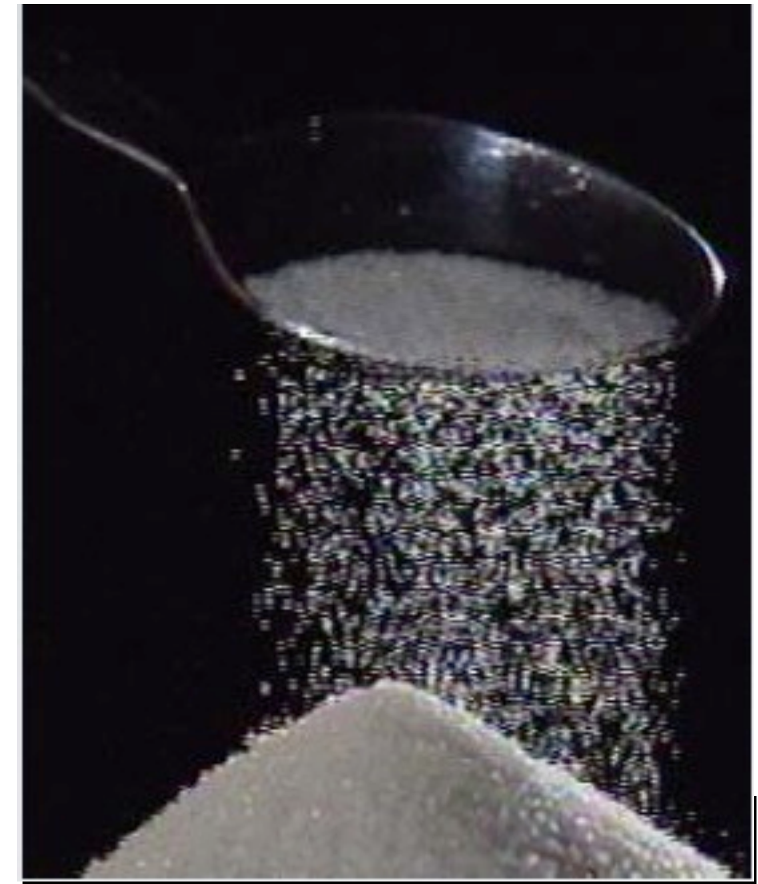
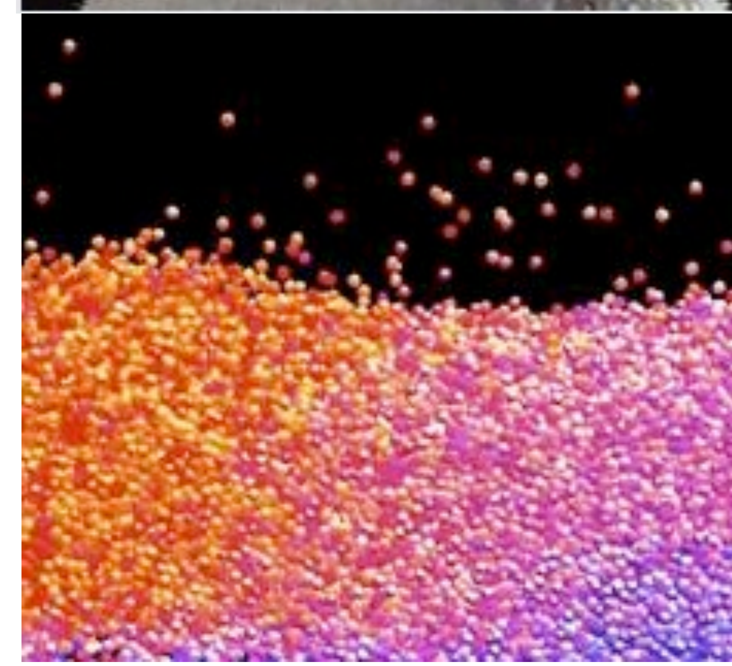
Fluctuations, Entropy, and "Temperature" of Jammed Granular Packings



Statistical systems out of thermal equilibrium



Acknowledgments
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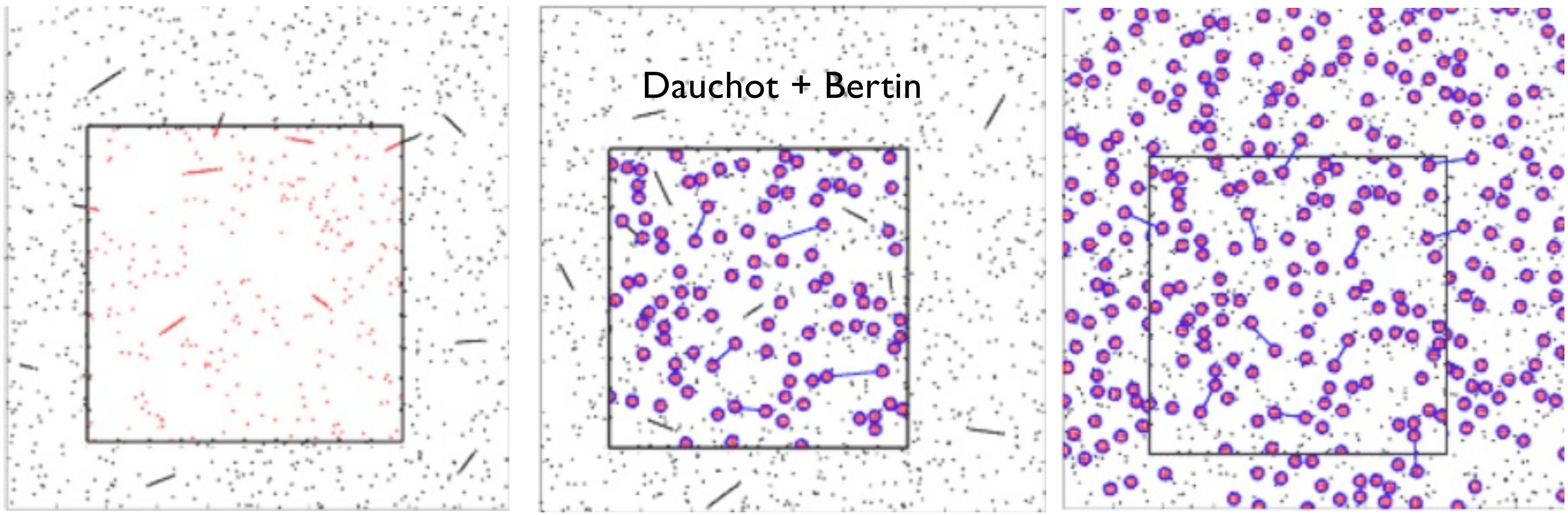
Out of Equilibrium

- Collection of macroscopic objects (athermal)
- Friction and dissipation
- Purely repulsive, contact interactions (Dry)
- **Need** energy input to maintain steady state (NESS)
- Jammed states (Mechanical equilibrium) end points of dynamical protocols
- Jammed states have to have a minimum number of contacts, Z_{iso}

Thermal system

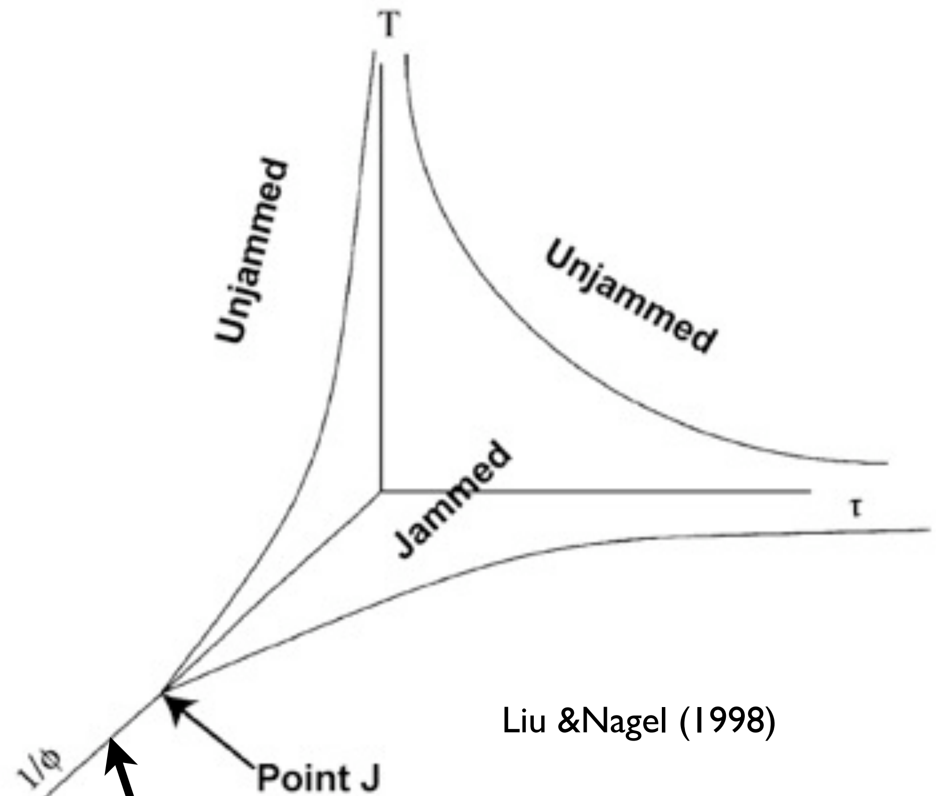
Difference in energy scale prevents thermalization

Granular bath

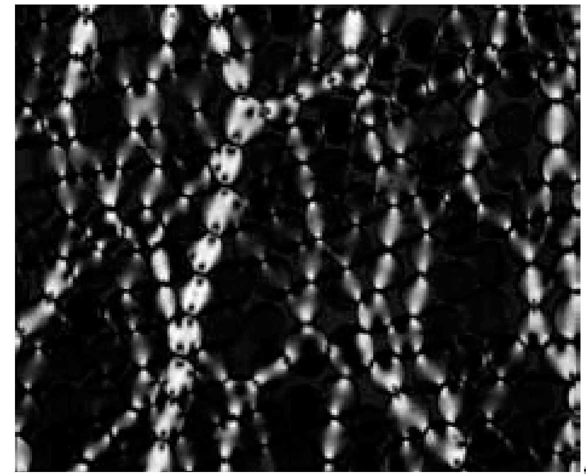


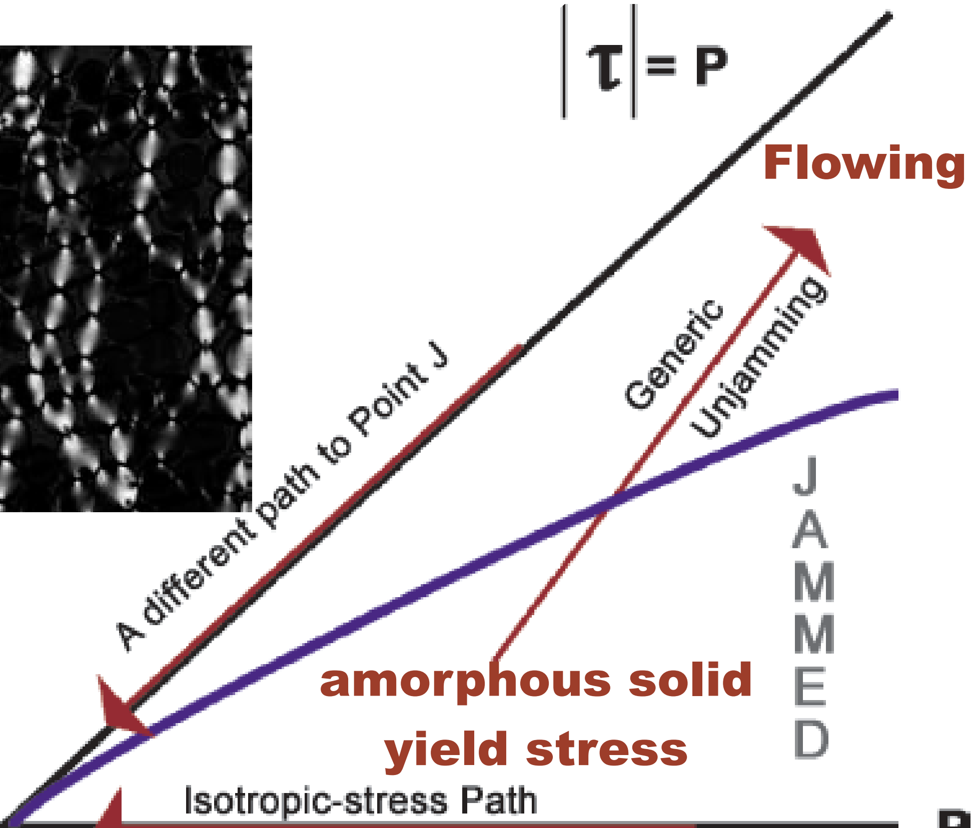
Unjamming/Jamming in Granular Matter

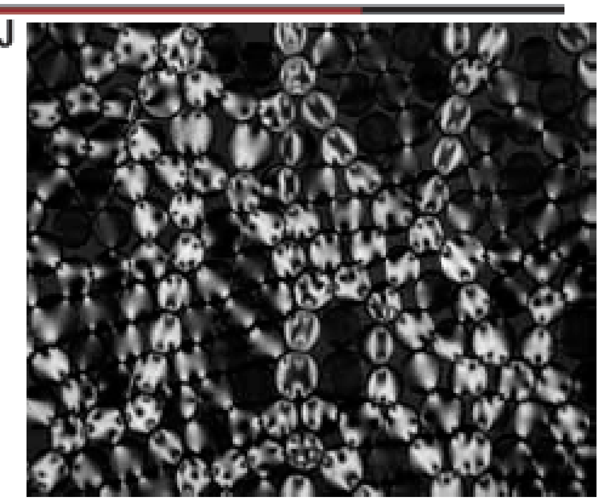
Purely repulsive, contact interactions (dry grains)



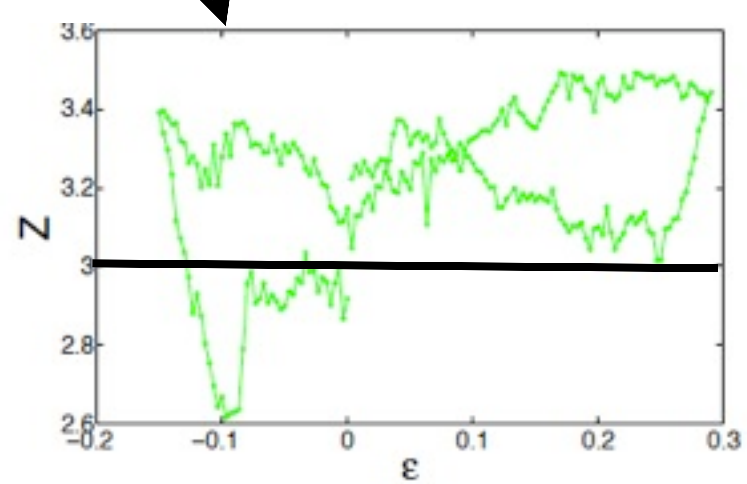
Liu &Nagel (1998)





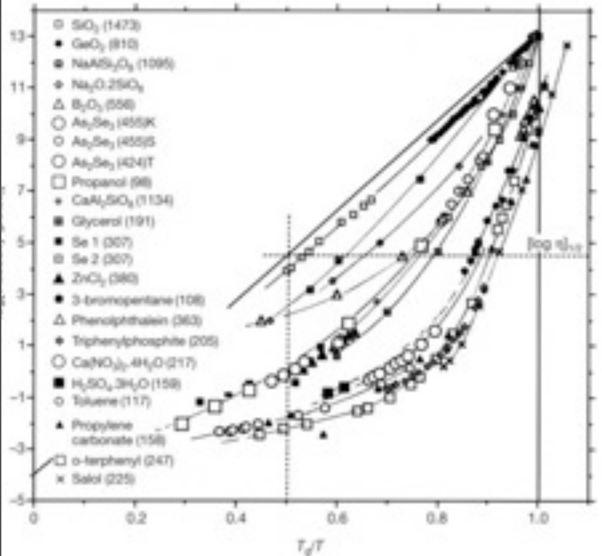


Jammed states:
$Z > Z_{iso}$



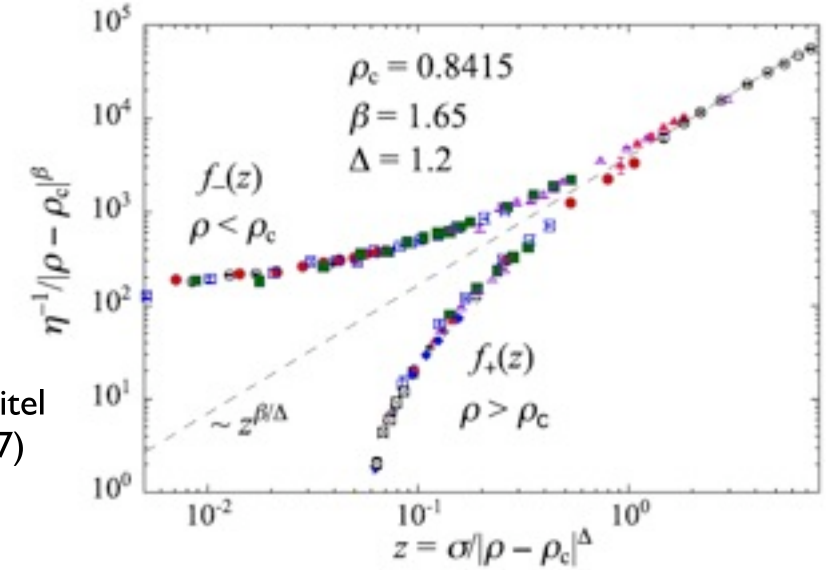
Point J is special
ISOTATIC: As many constraints as variables
A marginal solid that falls apart (sublimation?)

Jammed/Jamming vs Glass/Glass Transition



Angell + Martinez
Nature (2001)

Olsson+Teitel
PRL (2007)



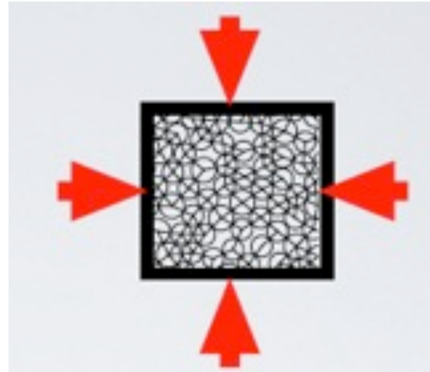
Similarities

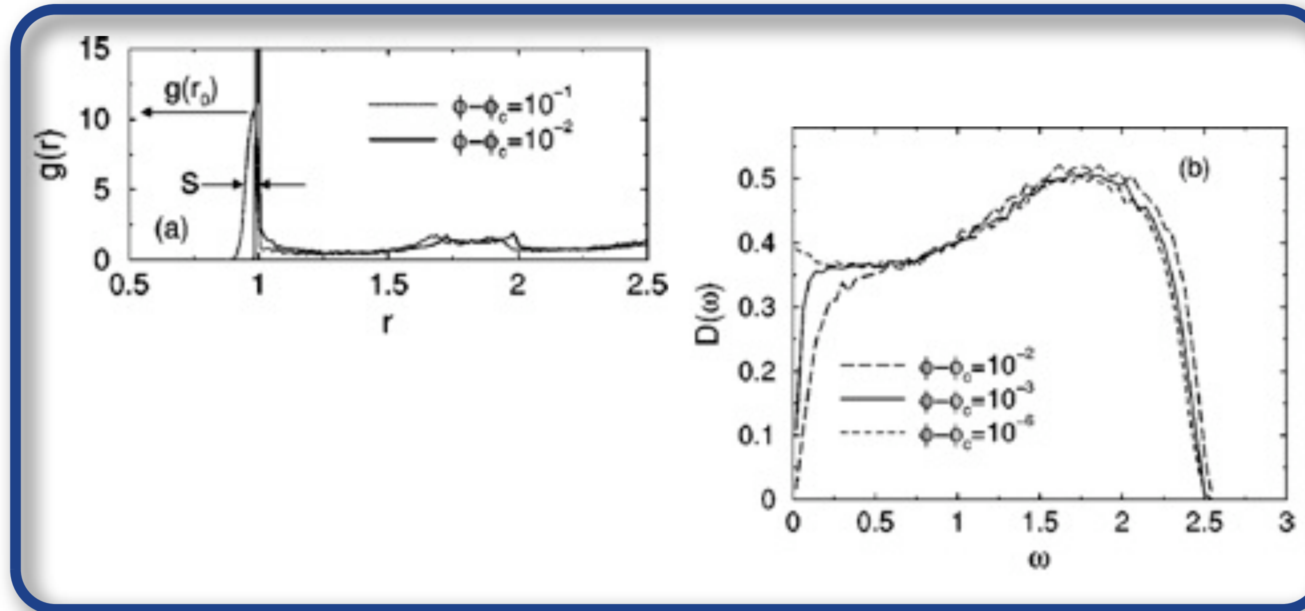
- Viscosity diverges, yield stress develops
- Many amorphous, metastable states
- Entropy of jammed states vanishes at Point J: like Kauzman ?
- Jamming/unjamming accompanied by slow dynamics (dynamical heterogeneities)
- Properties of jammed states near Point J similar to low temperature glasses ? Boson peak etc..

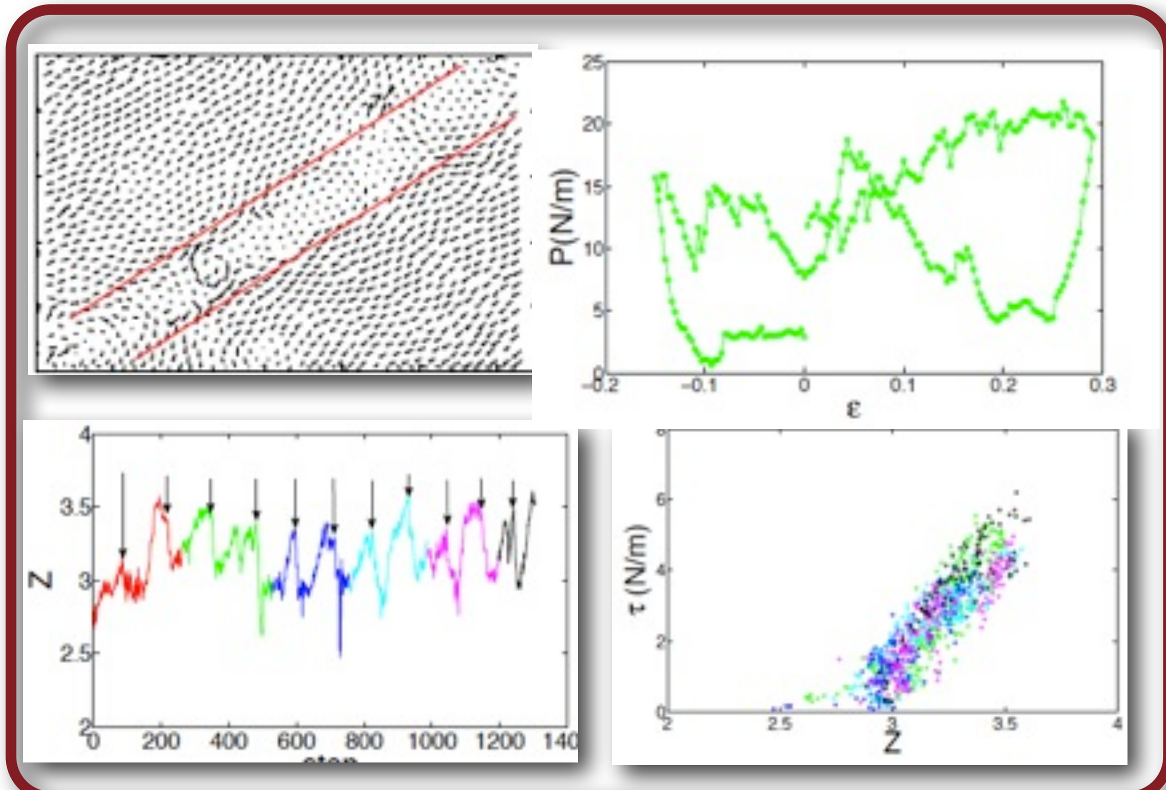
Differences

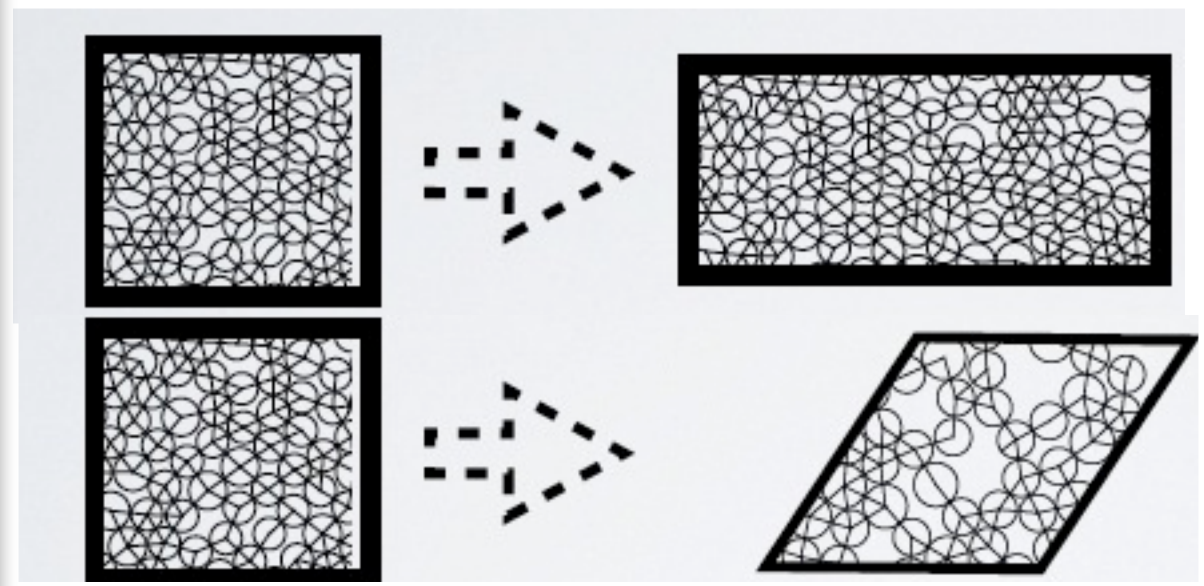
- Never in thermal equilibrium: flowing state maintained by shearing
- Unjamming: "falling out of mechanical equilibrium"
- Frictional forces implies indeterminacy of contact forces
- Mechanical equilibrium (T=0): force and torque balance on every grain is a constraint
- Fluctuations: sample to sample or by external driving
- Critical point (frictionless) is isostatic (a marginal solid)

Different protocols for creating jammed packings

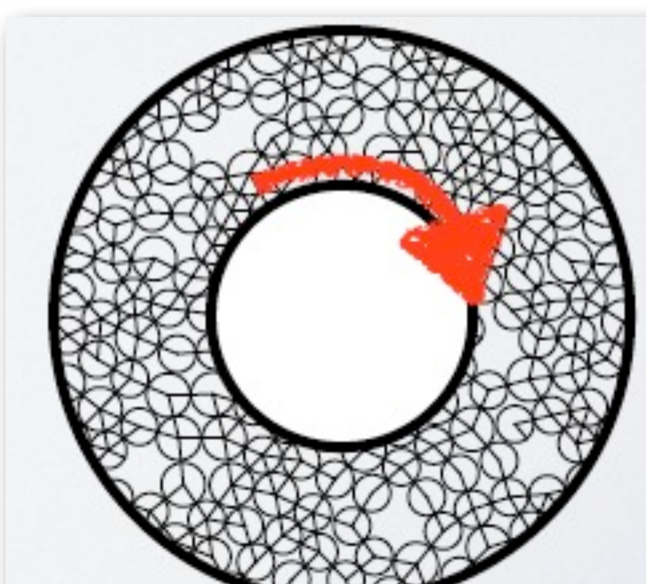


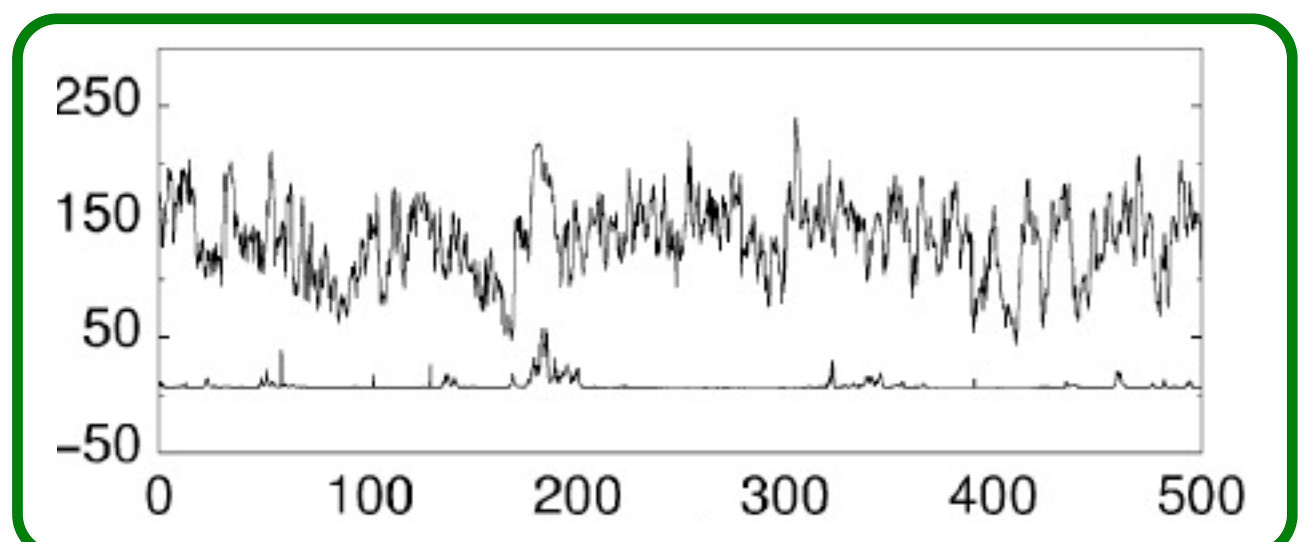






Cyclic





Questions:

•How do we analyze the statistics of fluctuations in jammed packings (sample to sample or quasistatic driving)

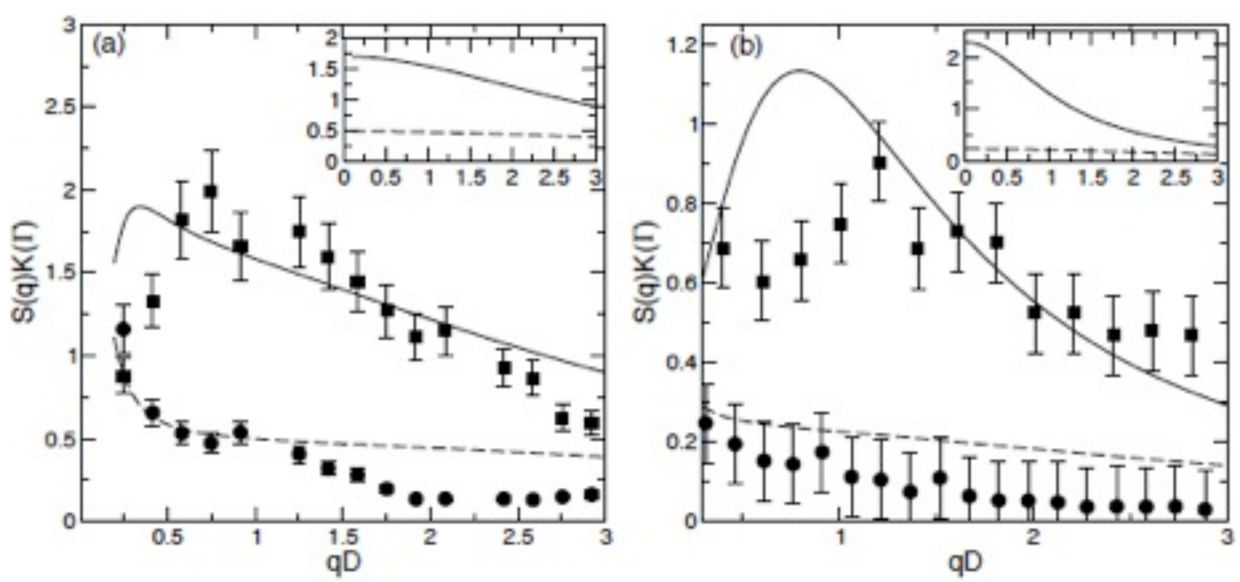
•Relation of fluctuations to response

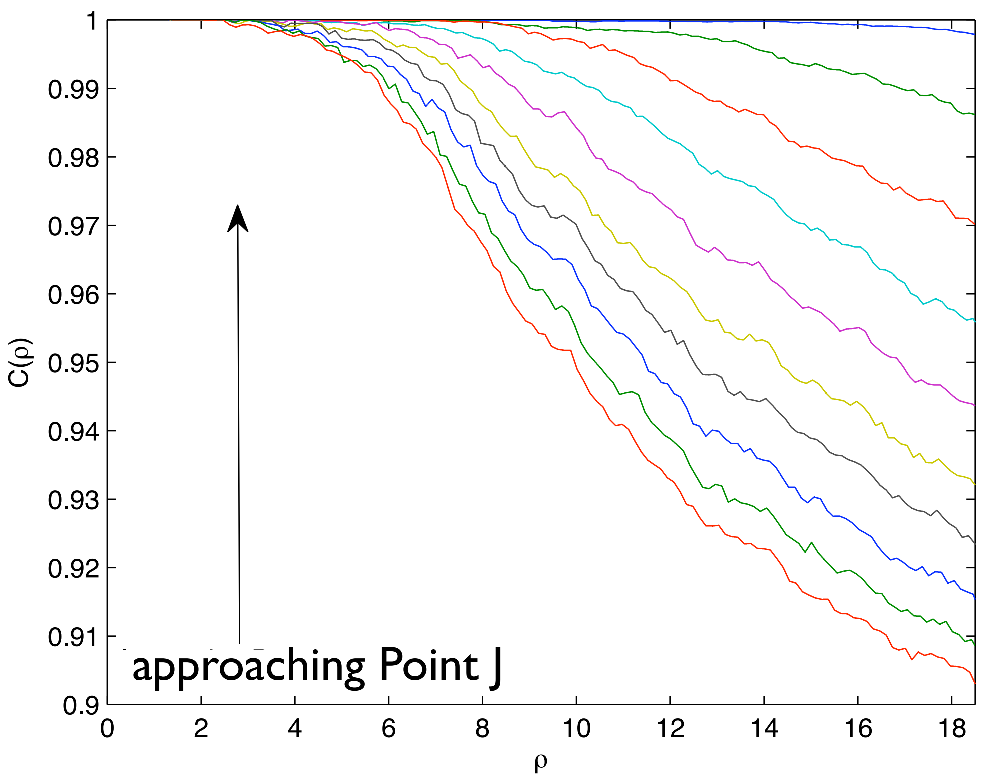
•Phase transitions ? Critical points ?

•We do not have the framework of equilibrium statistical mechanics

Generalization of idea by Edwards to a Stress ensemble

Coarse grained theories, correlation lengths





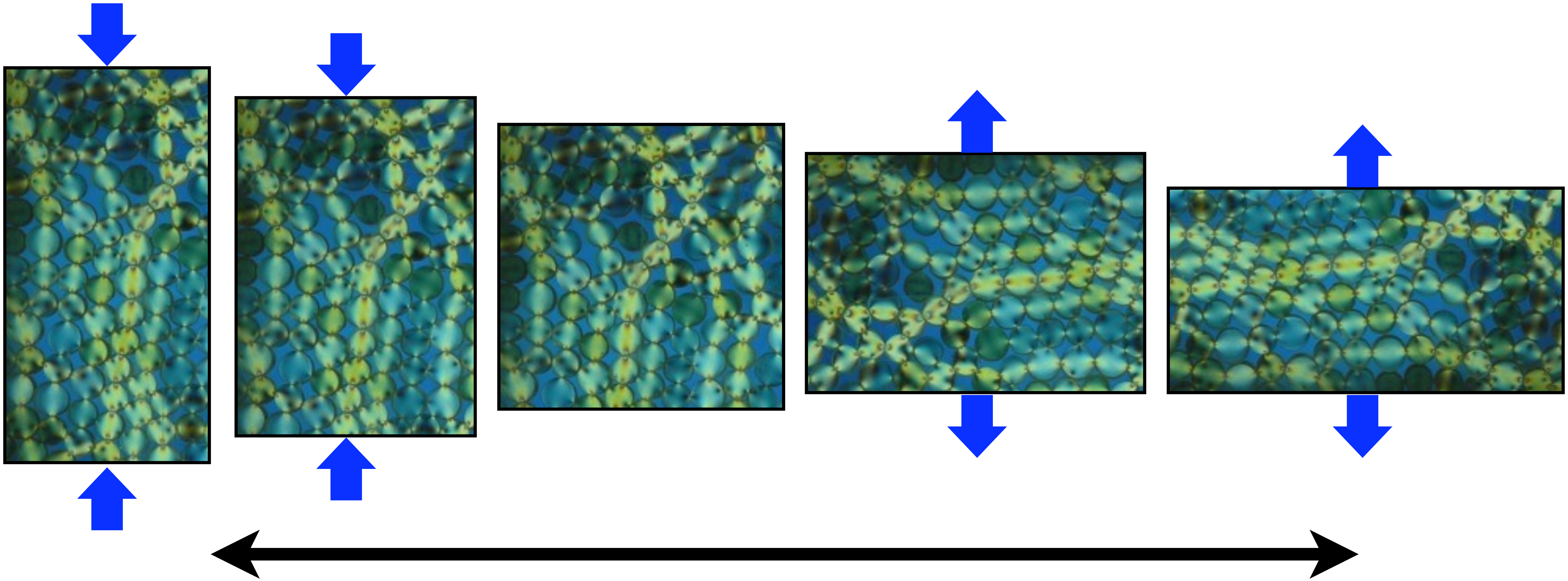
In this talk:

Analyze a particular set of experiments, connect it to the stress ensemble, and discuss implications (for glasses ?)

Cyclic Shear Experiments

(Zhang, Majmudar, Tordesillas & Behringer -- **Granular Matter** 2009)

- ★ 2-D bi-dispersed frictional disks Pure shear
- ★ Quasistatic process
- ★ Performed at fixed packing fraction
- ★ Contact forces can be resolved from photoelasticity
- ★ This protocol can jam packings below ϕ_J .



Quasistatic cyclic strain creating jammed packings

The force-moment tensor:

$$\hat{\Sigma} = \sum_{ij} \vec{r}_{ij} \vec{f}_{ij} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix}$$

The force-moment tensor is an extensive quantity calculated by summing over all of the contact forces and contact positions between grains. It differs from the stress tensor by a volume factor.

"Extensive" pressure:

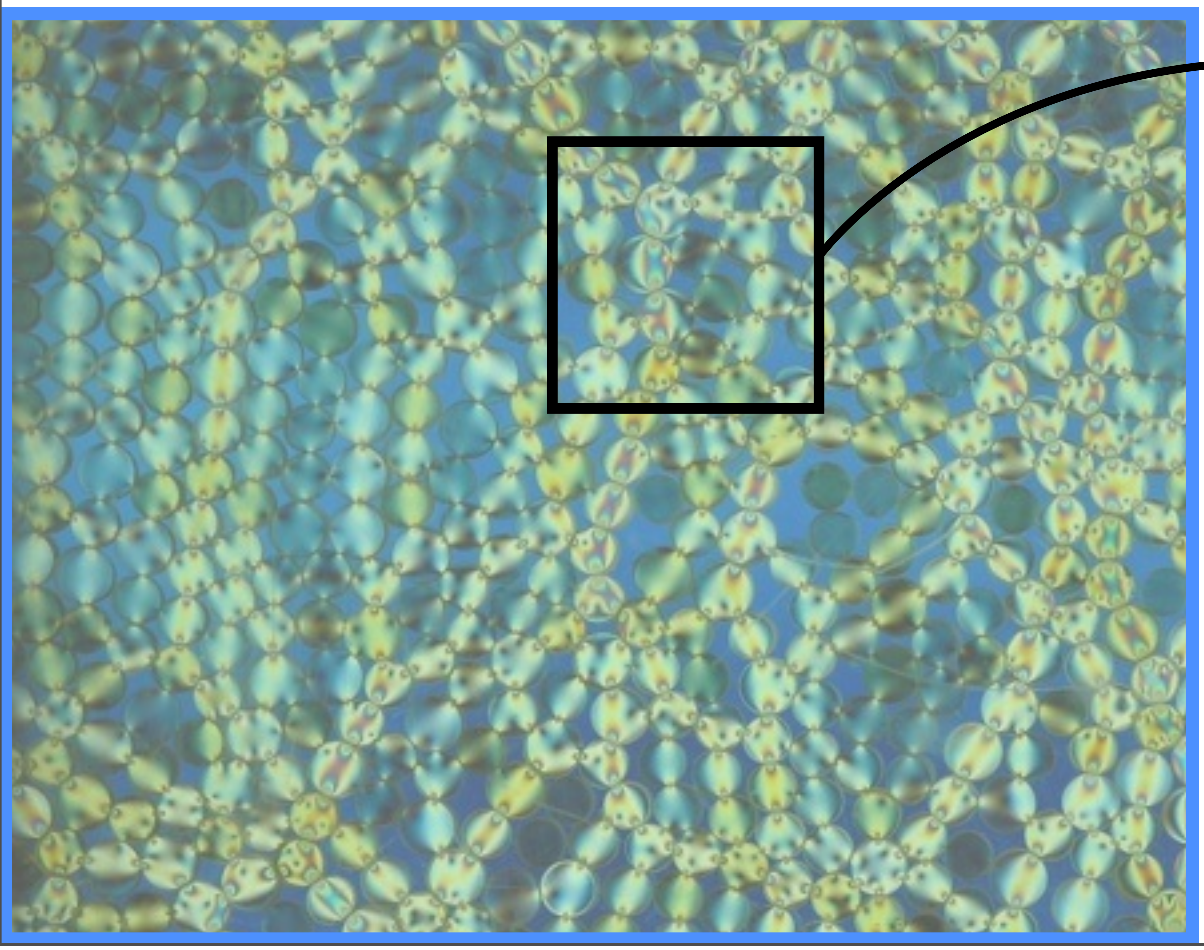
$$\Gamma = \frac{1}{2}\; Tr\hat{\Sigma}$$

"Extensive" deviatoric stress:

$$\tau = \frac{1}{2}\sqrt{(\Sigma_{xx} - \Sigma_{yy})^2 + 4\Sigma_{xy}^2}$$

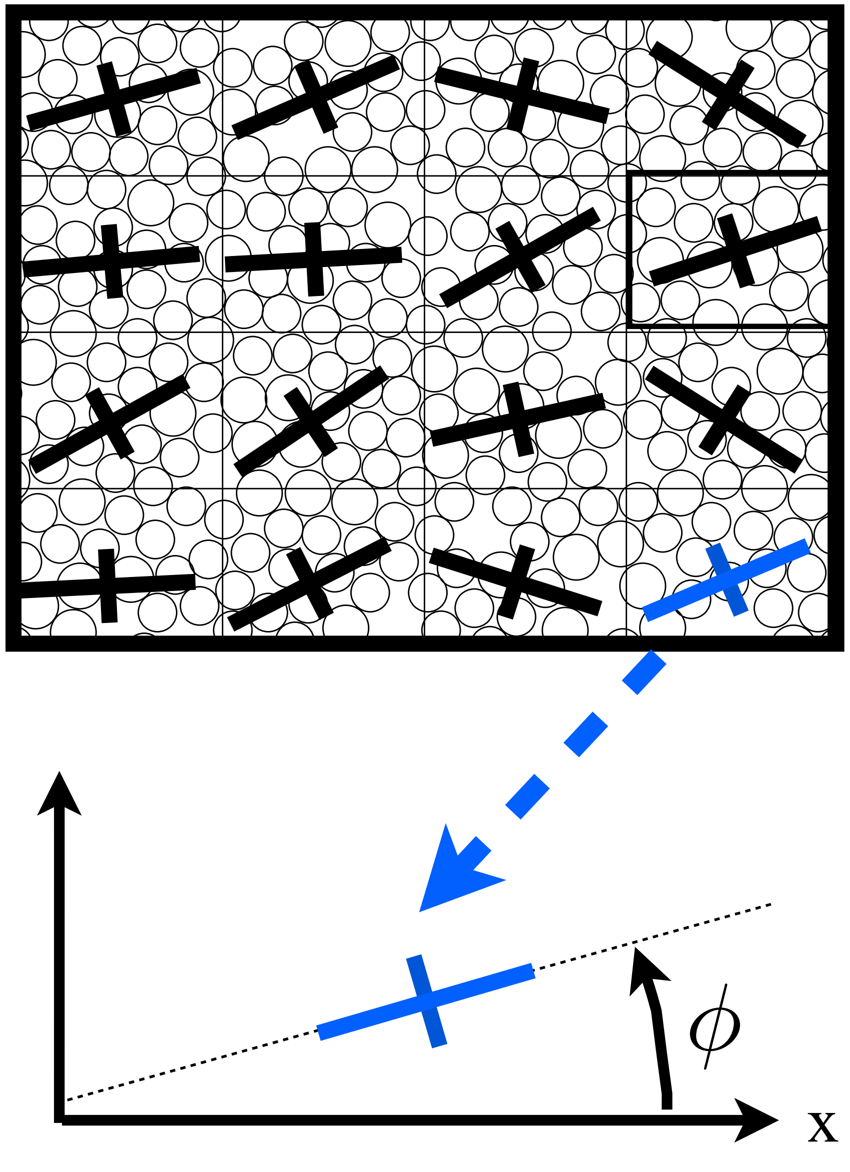
The extensive version of the pressure just the trace of the stress tensor; and the "extensive" deviatoric stress is just the difference between the two eigenvalues of the force-moment tensor. These are rotational invariants of the force-moment tensor.

Packing with
(N grains "extensive" pressure Γ_N)

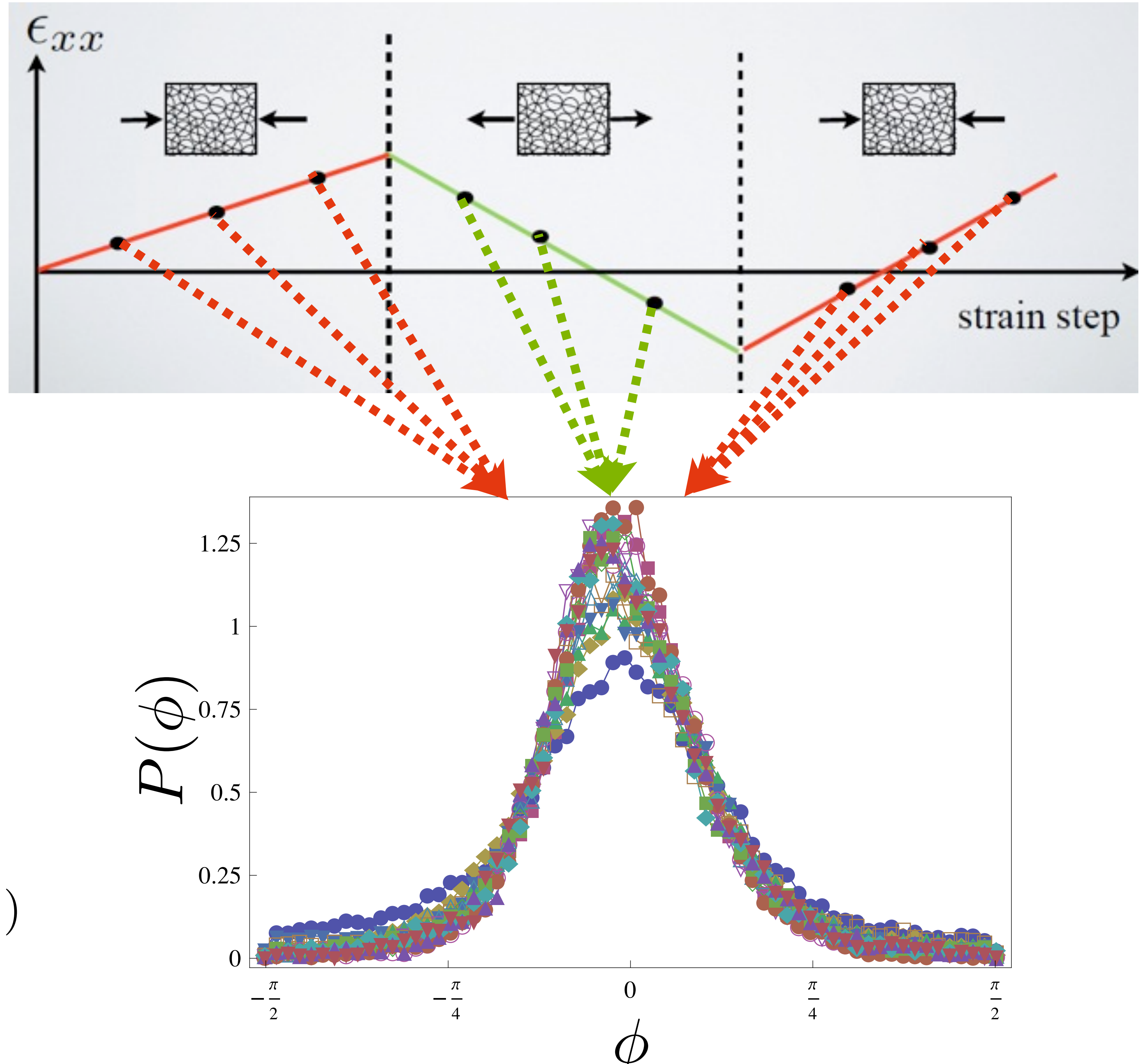


Coarse-grained subregion:
(m grains, force moment tensor $\hat{\Sigma}_m$)

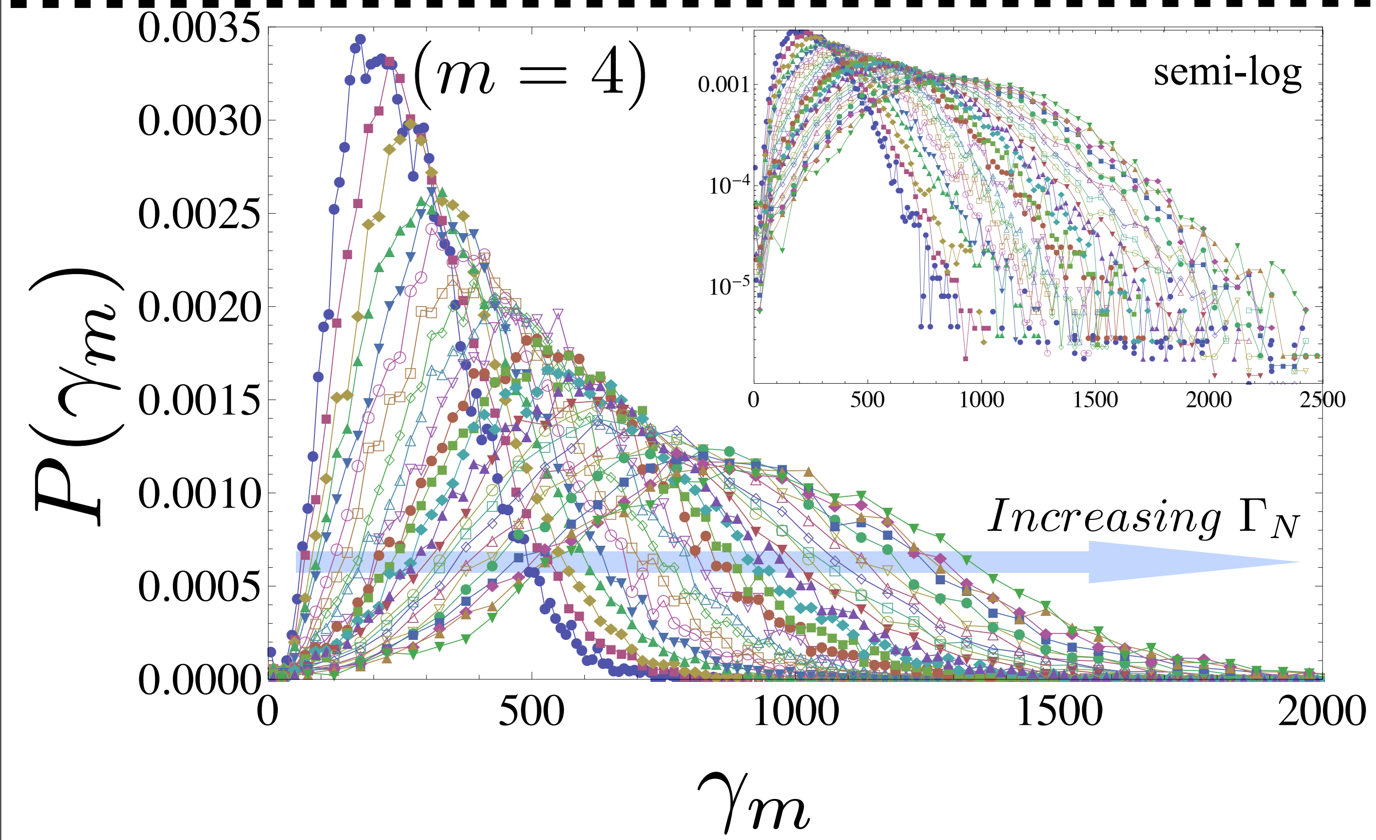
We are interested in looking at the local fluctuations of the force moment tensor. This can be analyzed by coarse-graining the system into subregions containing m number of grains. We repeat the analysis for multiple coarse-grain sizes (ranging from m=4 to m=70).

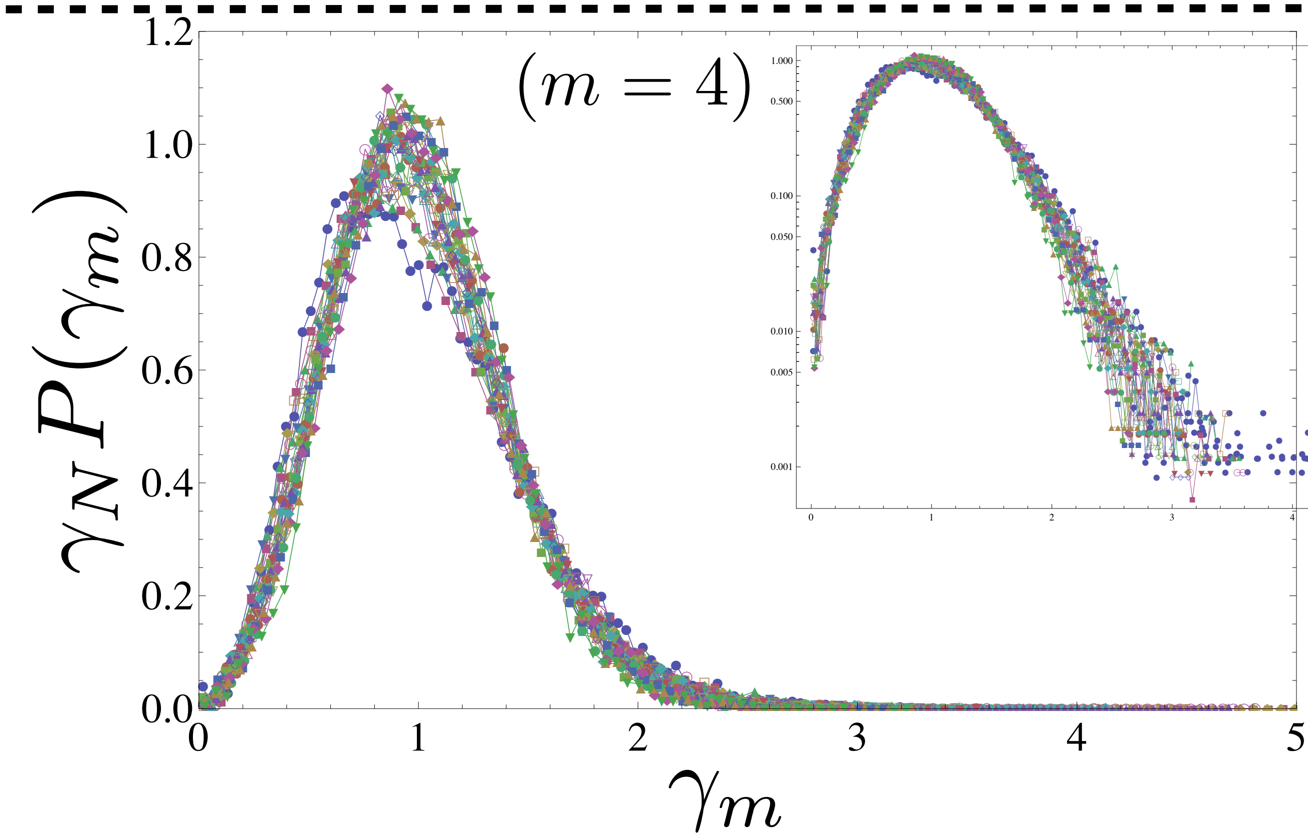


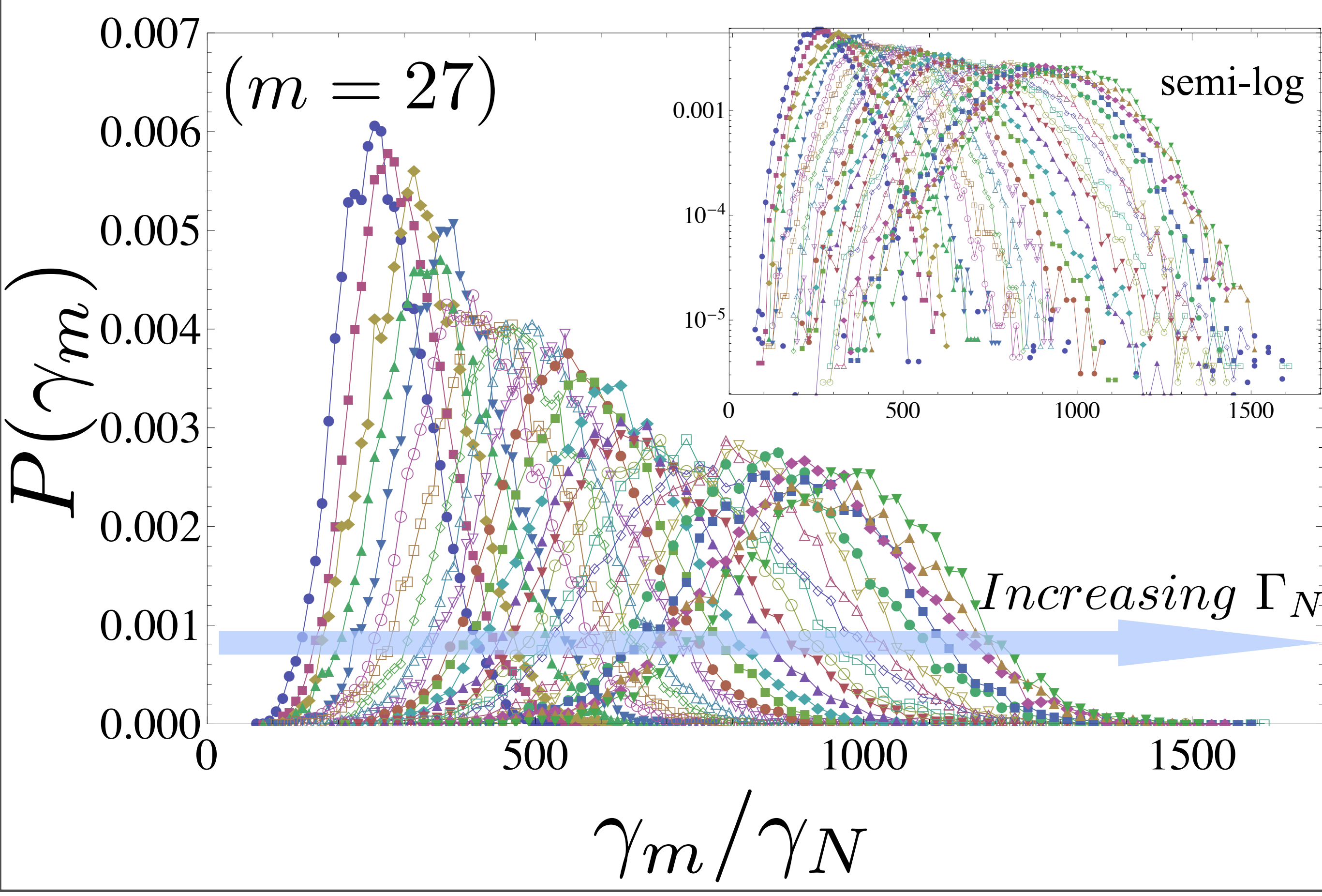
$$\phi = ArcTan(\frac{\Sigma^m_{xy}}{\Sigma^m_{xx} - \Sigma^m_{yy} + \tau^m})$$

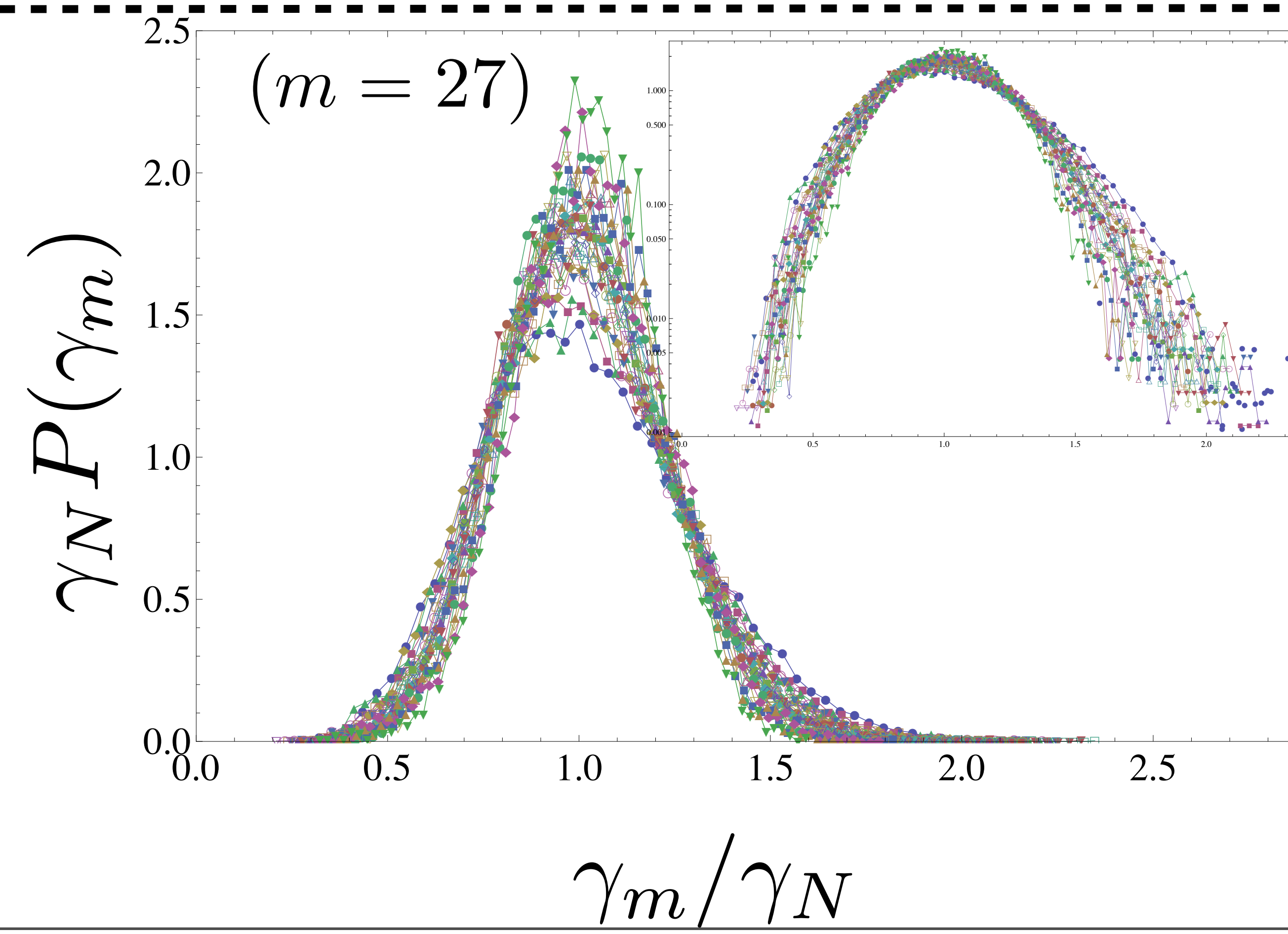


We bin the local pressure distributions by the external pressure. These distribution functions collapse onto a universal scaling function when the subregion (local) pressure is scaled by the external pressure. This holds for all coarse-graining sizes.

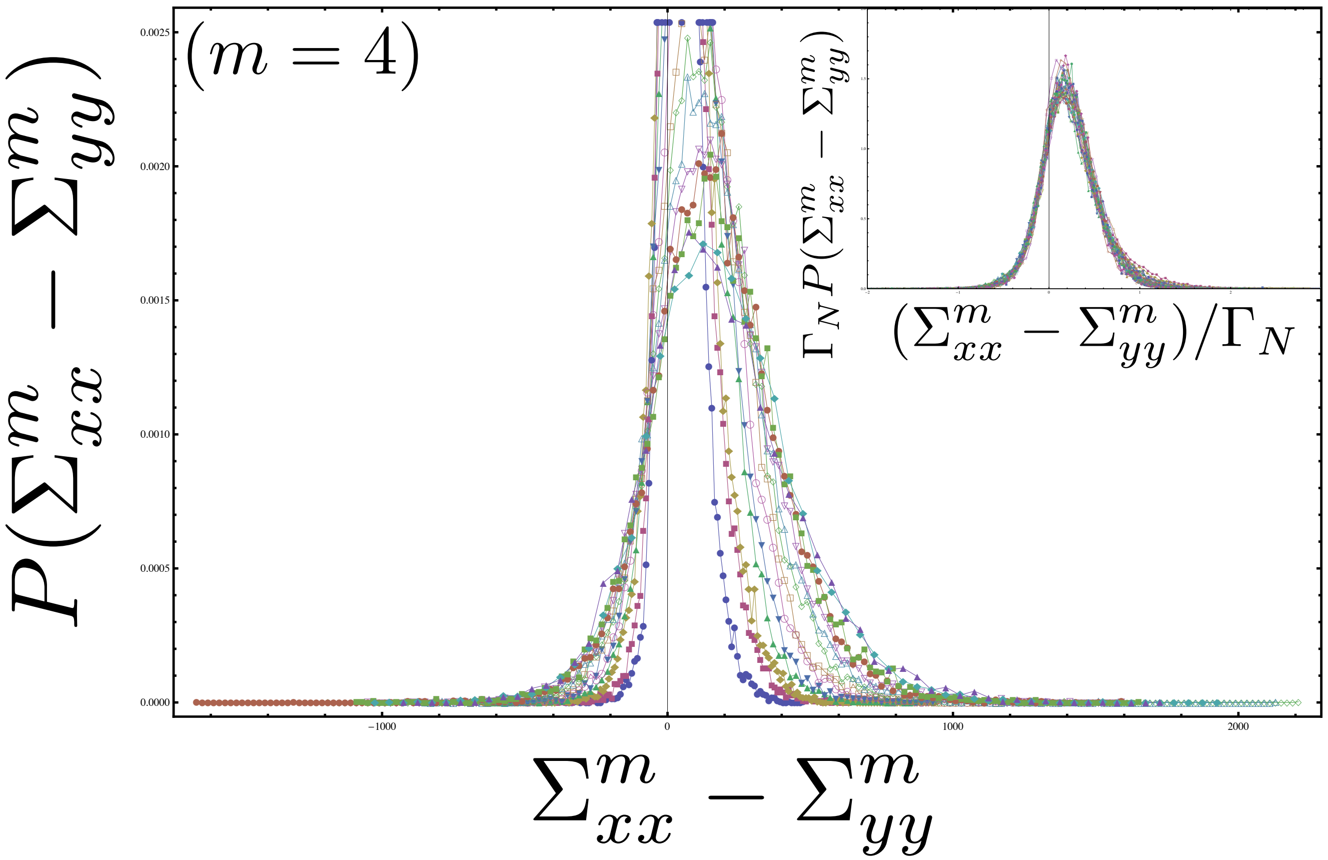


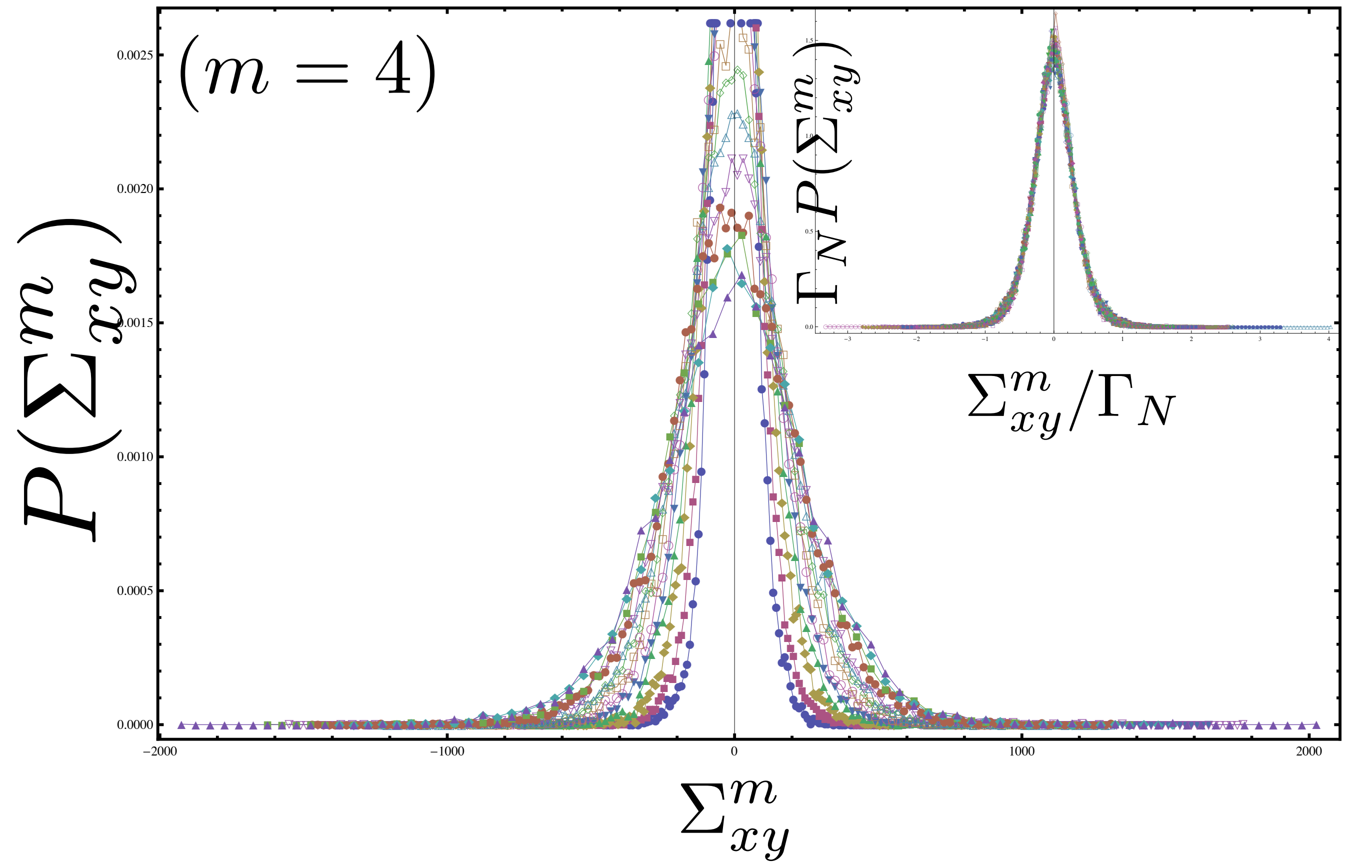






We find the same scaling collapse for all of the components of the force moment tensor:





Locally, the normal stress fluctuates about a mean which is proportional to the global applied pressure. Occasionally there are fluctuations of the local normal stress that is opposite of the global applied normal stress. The off-diagonal stress Σ_{xy} fluctuates about a mean of zero, this is the case for a pure-shear experiment. In a simple-shear experiment, however, the Σ_{xy} will have a non-zero mean.

Overall we find the following scaling form for the distributions.

$$P(\Sigma^m_{ij}) = \frac{1}{\Gamma_N} f^m_{ij}(\Sigma^m_{ij}/\Gamma_N)$$

where Γ_N is the global "extensive" pressure of the packing, and Σ^m_{ij} is the $(ij)^{th}$ component of the subregion (local) stress moment tensor.

Statistical Ensemble of Jammed states

- Many jammed packings correspond to a given set of macroscopic variables
- Is it possible to make a priori prediction of the probability of occurrence of a particular state?
- What are the state variables? Microscopic and Macroscopic?

Edwards' Postulate

1. Jammed (blocked) states control dynamics
2. For these blocked states, volume plays the role of the Hamiltonian in conservative systems
3. All blocked states with the same volume are equiprobable

$$\mathcal{P}_X(\mathbf{q}) = \frac{e^{-\mathcal{W}(\mathbf{q})/X}}{Z(X)}\Theta(\mathbf{q})$$

•Only mechanically stable configurations, q
•X, the compactivity is the analog of temperature
•Z(X) is the partition function, and generator of correlations

Test ?

$$\begin{aligned} p_X(V) &= \int \delta\left[V - \mathcal{W}(\mathbf{q})\right]\left(\frac{e^{-\mathcal{W}(\mathbf{q})/X}}{Z(X)}\right)\Theta(\mathbf{q})\,d\mathbf{q} \\ &= \int \delta\left[V - \mathcal{W}(\mathbf{q})\right]\left(\frac{e^{-V/X}}{Z(X)}\right)\Theta(\mathbf{q})\,d\mathbf{q}, \\ &= \frac{e^{-V/X}}{Z(X)}e^{S(V)}. \end{aligned}$$

$$\longrightarrow \quad \frac{p_{X_1}(V)}{p_{X_2}(V)} = \left(\frac{Z(X_2)}{Z(X_1)}\right)e^{V/X_2 - V/X_1}$$

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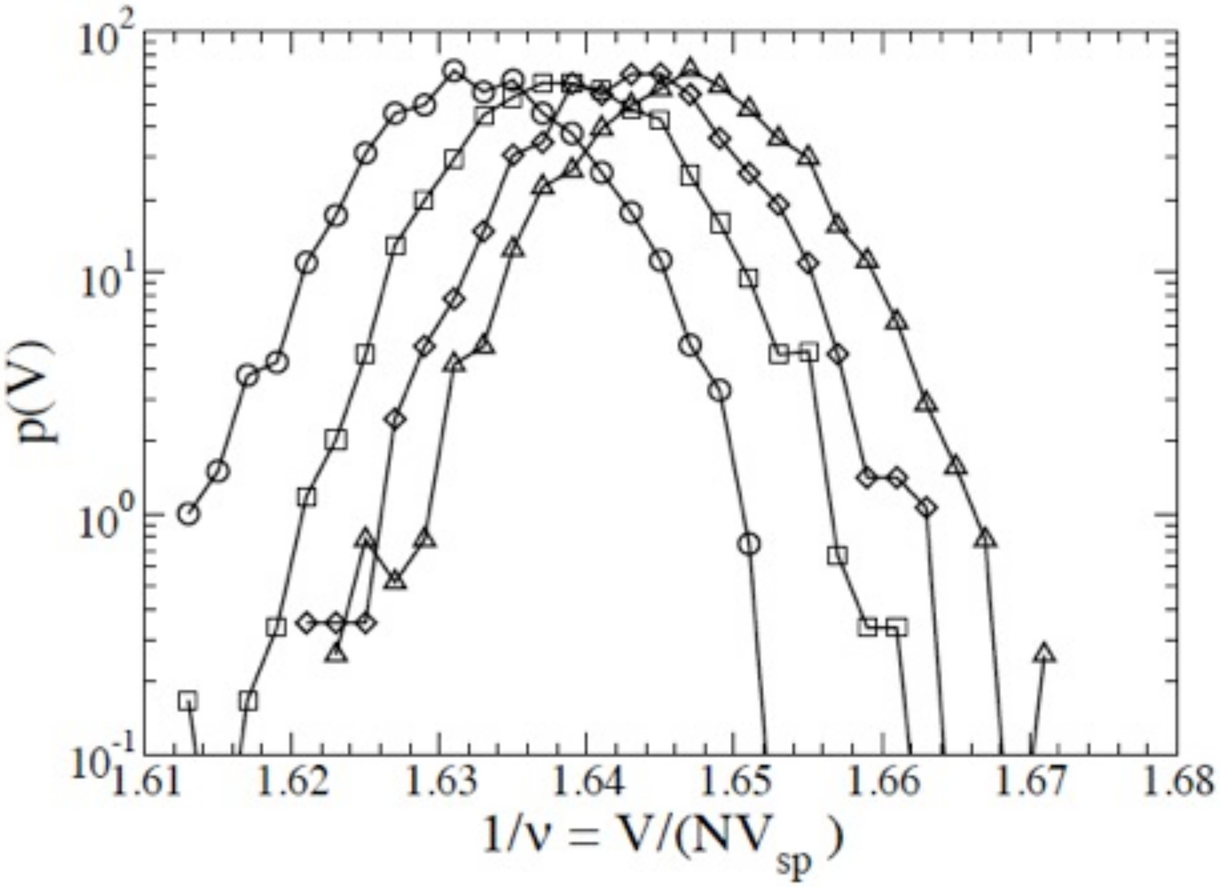
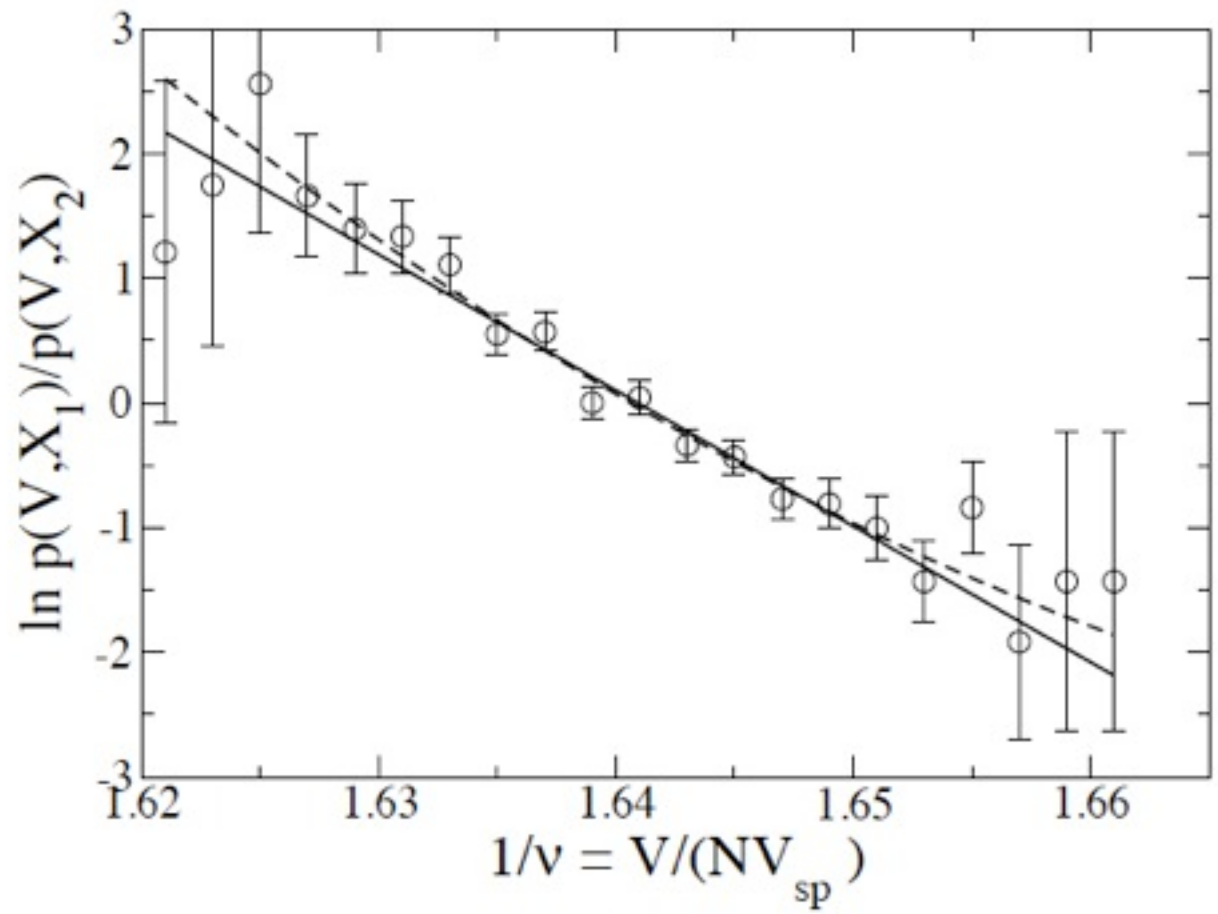
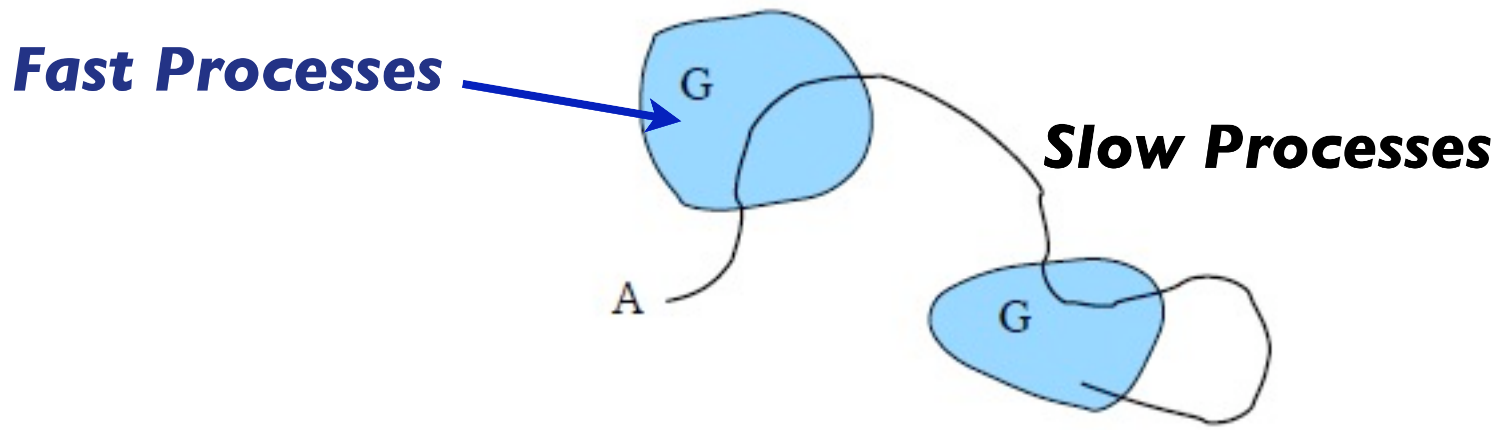




- Does not prove ergodicity or equiprobability
- Origin of this type of distribution in granular packings?
- Bertin & Dauchot (Phys Rev Lett 2006):
    - Dynamics conserves a quantity Q
    - Factorizability of weights of different states (much weaker condition than equiprobability)
- Is there a "conserved" quantity for jammed states?



Deepak Dhar + Joel Lebowitz:
Picocanonical ensemble

Stress-based statistical framework

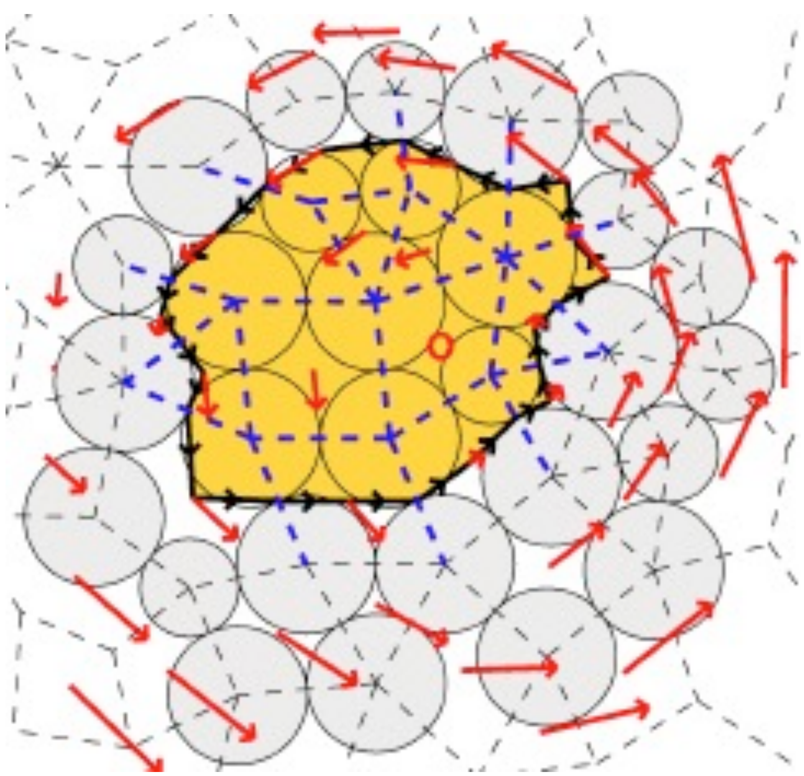
★ Granular materials are inherently out of equilibrium

★ They have energy scales much larger than *kT*

★ Energy is not conserved due to dissipative interactions.

Identifying the ***stress moment tensor*** as a **conserved quantity** in **mechanically stable** granular packings.

(Ball and Blumenfeld **PRL** 2002)



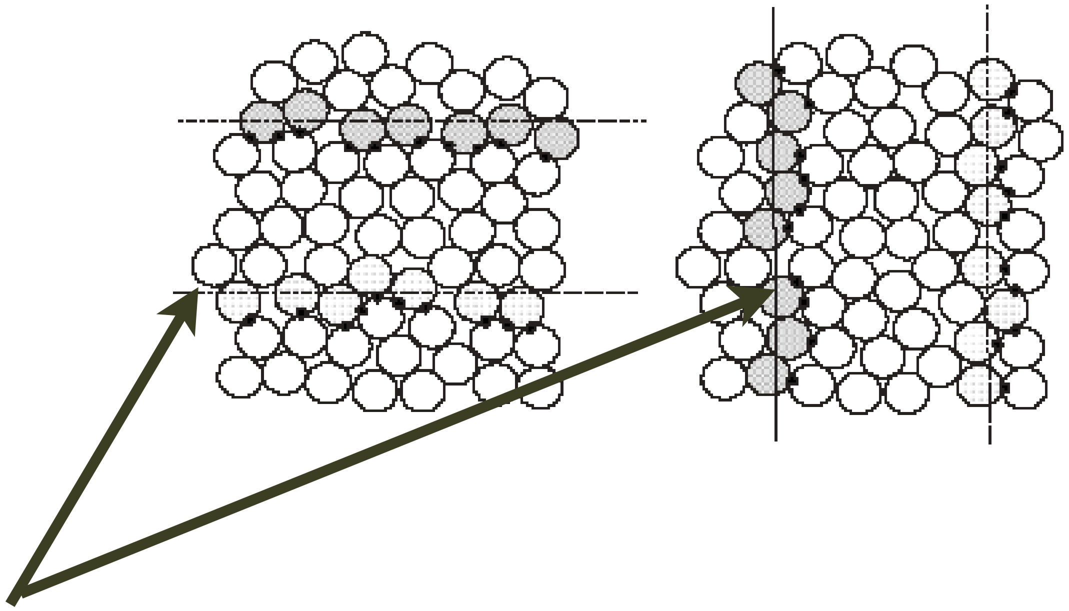
$$\hat{\Sigma} = \sum_{all\ \ grains} \vec{r}_{ij}\vec{f}_{ij} = \sum_{\mu \in boundary} \vec{r}_{\mu}\vec{h}_{\mu}$$

★ Using a height map formulation, the stress moment tensor of a cluster can be expressed as a quantity that only depends on boundary forces.

★ Therefore any local rearrangements of grains that respect force and torque balance cannot change the global value of $\hat{\Sigma}$.

★ We do statistical mechanics using $\hat{\Sigma}$, which will play the role of energy.

Topological Invariants of Mechanically Stable Packings (illustrated for planar packings)



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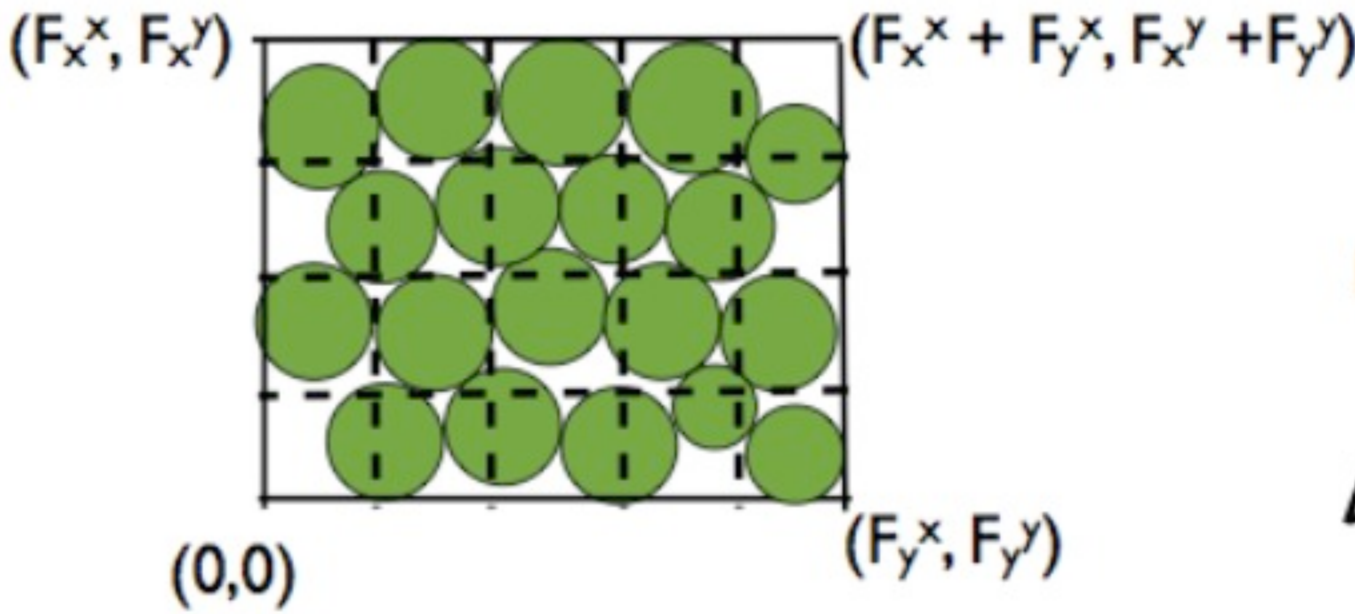
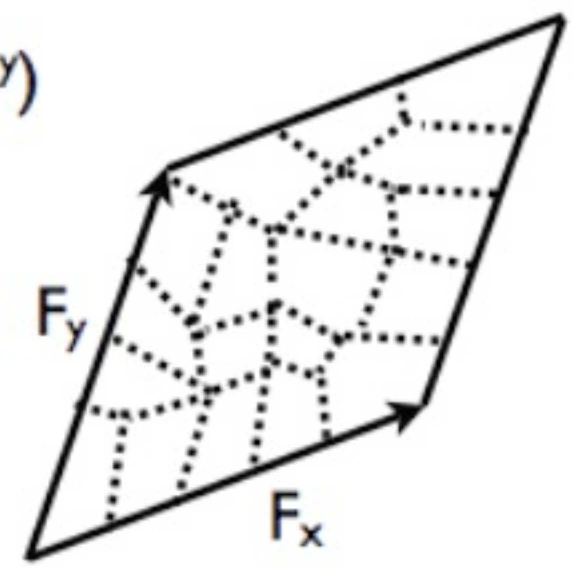
Total force normal to these lines have to remain the same as one translates the line



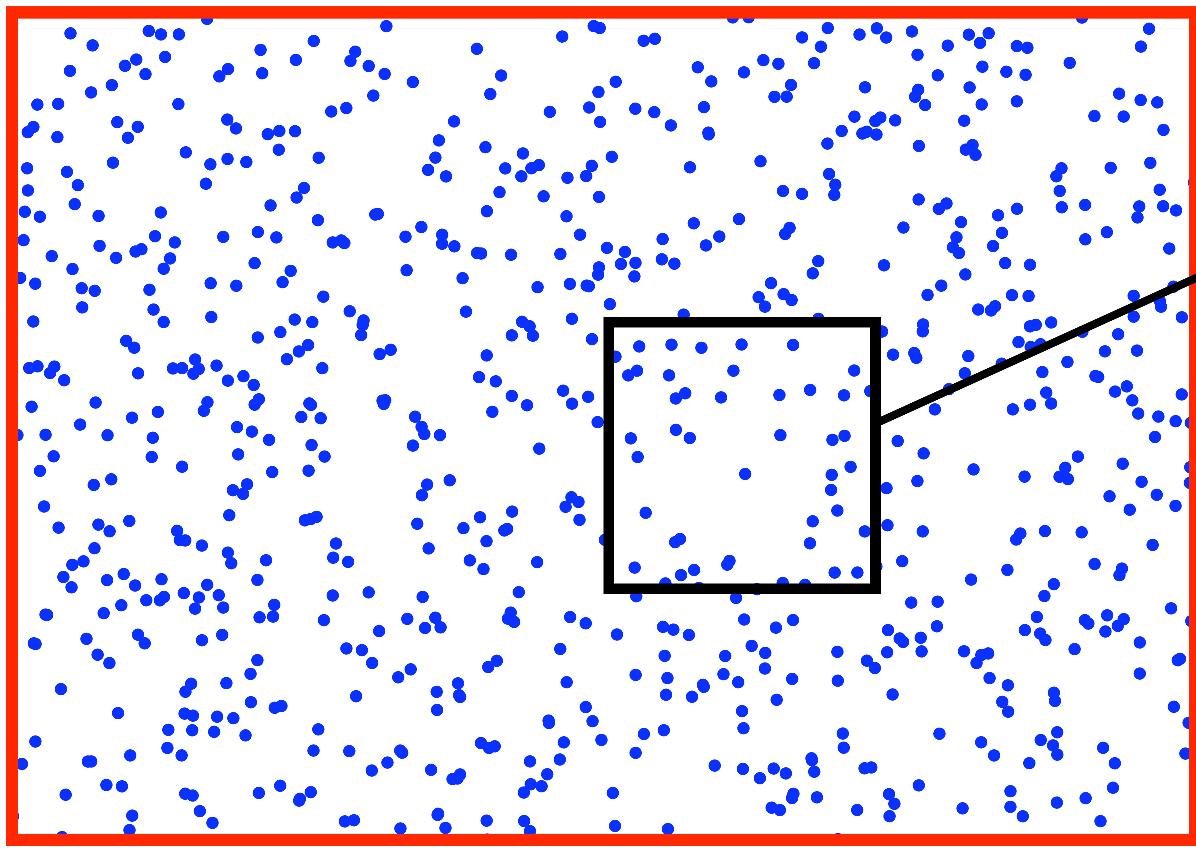


F_x, F_y scale linearly with system size

Review of the canonical ensemble for a thermal system

Thermal reservoir with $(E,\ V,\ N)$

subregion with $(E_m,\ V_m,\ m)$



★ Count the number of states and define entropy:

$$S(E,V,N) = k_B log[\Omega(E,V,N)]$$

★ Define the thermodynamic temperature:

$$\beta = \left.\frac{\partial S}{\partial E}\right|_{N,V}$$

★ Thermal equilibrium: all subregions of the reservoir have equal β.

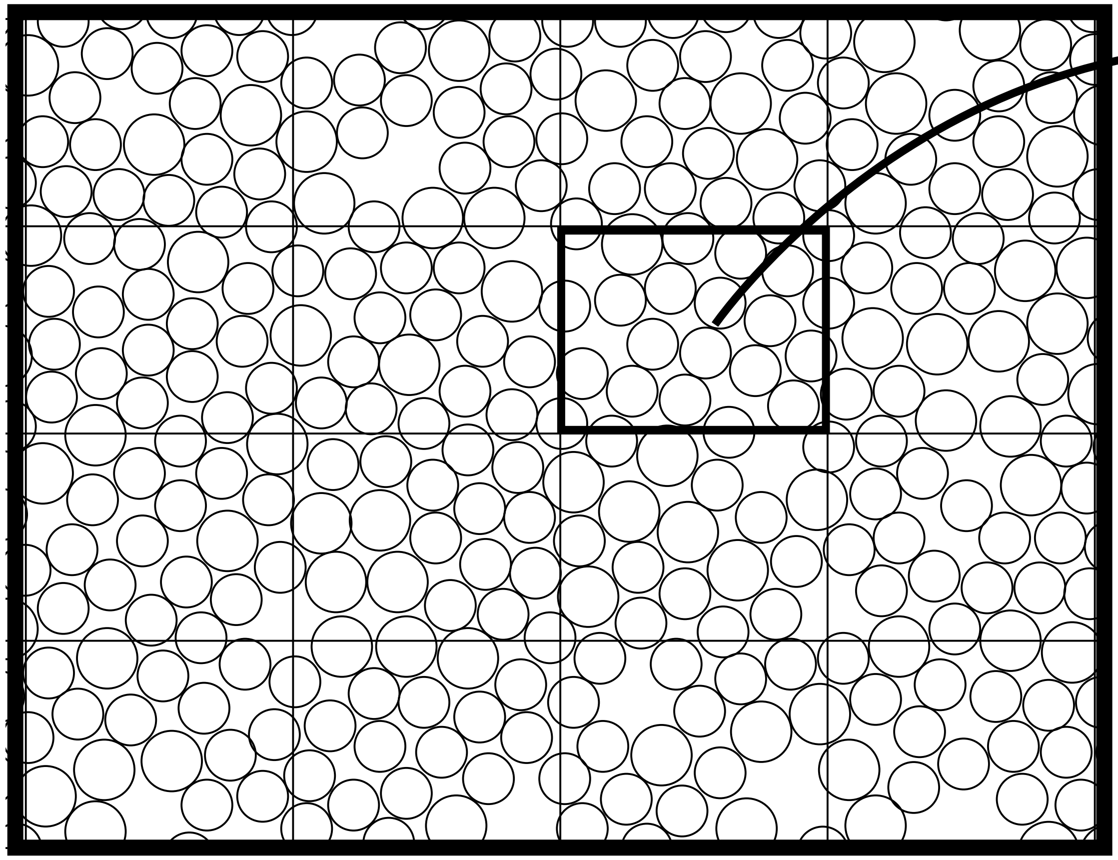
The distribution of energy is then given by:

$$P(E_m) = \frac{1}{Z}\Omega(E_m, V_m, m)e^{-\beta E_m}$$

An ensemble for granular materials

(Henkes & Chakraborty **PRL** 2007))

Packing with $(\hat{\Sigma}_N, V, N)$ $\hat{\Sigma}_N = \begin{pmatrix} \Sigma^N_{xx} & \Sigma^N_{xy} \\ \Sigma^N_{xy} & \Sigma^N_{yy} \end{pmatrix}$



subregion with

$(\hat{\Sigma}_m, V_m, m)$ $\hat{\Sigma}_m = \begin{pmatrix} \Sigma^m_{xx} & \Sigma^m_{xy} \\ \Sigma^m_{xy} & \Sigma^m_{yy} \end{pmatrix}$

★ Define entropy: Can include a non flat measure

$$S(\hat{\Sigma}_N, V, N) = log[\Omega(\hat{\Sigma}_N, V, N)]$$

★ Define the "granular temperature" called the :

Angoricity

$$\alpha_{ij} = \left.\frac{\partial S}{\partial \Sigma_{ij}}\right|_{N,V} \rightarrow \hat{\alpha} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} \\ \alpha_{xy} & \alpha_{yy} \end{pmatrix}$$

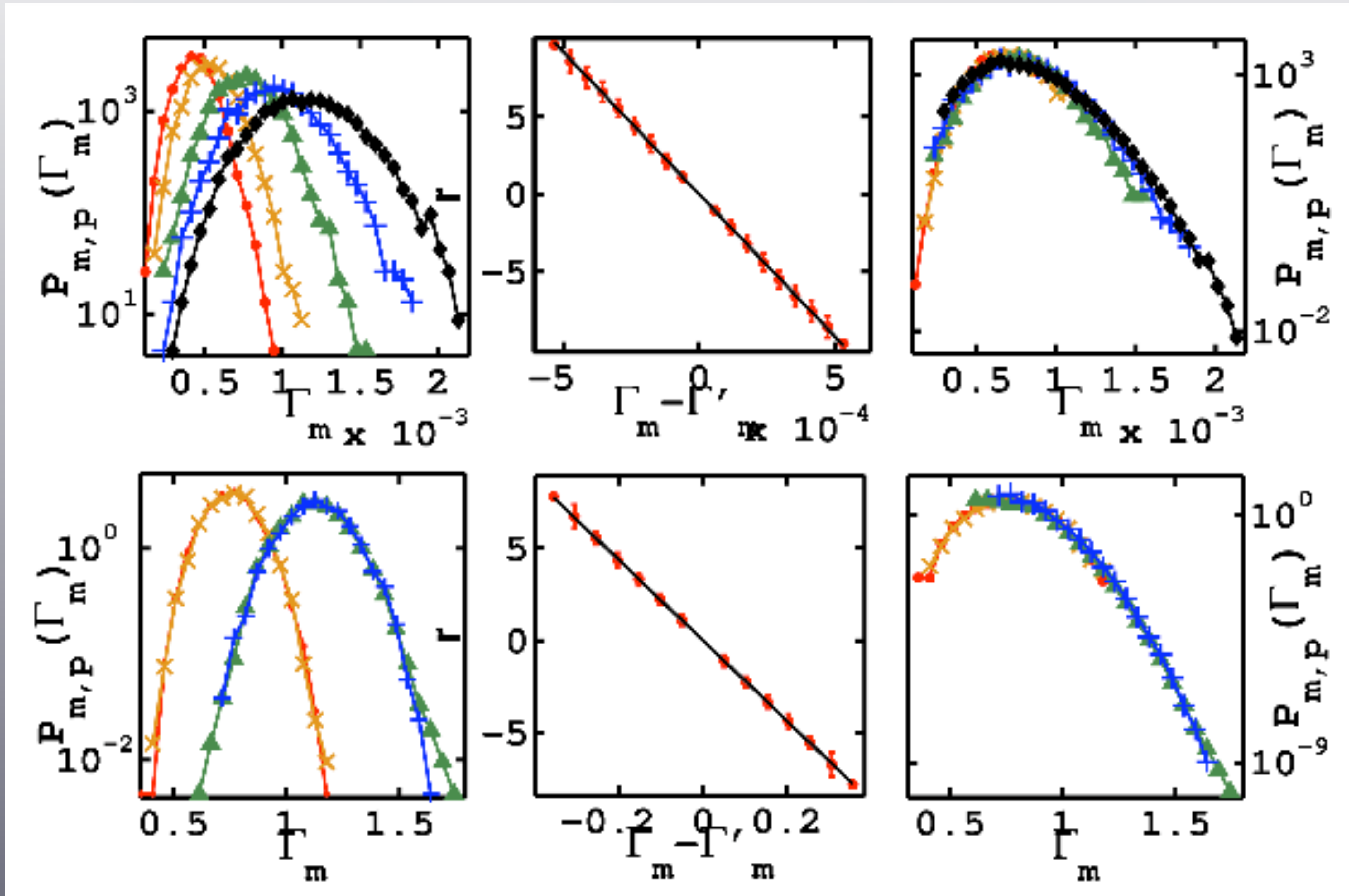
★ Mechanical equilibrium: all subregions of the reservoir have equal $\hat{\alpha}$.

The distribution of the stress moment tensor is then given by:

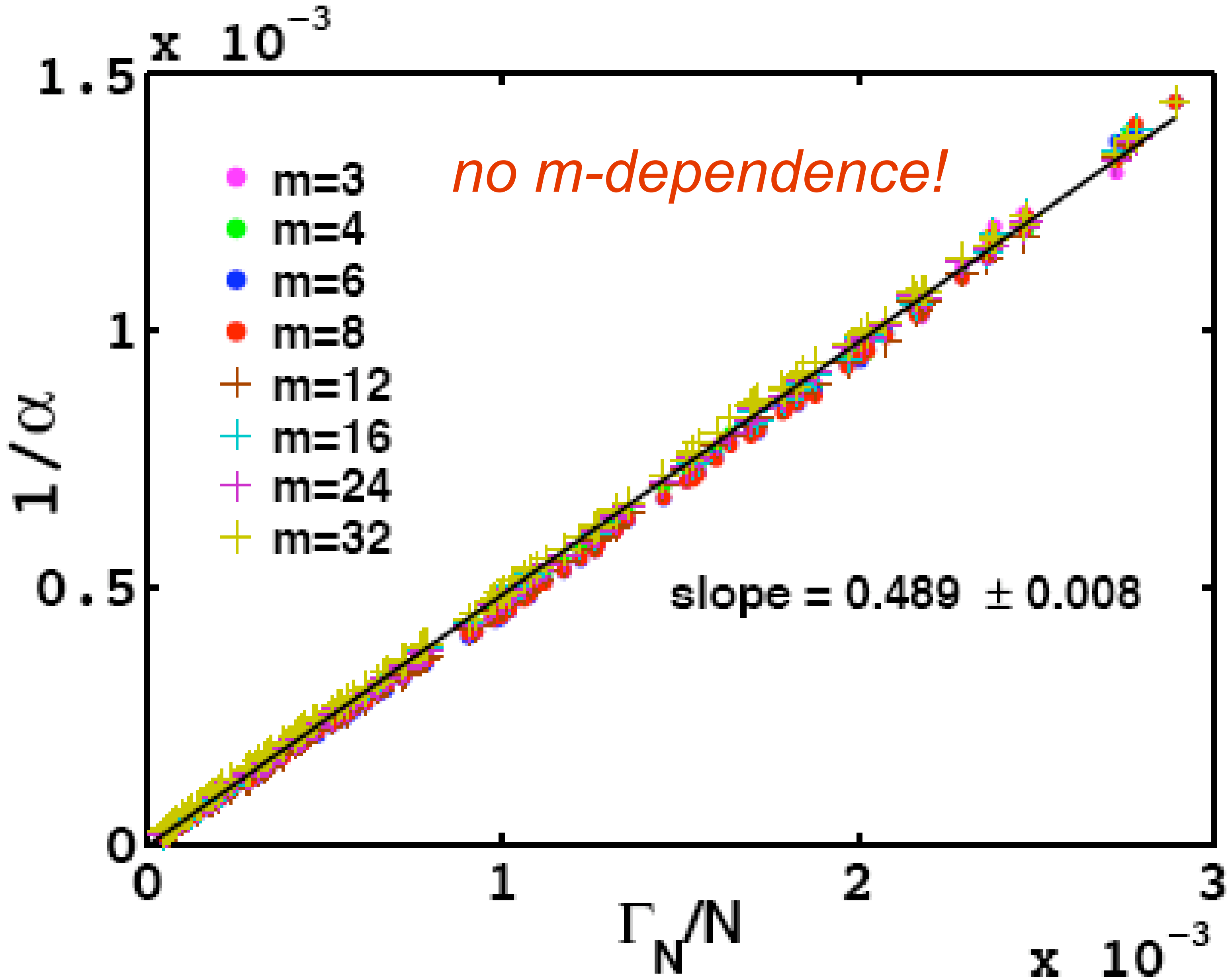
$$\begin{aligned} P(\hat{\Sigma}_m) &= \frac{1}{Z}\Omega(\hat{\Sigma}_m, V_m, m)e^{-\hat{\alpha}:\hat{\Sigma}_m} \\ &= \frac{1}{Z}\Omega(\hat{\Sigma}_m, V_m, m)e^{-\alpha_{xx}\Sigma_{xx}-\alpha_{yy}\Sigma_{yy}-2\alpha_{xy}\Sigma_{xy}} \end{aligned}$$

Simulations of frictionless disks (O'Hern)

Equality of alpha inside a configuration $\quad \delta\alpha_{p,q} = \alpha_p - \alpha_q$



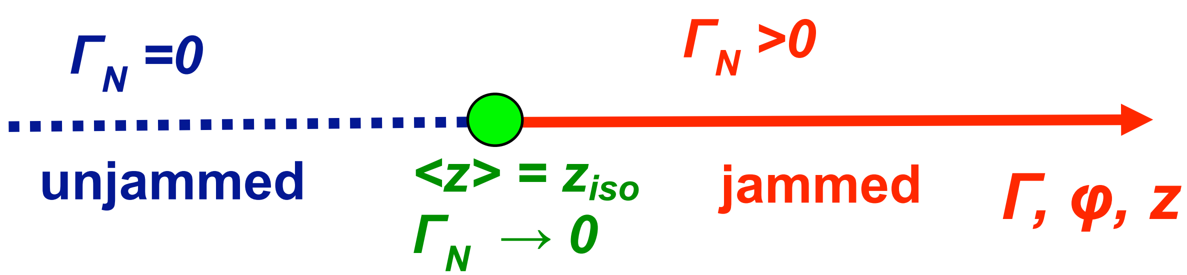
Equation of state close to unjamming (packings fall apart)



equation of state

$$\alpha = \frac{2N}{\Gamma_N}$$

$\Gamma_N \rightarrow 0$ at the jamming transition



FORCES BECOME INDEPENDENT IN THIS LIMIT AND P(F) IS EXPONENTIAL

Cyclic Shear Experiments

★ In the experimental data, we observed that all force-moment tensor components are controlled by the global pressure. Therefore, the angoricities must have the following property:

$$\alpha_{ij} \propto \frac{N}{\Gamma_N}$$

★ The entropy should only depend on rotationally invariant properties of the force-moment tensor:

$$S = S(\Gamma, \tau)$$

★ To lowest order, the entropy can be written:

$$\begin{aligned} S(N,\Gamma,\tau) &= a\ N\ log(\frac{\Gamma}{N}) - b\ N\ (\frac{\tau}{\Gamma})^2 + S_0 \\ &= a\ N\ log(\frac{\Sigma_{xx}+\Sigma_{yy}}{N}) - b\ N\ \frac{(\Sigma_{xx}-\Sigma_{yy})^2 + 4\Sigma_{xy}^2}{(\Sigma_{xx}+\Sigma_{yy})^2} + S_0 \end{aligned}$$

Using this entropy to write the distribution as:

$$\begin{aligned} P(\hat{\Sigma}^m) &= \frac{1}{\mathcal{Z}}\left(\frac{\Sigma_{xx}^m + \Sigma_{yy}^m}{m}\right)^{am} e^{-bm\frac{(\Sigma_{xx}^m - \Sigma_{yy}^m)^2 + 4(\Sigma_{xy}^m)^2}{(\Sigma_{xx}^m + \Sigma_{yy}^m)^2}} \\ &\quad \times exp(-\alpha_{xx}\Sigma_{xx}^m - \alpha_{yy}\Sigma_{yy}^m - 2\alpha_{xy}\Sigma_{xy}^m) \end{aligned}$$

- The experimental data of the distribution of all force-moment tensor components can be fitted to this prediction.
- Within this statistical framework, the distribution of major-axis angle can be shown to be independent of the pressure, strain, etc.

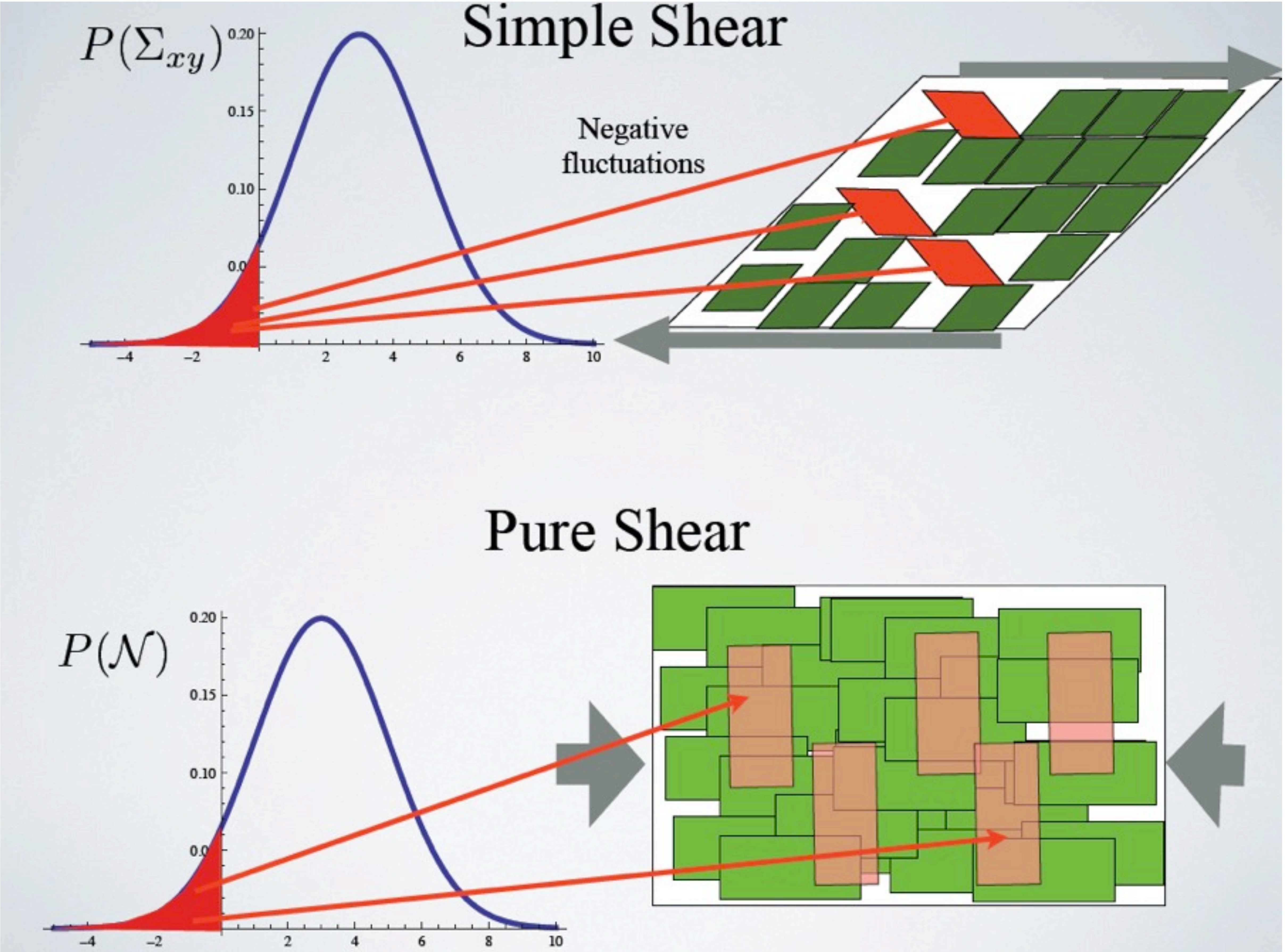
$$
\begin{aligned}
S(N,\Gamma,\tau) &= a\ N\ log(\frac{\Gamma}{N}) - b\ N\ (\frac{\tau}{\Gamma})^2 + S_0 \\
&= a\ N\ log(\frac{\Sigma_{xx}+\Sigma_{yy}}{N}) - b\ N\ \frac{(\Sigma_{xx}-\Sigma_{yy})^2 + 4\Sigma_{xy}^2}{(\Sigma_{xx}+\Sigma_{yy})^2} + S_0
\end{aligned}
$$

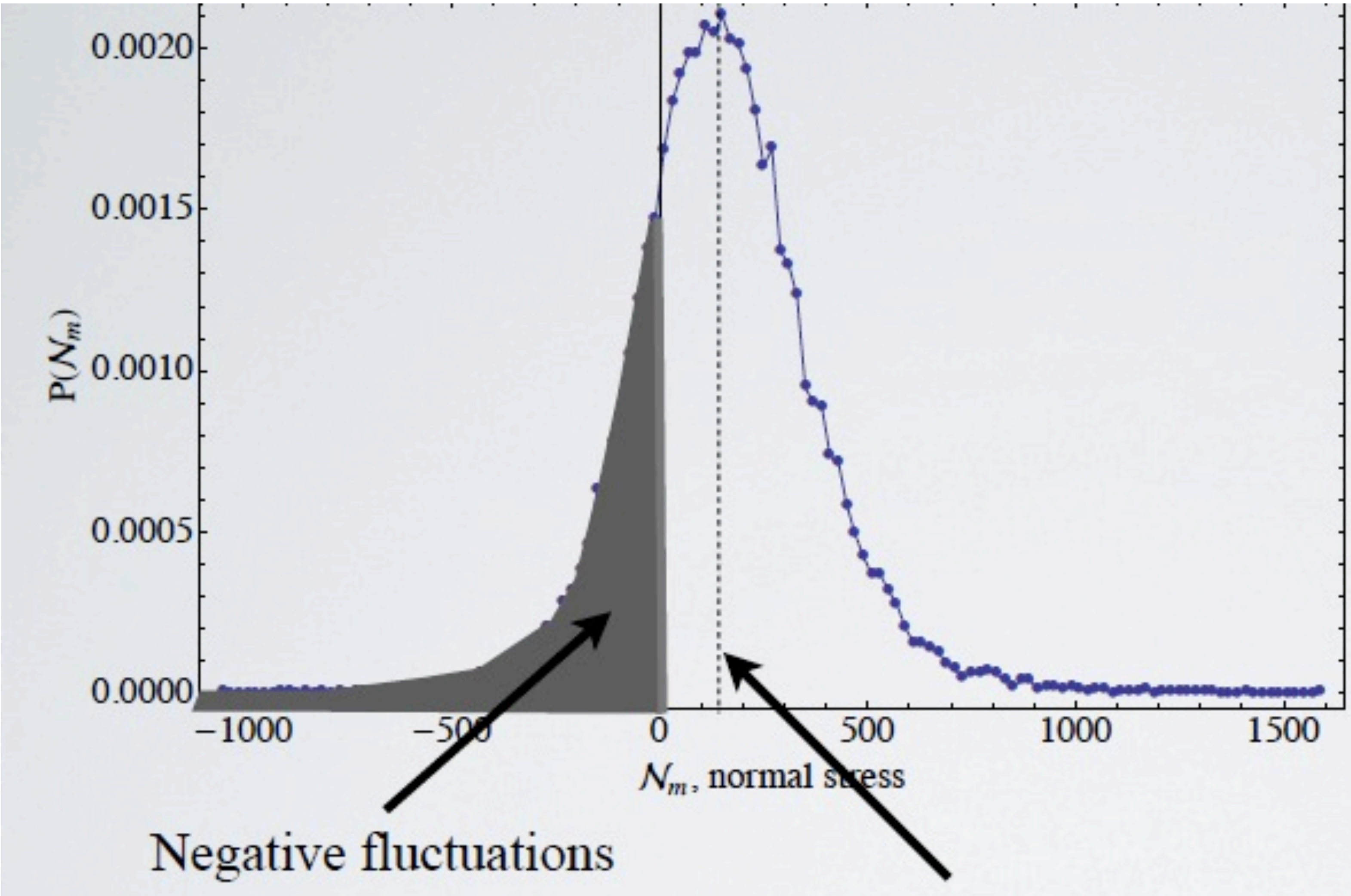
•Number density of jammed states vanishes at Point J
• As shear is increased at finite compression, the complexity goes to zero at a some critical shear stress
•True entropy vanishing transition at unjamming to flowing state (not point J)

$$
\begin{aligned}
P(\hat{\Sigma}^m) &= \frac{1}{\mathcal{Z}}\left(\frac{\Sigma_{xx}^m+\Sigma_{yy}^m}{m}\right)^{am} e^{-bm\frac{(\Sigma_{xx}^m-\Sigma_{yy}^m)^2+4(\Sigma_{xy}^m)^2}{(\Sigma_{xx}^m+\Sigma_{yy}^m)^2}} \\
&\quad \times exp(-\alpha_{xx}\Sigma_{xx}^m - \alpha_{yy}\Sigma_{yy}^m - 2\alpha_{xy}\Sigma_{xy}^m)
\end{aligned}
$$

•Specifying the imposed stress (at the boundary) determines the distribution of local stresses (very different from elastic systems): Field Theory for stress correlations

Fluctuations of Normal and Shear Stress



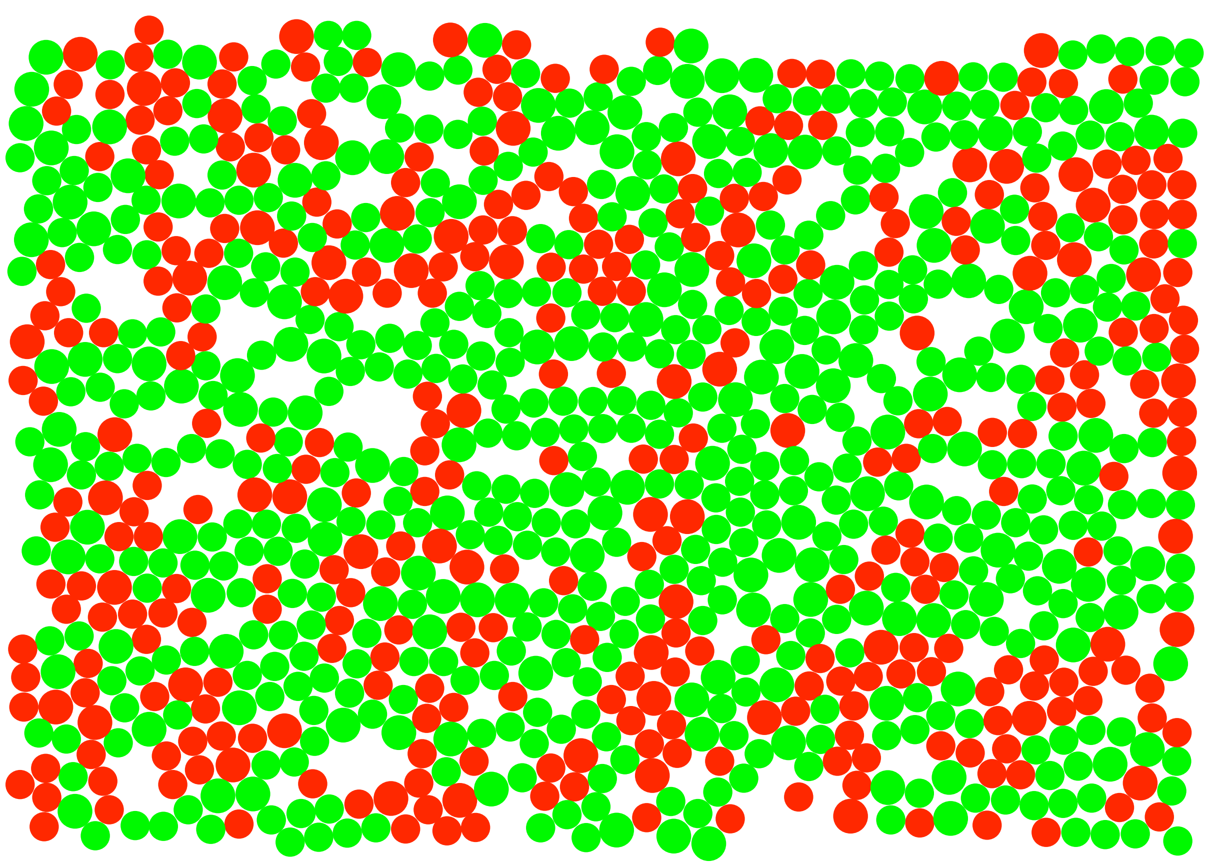


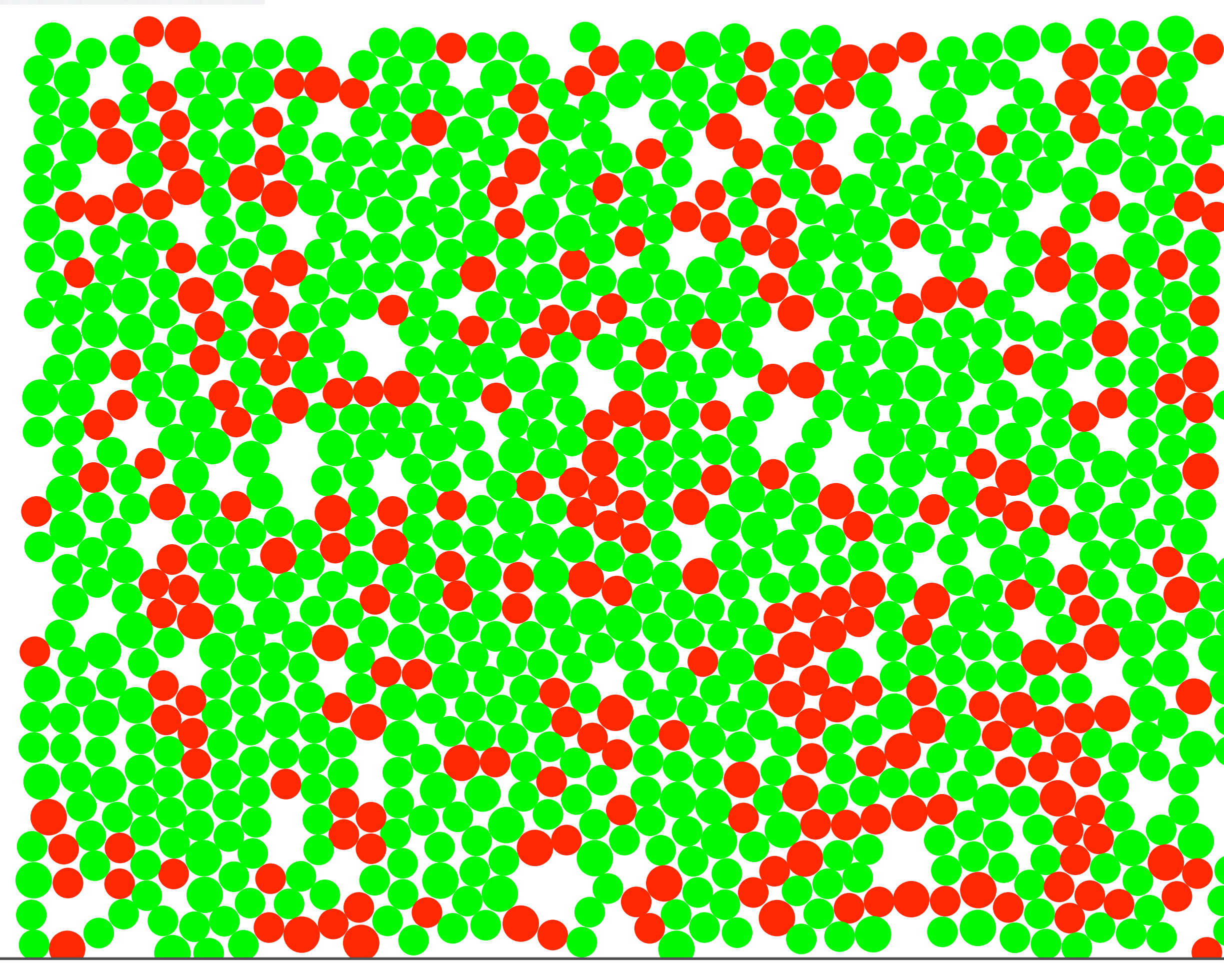
$$\frac{P(-N)}{P(+N)} \propto exp(-\frac{2N}{T_{eff}})$$

$$T_{eff} \propto \frac{N}{\Gamma_N}$$

Analog: Negative magnetization fluctuations in a positive field

Fluctuation relations in NESS



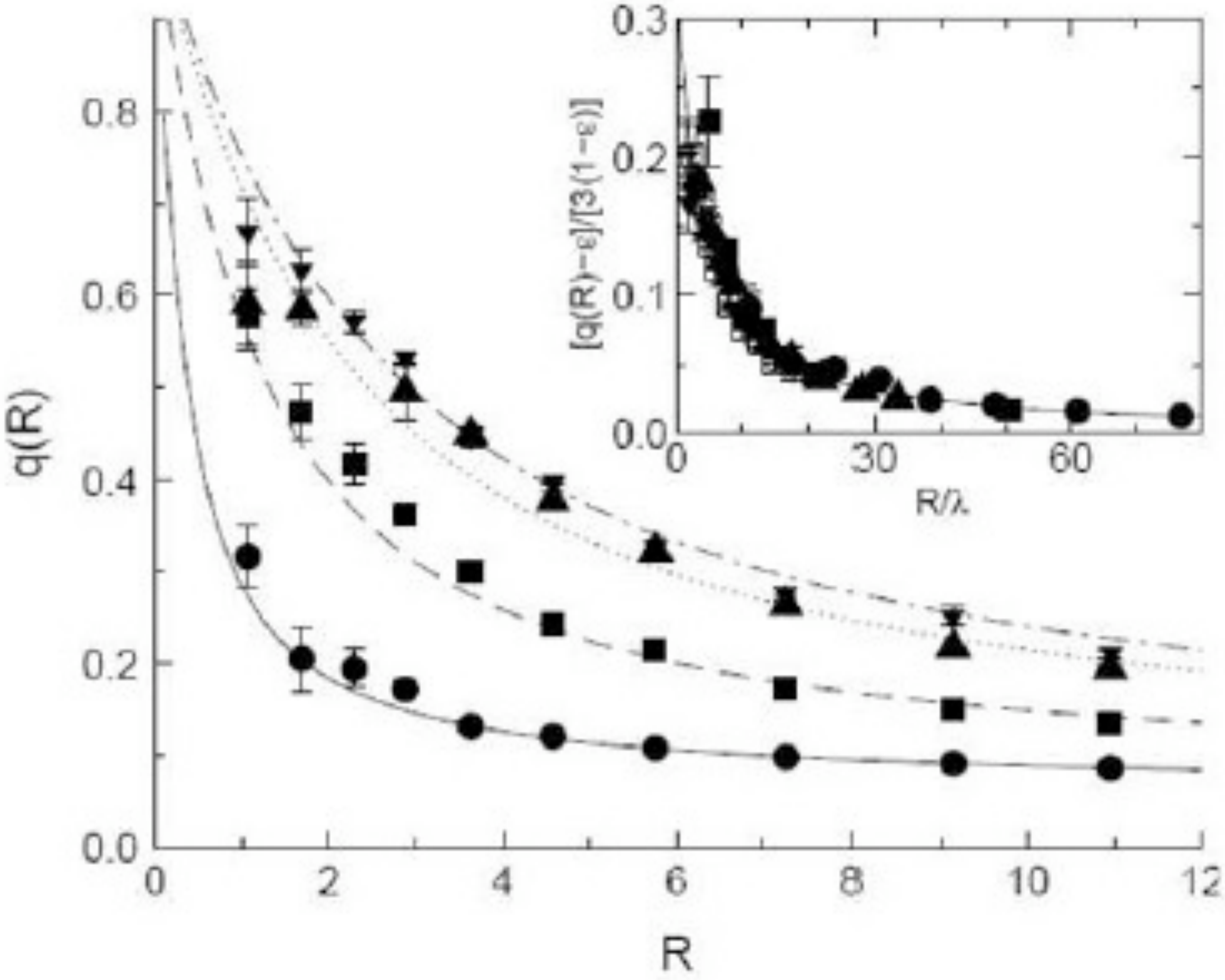


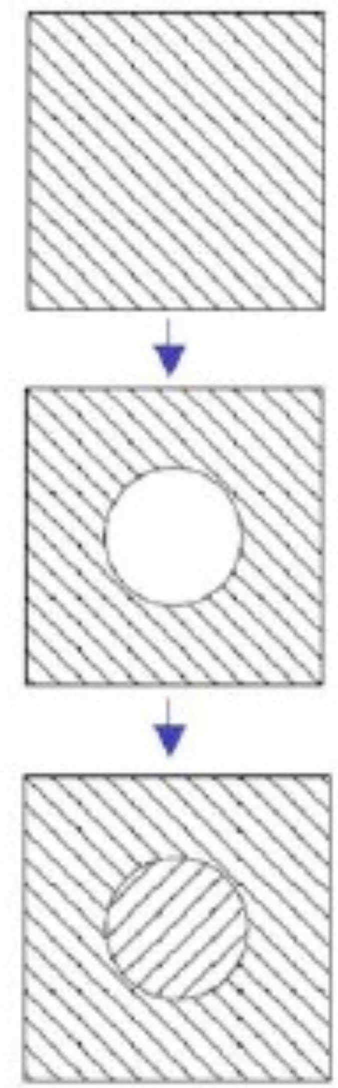
Connection to Complexity and Configurational Entropy

- Inherent structures akin to jammed states
- Configurational entropy: number density of inherent structures with a particular energy, or density, or free energy
- Meanfield models: Complexity is well defined because of infinite barrier between states
- Restricting ourselves to jammed states, creates diverging barriers between sectors with different $\hat{\Sigma}$. $S(\hat{\Sigma})$ is the analog of complexity

Point-To-Set Correlation Function

Lennard-Jones Glass-formers

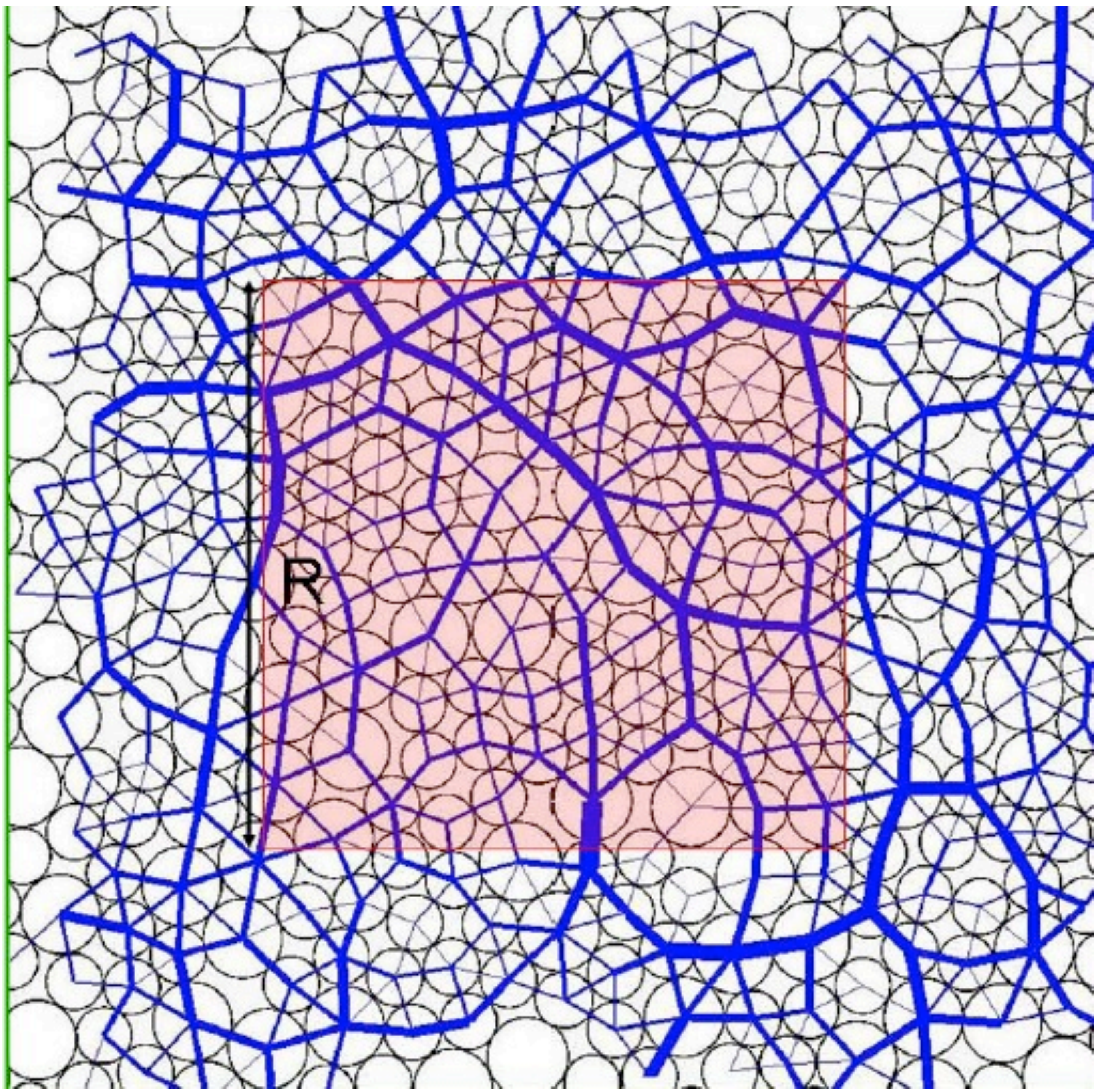


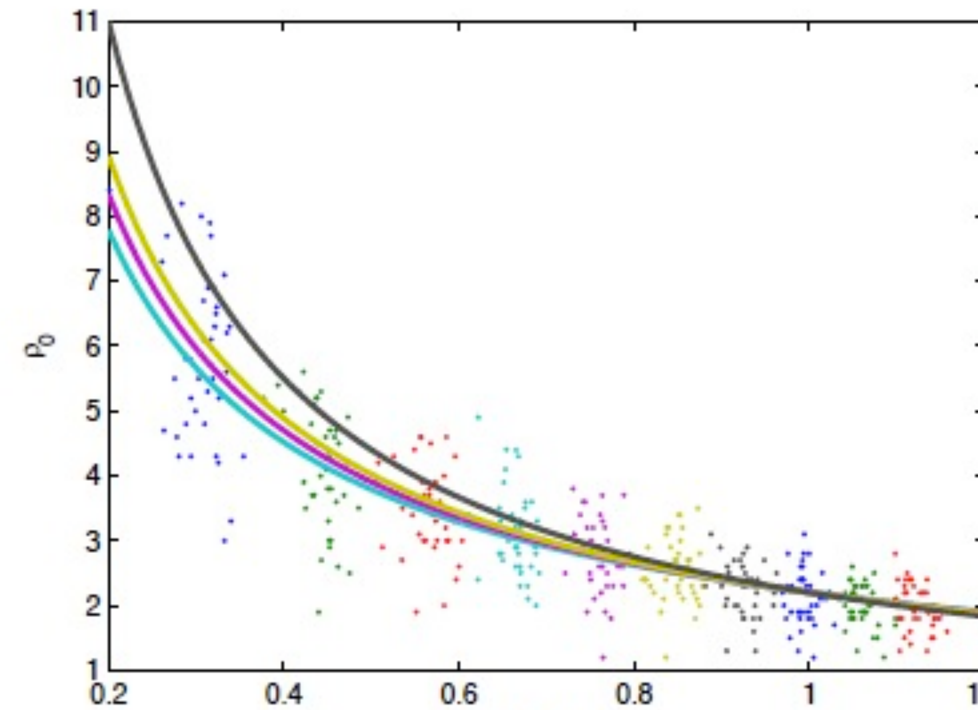


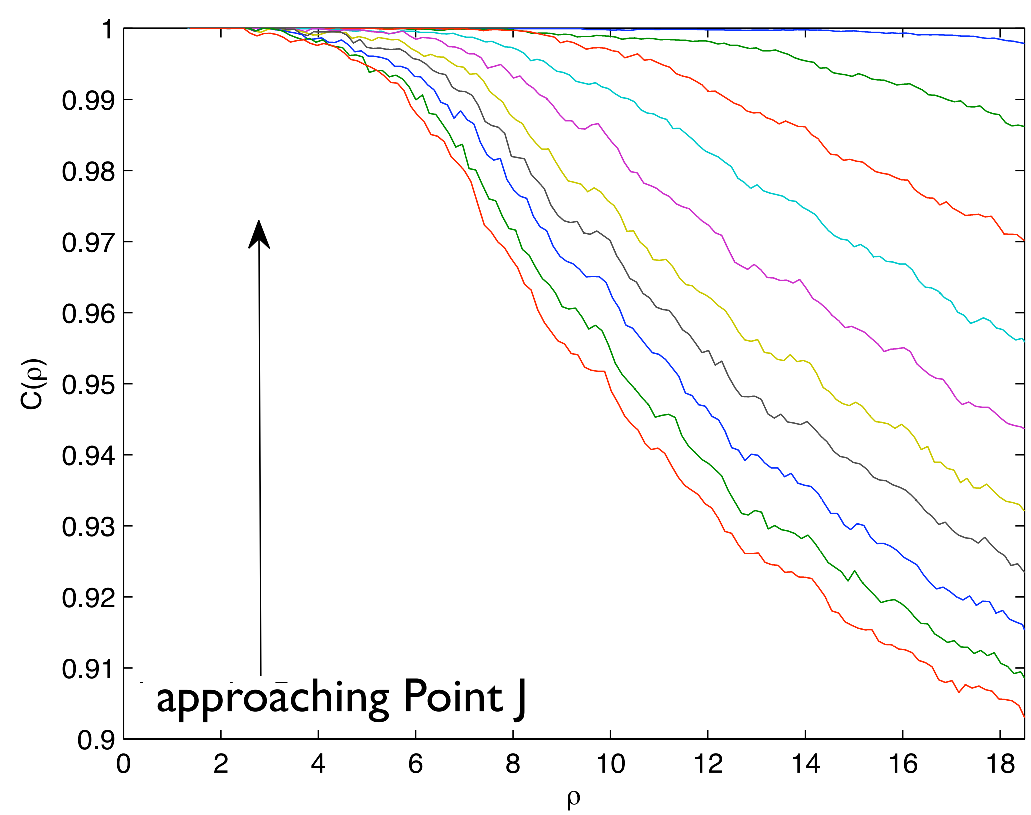
A. Cavagna arXiv:Cond-mat/0607817v2

Correlation function is defined as the overlap of a subregion within two inherent structures at the same temperature, and same boundary region.

Granular packings: $Z > Z_{iso}$ Keep geometry fixed and generate different force networks:(force-ensemble:Van Hecke)



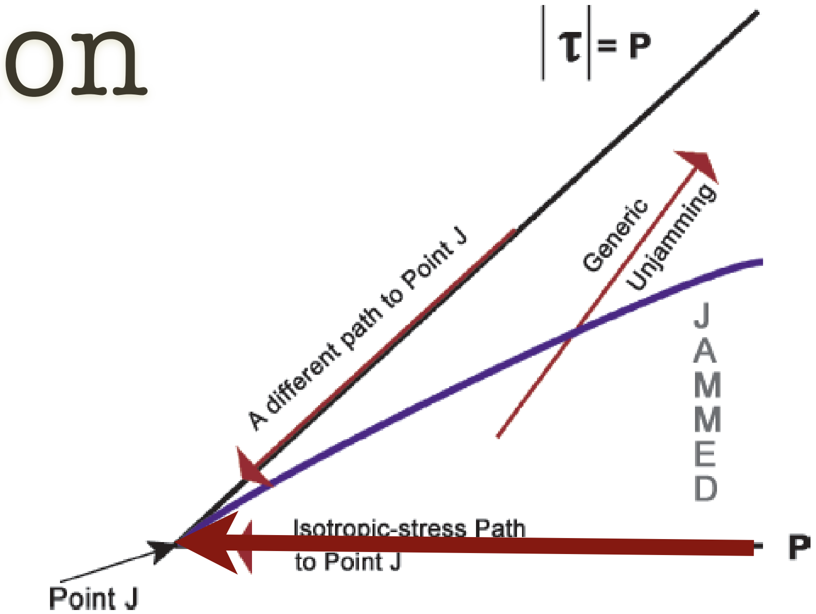




Conclusions

- Identifying a conserved quantity and assuming factorizability leads to a "canonical" ensemble.
- An intensive variable that plays the role of temperature in controlling stress fluctuations in athermal systems.
- Multiple temperature-like variables because conserved quantity is a tensor. Scale determined by compression only! The other varaibles are like fields whose sign matter.
- Constructed a field theory for stress fluctuations in granular packings
- Theory makes falsifiable predictions
- Predicts stress correlations based on "minimal" assumptions

Case 1: Isotropic Compression



$$\Gamma = \sum_{ij} r_{ij} f_{ij}$$

The effective theory can only involve second derivatives of the field ψ

Start with infinitely rigid grains, forces can be arbitrarily large, no force laws relating forces to positions

$$\mathrm{L}_\Gamma[\psi] = \int d^2r \Big[\frac{K(\Gamma)}{2} \Big\{ (\partial_x^2 \psi)^2 + (\partial_y^2 \psi)^2 + 2\partial_x^2\psi\partial_y^2\psi \Big\} + K'' \xi'^2 (\nabla^3 \psi)^2 \Big],$$

grain-scale length

- Looks like elasticity theory except the stiffness constants are completely determined by imposed stress.
- Origin is entropic: Stiffness related to entropy/density of states

Predictions

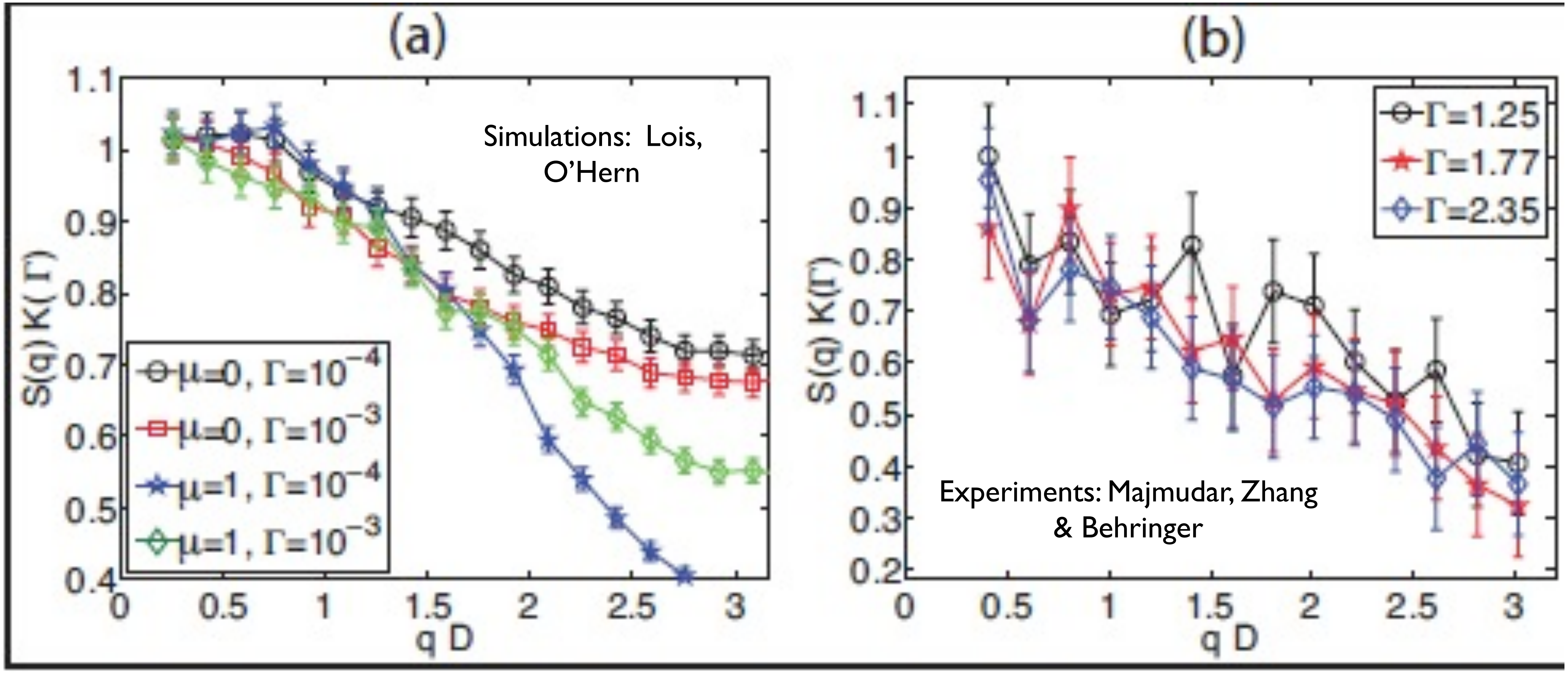
$$S(\mathbf{q}) = <|\delta\Gamma(\mathbf{q})|^2> = q^4 < |\psi(\mathbf{q})|^2 > = \frac{K^{-1}(\Gamma)}{1+\xi^2 q^2},$$

frictionless	$z_{\text{iso}} = 4$ in 2D
frictional	$z_{\text{iso}} = 3$ in 2D

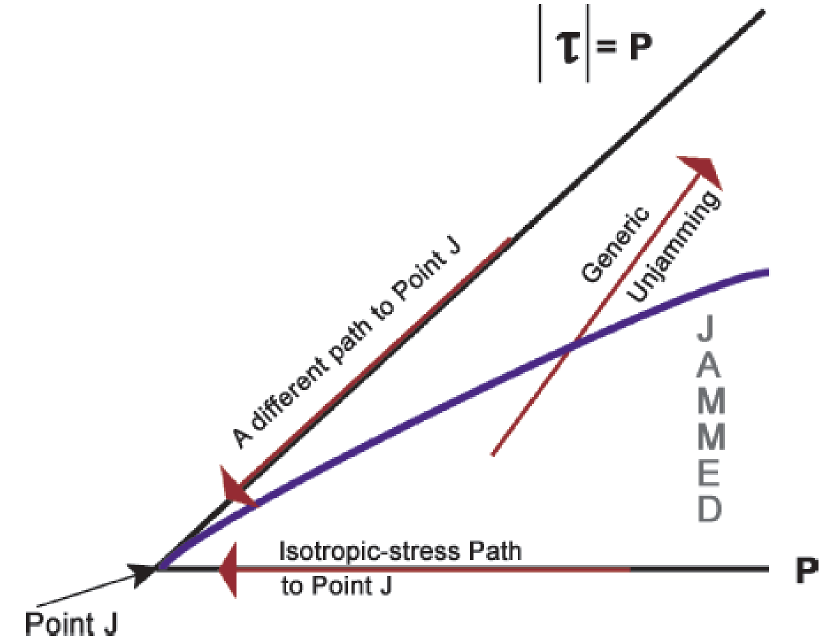
$$K(\Gamma) = (z_{\text{iso}}/2 + c(z - z_{\text{iso}}^2))/\Gamma^2$$

Grain scale length that does not depend on imposed stress
S(q) roughly independent of q scales with stiffness

Obtained from the fits to probability distributions of frictionless grains

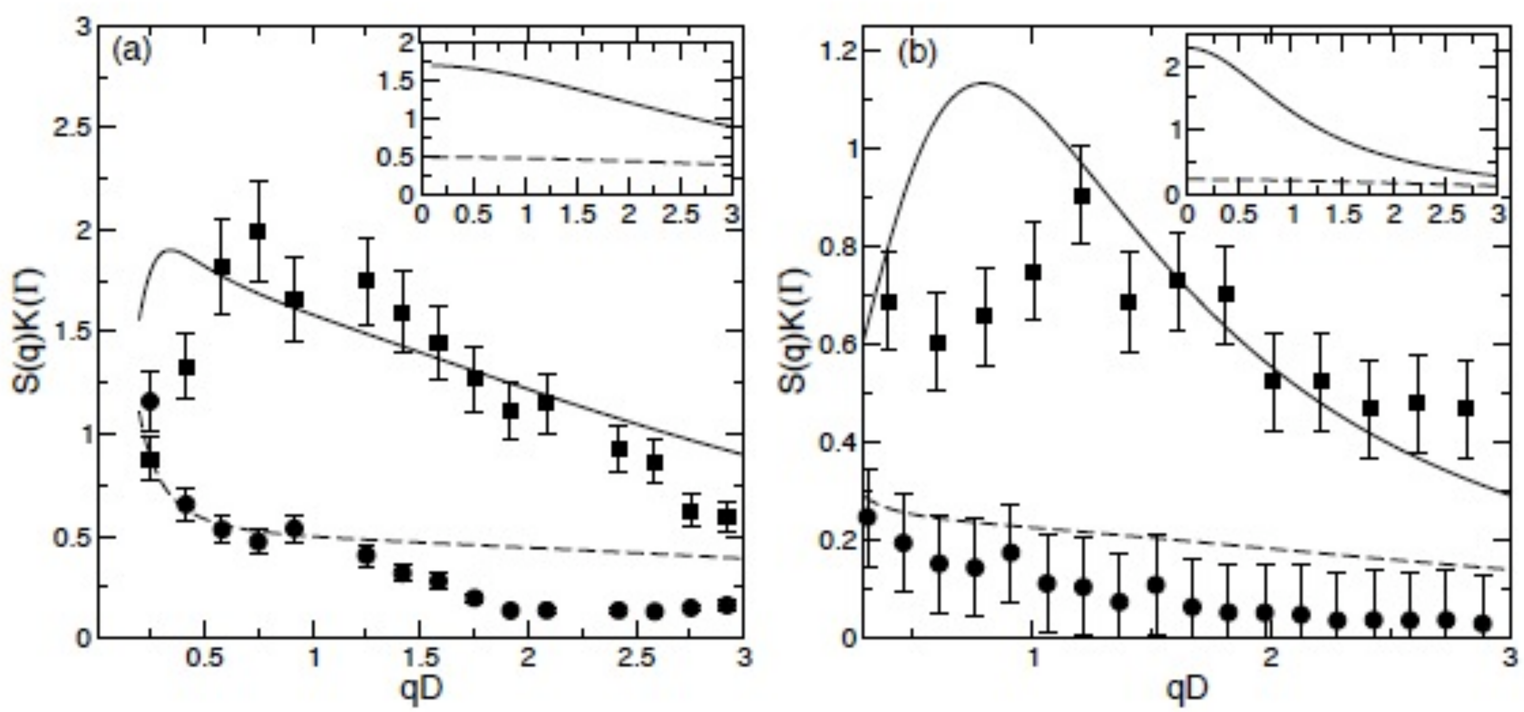


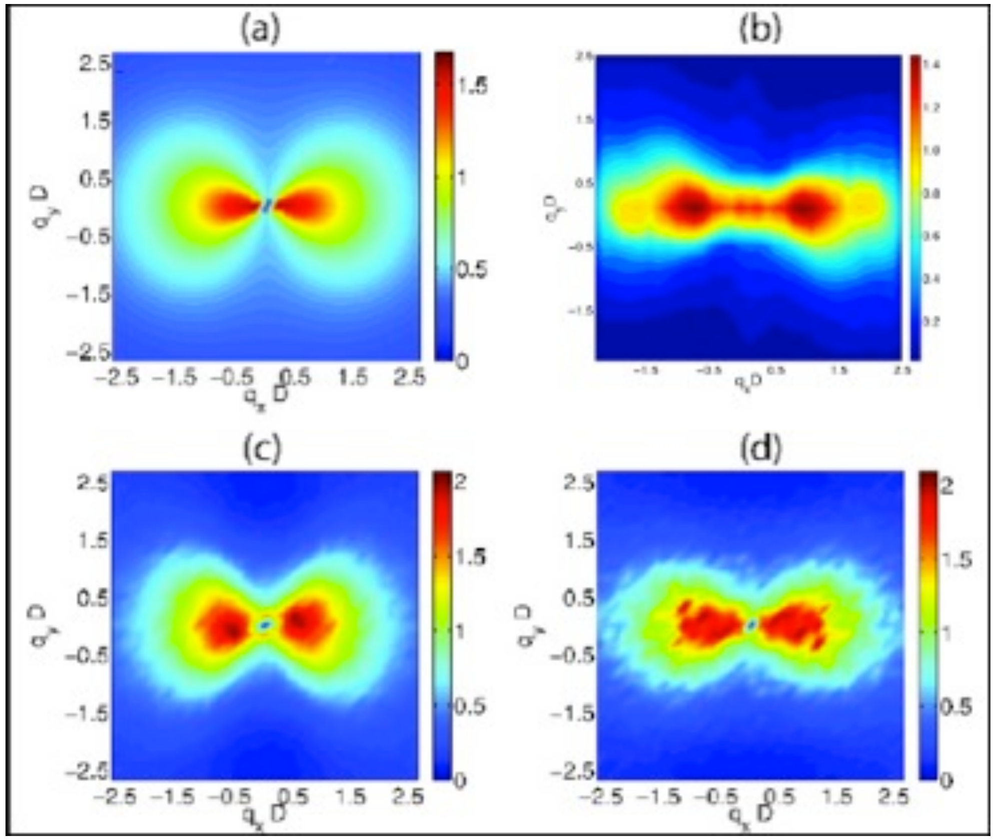
Pure Shear



$$L_{\tau,\Gamma}[\psi] = \int d^2r\Big[\frac{K(\Gamma+\tau)}{2}(\partial_x^2\psi)^2 + \frac{K(\Gamma-\tau)}{2}(\partial_y^2\psi)^2 + 2K'(\Gamma,\tau)\partial_x^2\psi\partial_y^2\psi + K''\xi'^2(\nabla^3\psi)^2\Big].$$

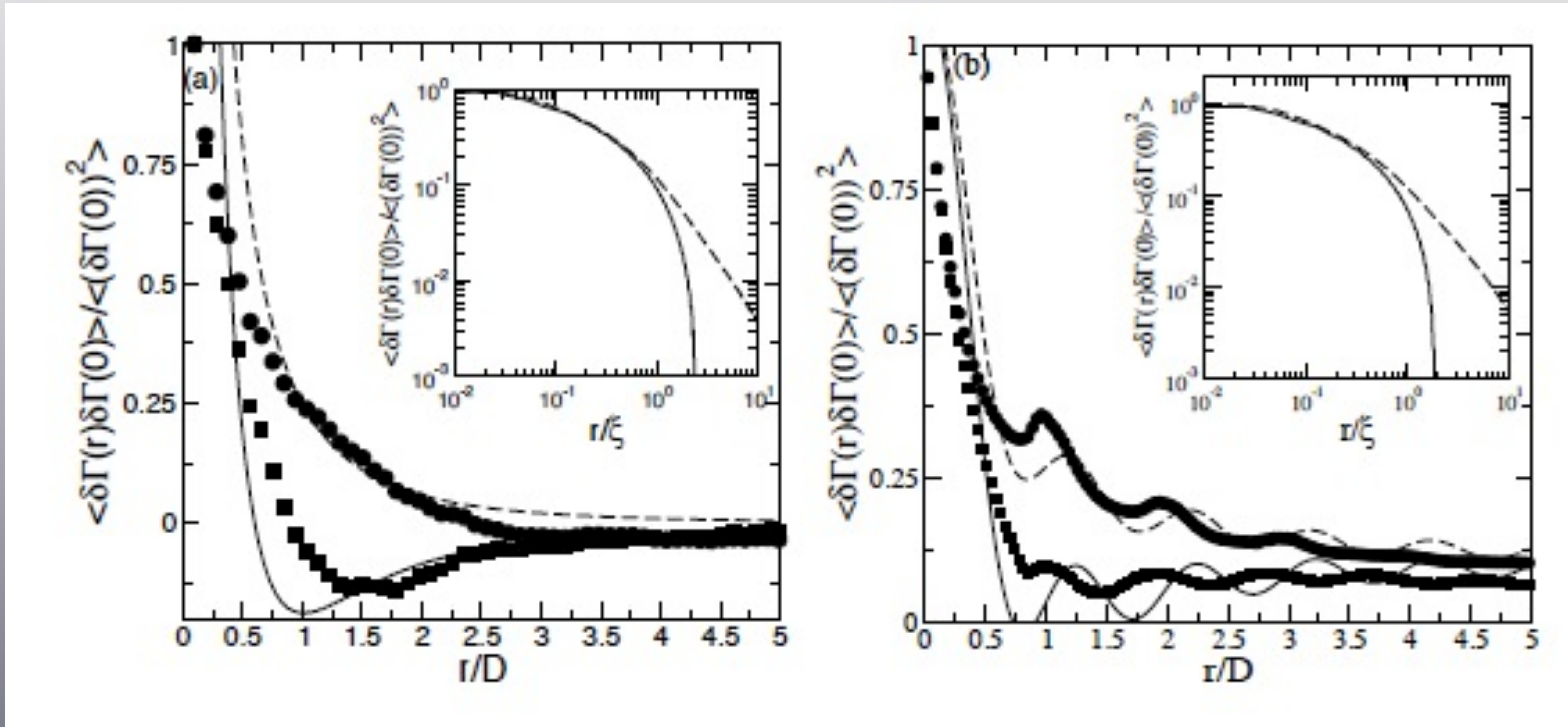
$$S(\mathbf{q})K(\Gamma) = q^4K(\Gamma)/\{K(\Gamma+\tau)q_x^4 + K(\Gamma-\tau)q_y^4 + 2K'q_x^2q_y^2 + K''\xi'^2q^6\}\ ,$$





Real- Space correlations decay more slowly in compressed direction

No growing correlation length!



Simulations: frictionless

Experiments

Velocity Correlations: Unjammed Side

While there has been some debate about the exponent, diverging length scales have been seen in correlations of particle displacement and velocities:

$$\xi \sim \delta\phi^{-\nu}$$

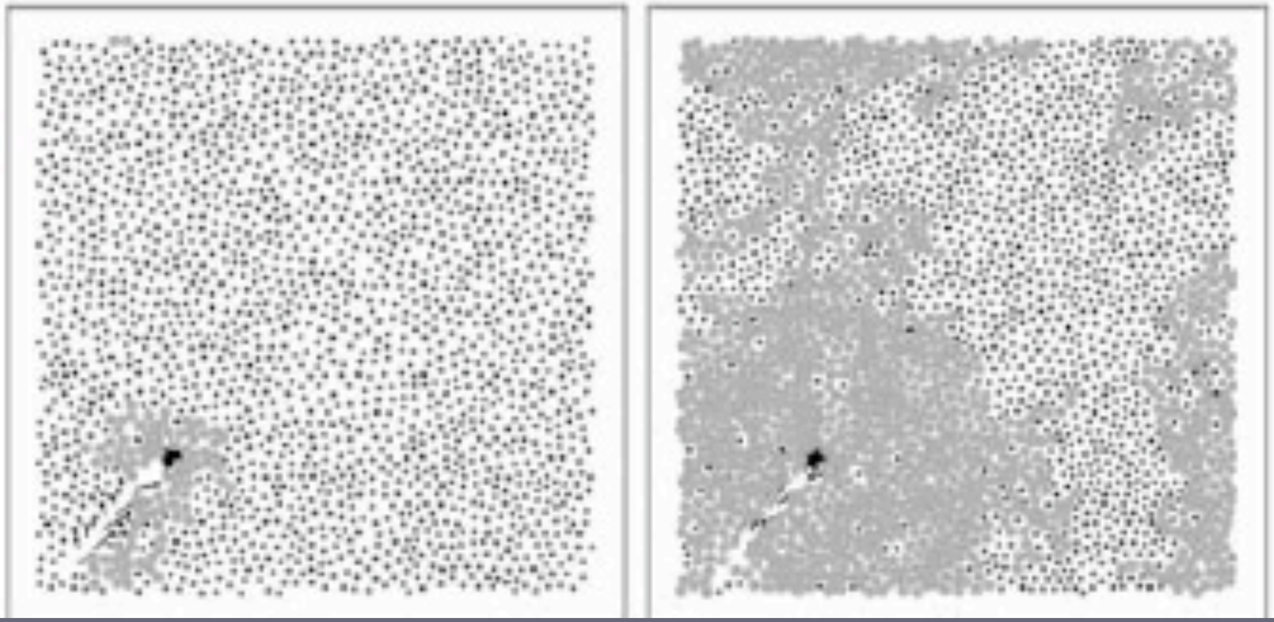
Heussinger and Barrat, PRL **102**, 218303 (2009): correlations in non-affine motions with exponent 0.8-1.0

$$G(r) = < u_{na}(r) u_{na}(0) >$$

Olsson and Teitel, PRL **99** 178001 (2007): correlations in tranverse velocities under shear with exponent 0.6

$$G(x) = < v_y(x) v_y(x + \delta x) >$$

J.A. Drocco et. al., PRL **95** 088001 (2005): measure the size of displaced grain clusters with over-damped dynamics. (a) packing fraction of 0.656 and (b) 0.811.

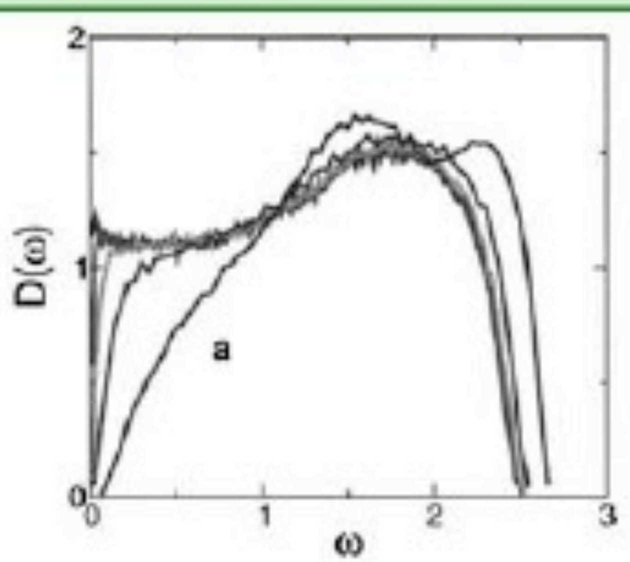


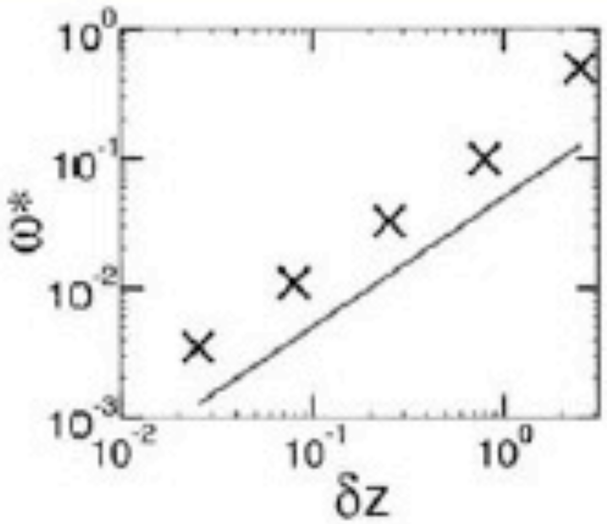
Critical exponents and Density of States: Jammed Side

C.S. O'hern et. al., PRE **68**, 011306 (2003): A length scale is extracted by using finite size scaling to collapse $\delta\Phi$ peaks for various system sizes, finding an exponent of 0.62-0.79.

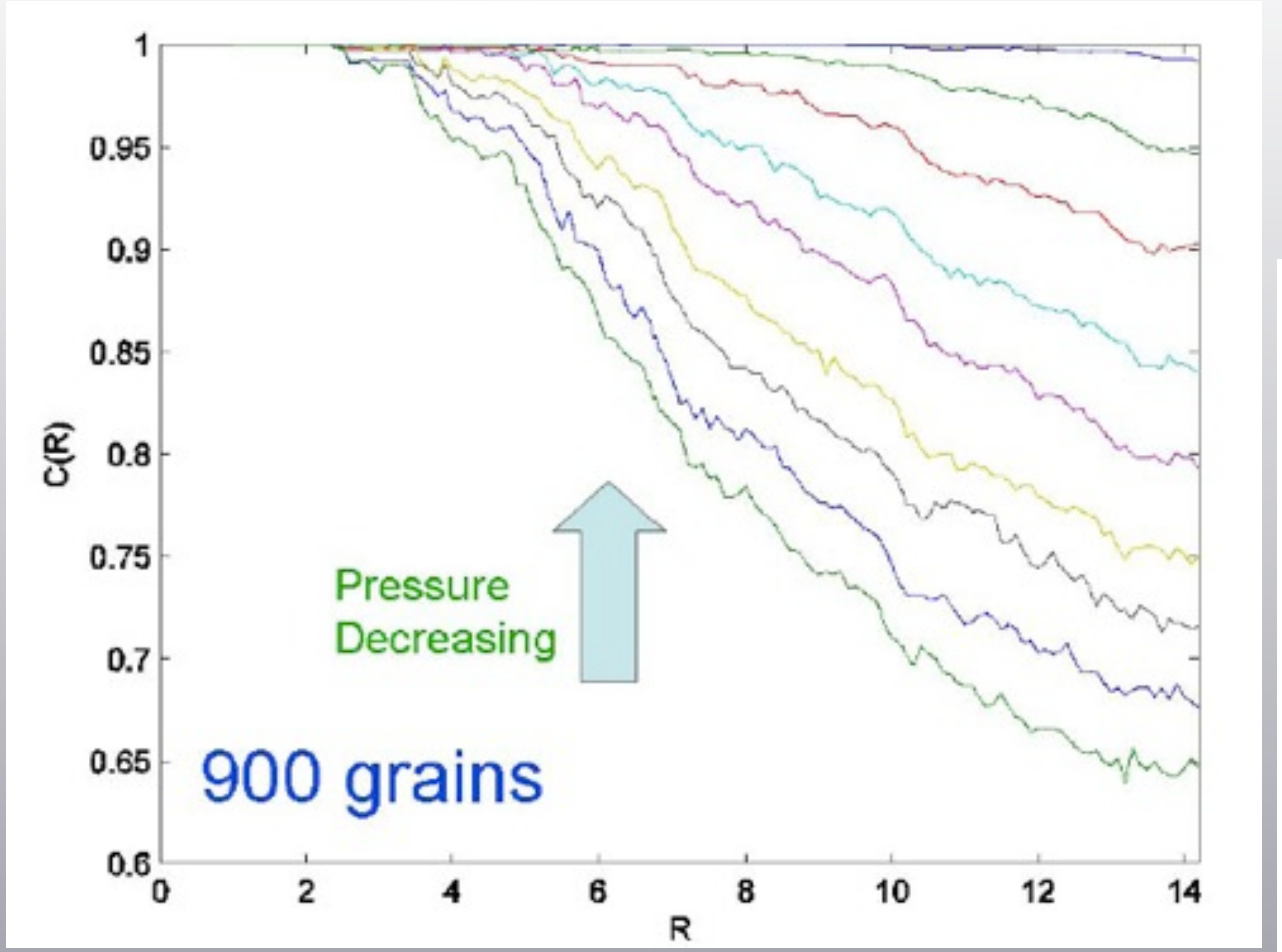
Similar results found by Teitel (unpublished): exponent of 0.72-0.86.

The length scale on the jammed side is related to the vibrational density of states for the jammed packings*. Specifically, $D(\omega)$ behaves Debye-like up to a critical frequency ω^*. Above ω^*, the lowest frequency modes project onto a subregion of the packing of characteristic size L. It is argued that $L \sim \delta z^{-1}$.



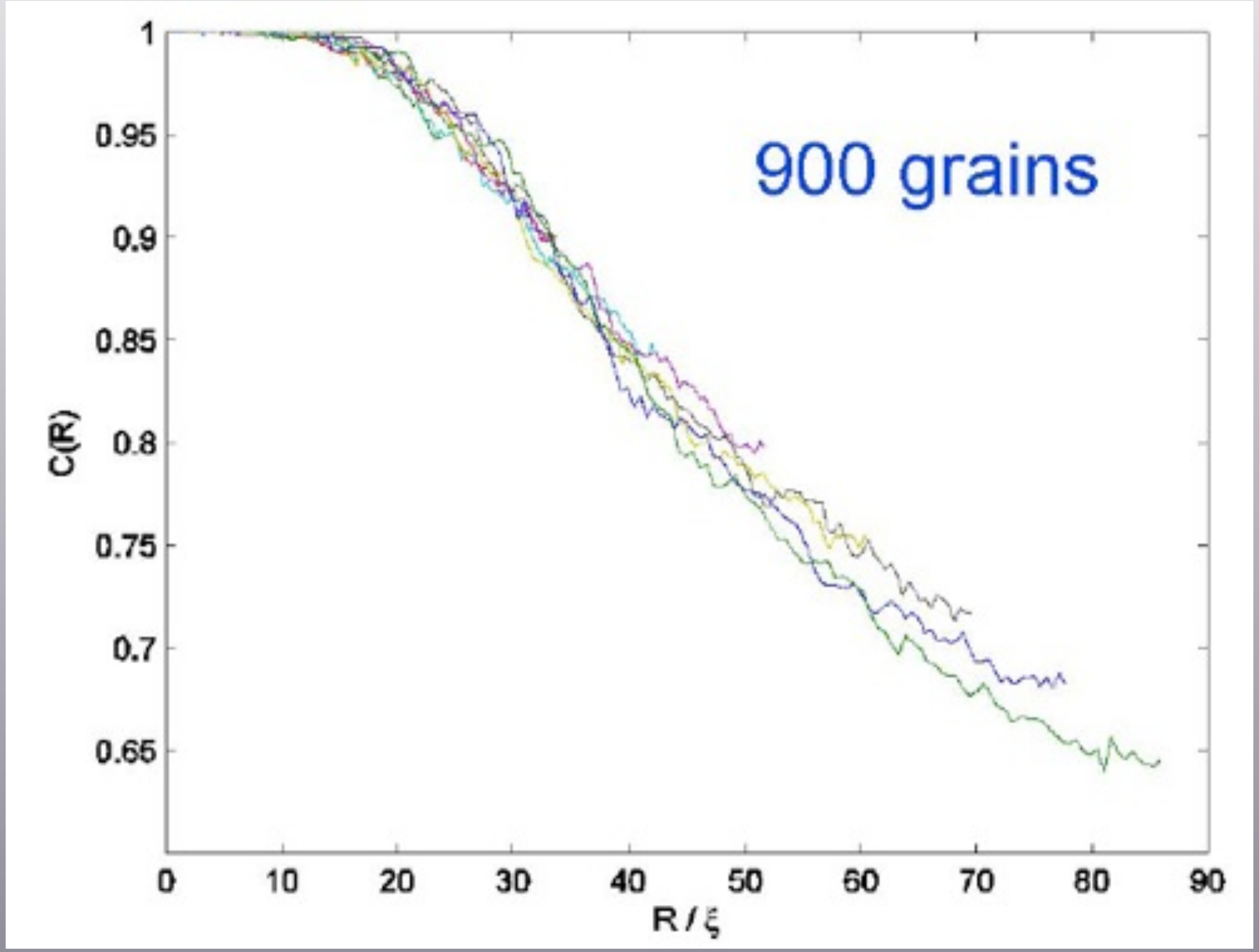


*Wyart et. al., PRE **72**, 051306 (2005)



$\xi \to P^{-\nu}$ With exponent of ~ 0.74

Scaling implies a diverging length scale!



Solution space shrinks as pressure goes to zero: Entropy vanishes

Field Theory: entropy vanishes and is only a function of shear/pressure

"Thermodynamics"

Blumenfeld & Edwards, (2007))
In differential form:

No body forces

$$\vec{\nabla} . \sigma = 0 \;\rightarrow\; \sigma = \text{"}\, \vec{\nabla} \times \hat{h} \,\text{"}$$

Use **generalized Stokes' theorem**

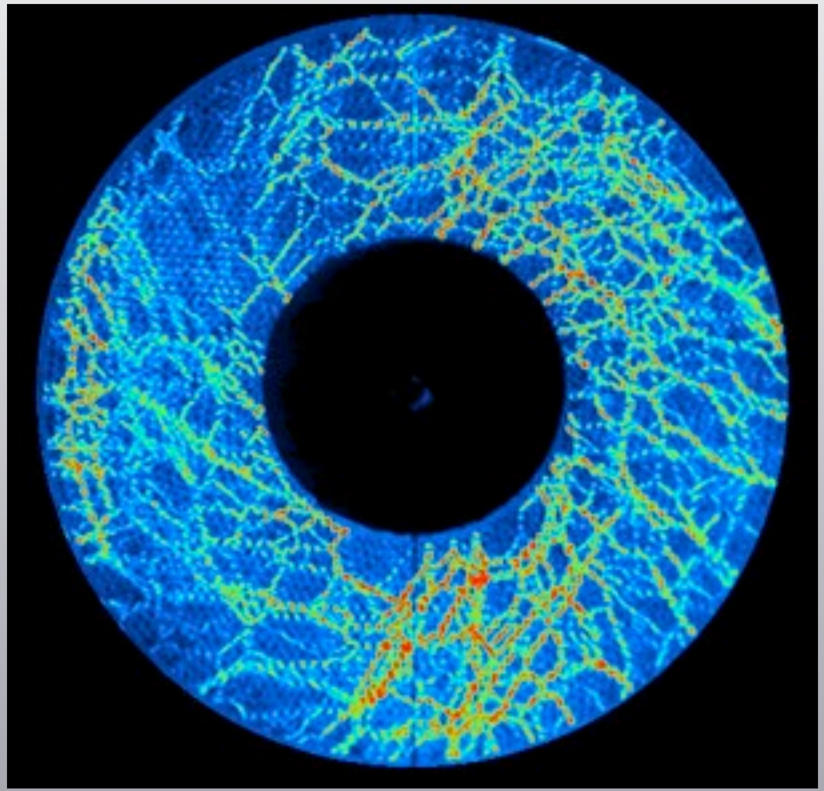
$$\int_V \sigma \, dV = \text{"} \int_{\partial V} n \times \hat{h} \, dS \text{"}$$

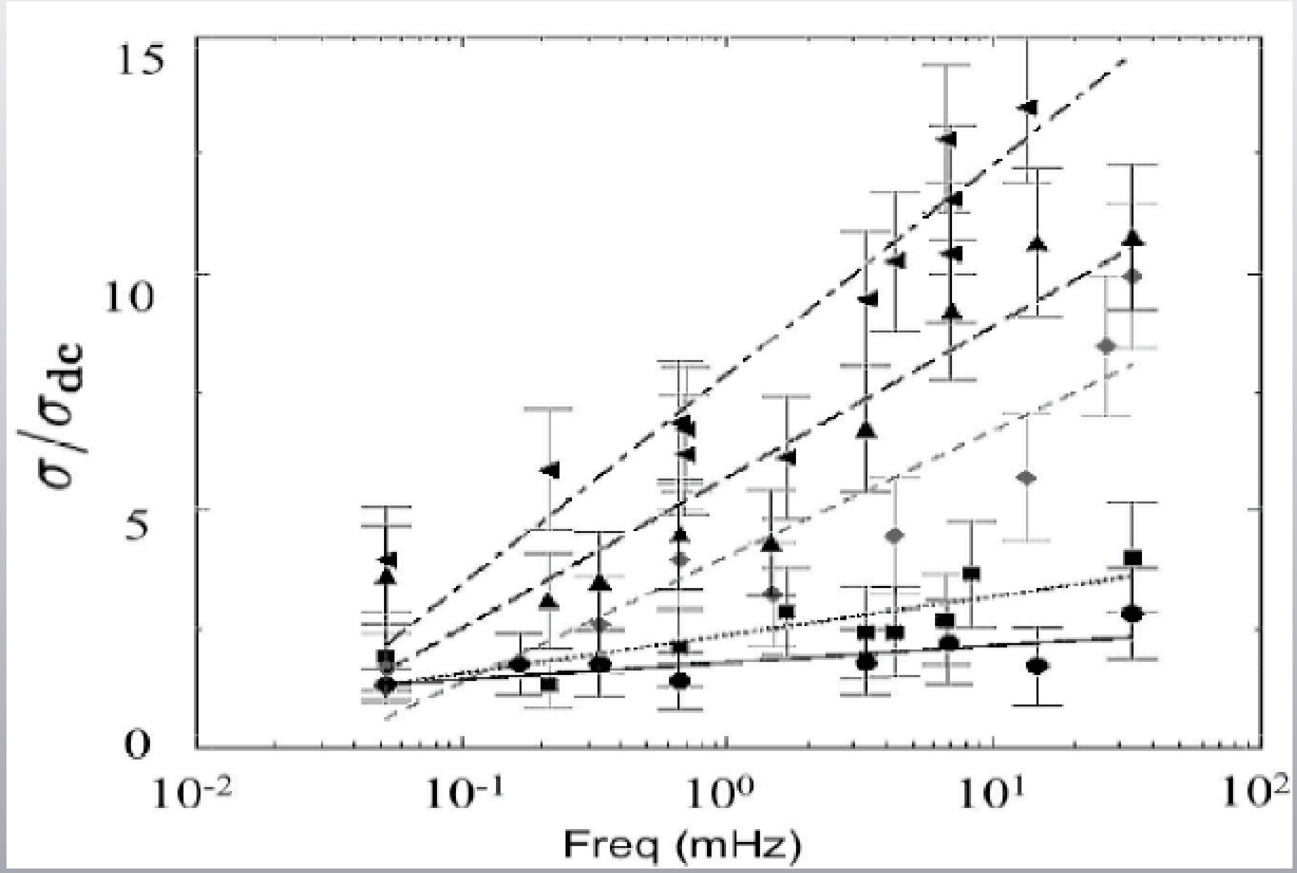
a boundary term

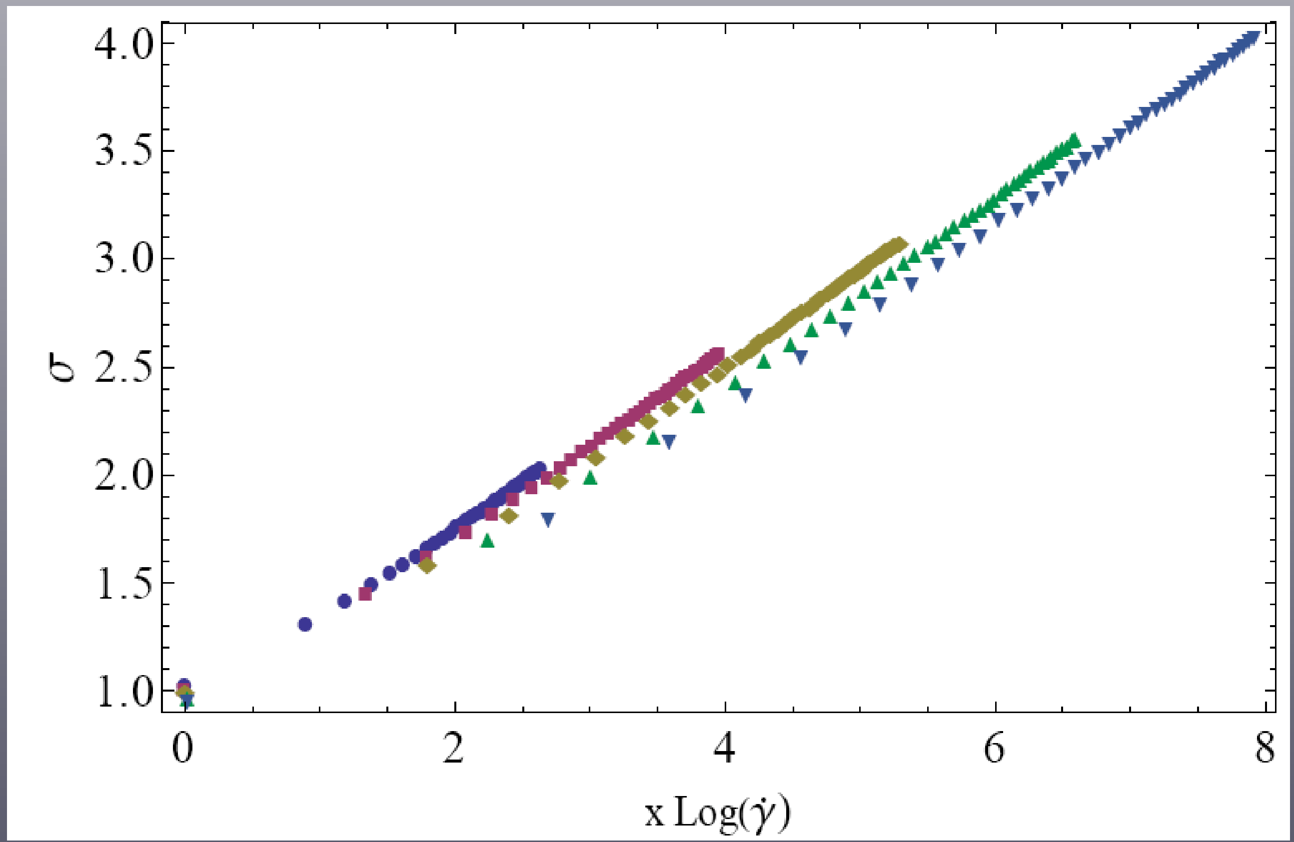
Extensive quantity $\Sigma = \int \sigma dV$ **, force-moment tensor, is invariant under local rearrangements in any dimension, for frictional and frictionless systems. Under periodic boundary conditions, this is a topological invariant.**

Conserved tensor - equivalent of energy

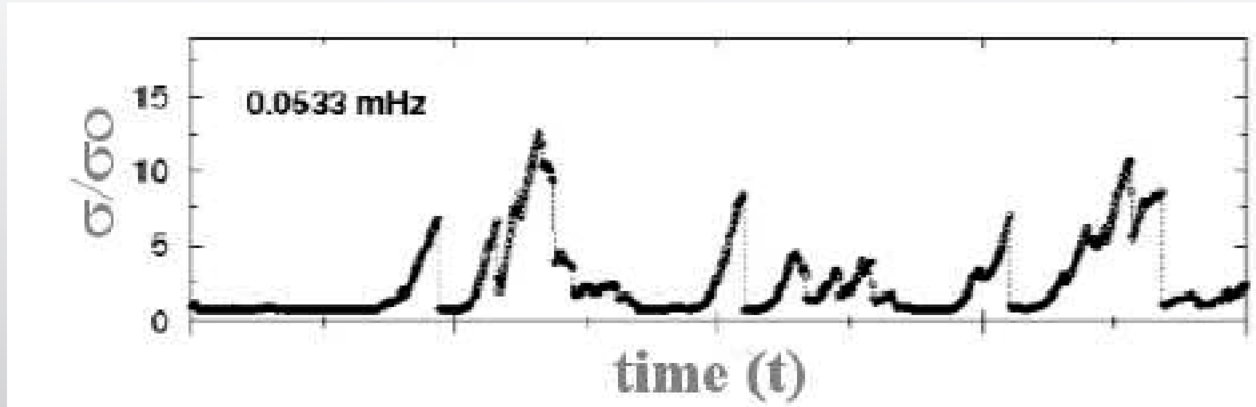
Stress Fluctuations and rheology (Response)



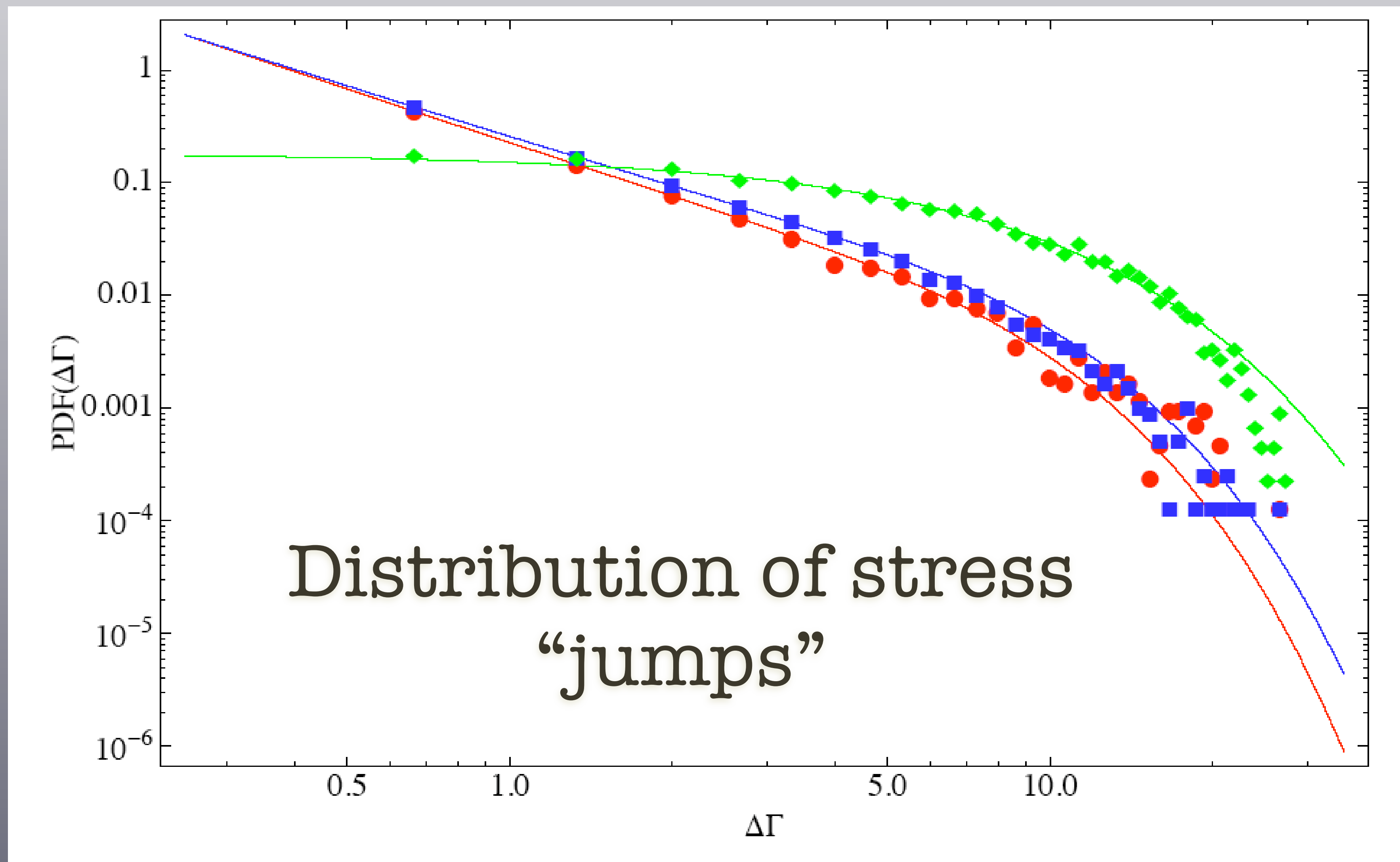




Activated Process aided by stress fluctuations

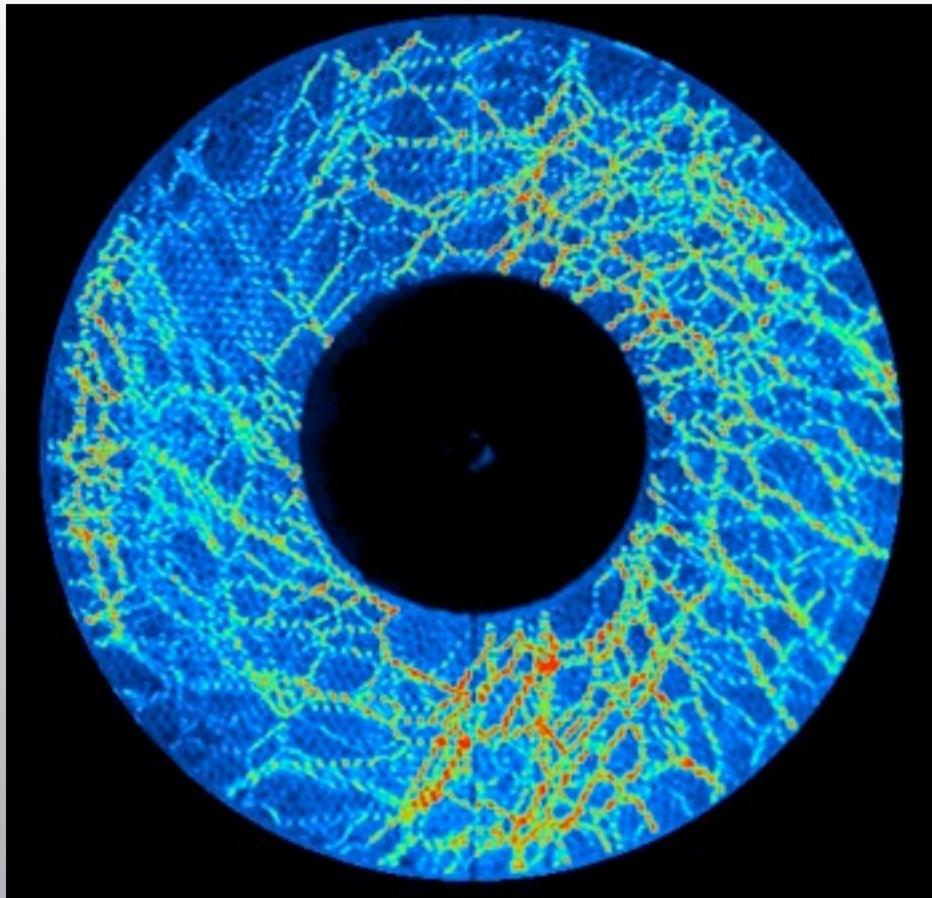


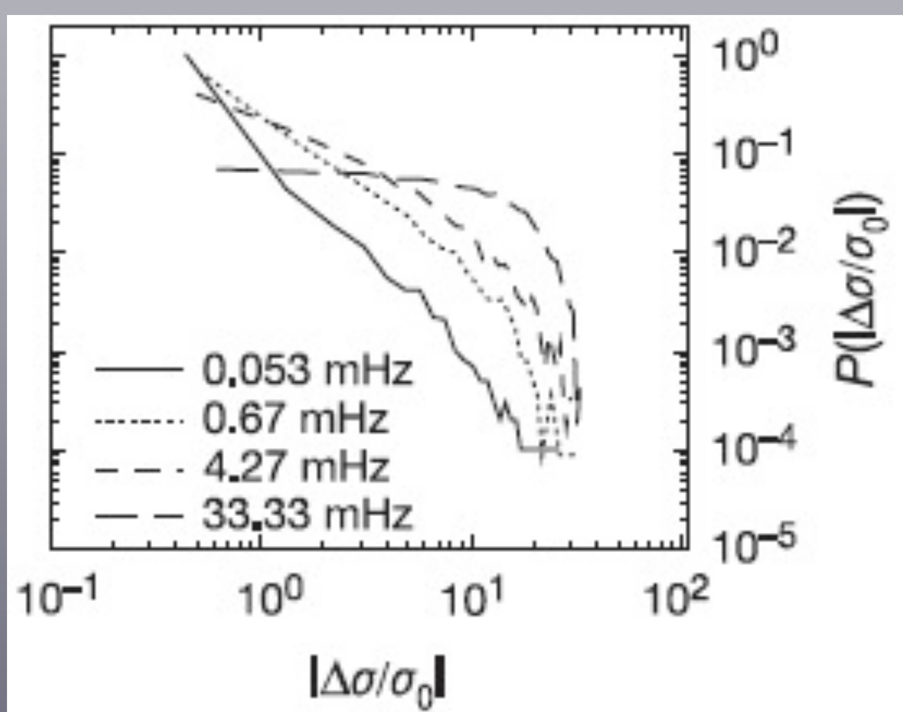
0.0533 mHz
σ/σ0
time (t)



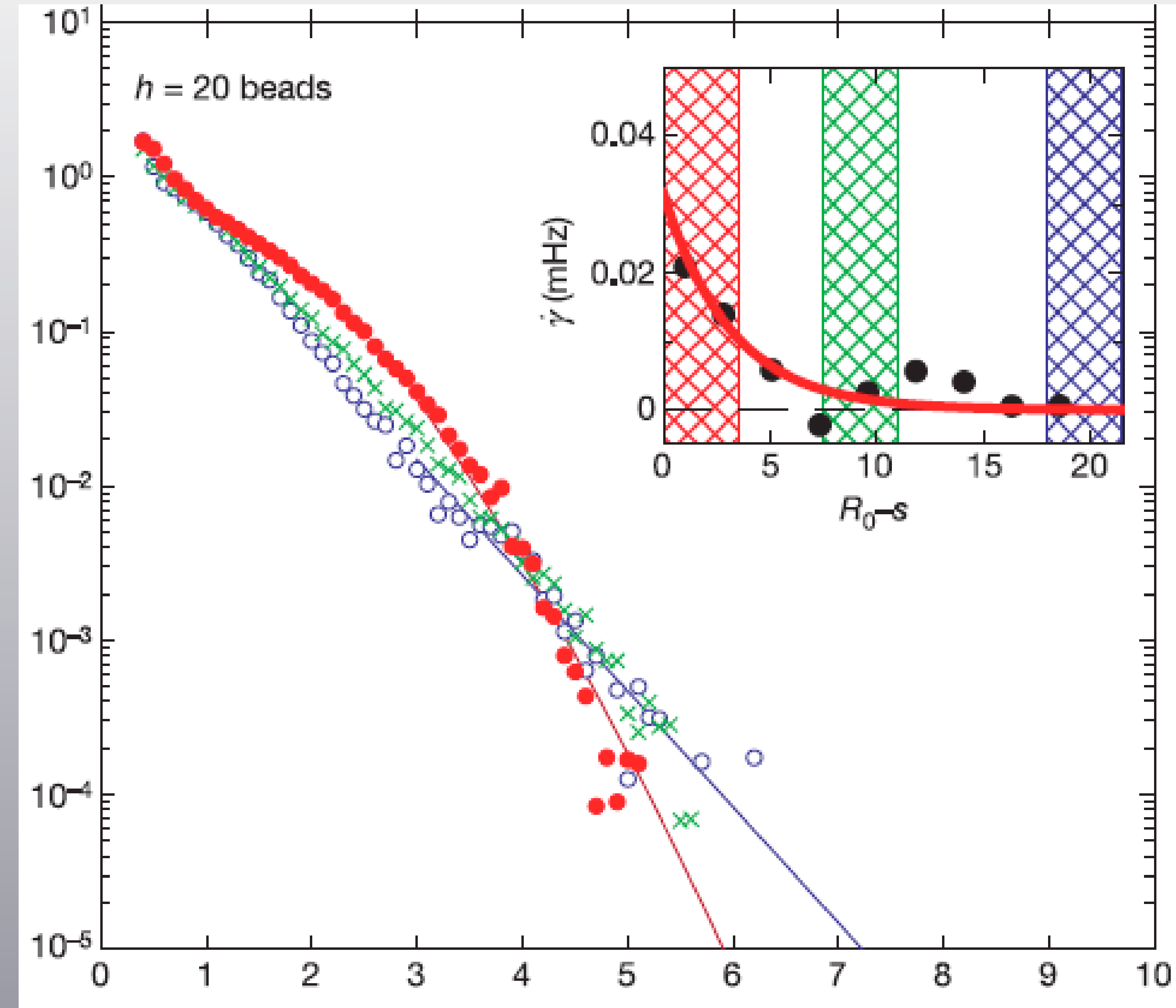
PDF(ΔΓ)
ΔΓ
Distribution of stress
“jumps”

Fluctuations and distributions in granular systems





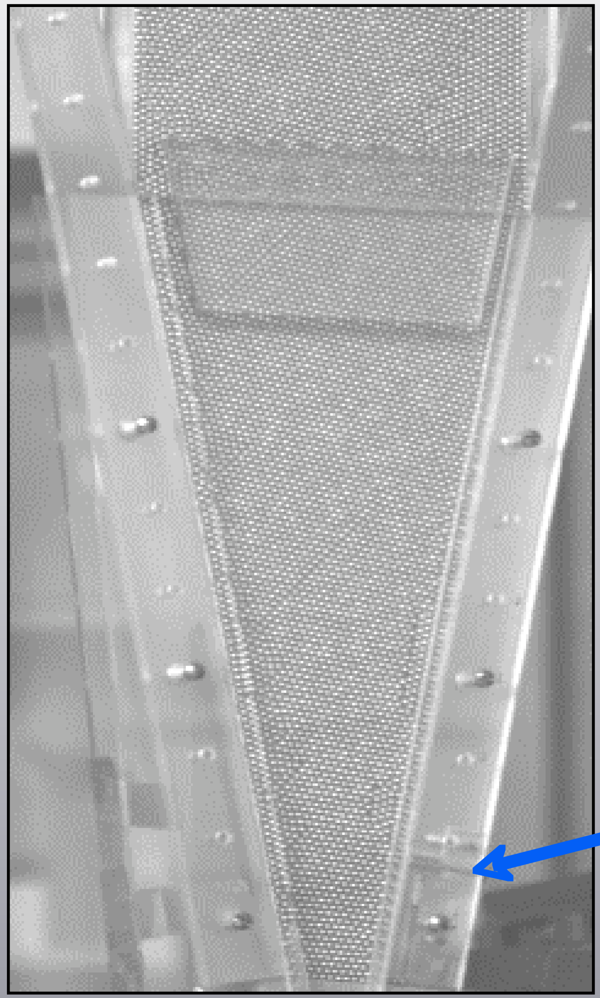
P(F)

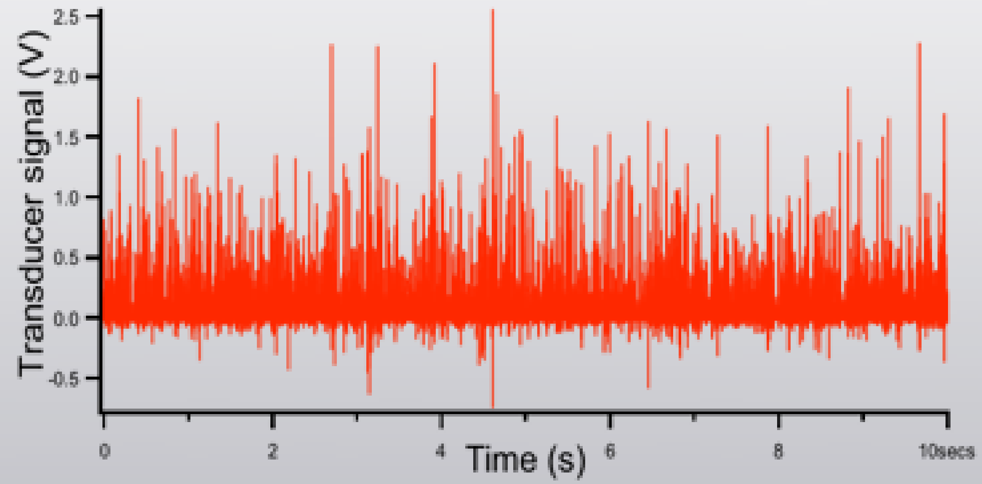


Eric I. Corwin, Heinrich M. Jaeger &
Sidney R. Nagel
Nature, June, 2005

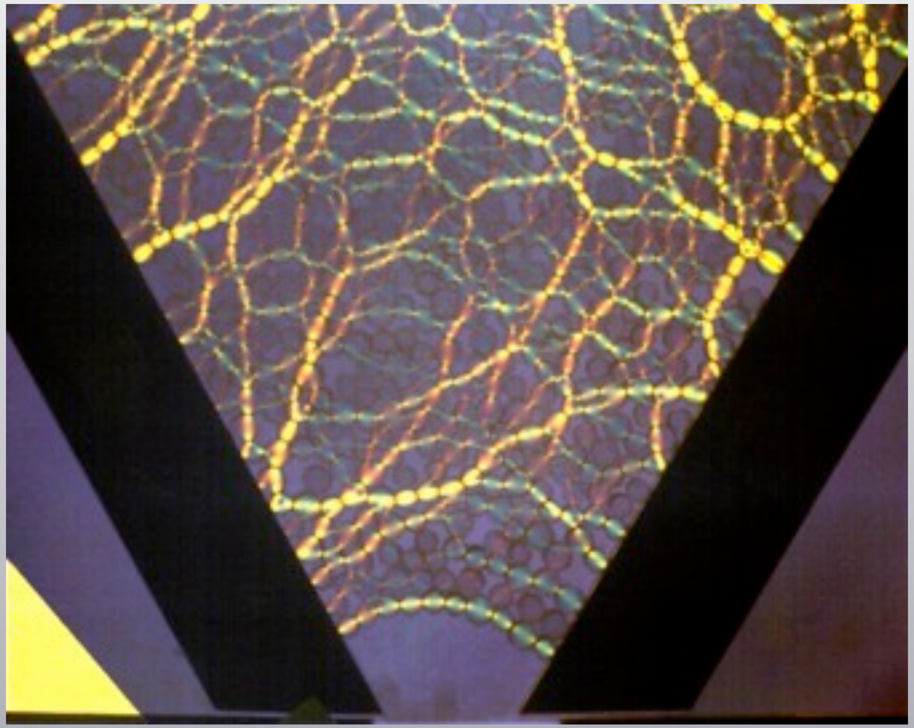
Varying Shear rate

Flowing systems (Finite kinetic energy)





Behringer group (Duke U.)

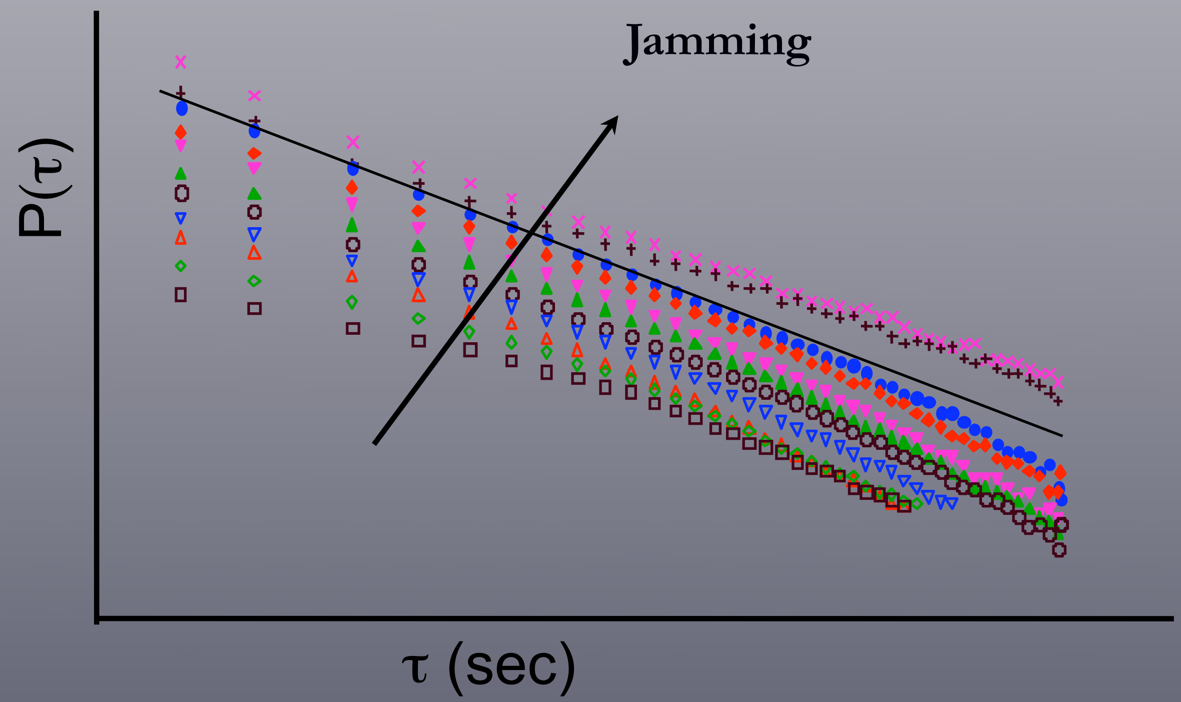


Transducer

- head size ≈d
- normal forces

- opening **a** - varied from 3d to 16d
- flow velocity constant as hopper drains
- packing very dense at all flow rates

Menon group (UMass, Amherst)



Theoretical Challenges

- Since tensile stresses are absent in dry granular materials they only remain intact via applied stress,
- The zero-state state is isostatic, where the number of degrees of freedom matches the number of constraints arising from force and torque balance. Believed to be extremely fragile
- Forces at the microscopic level are indeterminate due to friction and disorder
- Near isostaticity in particular, we expect fluctuations to be important, both within a single realization of a system, and from realization to realization.
- Granular materials are athermal, so that conventional energy-based statistical approaches are not appropriate
- What are appropriate state variables ?
- Is there a generalization of equilibrium statistical mechanics, that can predict fluctuations and response?

METASTABLE STATES

CONSERVED QUANTITY (BARRIER)

ENTROPY AND COMPLEXITY IN GLASSES

INTENSIVE QUANTITY

ENSEMBLE

COMPARISON TO EXPT

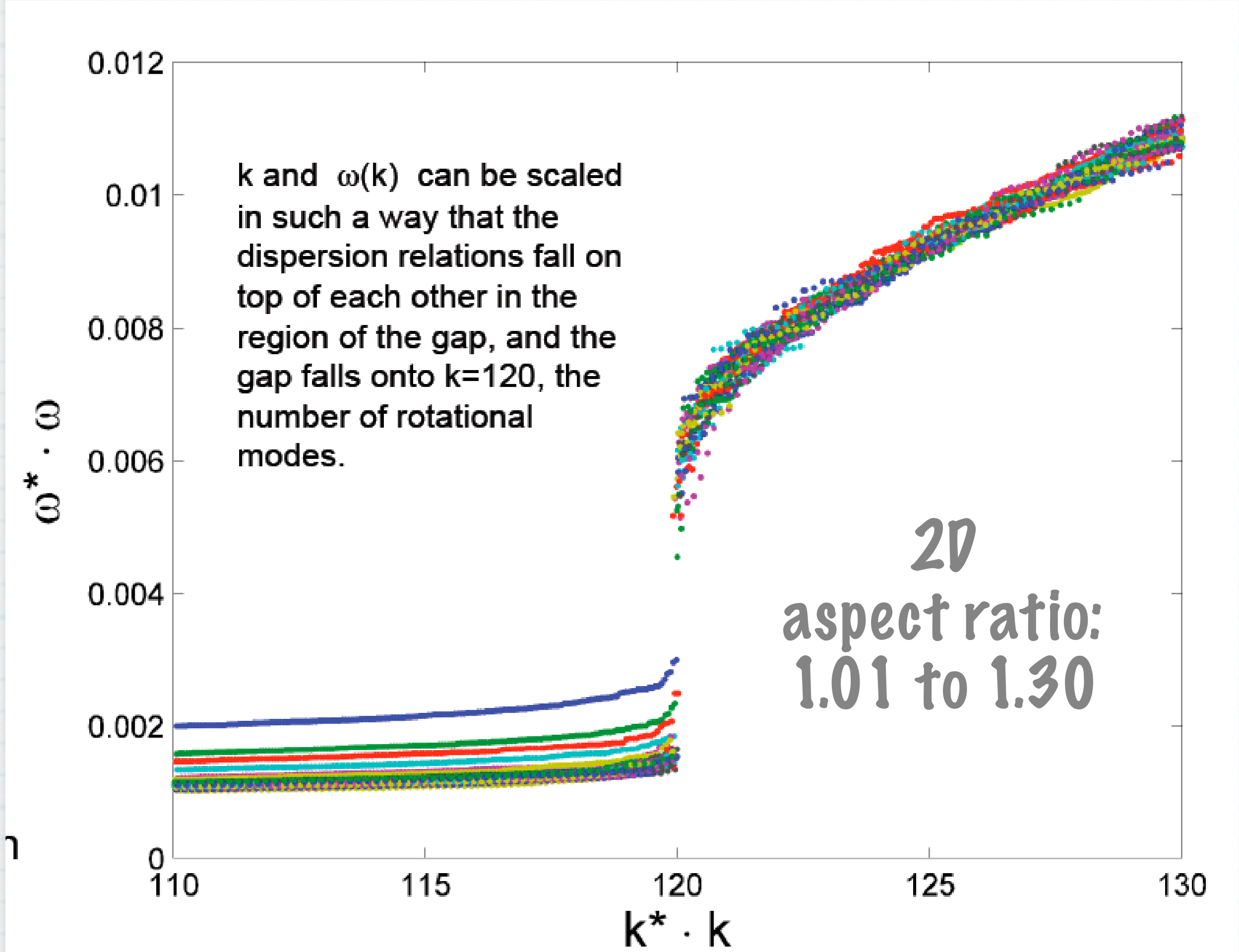
FLUCTUATION RELATION

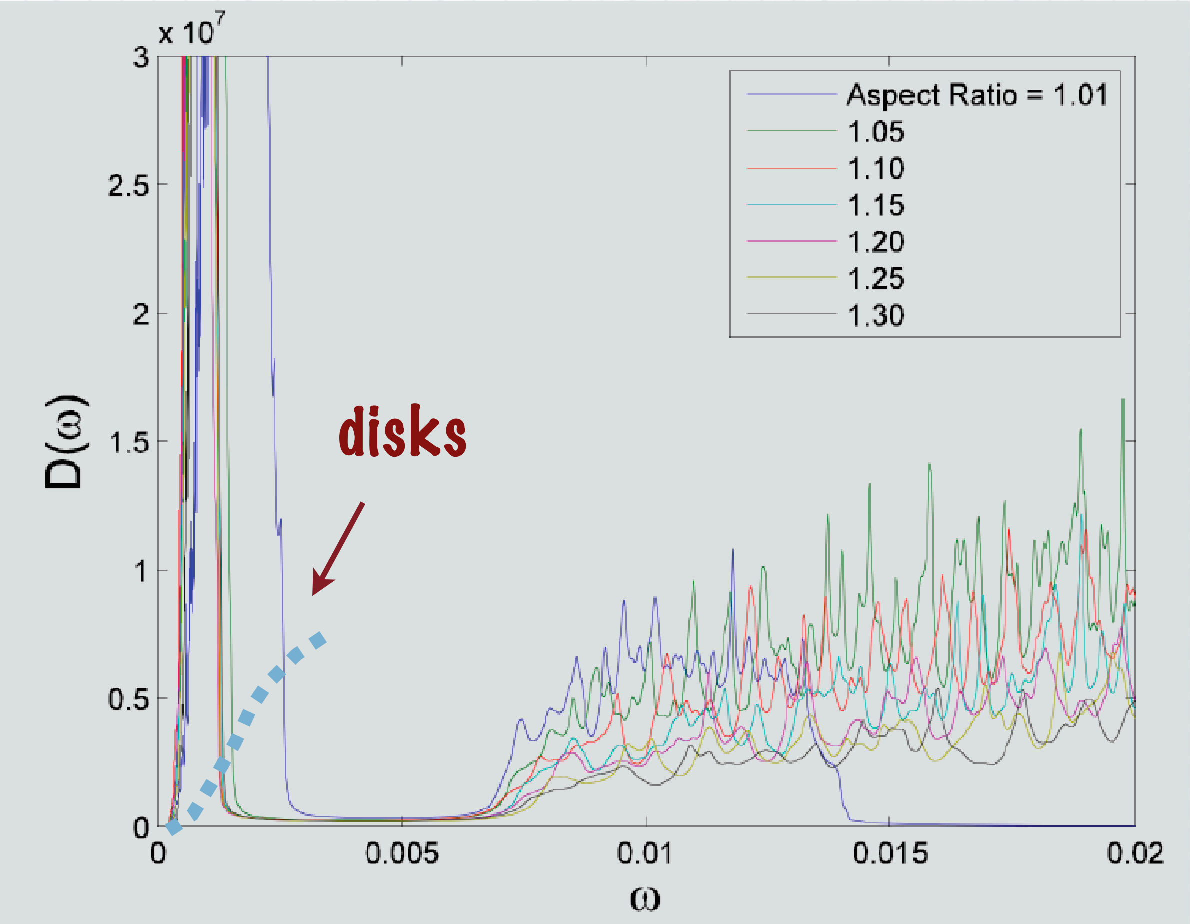
STZS

SHEAR BANDS

Density of States

* What are the low-lying modes of frictional packings?
* Any relationship to modes of packings of anisotropic grains?





Poster outside Rm 360 (with Corey O'Hern, Mitch Mailman, Carl Shreck)

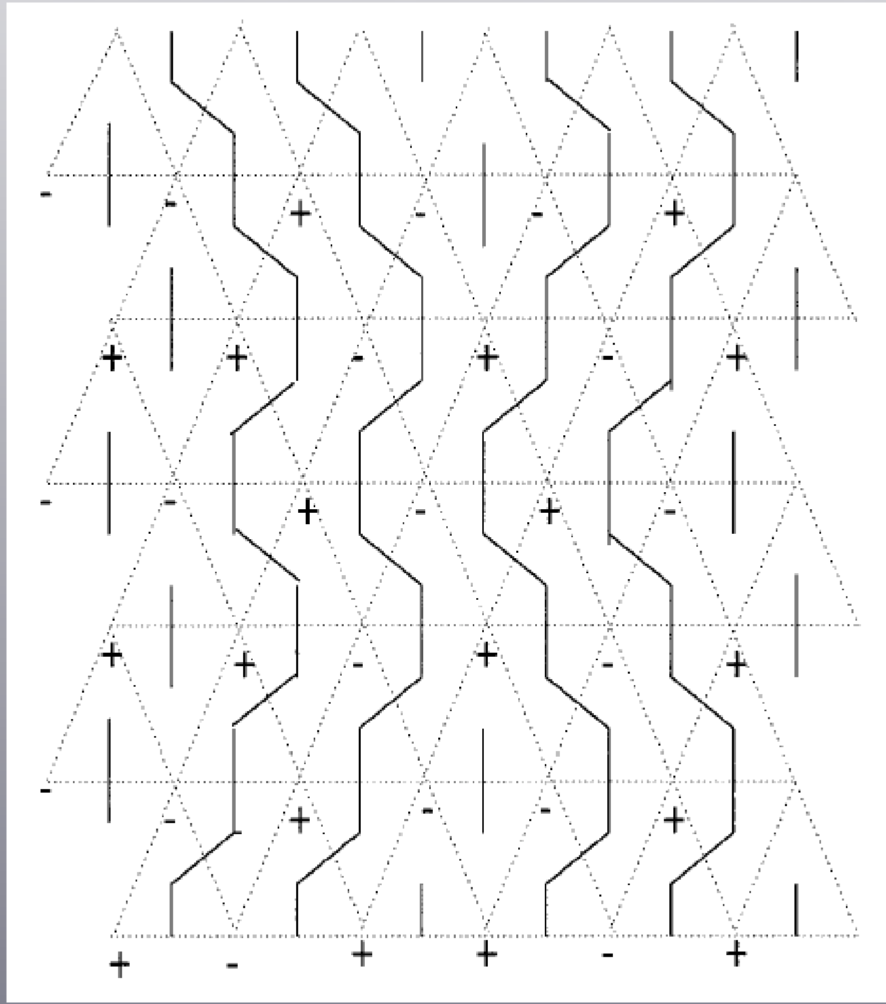
Many open questions:

1. Anomalous scaling of shear modulus related to long-range shear correlations? Correlations are controlled by external pressure, at least on the infinite compactivity line.
2. Hyperbolic equations for stress: connection to the predicted stress fluctuations ?
3. Indeterminacy of forces in frictional packings: does it lead to qualitative changes ?

Applications: Stress Fluctuations in Static Packings

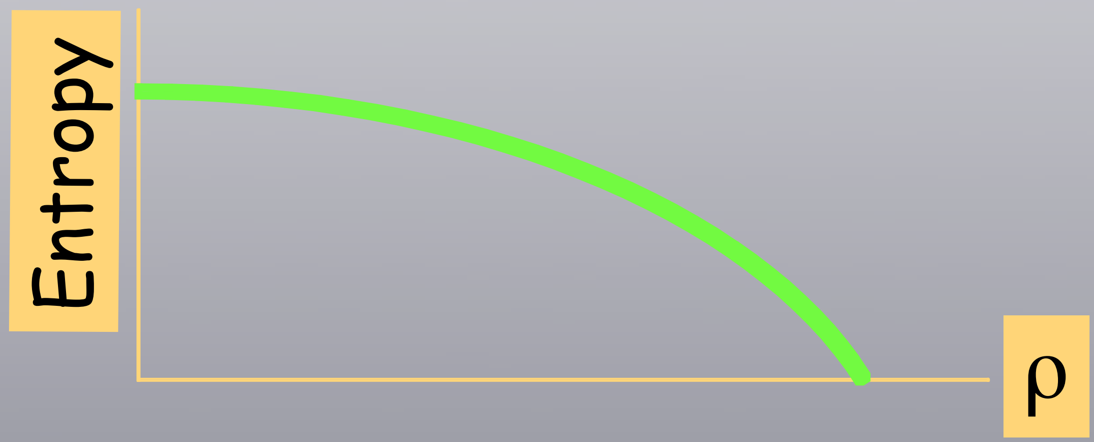
Construct Effective Theory

Analogy



Effective Theory for height fluctuations

For this system: Can calculate the entropy for a given, fixed number of strings

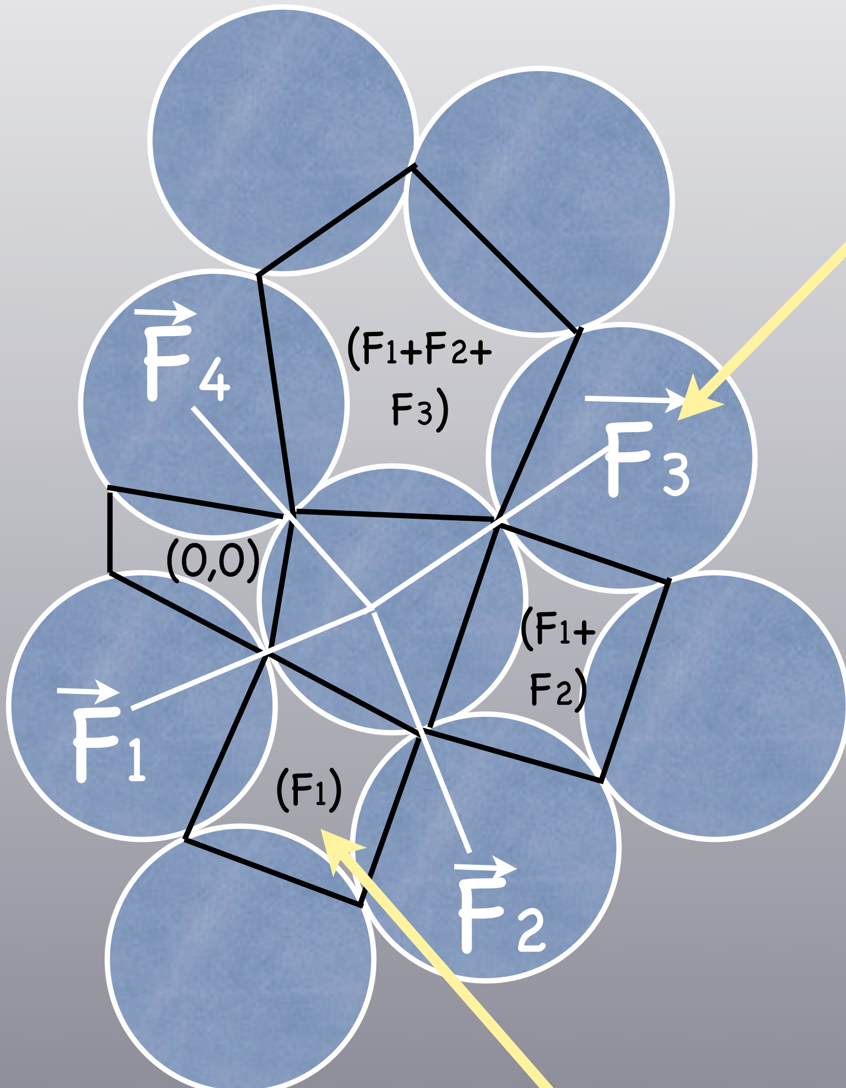


$$S(\rho) = L^2 \left\{ \frac{2\ln 2}{3}(1-\rho) + \frac{2}{\pi}\int_0^{\frac{\pi}{3}(1-\rho)} dx\, \ln[\cos x] \right\}$$

$$F_\rho[h] = K(\rho) \int d\mathbf{r} |\nabla h|^2$$

Height field in Planar Packings

Forces on central grain



- Height fields enforce force balance constraint
- Torques should also balance out
- For isotropic objects the above condition implies that
  - ★ height field is divergence free
  - ★ in 2D, a scalar field

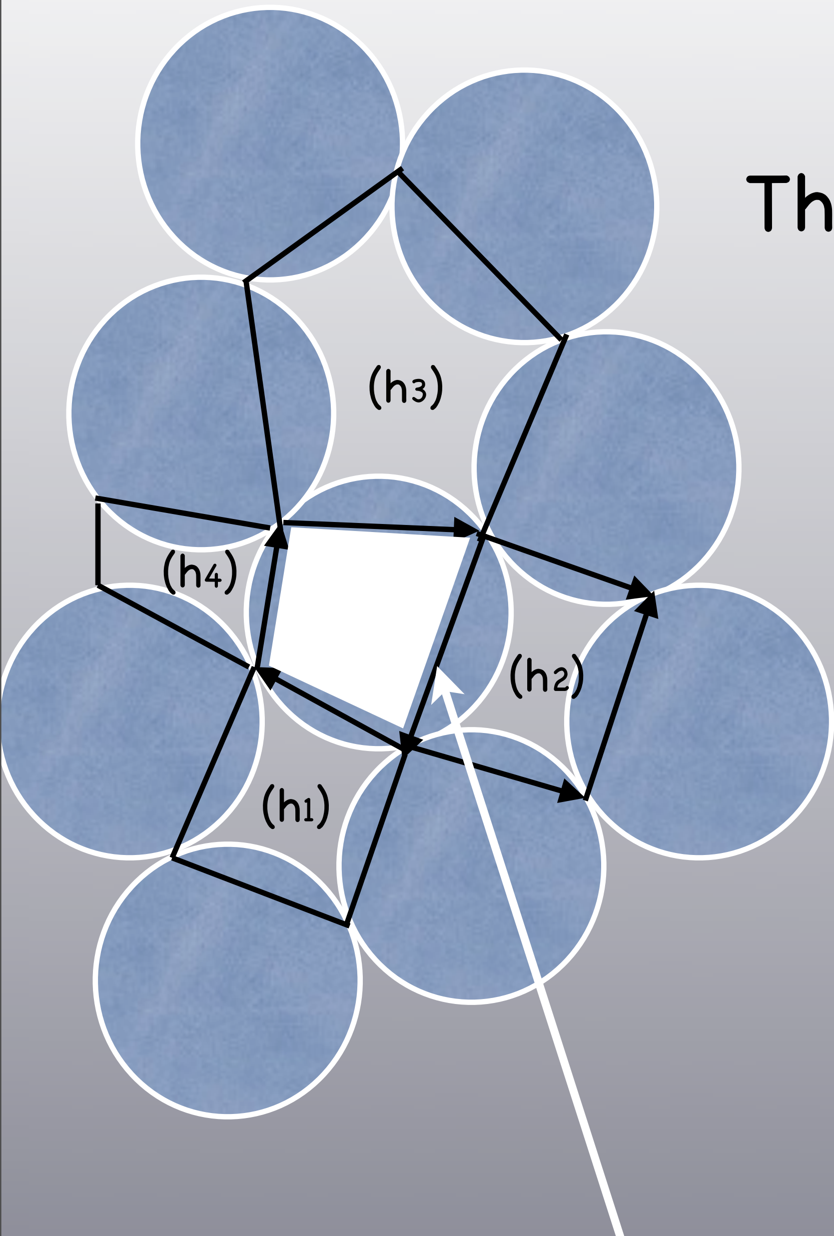
height fields (h) live on voids

$$h_x = \partial_y \psi \qquad h_y = -\partial_x \psi$$

Airy Stress Function

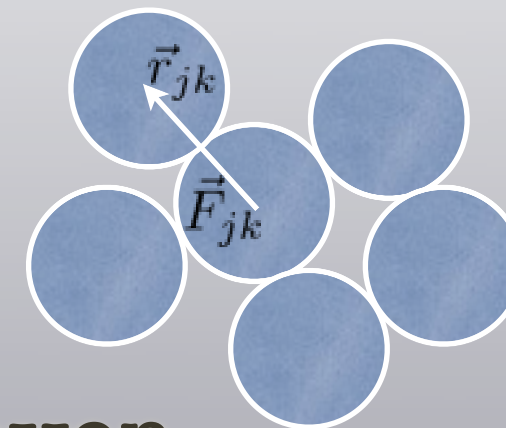
Ball & Blumenfeld (2002)

Microscopic and coarse-grained stress tensor



The microscopic stress tensor (force moment) of grains is defined as

$$\hat{\sigma}_j = \sum_k \vec{r}_{jk} \vec{F}_{jk}$$



Coarse-grain by summing over a few grains

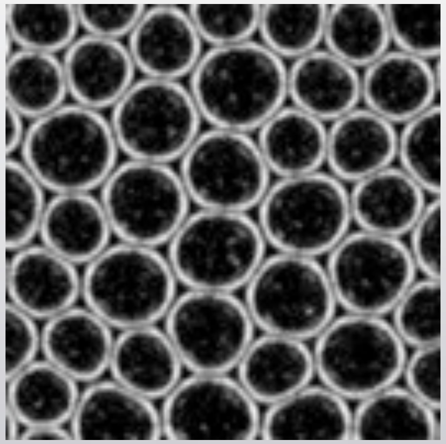
$$\hat{\sigma}(\vec{r}) = (1/A) \sum_{j \subset A} \sum_k \vec{r}_{jk} \vec{F}_{jk}$$

$$\hat{\sigma}_j = \sum_\mu \vec{R}_\mu \vec{h}_\mu$$

$$\hat{\sigma}(\vec{r}) = (1/A) \sum_{boundary} \vec{R}_b \vec{h}_b$$

$$\hat{\sigma}(\vec{r}) = \begin{bmatrix} \partial_y^2 \psi & -\partial_x \partial_y \psi \\ -\partial_x \partial_y \psi & \partial_x^2 \psi \end{bmatrix}$$

Take this approach over to granular packings



- Represent the packings by the coarse-grained field $\psi(\mathbf{r})$
- How many mechanically stable states ?

Replace # of strings by the tensor

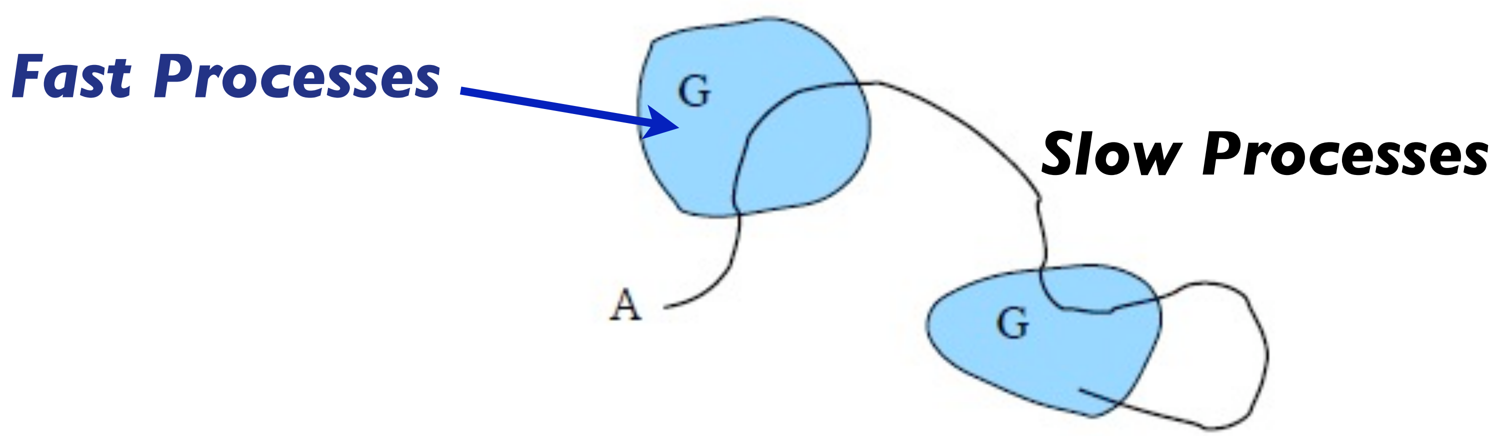
$$\hat{\Sigma} = M(\vec{F}_x, \vec{F}_y)$$

What is $S(\hat{\Sigma})$? $F_{\hat{\Sigma}}(\psi)$?

One crucial point: We are focussing on stress fluctuations, and assuming stress and density fluctuations are decoupled: true only for infinitely rigid grains, and/or packings close to jamming.

- Postulate 1 similar to recent ideas in systems with glassy dynamics
- Study in the limit where some very slow modes are turned off
- Structure of phase space/conserved quantities

Deepak Dhar + Joel Lebowitz:
Picocanonical ensemble



Different possibilities:

- **Perform many different experiments controlling which sector you are in, and measure fluctuations and averages in each sector**
- **Some dynamics that "occasionally" takes you out of a sector: dynamical averages**
- **Is there a natural partitioning of the space of granular packings that distinguishes between fast and slow variables?**