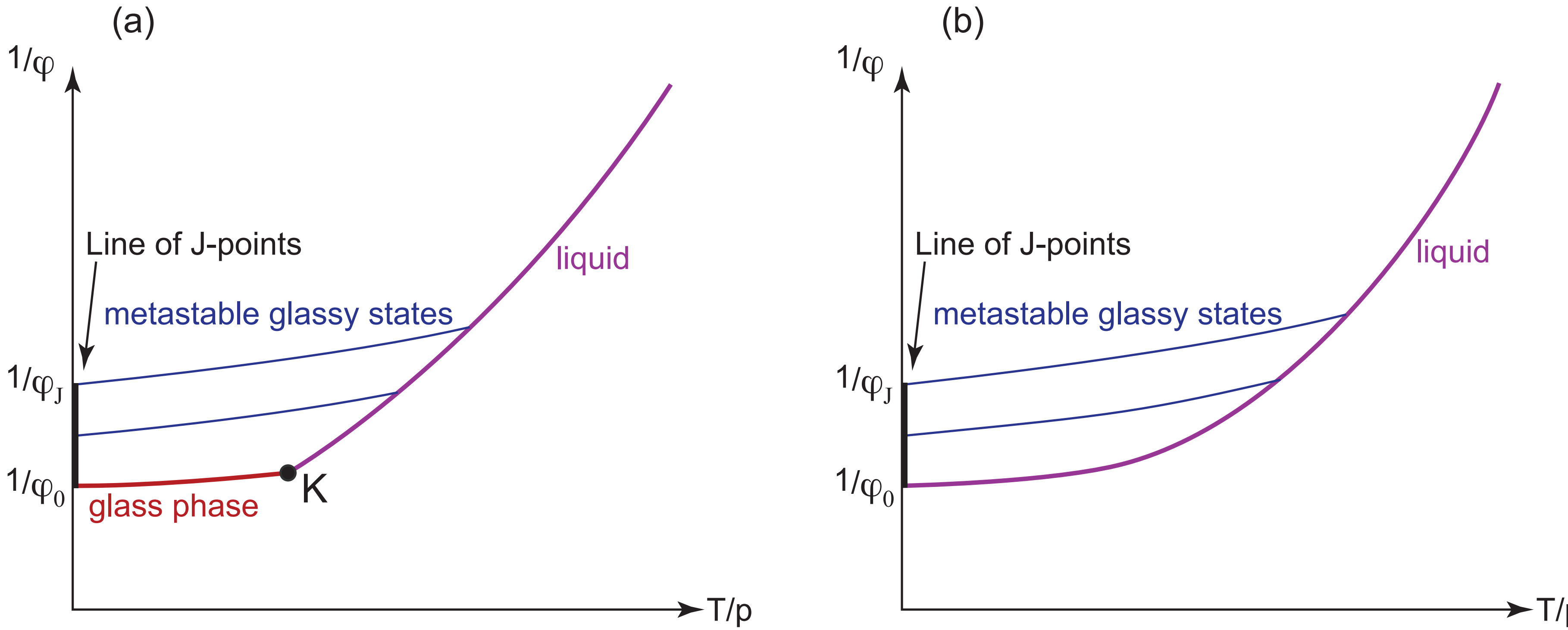


Hard spheres with reduced diameter σ_{eff}

- As $p\sigma^3/\varepsilon$ increases, σ_{eff} decreases
- As $T/p\sigma^3$ increases, σ_{eff} decreases

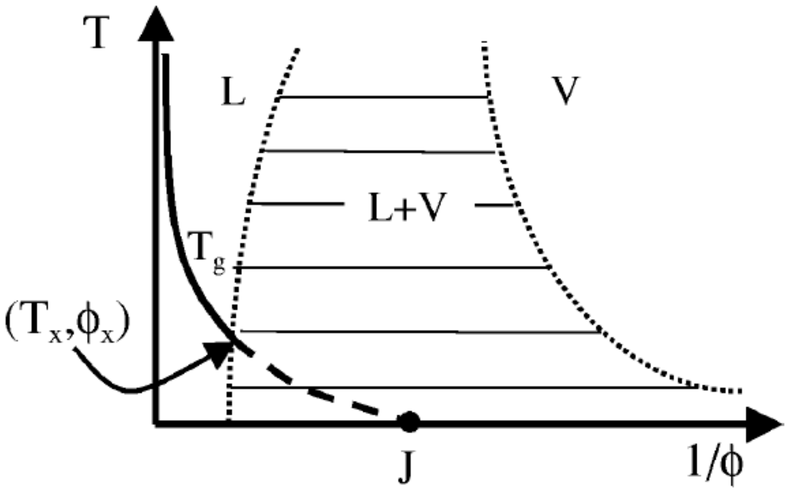
$$V(r) = \begin{cases} \dfrac{\varepsilon}{\alpha}\left(1-\dfrac{r}{\sigma}\right)^{\alpha} & r \le \sigma \\ 0 & r > \sigma \end{cases}$$

Jamming vs. Glass Transition



BUT: Real Liquids have Attractions

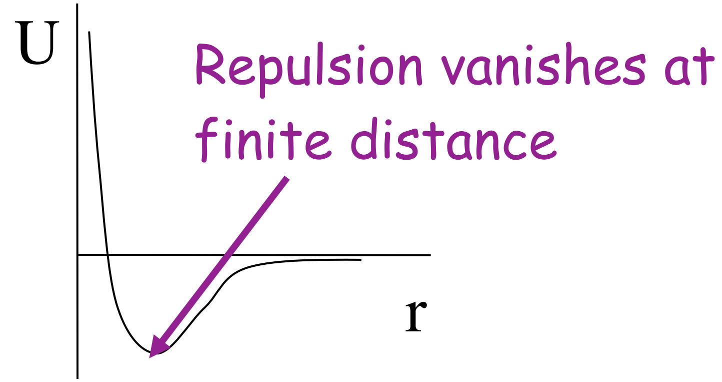
- Point J only exists for repulsive, finite-range potentials
- Attractions can lead to vapor-liquid transition which preempts Point J



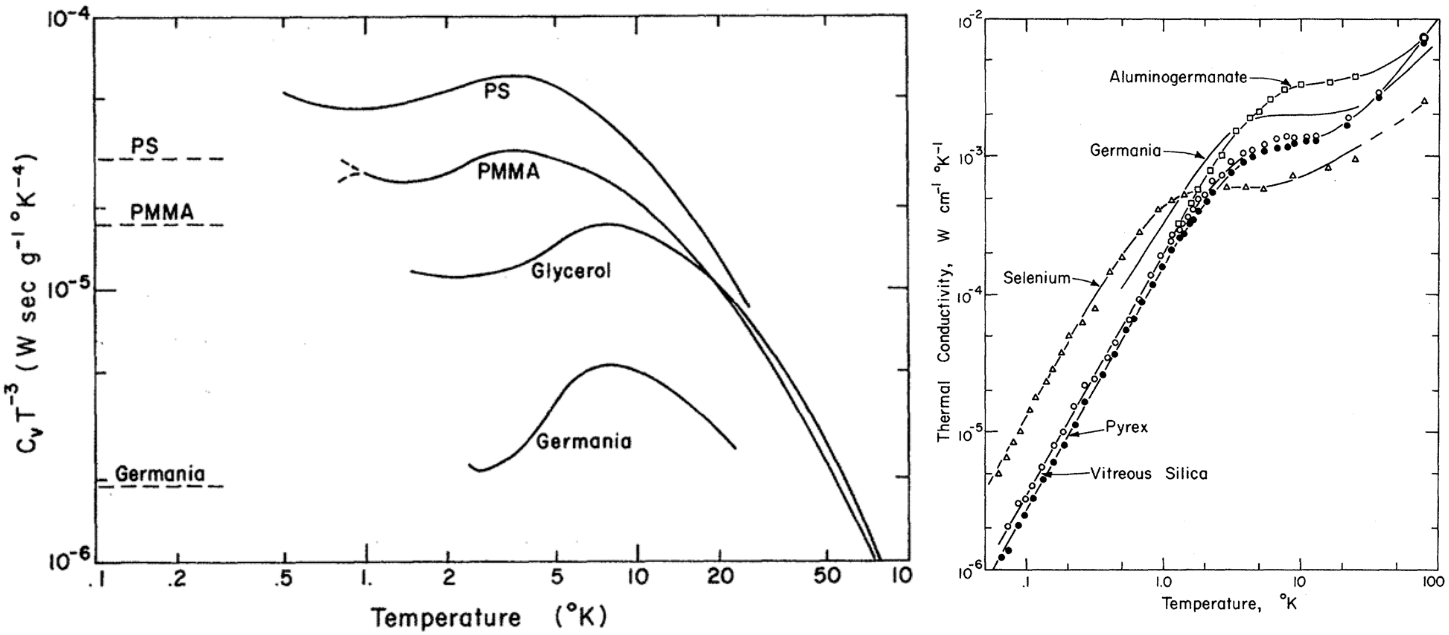
But there is hope:

Attractions serve to hold system at high enough density that repulsions come into play (WCA)

Behavior of liquids is controlled by repulsions, and attractions are perturbation



Similar Behavior in Jammed State



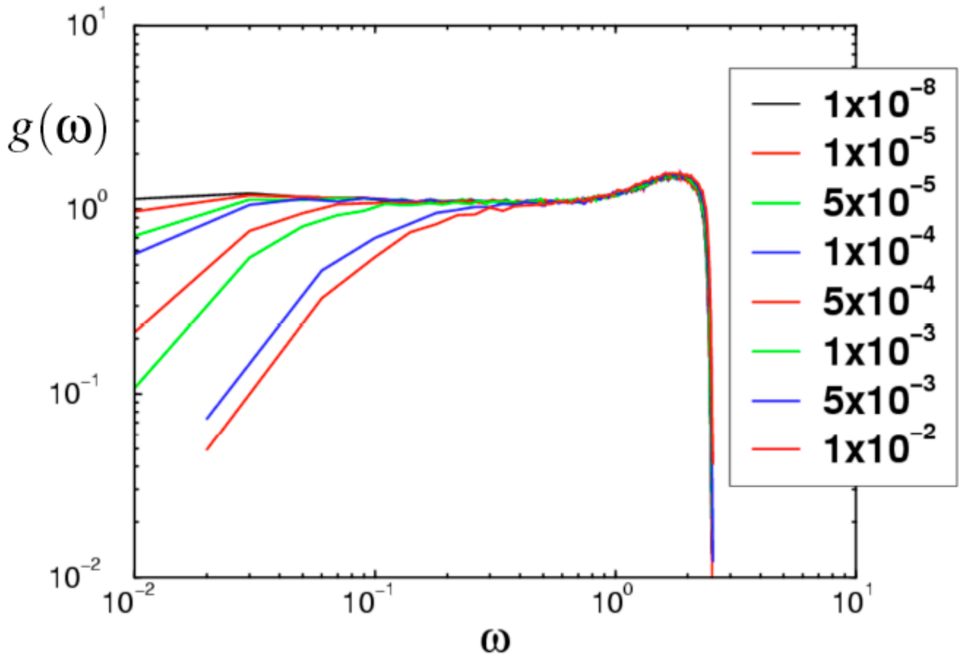
Zeller & Pohl, PRB 4, 2029 (1971).

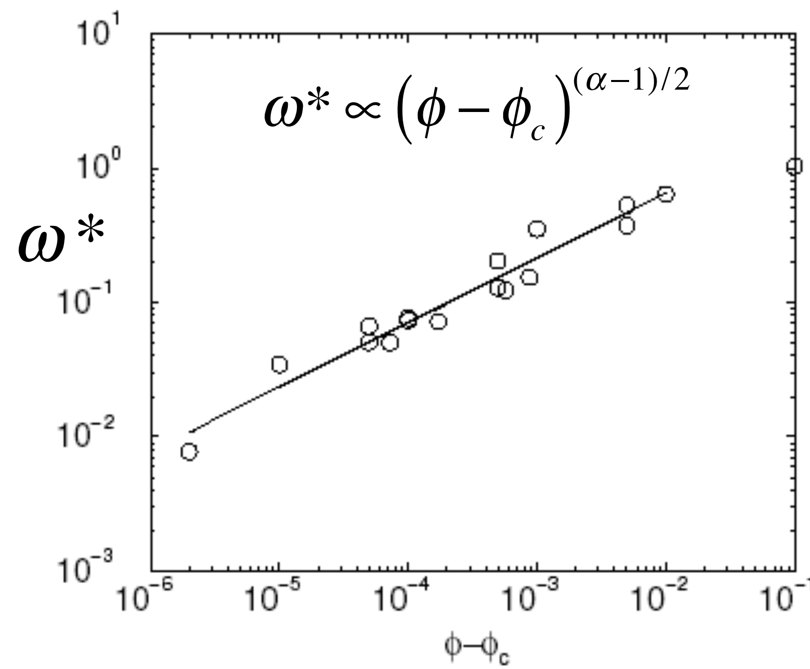
- Behavior of amorphous solids is
  - very different from that of crystals
  - same in all amorphous solids
  - still not understood

Marginally Jammed Solid

L. E. Silbert, A. J. Liu, S. R. Nagel, PRL **95**, 098301 ('05)

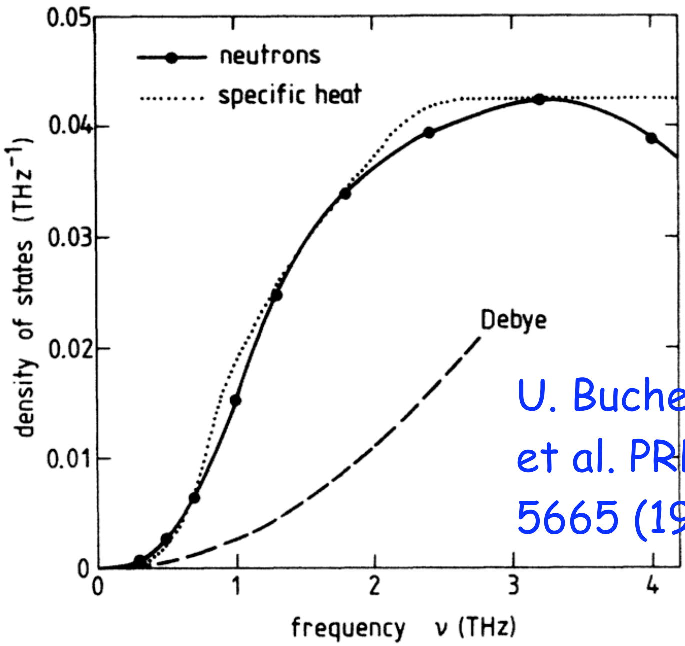
Density of Vibrational Modes



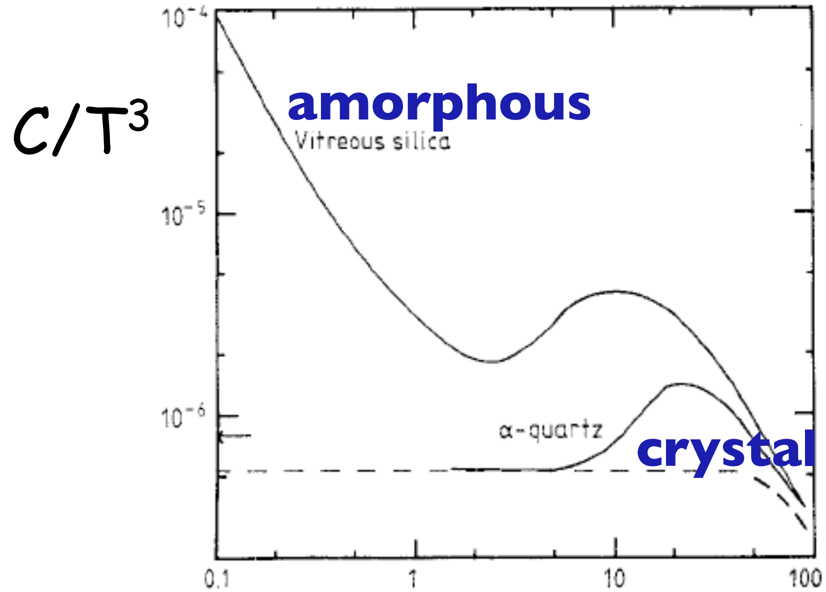


- Density of states is not Debye-like at low ω
- Result of isostaticity M. Wyart, S.R. Nagel, T.A. Witten, EPL **72**, 486 (05)
- Scaling of ω* is robust to systems near isostaticity Souslov, Liu, Lubensky, PRL **103**, 205503 (2009); Mao, Xu, Lubensky arXiv: 09092616

Boson Peak

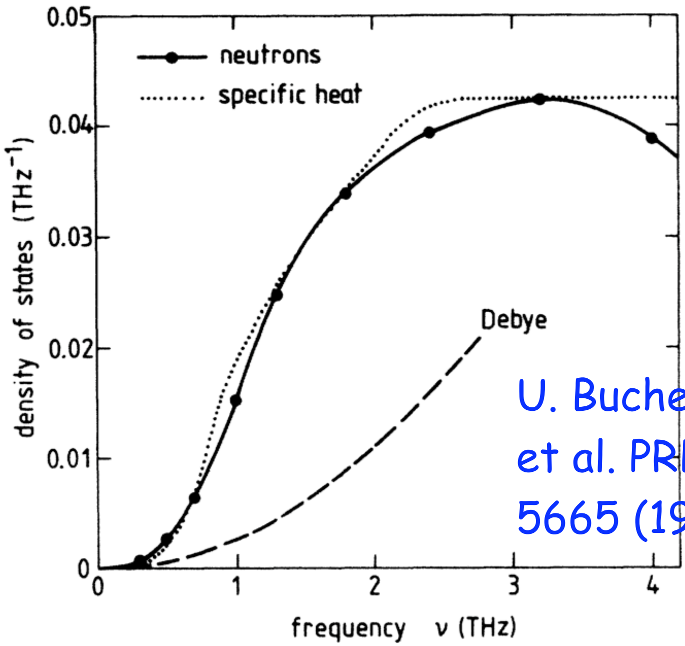


U. Buchenau, et al. PRB **34**, 5665 (1986)

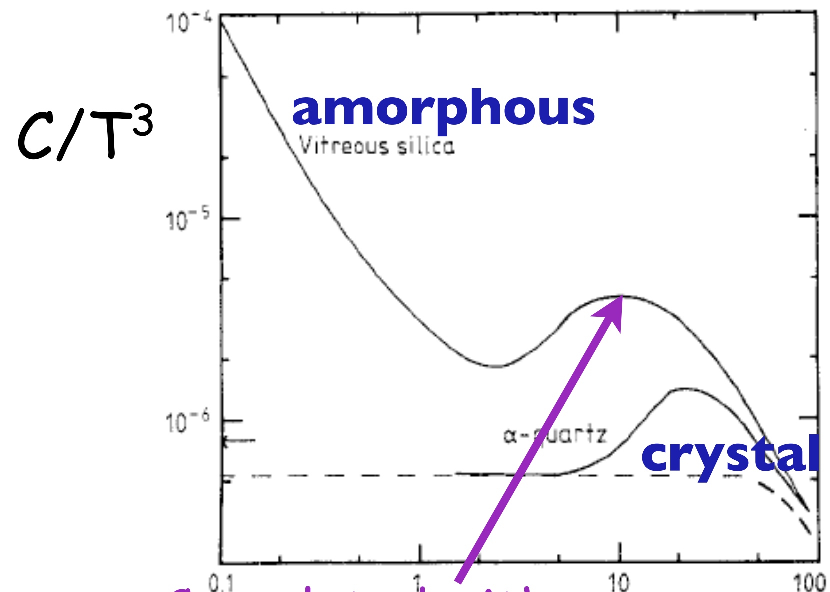


- Excess in density of states is tied to peak in heat capacity $C(T)/T^3$
- This frequency/temperature can be tuned in jammed packings by varying $\phi - \phi_c$

Boson Peak



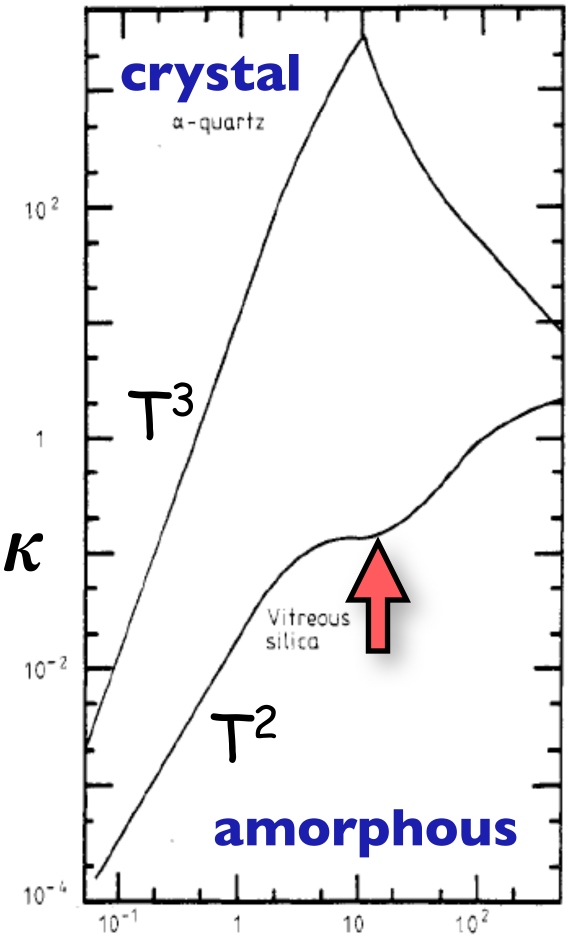
U. Buchenau, et al. PRB **34**, 5665 (1986)



Correlated with boson peak frequency at ω^*

- Excess in density of states is tied to peak in heat capacity $C(T)/T^3$
- This frequency/temperature can be tuned in jammed packings by varying $\phi - \phi_c$

Thermal Conductivity



$$\kappa = \frac{1}{V}\sum_i C_i(T) d_i$$

P. B. Allen and J. L. Feldman, PRB **48**,12581 (1993).

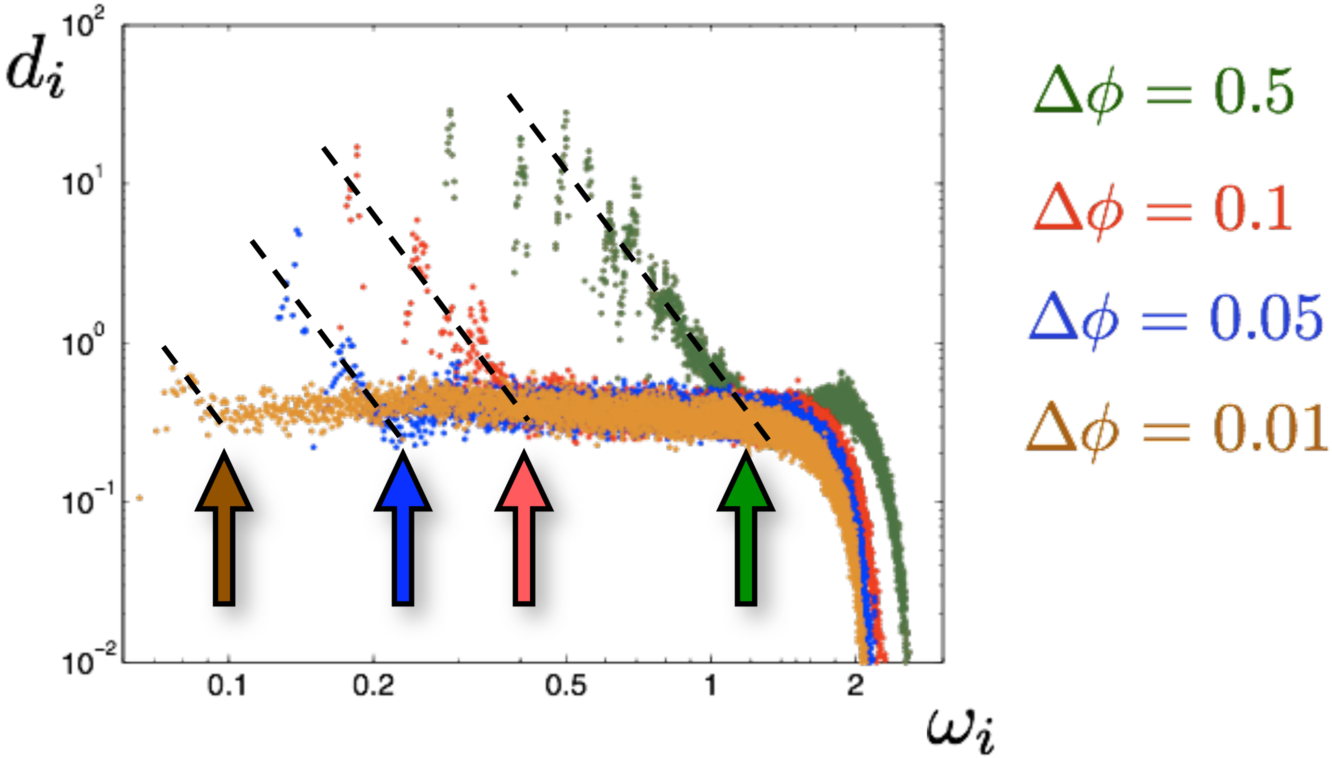
thermal conductivity — heat carried by mode i — diffusivity of mode i

Kubo formulation

$$d_i = \frac{\pi}{3\hbar^2\omega_i^{\,2}} \sum_{i\neq j} \left|S_{ij}\right|^2 \delta\left(\omega_i - \omega_j\right)$$

Kittel's 1949 hypothesis: rise in κ above plateau due to regime of freq-independent diffusivity

Ioffe-Regel Crossover

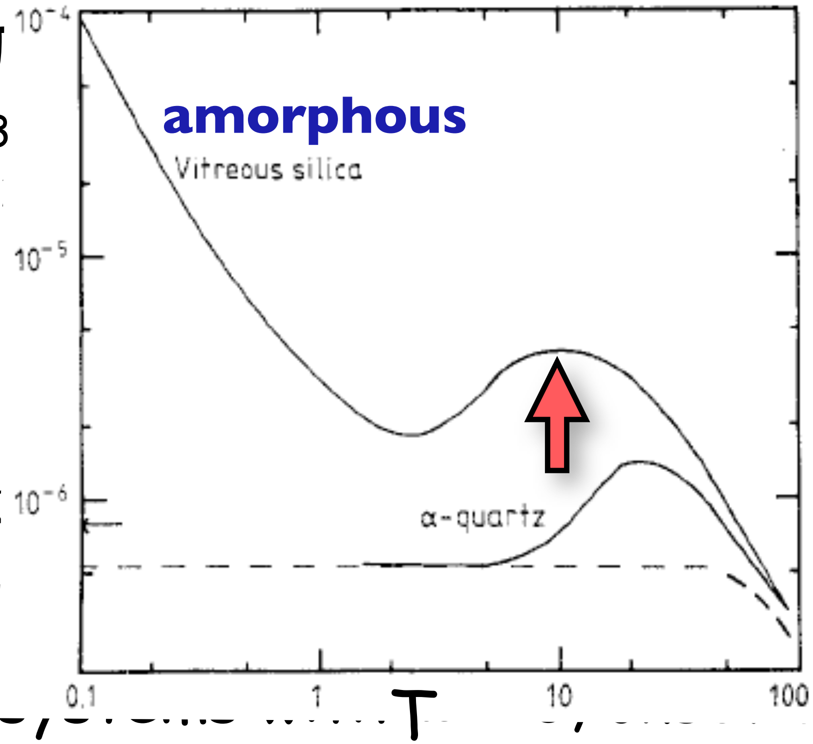


- Crossover from weak to strong scattering at ω^*
- At Point J, the diffusivity is flat down to ω=0
- Freq-indep diffusivity originates from Point J

N. Xu, V. Vitelli, M. Wyart, A. J. Liu, S. R. Nagel, PRL **102**, 038001 (2008).
V. Vitelli, N. Xu, M. Wyart, A. J. Liu, S. R. Nagel, arXiv:0908.2176

Consequences for Thermal Conductivity

- At J … rox
  C/T^3



ω_D

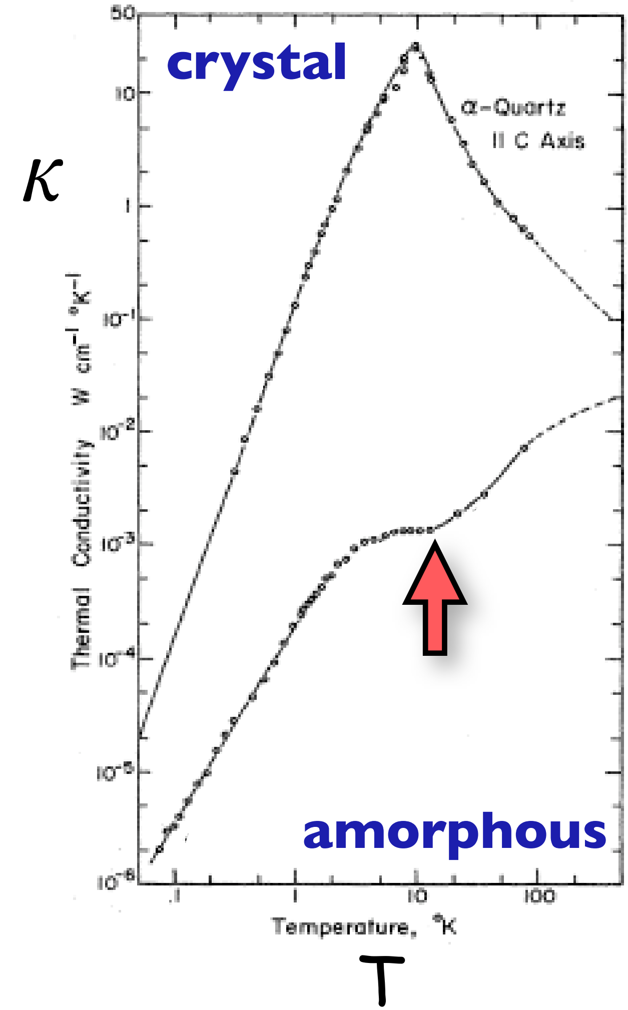
- So κ … satu
- For … f this

behavior occurs at $kT^* \sim \hbar\omega^*$

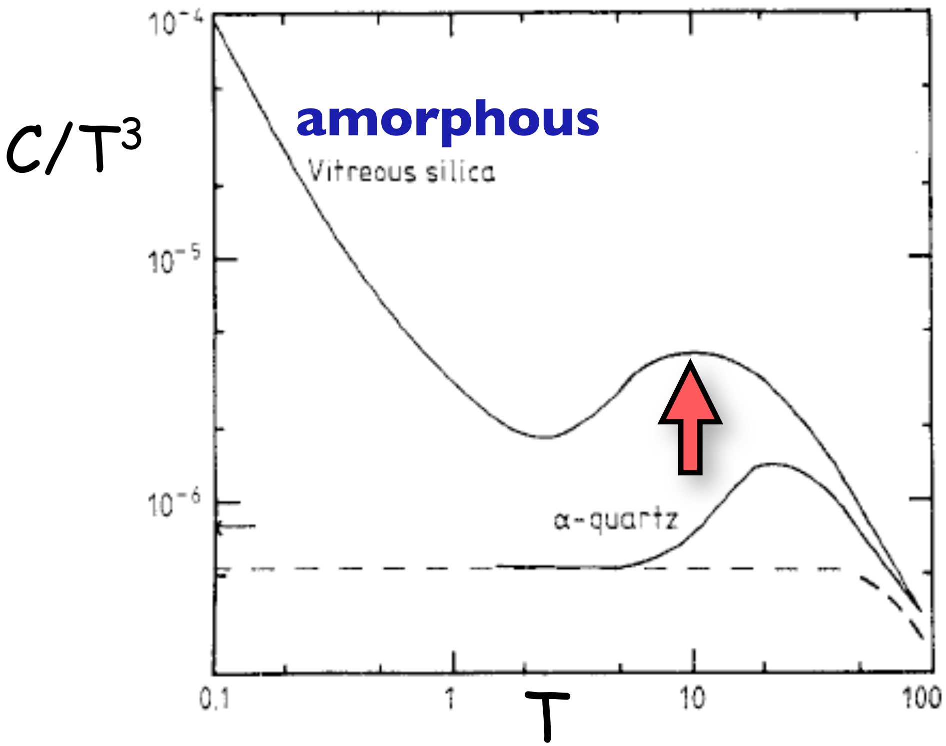
boson peak in C

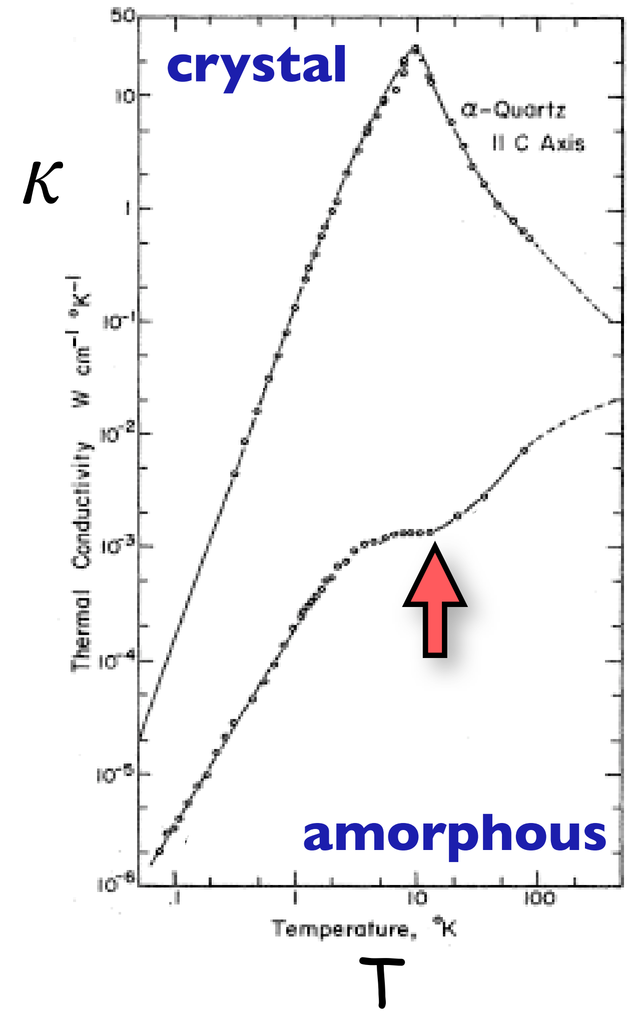
end of plateau in κ

tied together through ω^*



Consequences for Thermal Conductivity



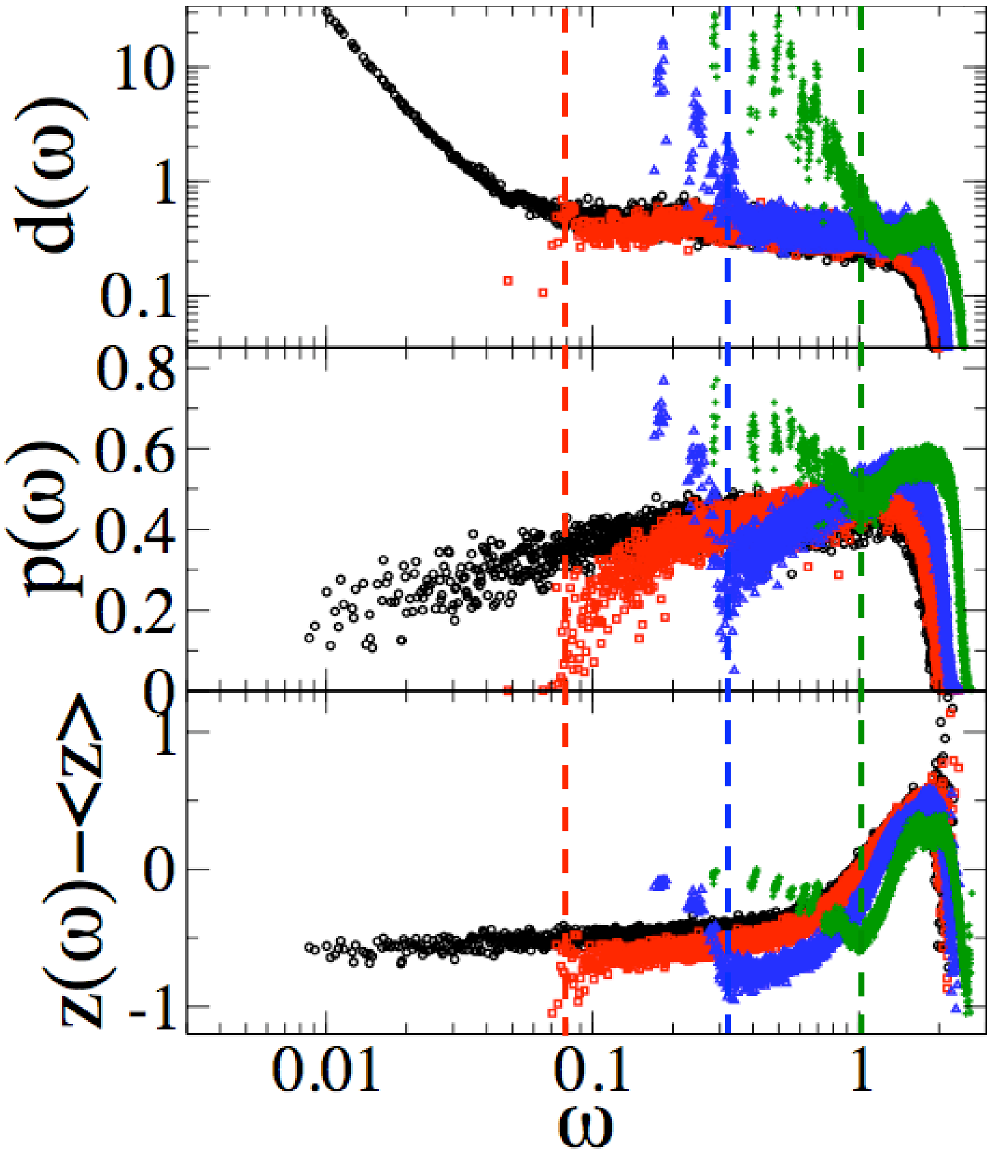


boson peak in C

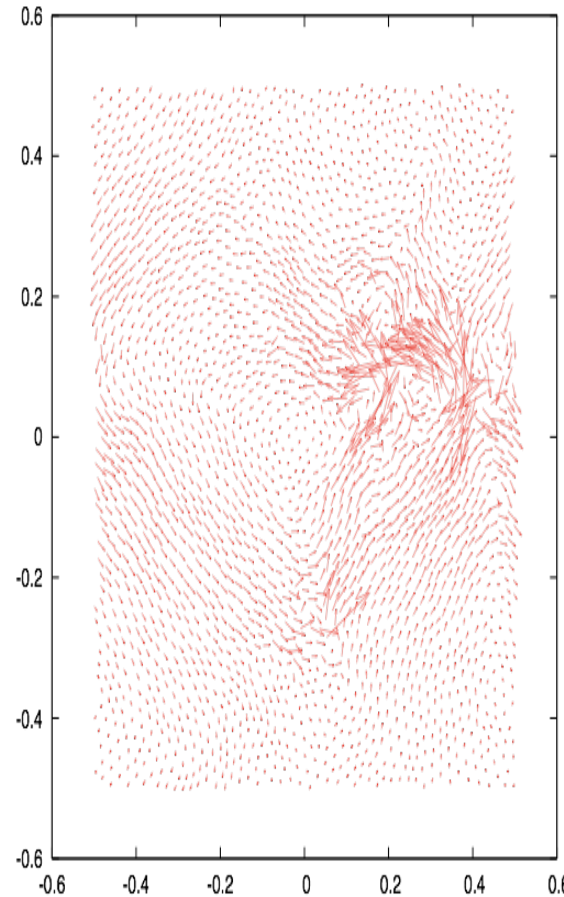
end of plateau in κ

tied together through ω^*

Quasilocalized Modes at ω*

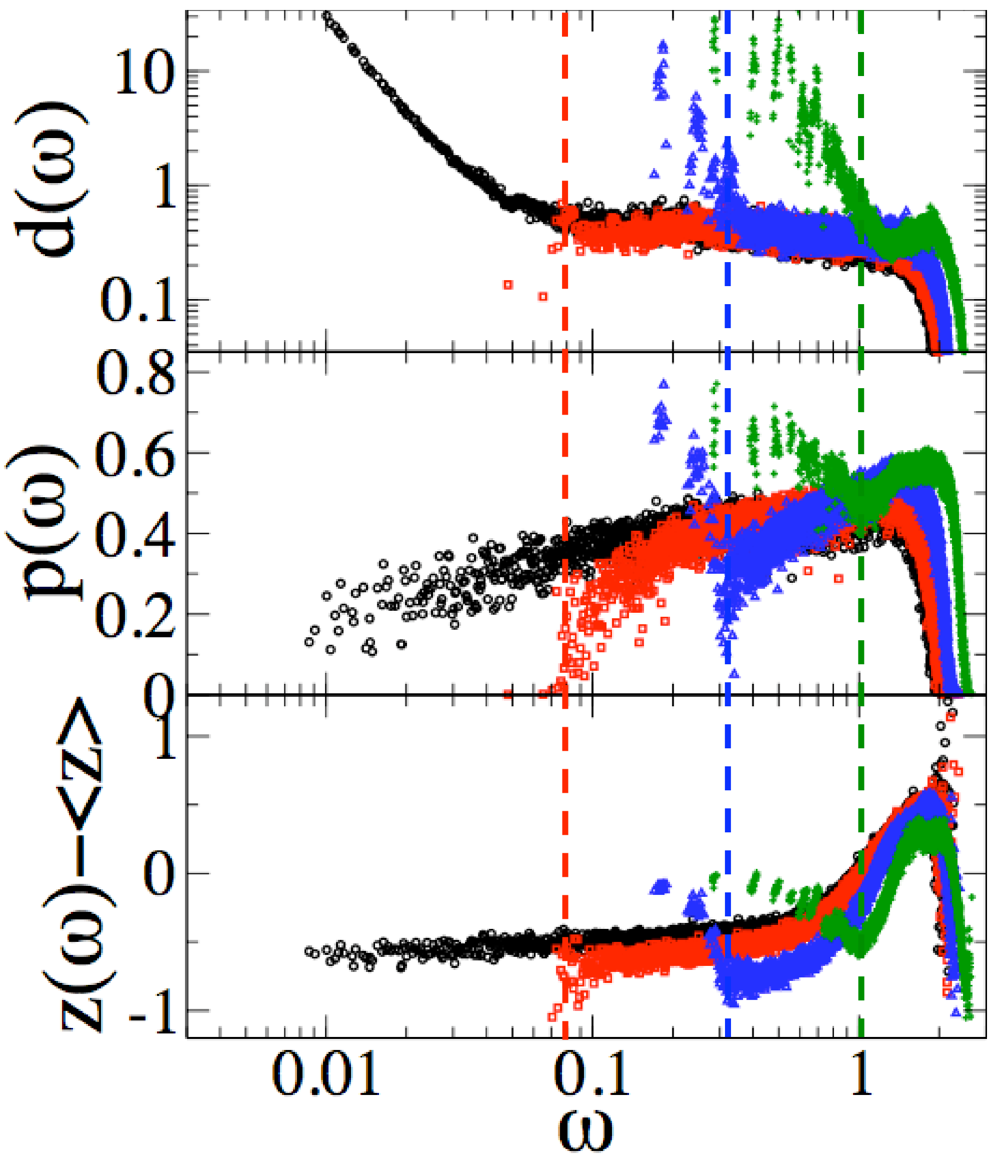


$$p_i = \frac{(\sum_\alpha |\epsilon_i(\alpha)|^2)^2}{\sum_\alpha |\epsilon_i(\alpha)|^4}$$



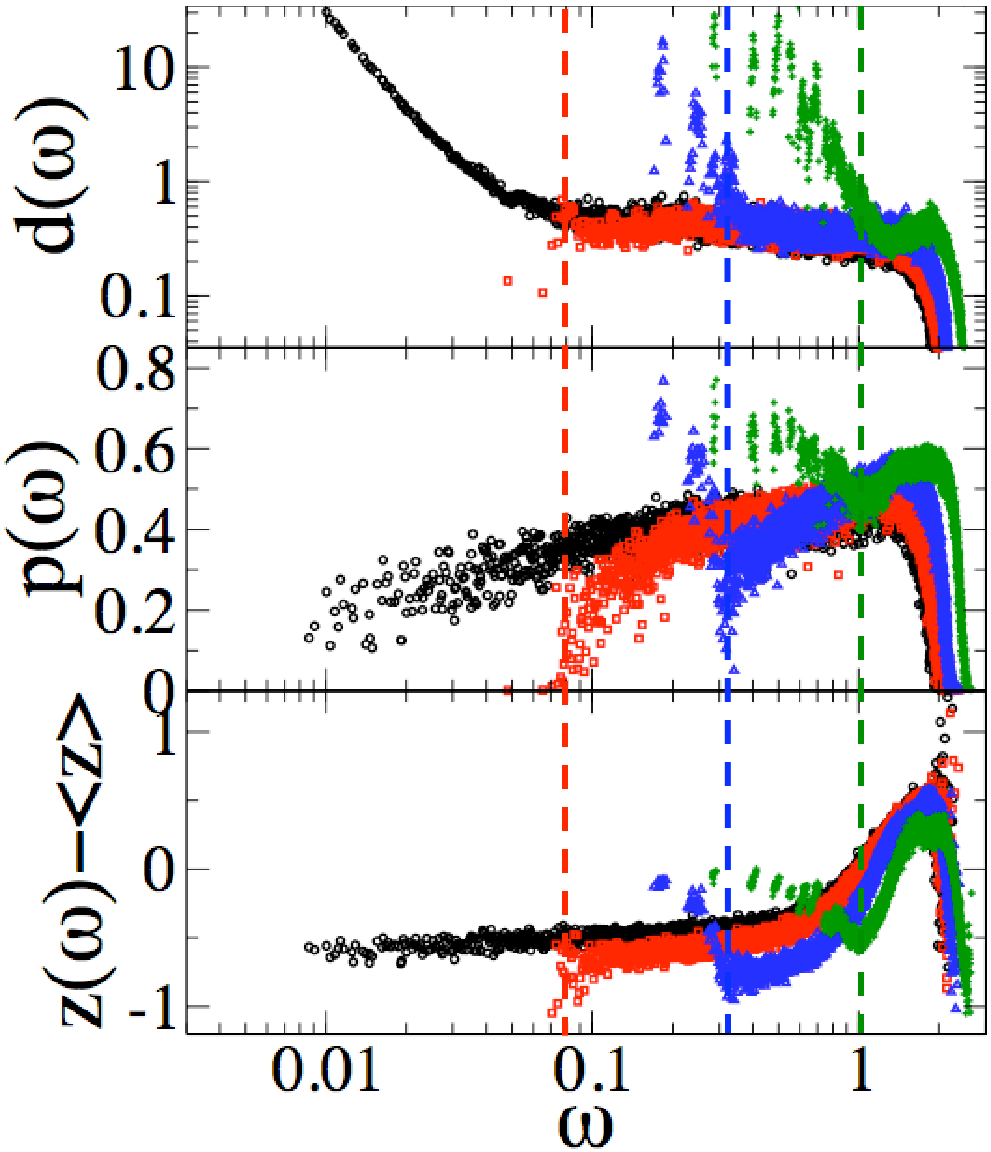
N. Xu, Vi Vitelli, A. J. Liu, S. R. Nagel, arXiv:0909.3710

Quasilocalized Modes at ω*



N. Xu, Vi Vitelli, A. J. Liu, S. R. Nagel, arXiv:0909.3710

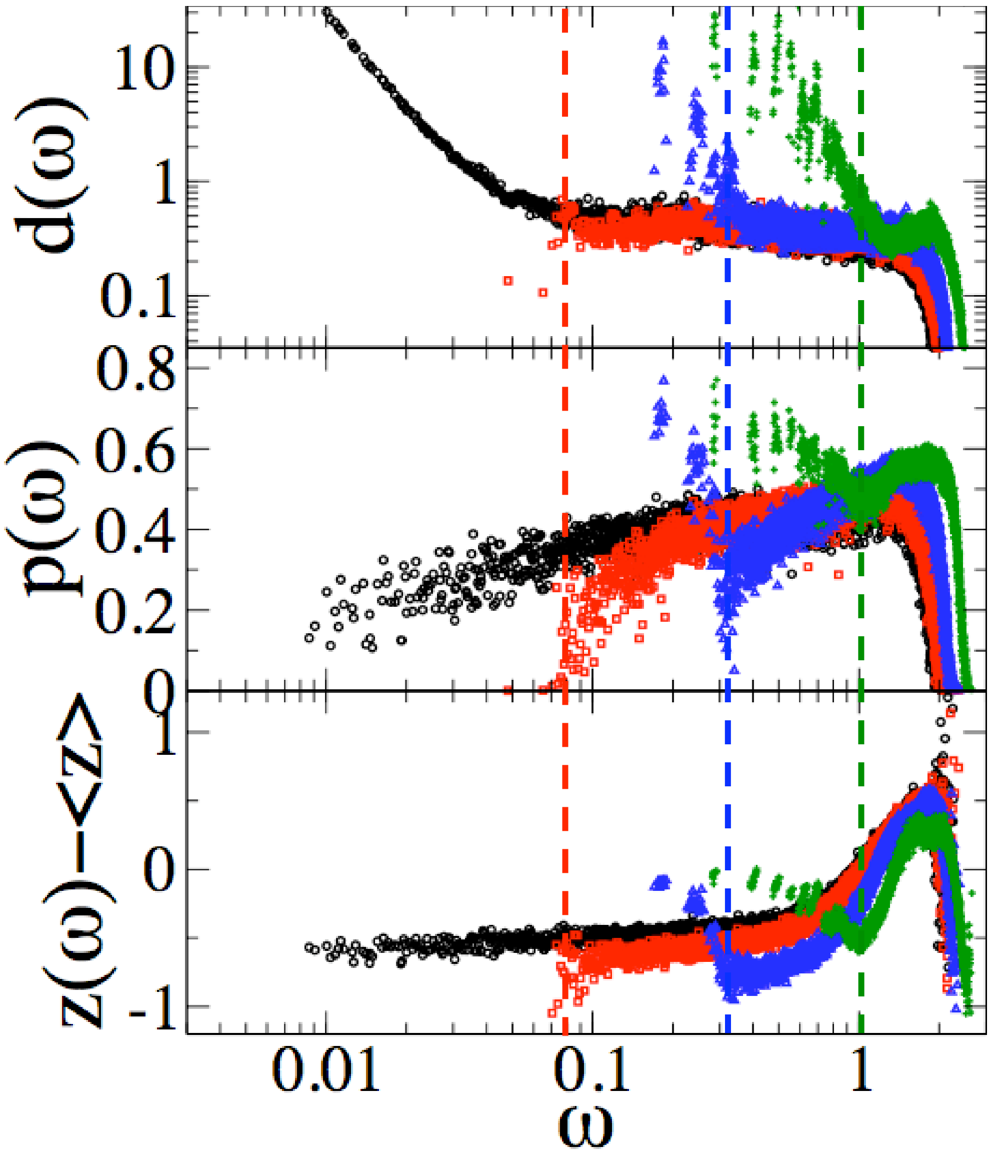
Quasilocalized Modes at ω*



- Modes become quasilocalized near Ioffe-Regel crossover

N. Xu, Vi Vitelli, A. J. Liu, S. R. Nagel, arXiv:0909.3710

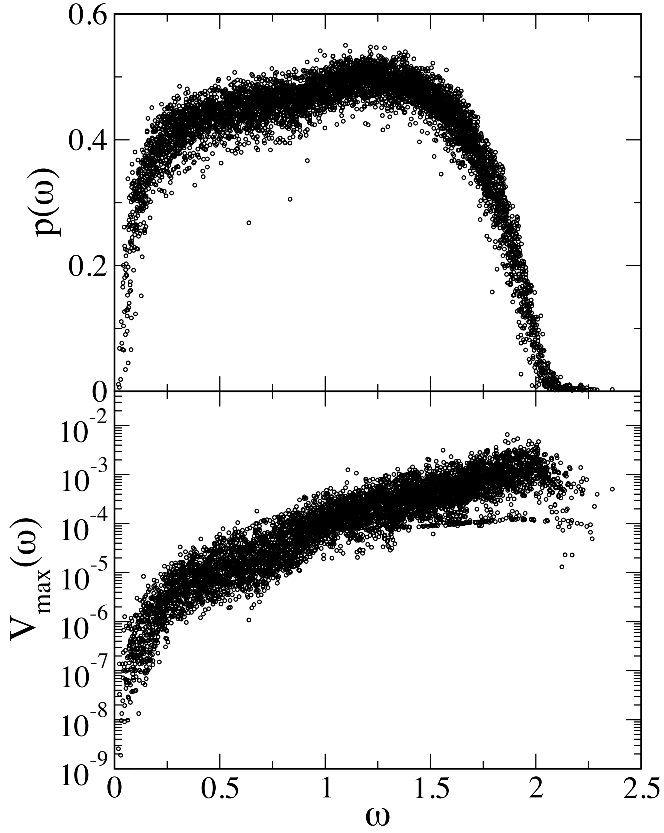
Quasilocalized Modes at ω*



- Modes become quasilocalized near Ioffe-Regel crossover
- High displacements occur in low-coordination regions

N. Xu, Vi Vitelli, A. J. Liu, S. R. Nagel, arXiv:0909.3710

Anharmonicity

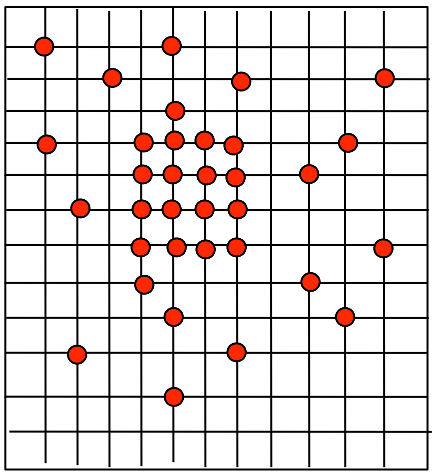


- The low-frequency quasi-localized modes have the lowest energy barriers to rearrangement
  - two-level systems?
  - STZ's?

N. Xu, Vi Vitelli, A. J. Liu, S. R. Nagel, arXiv:0909.3710

K-Core Percolation

- Culling process
  - Occupied sites with fewer than k occupied neighbors become vacant
- Repeat culling process until no more can be removed
- Remaining occupied sites called the k-core



k=3

K-Core Percolation

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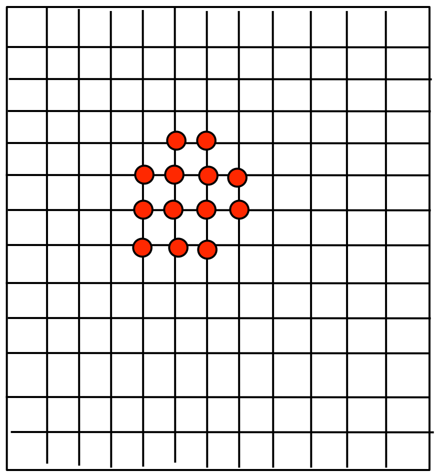

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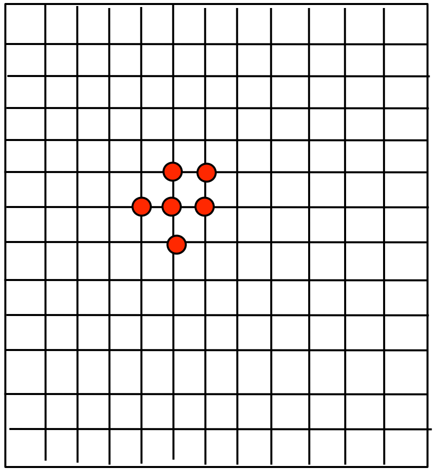

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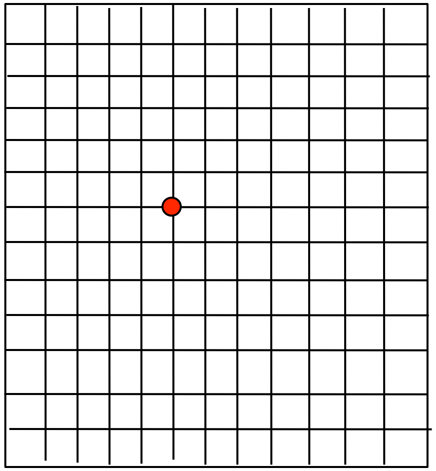

k=3

K-Core Percolation

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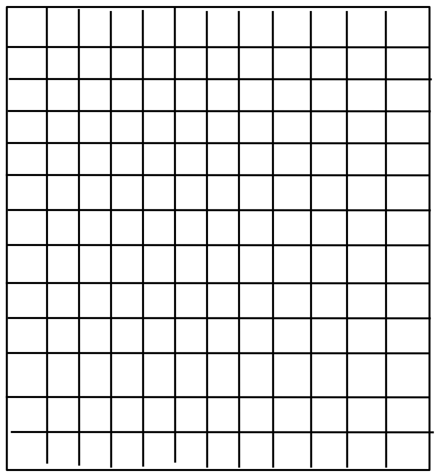


k=3

Jamming vs k-Core (Bootstrap) Percolation

J. M. Schwarz, A. J. Liu, L. Chayes, EPL **73**, 560 (2006).

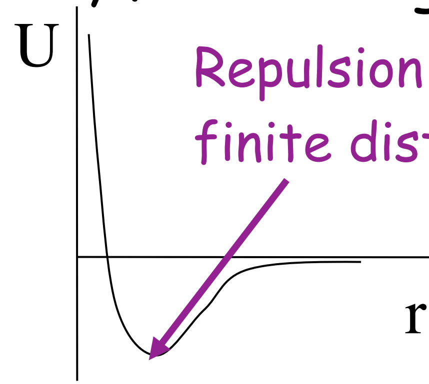
- Jammed configs at T=0 are mechanically stable
- For particle to be locally stable, it must have at least d +1 overlapping neighbors in d dimensions
- Each of its overlapping nbrs must have at least d+1 overlapping nbrs, etc.
- At $\phi > \phi_c$ all particles in load-bearing network have at least d+1 neighbors

- Consider lattice with coord. # Z_{max} with sites indpendently occupied with probability p
- For site to be part of "k-core", it must be occupied and have at least k=d+1 occupied neighbors
- Each of its occ. nbrs must have at least k occ. nbrs,etc
- Look for percolation of k-core

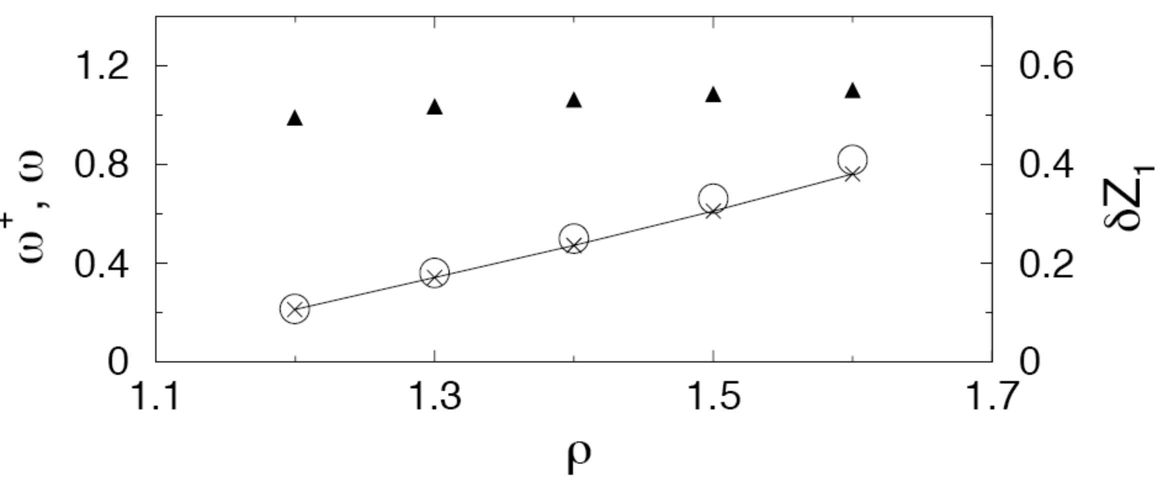
Long-Ranged Interactions/Attractions

- Point J only exists for repulsive, finite-range potentials
- Real liquids have attractions

Attractions serve to hold system at high enough density that repulsions come into play (WCA)



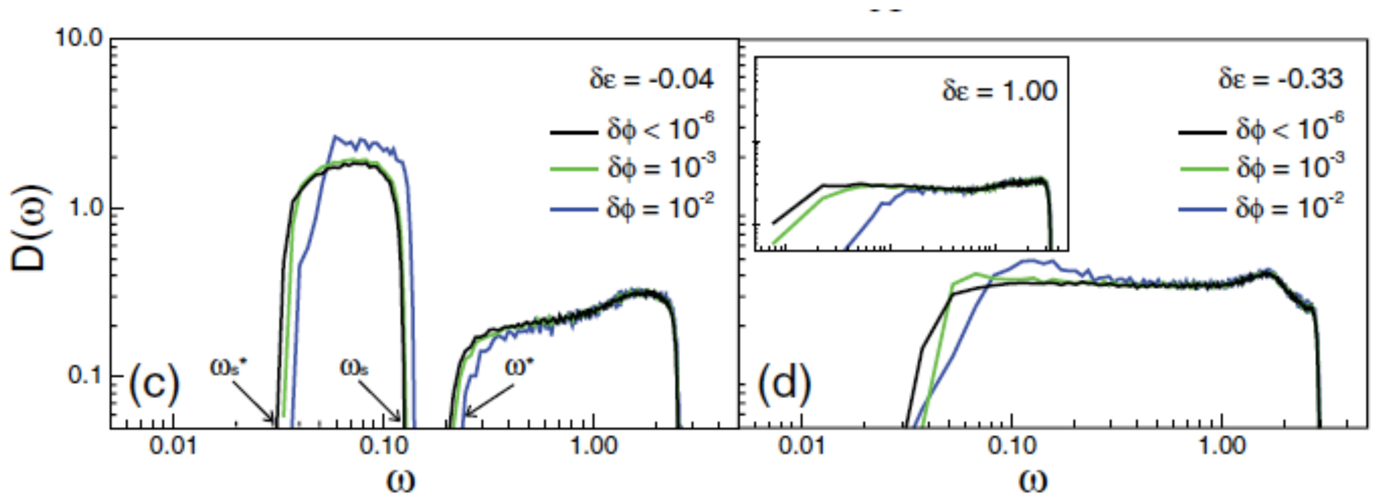
- Excess vibrational modes (boson peak) believed responsible for unusual low temp properties of glasses
- These modes derive from the excess modes near Point J

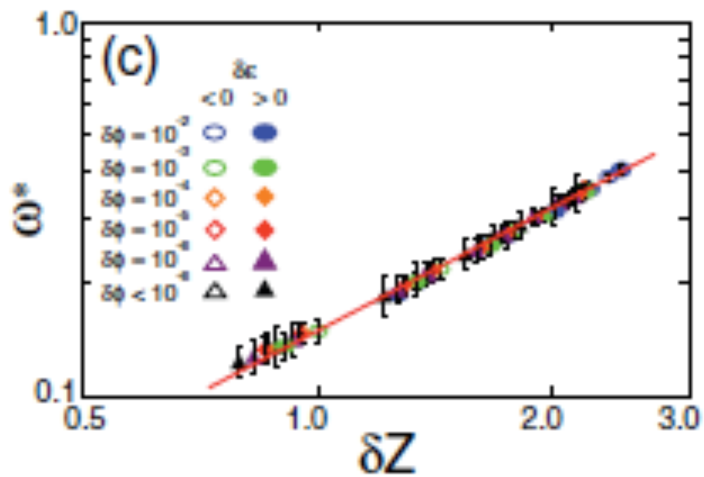


N. Xu, M. Wyart, A. J. Liu, S. R. Nagel, PRL **98**, 175502 (2007).

Effect of Particle Shape

- Ellipsoids



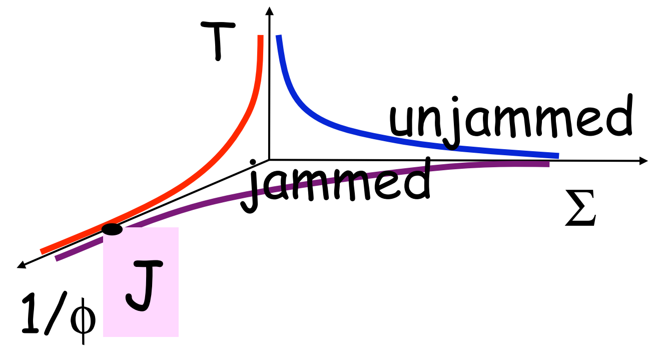


Z. Zeravcic, N. Xu, A. J. Liu, S. R. Nagel, W. van Saarloos, EPL, **87**, 26001 (2009).

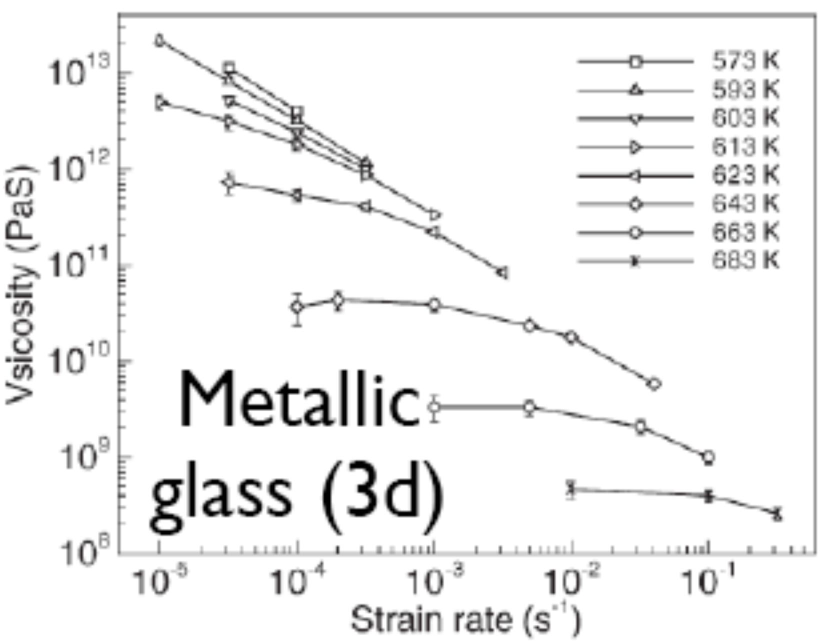
- Introduce new rotational band but onset of translational band still scales as for spheres
- Ellipsoids controlled by Point J for spheres

What happens at Σ>0?

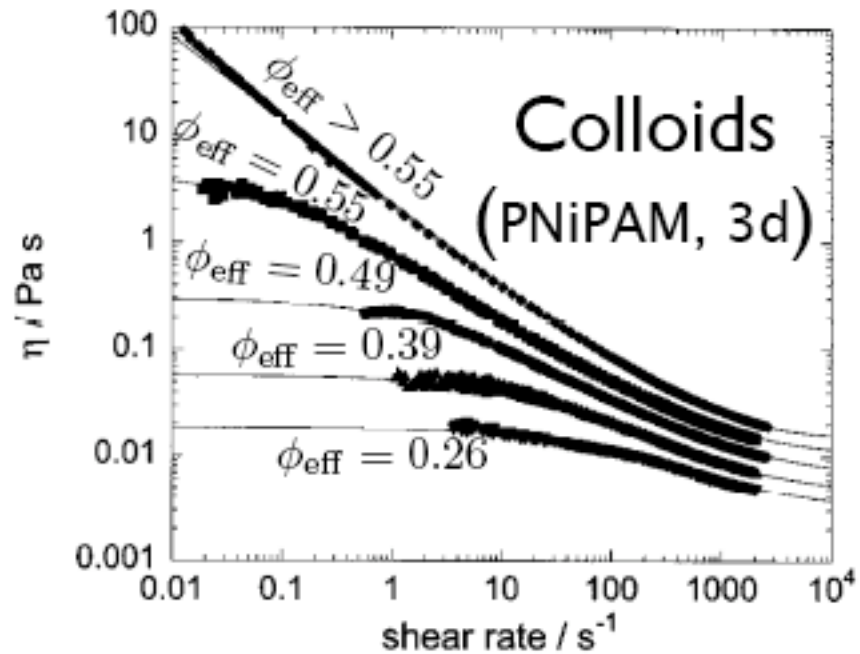
Look at rheology: stress Σ vs. T, p, $\dot{\gamma}$



Glasses & colloidal glasses shear thin--is there a connection?



J. Lu, et al, Acta Mater. **51** 3429 (2003)



H. Senff & W. Richtering, JCP **111** 1705 (1999)

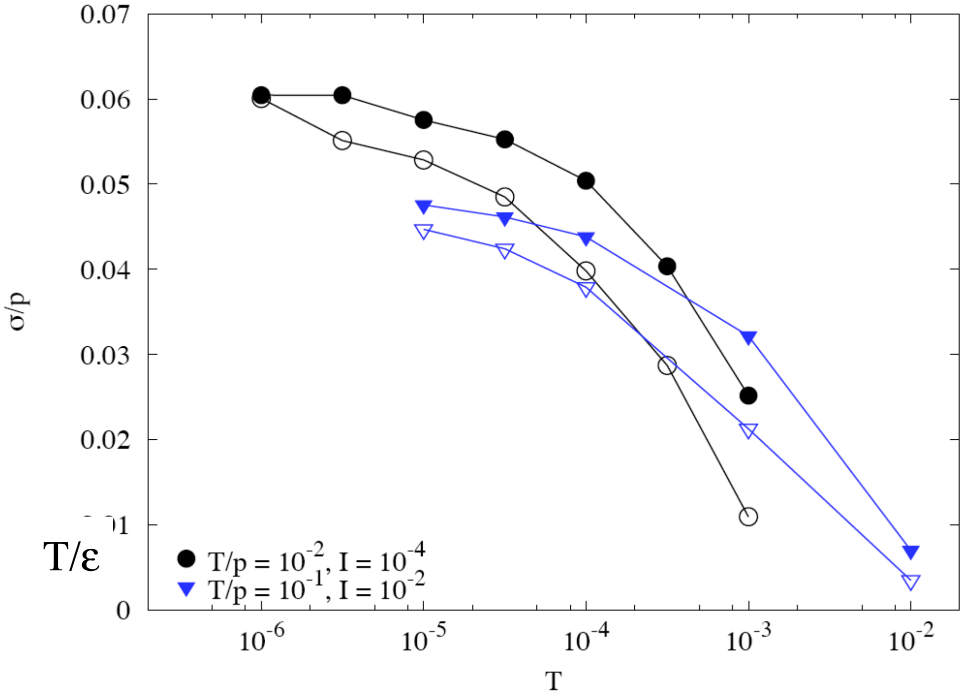
Dimensional Analysis

T. Haxton (2D)

$$\frac{\Sigma\sigma^2}{\varepsilon} = H\left(\frac{T}{\varepsilon}, \dot{\gamma}\sqrt{\frac{m\sigma^2}{\varepsilon}}, \frac{p\sigma^2}{\varepsilon}\right)$$

or equivalently,

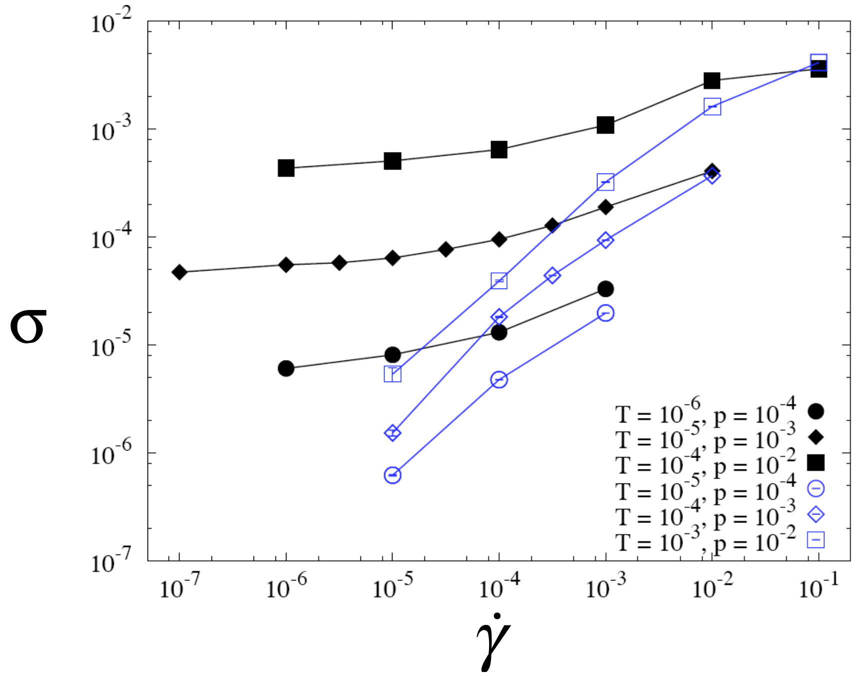
$$\frac{\Sigma}{p} = G\left(\frac{T}{p\sigma^2}, I = \dot{\gamma}\sqrt{\frac{m}{p}}, \frac{p\sigma^2}{\varepsilon}\right)$$

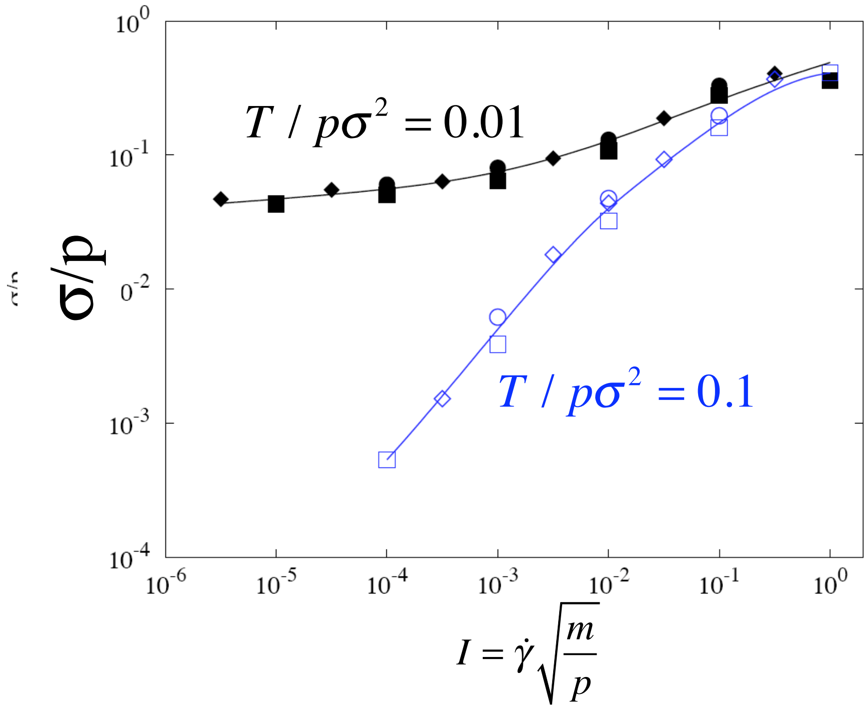


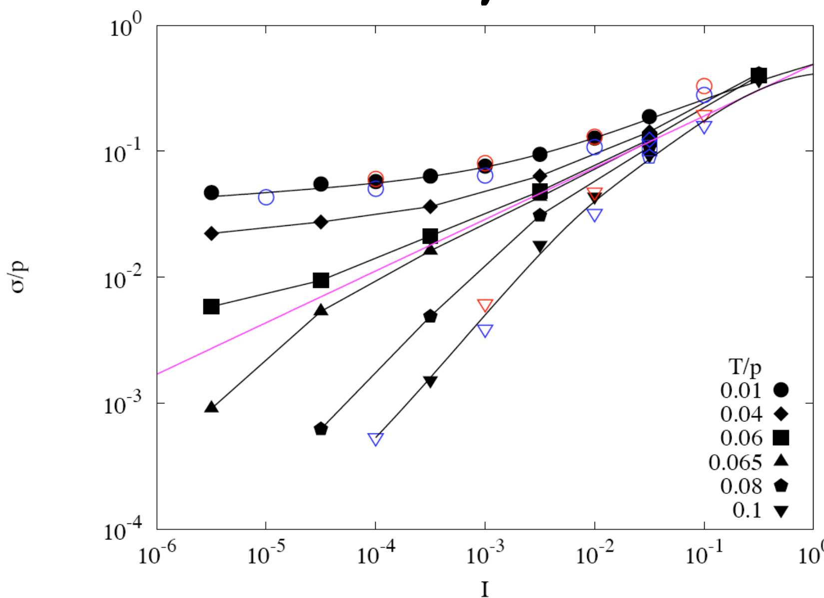
- Stress is indep of potential for small T, p

$$\lim_{p\sigma^2/\varepsilon\to 0}\frac{\Sigma}{p} = F\left(\frac{T}{p\sigma^2}, I \equiv \dot{\gamma}\sqrt{\frac{m}{p}}\right)$$

Data Collapse for Rheology





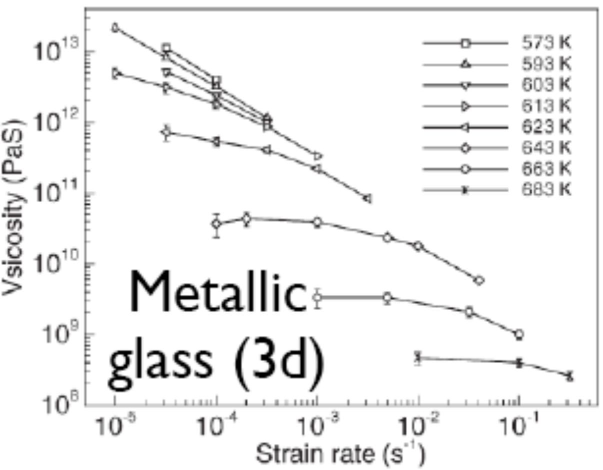


$$\lim_{p\sigma^2/\varepsilon \to 0} \frac{\Sigma}{p} = F\left(\frac{T}{p\sigma^2}, I = \dot{\gamma}\sqrt{\frac{m}{p}} \right)$$

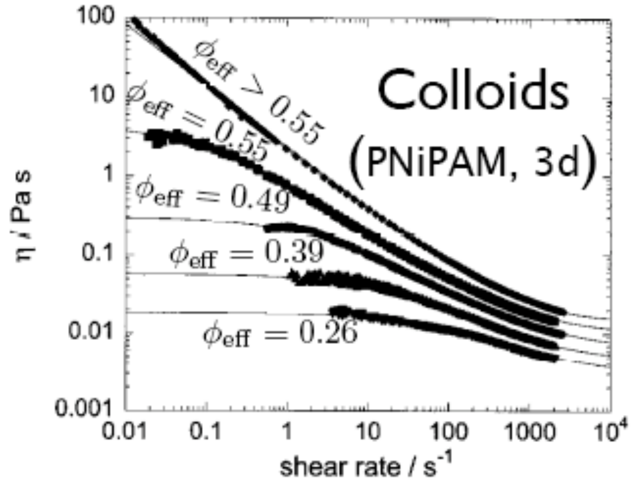
High $T/p\sigma^2$, low I: viscous

Low $T/p\sigma^2$, low I: elastic

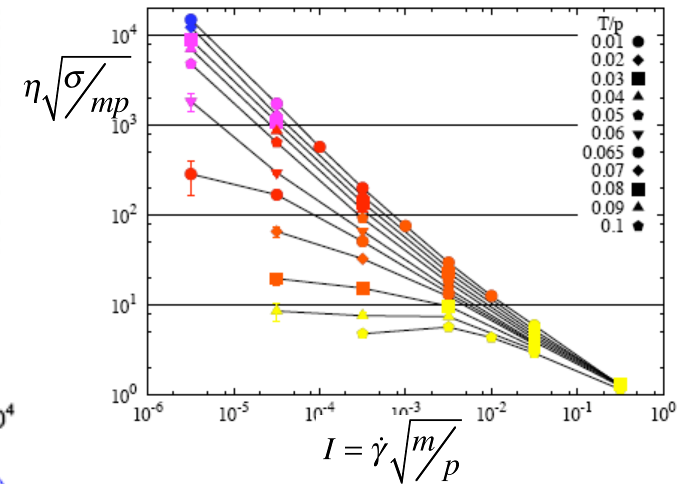
Consequences of Rheology Collapse



J. Lu, et al, Acta Mater. **51** 3429 (2003)



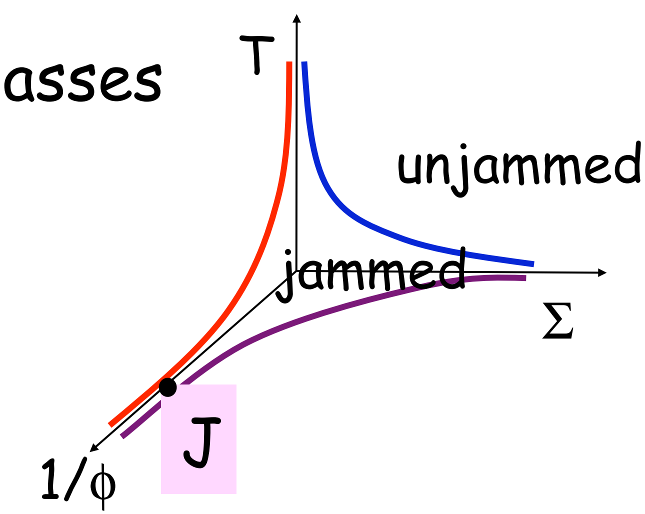
H. Senff & W. Richtering, JCP **111** 1705 (1999)



Equivalence of rheology for colloids and glasses

$$\lim_{p\sigma^2/\varepsilon\to 0}\frac{\Sigma}{p} = F\left(\frac{T}{p\sigma^2}, I = \dot{\gamma}\sqrt{\frac{m}{p}}\right)$$

$$\lim_{p\sigma^2/\varepsilon\to 0}\tau\sqrt{\frac{p}{m}} = \tilde{F}\left(\frac{T}{p\sigma^2}, \frac{\Sigma}{p}\right)$$



describes jamming surface: universal near Point J!