Why you need a functional RG to survive in a disordered world

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KITP, Mai 2003

References: PRL 86 (2001) 1785: 2 loop
PRL 89 (2002) 125702: large N
cond-mat/0302322 : intro + review

$\vdots$

http://theory.itp.ucsb.edu/∼wiese/

Statistical Mechanics

PURE SYSTEMS: pretty well understood

↓ add disorder

STRONGLY DISORDERED SYSTEMS

???

- disorder dominates over entropy
- what is the ground state?
- metastability
- very slow dynamics

Examples

- glasses : spin-glass, vortex-glass, electron-glass, structural glass
- random field magnet
- elastic systems in disorder

Current understanding of disordered systems

Still many puzzles despite 30 years of research. . .

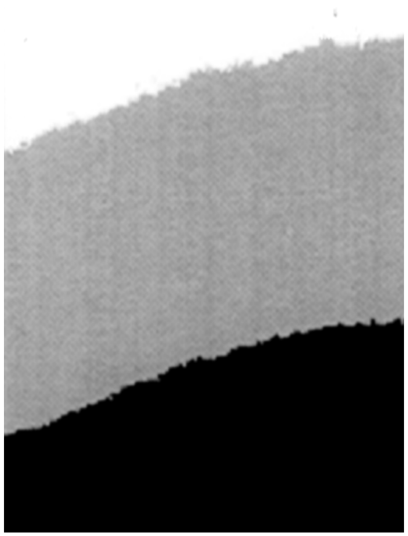
- simulations
- very few exact solutions
- phenomenological models (droplet-picture)
- mean-field approximation
  unclear of whether that applies to any real physical system

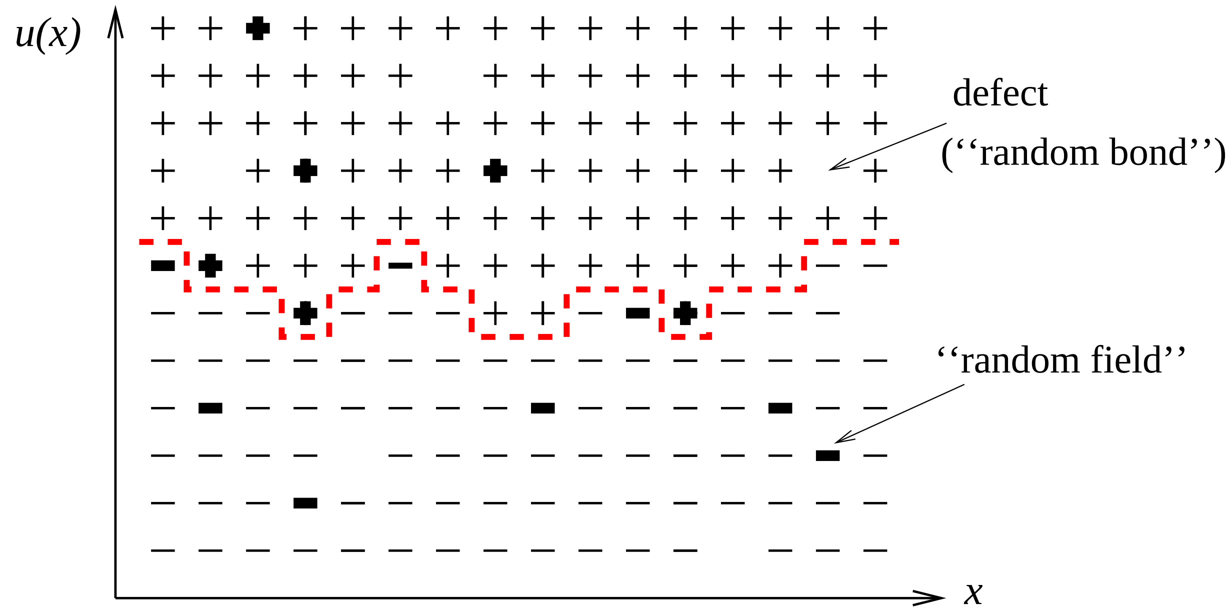
Recent advances

- for elastic manifolds in random media
- advantage: approachable by other (analytical) methods, while containing all ingredients of strongly disordered systems

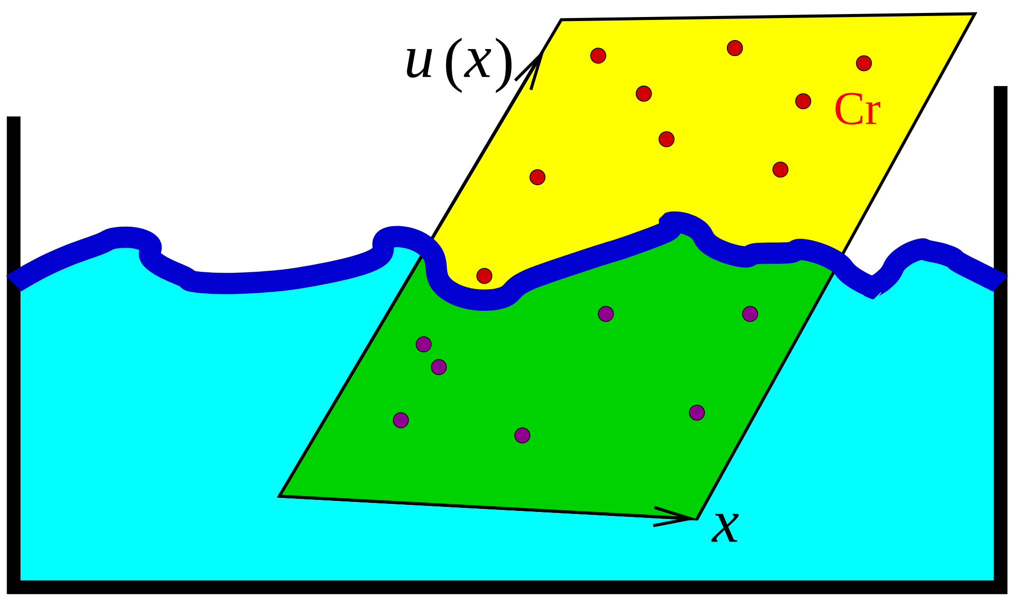
Physical Realizations

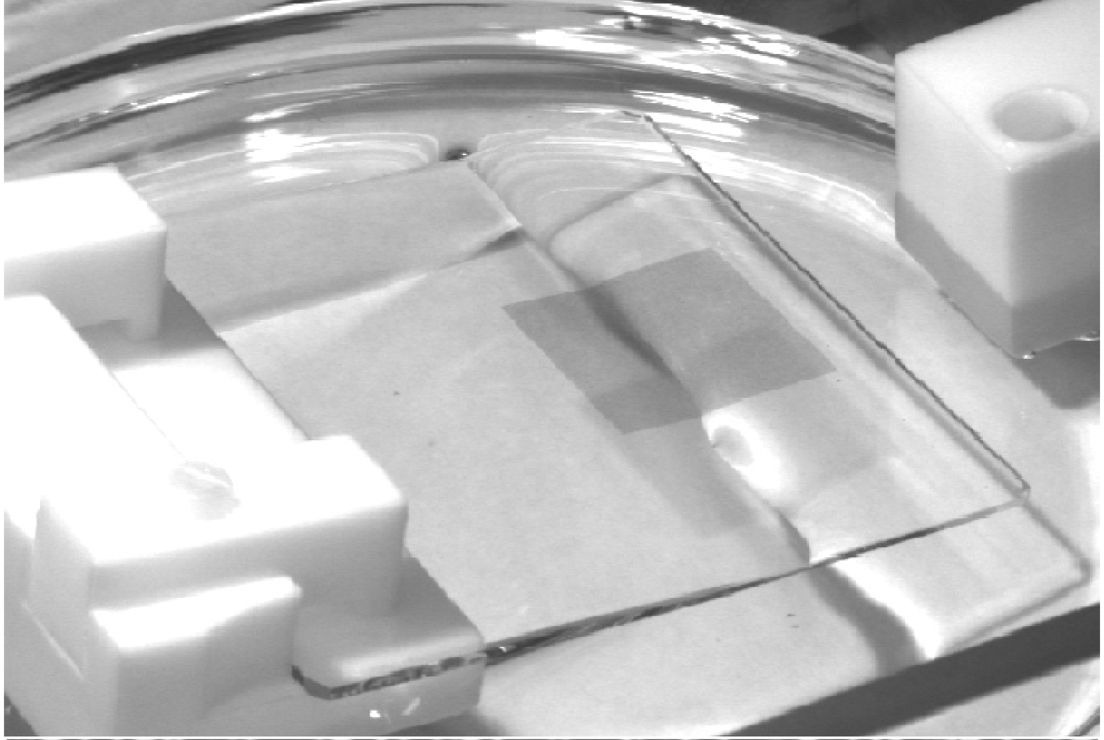
Domain-walls in magnets

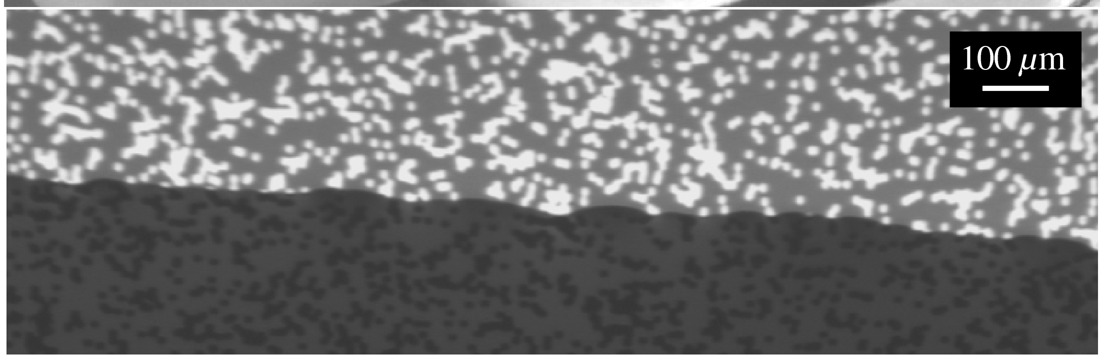


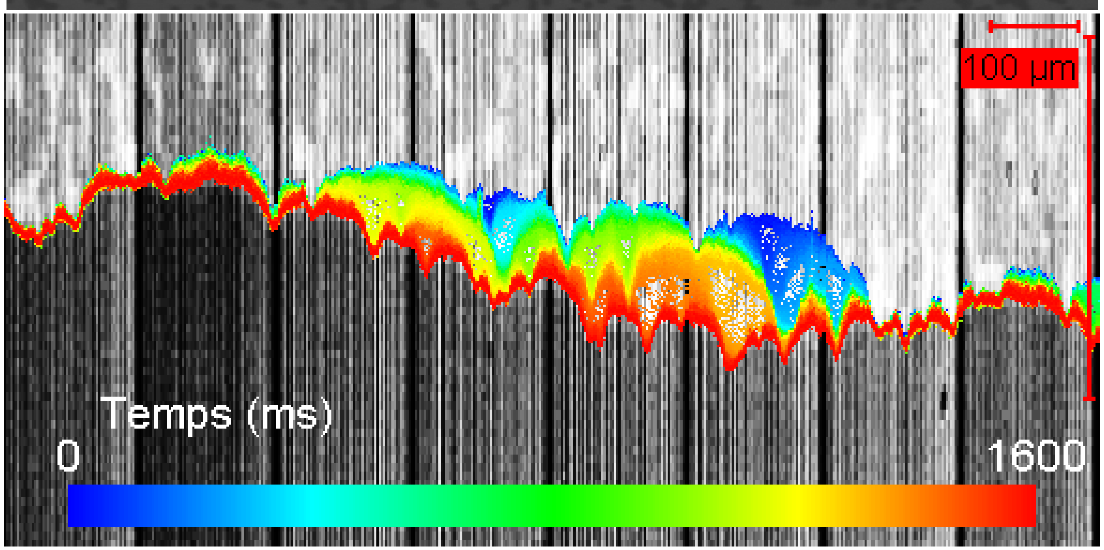


Contact line of liquid Helium/water







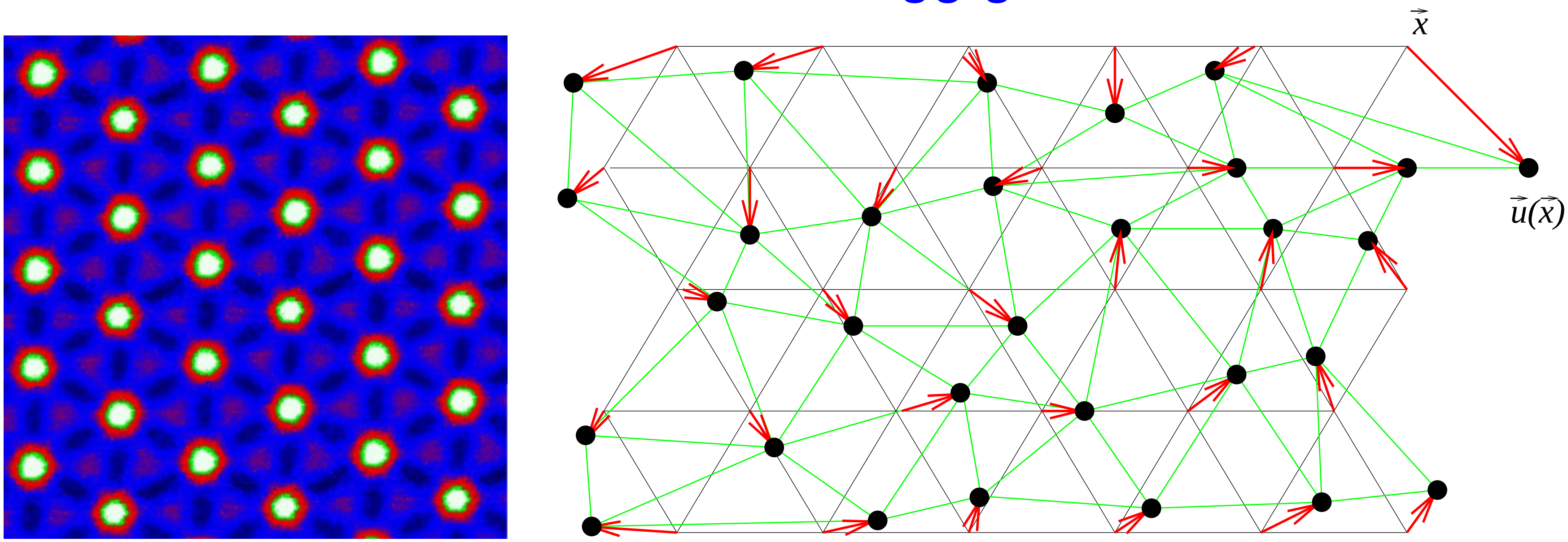


Depinning of contact line

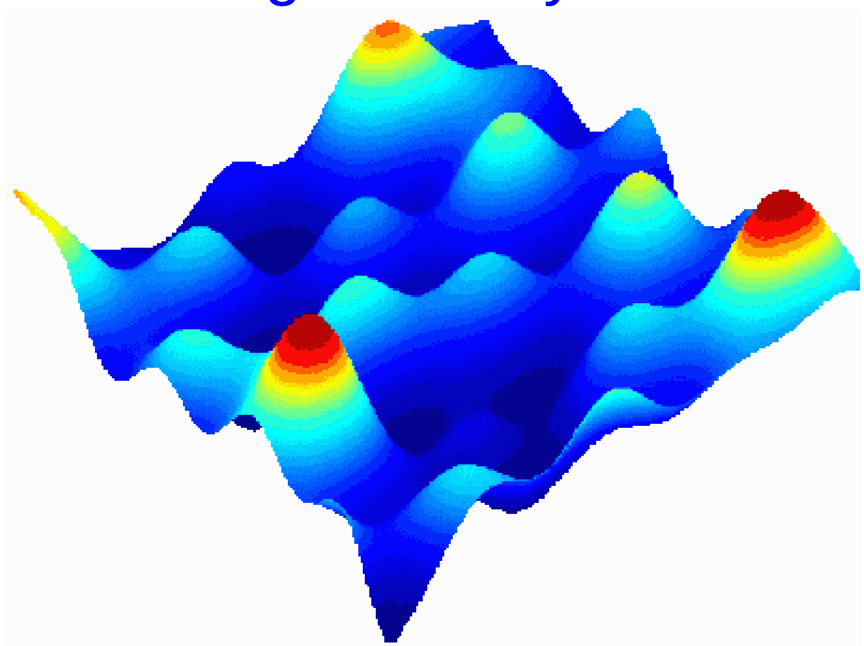
Eur. Phys. J. A 8 (2002) 437

Pictures courtesy of
S. Moulinet, E. Rolley

Vortex-lattice/Bragg glass

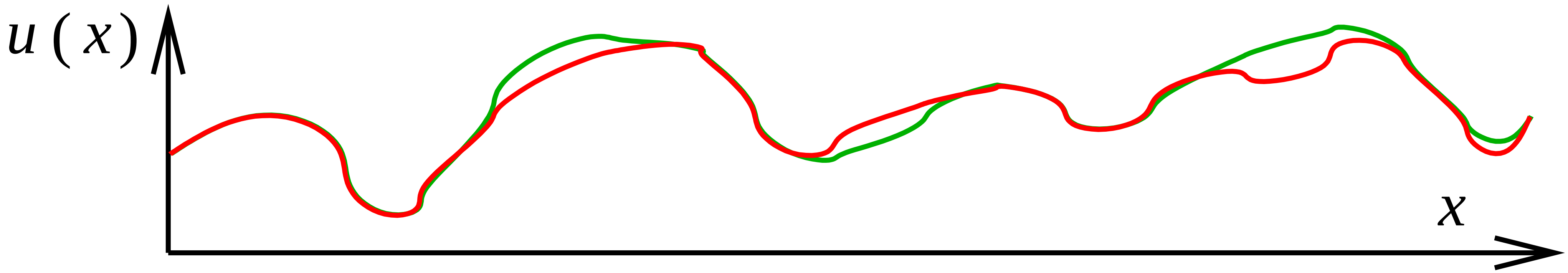


Charge Density wave



Cracks, earthquakes, directed polymer (KPZ),. . .

Model and Observables



Displacement field $\quad x \in \mathbb{R}^d \longrightarrow u(x) \in \mathbb{R}^N$

↑ coordinate $\qquad$ ↑ displacement-field

Elastic energy: $$\mathscr{H}_{\mathrm{el}} = \int \mathrm{d}^d x \frac{1}{2}\left[\nabla u(x)\right]^2$$

Disorder energy $$\mathscr{H}_{\mathrm{DO}} = \int \mathrm{d}^d x V(u(x), x)$$

with correlations ($N = 1$) $$\overline{V(u,x)V(u',x')} = \delta^d(x-x')R(u-u')$$

Observables

roughness ζ $$\overline{\left[u(x)-u(x')\right]^2} \sim |x-x'|^{2\zeta}$$

full probability-distribution function

How to treat disorder? The replica-trick

Example: free energy averaged over disorder

$$\overline{\mathscr{F}} = \overline{\ln \mathscr{Z}}$$

But how to calculate ?

$$\ln \mathscr{Z} = \lim_{n\to 0} \frac{1}{n}\left(e^{n \ln \mathscr{Z}} - 1\right) = \lim_{n\to 0} \frac{1}{n}\left(\mathscr{Z}^n - 1\right)$$

n times replicated system

"Replica Hamiltonian"

$$\mathscr{H}[u] = \frac{1}{T}\sum_{a=1}^{n}\int d^d x \frac{1}{2}\left[\nabla u_a(x)\right]^2 - \frac{1}{2T^2}\sum_{a,b=1}^{n}\int d^d x R(u_a(x) - u_b(x))$$

Dynamic formulation or supersymmetry could be used instead.
(No problem $n \to 0$.)

The problem in the treatment of disorder: dimensional reduction

“Theorem” (Efetov, Larkin 1977): A d-dimensional disordered system at zero temperature ($T = 0$) is equivalent to all orders in perturbation theory to a pure system in $d-2$ dimensions at finite temperature. (“Holds” under quite general assumptions.)

Example: Elastic manifolds in disorder

The thermal 2-point function becomes

$$\left\langle [u(x)-u(0)]^2 \right\rangle \sim |x|^{2-d} \quad \longrightarrow \quad \overline{[u(x)-u(0)]^2} \sim x^{4-d}$$

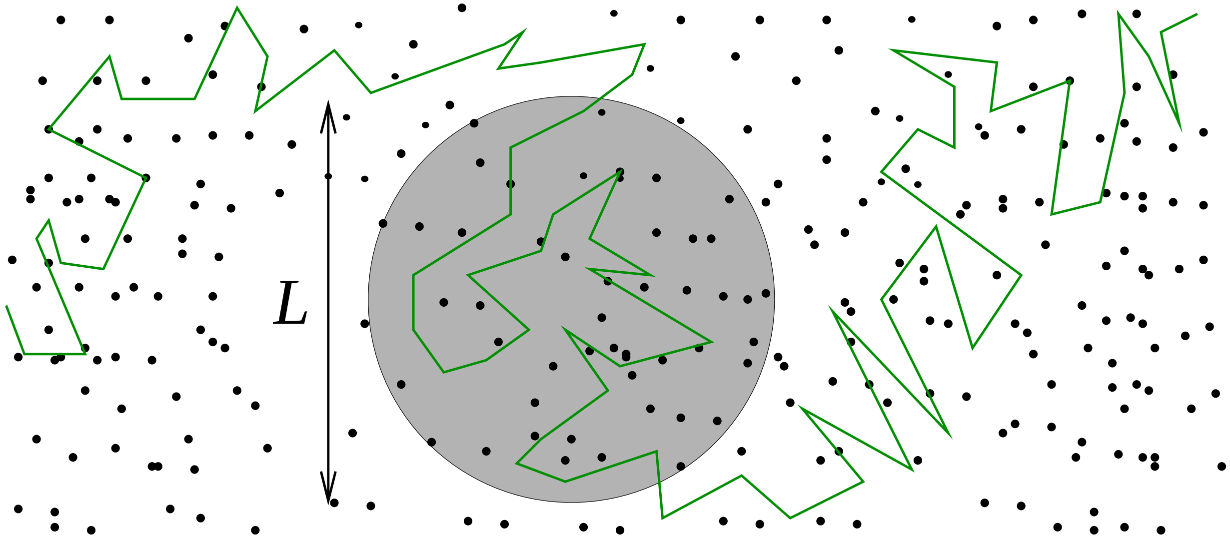
roughness exponent

$$\zeta = \frac{4-d}{2}$$

Counter-example:

3d disordered Ising-model at $T = 0$ is ordered; in contrast to the 1d Ising-model without disorder at $T > 0$.

The Larkin-length



Be the disorder force F_x gaussian, with correlation length r. Typical energy due to disorder on segment

$$\mathscr{E}_{\mathrm{DO}} = \bar{f} \left(\frac{L}{r} \right)^{d/2}$$

Elastic energy $$\mathscr{E}_{\mathrm{el}} = c L^{d-2}$$

Balance energies $\mathscr{E}_{\mathrm{DO}} = \mathscr{E}_{\mathrm{el}}$ at $L = L_c$ (Larkin-length)

$$L_c = \left(\frac{c^2}{\bar{f}^2} r^d \right)^{\frac{1}{4-d}}$$

$d < 4$: Membrane pinned by disorder on scales larger than the L_c

Why YOU need a functional RG to survive in a disordered world

Old idea: Wegner, Houghton (1973)

For disordered systems: D. Fisher (1985)

Larkin's argument:

$d = 4$ is critical dimension

Make an $\varepsilon = 4 - d$ expansion

Dimensional reduction says:

$$\zeta = \frac{4-d}{2}$$

Even though wrong for $d < 4$, it correctly says: field is marginal in $d = 4$.

NEED FOR A FUNCTIONAL RENORMALIZATION GROUP!

Functional renormalization group (FRG)

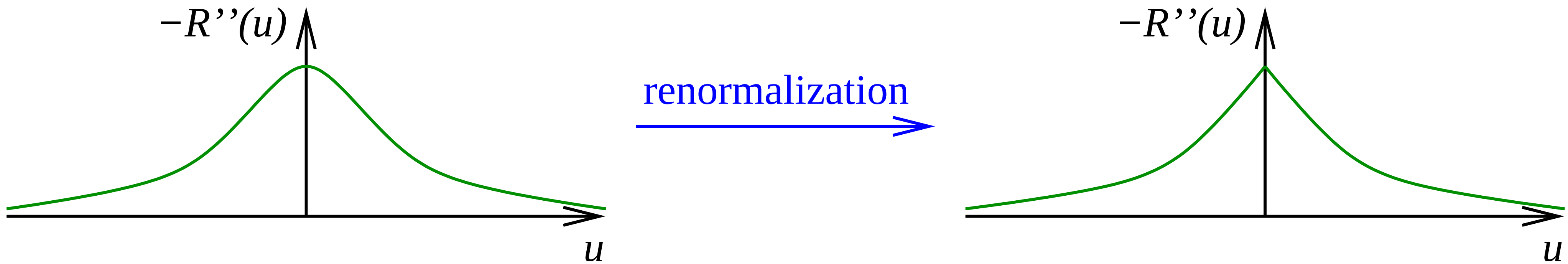
(D. Fisher 1986)

$$\mathscr{H}[u] = \int_x \frac{1}{2T} \sum_{a=1}^{n} [\nabla u_a(x)]^2 - \frac{1}{2T^2} \sum_{a,b=1}^{n} R(u_a(x) - u_b(x))$$

Functional renormalization group equation (FRG) for the disorder correlator $R(u)$:

$$\partial_\ell R(u) = (\varepsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0)$$

Solution for force-force correlator $-R''(u)$:



Cusp: $R''''(0) = \infty$ appears after finite RG-time (at Larkin-length)

$R''_{L>L_c}(0) \neq$ dim.red.
eventhough formally
$\partial_\ell R''(0) = (\varepsilon - 2\zeta)R''(0)$
($\equiv$ dim.red.)

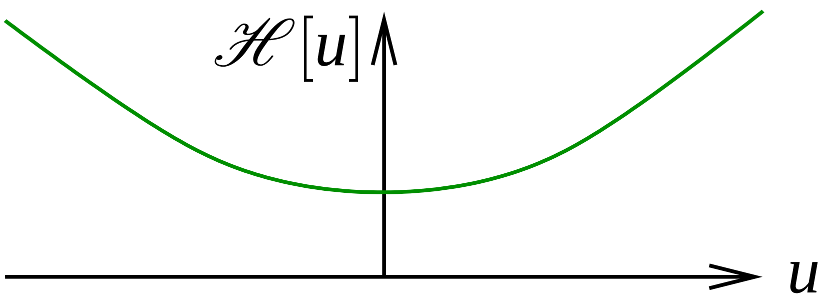
Renormalization of whole function overcomes dimensional reduction

Why is a cusp necessary?

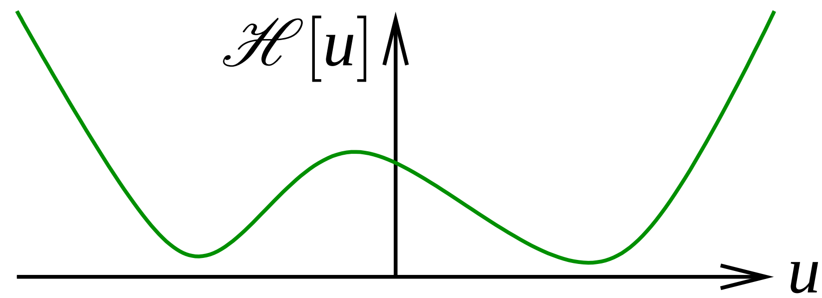
Consider simple model with one mode

$$\mathscr{H}[u] = \frac{1}{2} q^2 u^2 + \sqrt{\varepsilon}\, V(u)$$

Physics beyond the Larkin-length L_c: multiple minima



before L_c



after L_c

This implies that for all ε and some u

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} \mathscr{H}[u] = q^2 + \sqrt{\varepsilon}\, V''(u) < 0$$

Thus $$R''''(0) = \overline{V''(u) V''(u')}\Big|_{u=u'} = \infty$$

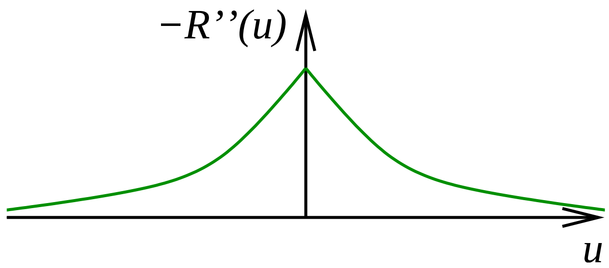
Beyond leading order (1 loop) ???

As a consistent theory, it should

- allow for systematic corrections beyond 1 loop
- be renormalizable
- and thus make universal predictions.

A puzzle since 1986 . . .

Next order involves $R'''(0) = ?$



$$\lim_{\substack{u>0 \\ u\to 0}} R'''(u) = -\lim_{\substack{u<0 \\ u\to 0}} R'''(u)$$

Solution of the puzzle

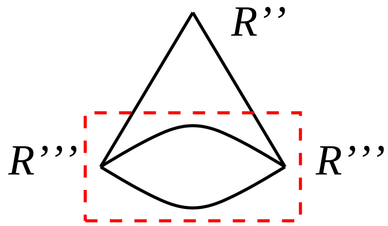
- 2-loop statics: PRL 76 (2001) 1785
- 2-loop driven dynamics: PRL 76 (2001) 1785, cond-mat/0205108
- large N: cond-mat/0109204

2 loop statics

$$\partial_\ell R(u) = (\varepsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(u)R''(0)$$
$$+\frac{1}{2}\left[R''(u) - R''(0)\right]R'''(u)^2 - \frac{1}{2}R'''(0^+)^2 R''(u)$$

Result of sloop-algorithm, recursive construction, super-symmetry, background method. Only result consistent with:

- renormalizability



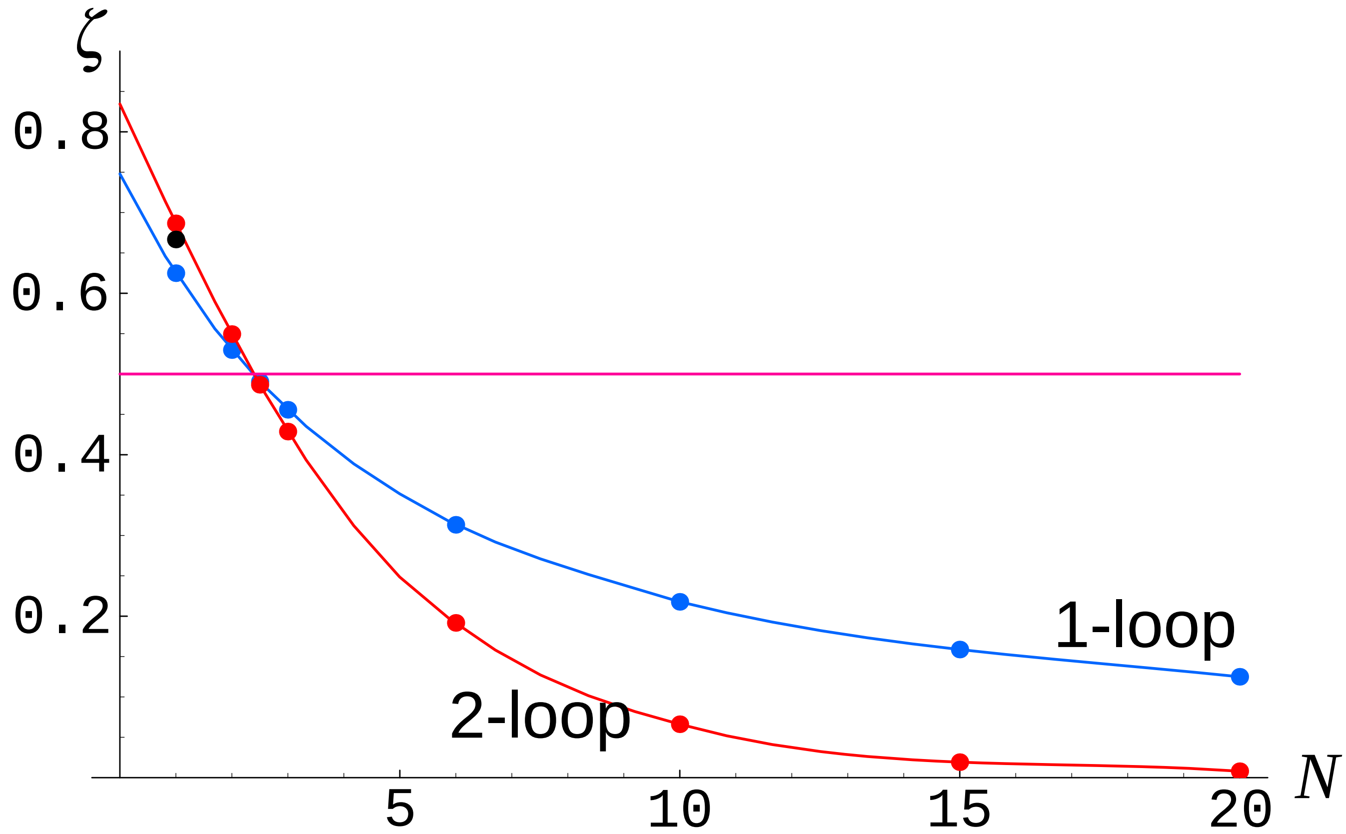
- potentiality (forces are gradient of a potential)

Solution for the fixed point

- periodic case: $A_d = \frac{\varepsilon}{18} + \frac{7\varepsilon^2}{108}$ (universal amplitude)
- random field $\zeta = \frac{\varepsilon}{3}$ (exact)
- random bond $\zeta = 0.20829804\varepsilon + 0.006858\varepsilon^2$

roughness ζ for random bond			
d	ε	ε^2	simulation
3	0.208	0.215 ± 0.003	0.22 ± 0.01
2	0.417	0.444 ± 0.007	0.41 ± 0.01
1	0.625	0.687 ± 0.02	$2/3$

2 loops, N components



$$\zeta = \frac{1}{z_{\mathrm{KPZ}}}$$

$$z_{\mathrm{KPZ}} + \zeta_{\mathrm{KPZ}} = 2$$

$$\partial_\ell R(u) = (\varepsilon - 4\zeta)R(u) + \zeta u R'(u) + \frac{1}{2}R''(u)^2 - R''(0)R''(u) + \frac{N-1}{2}\frac{R'(u)}{u}\left(\frac{R'(u)}{u} - 2R''(0)\right)$$

$$+\frac{1}{2}\left(R''(u) - R''(0)\right) R'''(u)^2 + \frac{N-1}{2}\frac{\left(R'(u)-uR''(u)\right)^2 \left(2R'(u)+u(R''(u)-3R''(0))\right)}{u^5}$$

$$-R'''(0^+)^2 \left[\frac{N+3}{8}R''(u) + \frac{N-1}{4}\frac{R'(r)}{u}\right]$$

Solution at large N

$\vec{u}(x) \in \mathbb{R}^N$, e.g. directed polymer in N dimensions

Calculate free energy in presence of an external field; do Legendre-transform; obtain self-consistent equation for effective action (exact)

$$\tilde{R}'(u^2) = R'\left(u^2 + 2TI_1 + 4I_2(\tilde{R}'(u^2) - \tilde{R}'(0))\right)$$

$R(\ldots)$ bare disorder
$\tilde{R}(\ldots)$ effective (renormalized) disorder
$T =$ temperature
$I_n = \int \frac{\mathrm{d}^d k}{(k^2+m^2)^n}$

Functional renormalization group equation (FRG)

$$-m\frac{\partial}{\partial m}\tilde{R}(x) = (\varepsilon - 4\zeta)\tilde{R}(x) + 2\zeta x\tilde{R}'(x) + \frac{1}{2}\tilde{R}'(x)^2 - \tilde{R}'(x)\tilde{R}'(0) + \frac{\varepsilon T\tilde{R}'(x)}{\varepsilon + \tilde{R}''(0)}$$

complicated non-linear partial differential equation: solved analytically; cusp under analytical control.

Replica Symmetry Breaking (RSB)

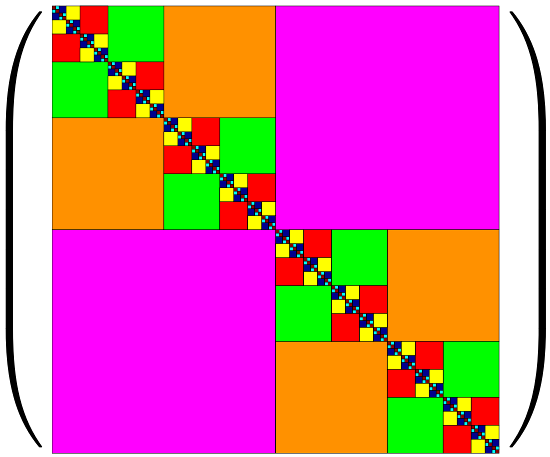
No symmetry-breaking field (Mézard, Parisi 1992). Gaussian variational ansatz exact at $N = \infty$:

$$R\left((u_a - u_b)^2\right) = \sigma_{ab}\, u_a\, u_b$$

RS: $\sigma_{ab} = \sigma \; \forall \; a \neq b$: dimensional reduction

RSB:

$$\sigma_{ab} =$$



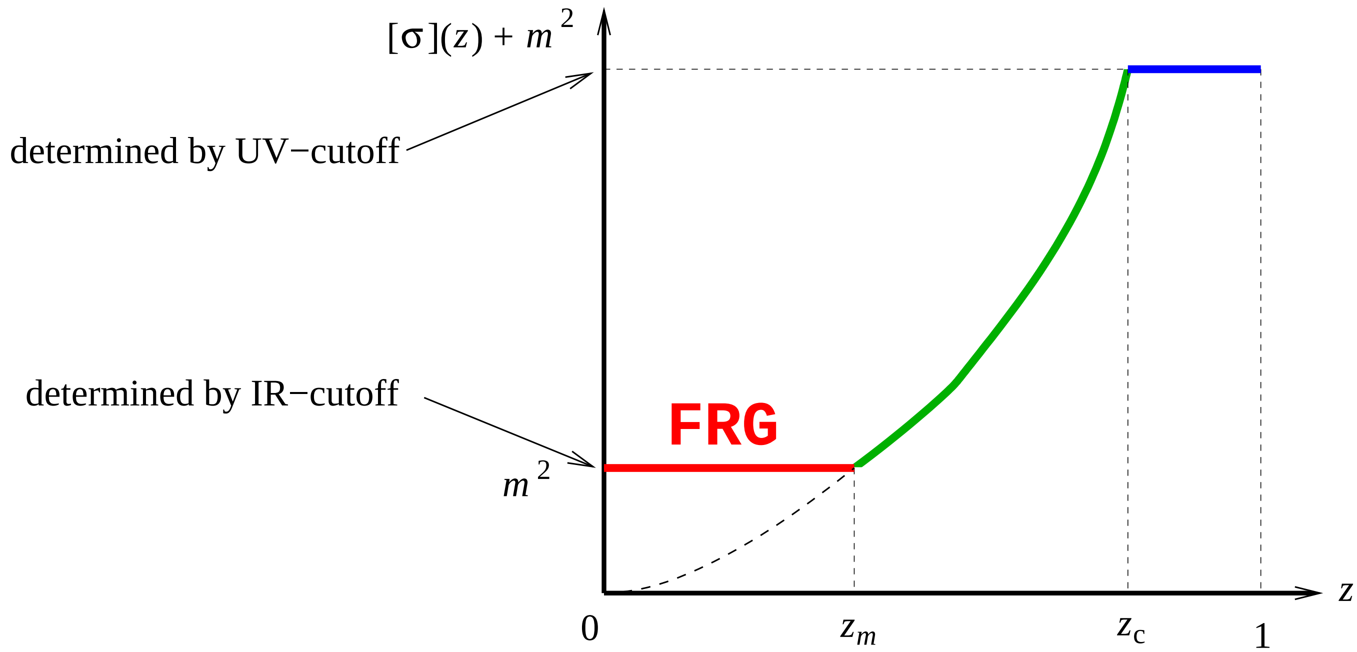
Infinit-step RSB: $\sigma_{ab} \to [\sigma](z)$, $z \in [0,1]$

$z =$ overlap, $\begin{cases} z = 0 & \text{distant states} \\ z = 1 & \text{nearby states} \end{cases}$

Observables are constructed out of $[\sigma](z)$

$$\langle u_k u_{-k} \rangle = \frac{1}{k^2 + m^2} \left(1 + \int_0^1 \frac{\mathrm{d}z}{z^2} \frac{[\sigma](z)}{k^2 + m^2 + [\sigma(z)]} \right)$$

RSB and FRG, the relation



- FRG gives the contribution of the RSB-states with minimal overlap
- RSB: spontaneous symmetry breaking
- FRG: explicit symmetry breaking by applied field
- green part is RG-invariant: $m\frac{\mathrm{d}}{\mathrm{d}m}([\sigma](z)+m^2)=0$
- RSB-curve can be scanned by varying m^2
- RSB-reconstruction-formula (out of FRG-objects)

$$\left\langle u^a u^b \right\rangle\Big|_{k=0} = \frac{\tilde{R}'_m(0)}{m^4} + \int_m^{m_c} \frac{\mathrm{d}\tilde{R}'_\mu(0)}{\mu^4} + \frac{1}{m_c^2} - \frac{1}{m^2}$$

- no hierarchic matrix was ever inverted!

$1/N$-calculations

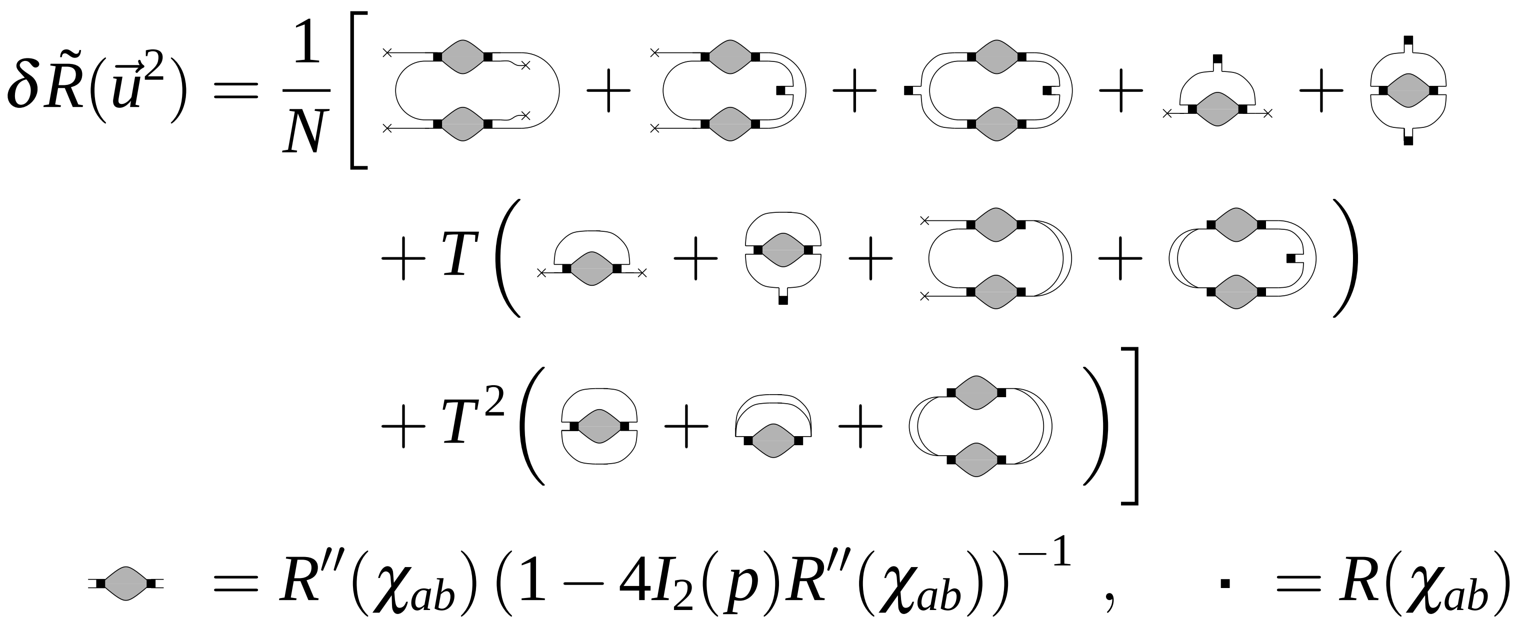
Results are exact in dimension d

Physically interesting for directed polymer $d = 1$ (equivalent to KPZ).

However, at $N = \infty$: roughness $\zeta = 0$.

Non-trivial exponent needs $1/N$-expansion.

Renormalized disorder at order $1/N$:



The β-function at order $1/N$

$$-m\frac{\partial}{\partial m}\tilde{R}(x) = (\varepsilon-4\zeta)\tilde{R}(x) + 2\zeta x\tilde{R}'(x) + \frac{1}{2}\tilde{R}'(x)^2 - \tilde{R}'(x)\tilde{R}'(0) + \frac{\varepsilon T\tilde{R}'(x)}{\varepsilon+\tilde{R}''(0)} + \frac{1}{N}\times\Big[$$

$$\frac{i_4[p](\tilde{b}'[0]-\tilde{b}'[x])^2\tilde{b}''[x](2\epsilon+(1+\epsilon i_2-\epsilon i_2[p])\tilde{b}''[x])}{(1+(i_2-i_2[p])\tilde{b}''[x])^2} + \frac{2i_3[p]^2(\tilde{b}'[0]-\tilde{b}'[x])^2\tilde{b}''[x]^2(3\epsilon+(2+\epsilon i_2-\epsilon i_2[p])\tilde{b}''[x])}{(1+(i_2-i_2[p])\tilde{b}''[x])^3} -$$

$$(2i_0[p]^2(\tilde{b}'[0]-\tilde{b}'[x])\tilde{b}''[x](x\epsilon+\tilde{b}'[0]-\tilde{b}'[x]+(x+(i_2-i_2[p])(\tilde{b}'[0]-\tilde{b}'[x]))\tilde{b}''[x]))/(1+(i_2-i_2[p])\tilde{b}''[x])^2 -$$

$$(4i_0[p]i_3[p](\tilde{b}'[0]-\tilde{b}'[x])\tilde{b}''[x]^2(2x\epsilon+\tilde{b}'[0]-\tilde{b}'[x]+(2x+(i_2-i_2[p])(\tilde{b}'[0]-\tilde{b}'[x]))\tilde{b}''[x]))/(1+(i_2-i_2[p])\tilde{b}''[x])^3 +$$

$$(2xi_0[p]^2\tilde{b}''[x]^2(x\epsilon+2\tilde{b}'[0]-2\tilde{b}'[x]-(-2x+(i_2-i_2[p])(x\epsilon-2\tilde{b}'[0]+2\tilde{b}'[x]))\tilde{b}''[x]))/(1+(i_2-i_2[p])\tilde{b}''[x])^3 +$$

$$t((i_3[p]\tilde{b}'[x]\tilde{b}''[0](-2\epsilon-2(2+2\epsilon i_2+\epsilon i_2[p])\tilde{b}''[0]-(2\epsilon i_2^2+i_2[p]-\epsilon i_2[p]^2+2i_2(5+\epsilon i_2[p]))\tilde{b}''[0]^2-2(4i_2^2-i_2[p]^2)\tilde{b}''[0]^3-i_2(i_2-i_2[p])(2i_2+i_2[p])\tilde{b}''[0]^4))/$$

$$(2(1+i_2\tilde{b}''[0])^2(1+(i_2-i_2[p])\tilde{b}''[0])^2)+\frac{1}{2}\left(\frac{\tilde{b}'[x]\tilde{b}''[0](2+(2i_2-i_2[p])\tilde{b}''[0])(i_0[p]+i_3[p]\tilde{b}''[0])}{(1+(i_2-i_2[p])\tilde{b}''[0])^2}\right. - (\epsilon(xi_0[p]+i_3[p](-\tilde{b}'[0]+\tilde{b}'[x]))\tilde{b}''[x](2+(2i_2-i_2[p])\tilde{b}''[x]))/$$

$$(1+(i_2-i_2[p])\tilde{b}''[x])^2 + \frac{\tilde{b}'[0]\tilde{b}''[x](2+(2i_2-i_2[p])\tilde{b}''[x])(i_0[p]+i_3[p]\tilde{b}''[x])}{(1+(i_2-i_2[p])\tilde{b}''[x])^2} - \frac{\tilde{b}'[x]\tilde{b}''[x](2+(2i_2-i_2[p])\tilde{b}''[x])(i_0[p]+i_3[p]\tilde{b}''[x])}{(1+(i_2-i_2[p])\tilde{b}''[x])^2} +$$

$$\left.(\tilde{b}''[x](\epsilon+\tilde{b}''[x])(2xi_0[p](1+i_2\tilde{b}''[x])-i_3[p](\tilde{b}'[0]-\tilde{b}'[x])(4+(2i_2-i_2[p])\tilde{b}''[x](3+(i_2-i_2[p])\tilde{b}''[x]))))/(1+(i_2-i_2[p])\tilde{b}''[x])^3\right) +$$

$$\left(i_2[p]^2\tilde{b}''[0]\left(-\frac{\tilde{b}'[x]\tilde{b}''[0]^2(1+i_2\tilde{b}''[0])(i_0[p]+i_3[p]\tilde{b}''[0])}{(1+(i_2-i_2[p])\tilde{b}''[0])^2} + \frac{\epsilon(xi_0[p]+i_3[p](-\tilde{b}'[0]+\tilde{b}'[x]))(1+i_2\tilde{b}''[0])\tilde{b}''[x]^2}{(1+(i_2-i_2[p])\tilde{b}''[x])^2} - \frac{\tilde{b}'[0](1+i_2\tilde{b}''[0])\tilde{b}''[x]^2(i_0[p]+i_3[p]\tilde{b}''[x])}{(1+(i_2-i_2[p])\tilde{b}''[x])^2}\right.\right. +$$

$$\frac{\tilde{b}'[x](1+i_2\tilde{b}''[0])\tilde{b}''[x]^2(i_0[p]+i_3[p]\tilde{b}''[x])}{(1+(i_2-i_2[p])\tilde{b}''[x])^2} + (\tilde{b}''[x]^2(xi_0[p](-2(1+i_2\tilde{b}''[0])(\epsilon+\tilde{b}''[x])-(\epsilon+\tilde{b}''[0])(1+(i_2-i_2[p])\tilde{b}''[x]))+i_3[p](\tilde{b}'[0]-\tilde{b}'[x])$$

$$\left.\left.(2(1+i_2\tilde{b}''[0])(\epsilon+\tilde{b}''[x])-(-1+(-i_2+i_2[p])\tilde{b}''[x])(2\epsilon+\tilde{b}''[0]+\tilde{b}''[x]+i_2\tilde{b}''[0](\epsilon+\tilde{b}''[x])))))/(1+(i_2-i_2[p])\tilde{b}''[x])^3\right)\right)/(2(1+i_2\tilde{b}''[0])^2)\Big) +$$

$$t^2\Big(-(i_2[p]\tilde{b}''[x]^2(-4+4\epsilon i_2-5\epsilon i_2[p]+(4\epsilon i_2^2+i_2[p](2+3\epsilon i_2[p])-i_2(4+7\epsilon i_2[p]))\tilde{b}''[x]))/(32(1+(i_2-i_2[p])\tilde{b}''[x])^3) -$$

$$(i_2[p]^2\tilde{b}''[0]\tilde{b}''[x](\epsilon+\tilde{b}''[0]+(1+\epsilon(i_2+i_2[p])-(i_2[p]+i_2(-3+\epsilon i_2-2\epsilon i_2[p]))\tilde{b}''[0])\tilde{b}''[x]+(i_2+i_2[p]-i_2^2(-2+\epsilon i_2-\epsilon i_2[p])\tilde{b}''[0])\tilde{b}''[x]^2))/$$

$$(16(1+i_2\tilde{b}''[0])^2(1+(i_2-i_2[p])\tilde{b}''[x])^3) + \frac{1}{8}\left(\epsilon\,\mathrm{Log}\left[1-\frac{i_2[p]\tilde{b}''[x]}{1+i_2\tilde{b}''[x]}\right] + \frac{i_2[p]\tilde{b}''[x](\epsilon+\tilde{b}''[x])}{(1+i_2\tilde{b}''[x])(1+(i_2-i_2[p])\tilde{b}''[x])}\right) +$$

$$(i_2[p]^3\tilde{b}''[0]^2\tilde{b}''[x](-\epsilon(1+i_2\tilde{b}''[0])(1+(i_2-i_2[p])\tilde{b}''[x])(-2+(-2i_2+3i_2[p])\tilde{b}''[x])+2(-(1+i_2\tilde{b}''[0])(\epsilon+\tilde{b}''[x])(1+(i_2-2i_2[p])\tilde{b}''[x]) -$$

$$(\epsilon+\tilde{b}''[0])(2+(2i_2-3i_2[p])\tilde{b}''[x])(1+(i_2-i_2[p])\tilde{b}''[x]))))/(32(1+i_2\tilde{b}''[0])^3(1+(i_2-i_2[p])\tilde{b}''[x])^3) - \frac{i_2 i_2[p]\tilde{b}'[x](2+(2i_2+i_2[p])\tilde{b}''[0])g^{(3)}[0]}{8(1+i_2\tilde{b}''[0])^3(1+(i_2-i_2[p])\tilde{b}''[0])}\Big) +$$

$$\Big((t^2 i_2 i_2[p]\tilde{b}'[x]\tilde{b}''[0](-2+6\epsilon i_2-3\epsilon i_2[p]-2(i_2[p]+3\epsilon i_2(-2i_2+i_2[p]))\tilde{b}''[0]+(6\epsilon i_2^3+i_2[p]^2+i_2^2(6-3\epsilon i_2[p])-i_2 i_2[p](4+3\epsilon i_2[p]))\tilde{b}''[0]^2 +$$

$$2i_2(i_2-i_2[p])(2i_2+i_2[p])\tilde{b}''[0]^3))/(8(1+i_2\tilde{b}''[0])^4(1+(i_2-i_2[p])\tilde{b}''[0])^2) + t^2\Big(\frac{i_2[p]^3\tilde{b}'[x]\tilde{b}''[0]^2(-1-(i_2-2i_2[p])\tilde{b}''[0])}{16(1+i_2\tilde{b}''[0])^2(1+(i_2-i_2[p])\tilde{b}''[0])^3} +$$

$$\frac{i_2[p]\tilde{b}'[x]}{8(1+i_2\tilde{b}''[0])(1+(i_2-i_2[p])\tilde{b}''[0])} - \frac{i_2[p]\tilde{b}'[x](-2+(-2i_2+i_2[p])\tilde{b}''[0])}{16(1+(i_2-i_2[p])\tilde{b}''[0])^3} - \frac{i_2[p]^2\tilde{b}'[x]\tilde{b}''[0](1+(i_2+i_2[p])\tilde{b}''[0])}{16(1+i_2\tilde{b}''[0])(1+(i_2-i_2[p])\tilde{b}''[0])^3}\Big)\Big)\tilde{b}^{(3)}[0]\Big] + O\left(\frac{1}{N^2}\right)$$

... remains to be analyzed

Summary

- higher order calculations: very cumbersome, but under control
- exact solution of the large-N limit
- cusp analytically under control
- precise relation to RSB
- $1/N$-expansion

Outlook

- random field
- dynamics: 2-loop calculations necessary to account for experimental and numerical data
- anisotropic depinning, relation to branching processes

COME TO  NEXT WEEK!