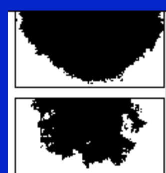
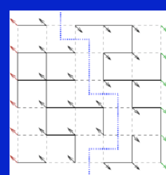
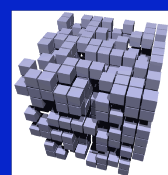
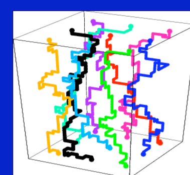
UNIVERSITÄT DES SAARLANDES

Disordered Systems, Ground States and Combinatorial Optimization in Statistical Physics

H. Rieger

Santa Barbara, 5/23/03

Collaborators

J.-D. Noh, F.O. Pfeiffer, R. Schorr

Universität des Saarlandes

V. Petäjä, M. Alava

Helsinki University of Technology

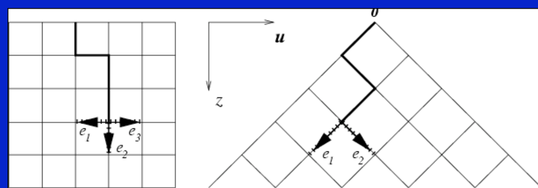
A. Hartmann

Universität Göttingen

J. Kisker, U. Blasum

Universität zu Köln

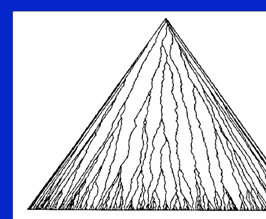
A simple combinatorial optimization problem:
The (directed) polymer model:



Lattice graph Non-directed directed

Random bond energies: $e_i \in [0,1]$

Total energy: $E = \sum_{i \in path} e_i$

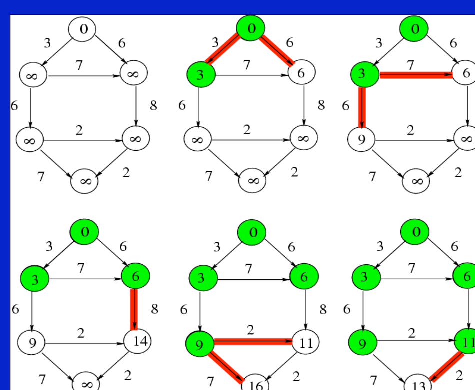


Find ground state – i.e. optimal path
From top node to a bottom nodes:

Collection of optimal directed polymers

Dijkstras algorithm
for shortest paths in general graphs

Start node: s
Minimal distance (energy) from s to j: d(j)
Predecessor of j: pred(j)



```
algorithm Dijkstra
begin
  S:={s}, S'=N\{s};
  d(s):=0, pred(s):=0;
  while |S|<|N| do
  begin
    choose (i,j):
      d(j):=min_{k,m}{d(k)+c_{km}|k in S, m in S'};
    S'=S'\{j}; S=S+{j};
    pred(j):=i;
  end
end
```

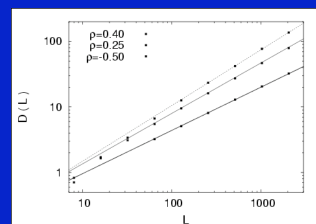
Performance $O(N^2)$,
with heap reshuffling $O(N \log(N))$

Optimal paths with correlated disorder

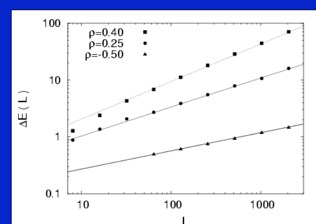
Isotropically correlated disorder: $\langle e_i e_{i+r}\rangle \sim r^{2\rho-1}$

Universal geometrical properties:

Roughness:
$D(L) = \langle x^2\rangle - \langle x\rangle^2 \sim L^{\zeta}$



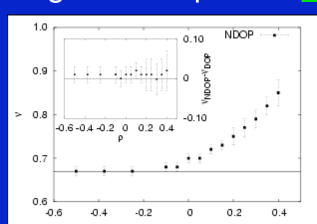
Energy fluctuations:
$\Delta E(L) = \langle E^2\rangle - \langle E\rangle^2 \sim L^{\omega}$

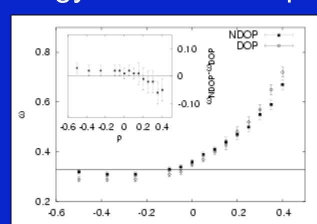


Optimal paths with correlated disorder (2)

2d: Roughness exponent ζ Energy fluctuation exp. ω

$\langle e_i e_{i+r}\rangle \sim r^{2\rho-1}$



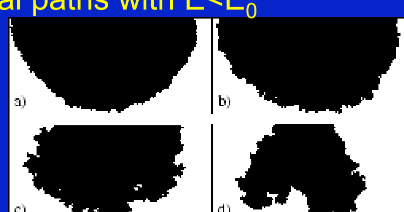


Optimal paths with $E<E_0$

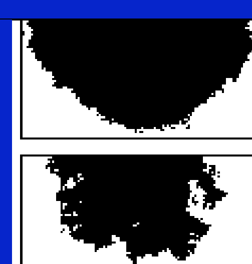
$\rho<0$

2d

$\rho=0.4$



3d



From one line to many lines

Continuum model for N interacting elastic lines in a random potential

$$H = \sum_{i=1}^{N} \int_0^H dz \left\{ \frac{\gamma}{2} \left(\frac{dr_i}{dz} \right)^2 + V_{rand}[r_i(z), z] + \sum_{j(\neq i)} V_{int}[r_i(z) - r_j(z)] \right\}$$

Strong disorder: $V_{rand} \gg \gamma$, V_{int} short ranged, hard core

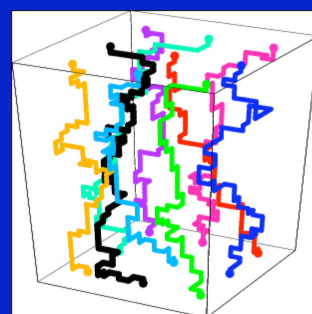
$$\rightarrow \quad H = \sum_{i(bond)} e_i n_i$$

$n_i = 0,1$

$e_i \in [0,1]$

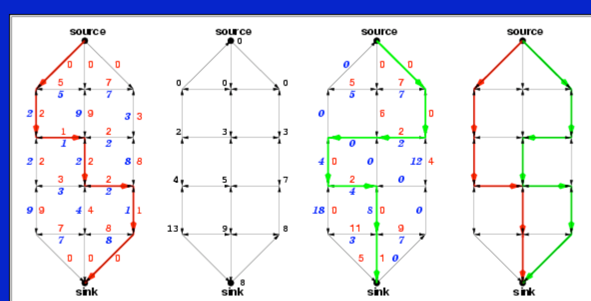
Ground state of N-line problem: Minimum Cost Flow problem

Example conf. for 9 lines:



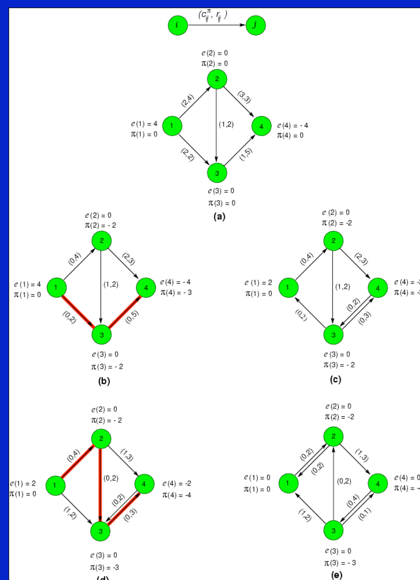
Successive shortest path algorithm

Find successively shortest paths from s to t in the residual network $G_r(n)$:
Use node potentials $\pi(i)$
$\rightarrow$ reduced energies $e^{\pi}_{ij}(n)$ all positive
Use Dijkstra's algorithm




```
begin
  n:=0; π(i)=0; G_r(n):=G;
  for line-counter = 1 to N do
    compute reduced energies e^π_ij(n)=e_ij + π(i)- π(j);
    find the shortest path distance d(i) from s to all other nodes i
        in the residual network G_r(n) w.r.t. to the reduced energies e^π_ij(n);
    increase flow on the shortest path from s to t by one unit;
    compute π(i):=π(i)-d(i);
  end
end
```

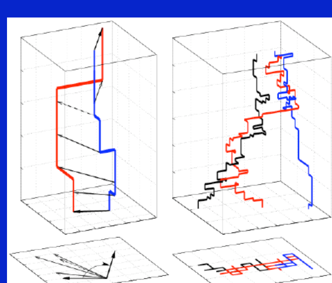
Successive shortest path algorithm (2)



Entanglement of elastic lines in a disordered environment

Magnetic flux lines in type II superconductors:

Pure case:	Abrikosov-FL lattice
Weak disorder:	Bragg glass phase
Strong disorder:	Topological defects FL entanglement



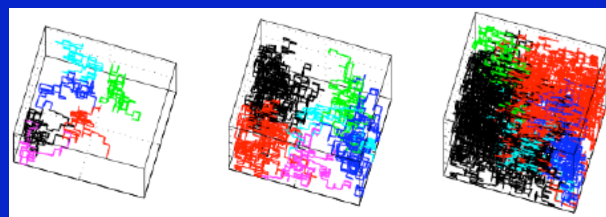
Definition of two-line-entanglement

Check winding angle of line A and B:
if >2□ : A and B are entangled.

Definition of entangled clusters:

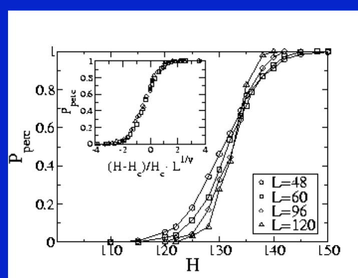
Entangled lines form clusters or bundles:
A @ B and B @ C □ A & B & C in one bundle

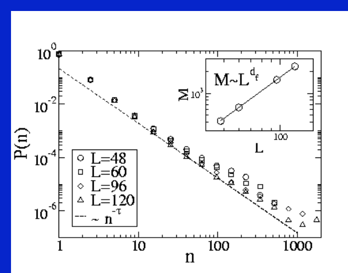
Entanglement transition of elastic lines



H =64 H =96 H=128

Conventional 2d percolation transition



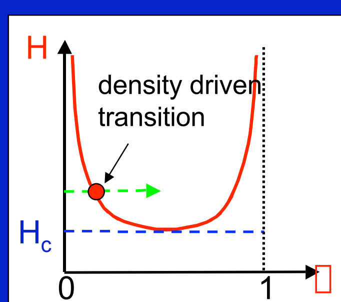


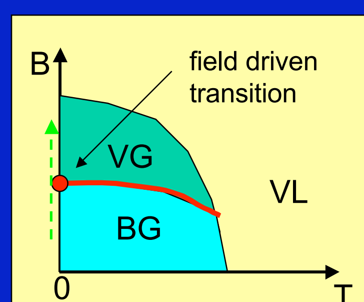
d_f=1,896

□=4/3

□=2,055

Entanglement transition of elastic lines (2)





Thick samples ($H>H_c$):
Entanglement transition;

Thin samples ($H<H_c$):
Disentangled lines – rods.

Magnetic field (B) driven
Bragg glass (BG)
to vortex glass (VG) transition
in disordered high-T_c
superconductors;
n.b.: line density □ B

Loop percolation of magnetic flux lines in HTC (1)

Vortex-Glass Model

$$H = -\frac{1}{2}\sum_{i,j}(n_i - b_i)G_\lambda(i-j)(n_j - b_j)$$

$$n_i = \ldots, -2, -1, 0, 1, 2, \ldots \qquad \sum n_i = 0$$

$$b_j = \sum_{plaq.} A_{ij} \quad , \quad A_{ij} \in [0, 2\pi\sigma]$$

$$G_\lambda(i-j) \sim \exp(-r_{ij}/\lambda)/r_{ij}$$

Strong screening limit: $\lambda \to 0$ $\Rightarrow$ $H = \sum_i (n_i - b_i)^2$

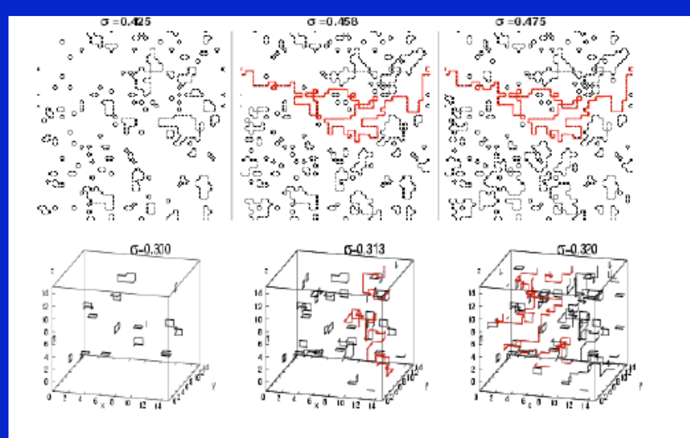
Ground state: Minimum cost flow problem
Successive shortest path algorithm

Note: σ = strength of disorder

Loop percolation of magnetic flux lines in HTC (2)

2d

3d

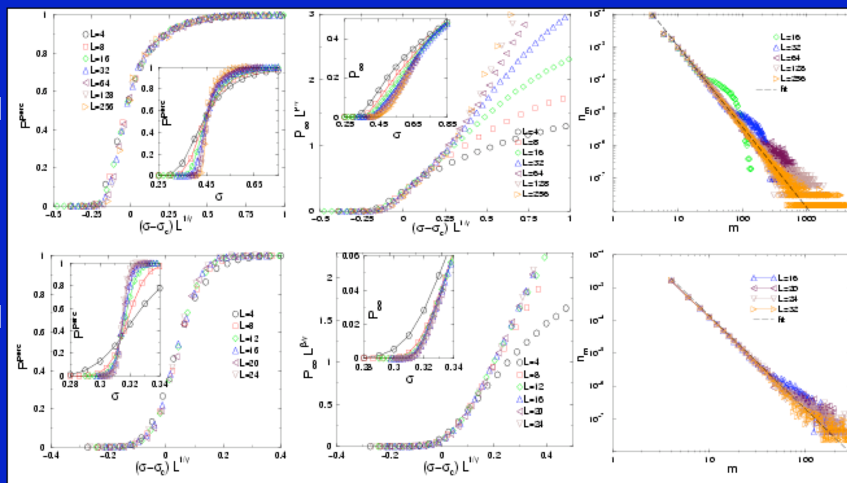


Loop percolation of magnetic flux lines in HTC (3)

P_{perc} | P_{pc} | $P(n)$

2d

3d



2d: □=3.3 □=1.8 □=2.45

3d: □=1.05 □=1.4 □=2.85

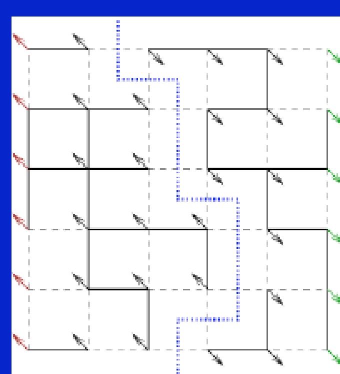
Another combinatorial optimization problem: Interfaces in random bond Ising ferromagnets

$$H = -\sum_{i} J_{ij} S_i S_j \qquad J_{ij} \geq 0, \quad S_i = \pm 1$$

Find for given random bonds J_{ij} the ground state configuration $\{S_i\}$ with fixed +/- b.c.

□ Find interface (cut) with minimum energy

$S_i = +1$

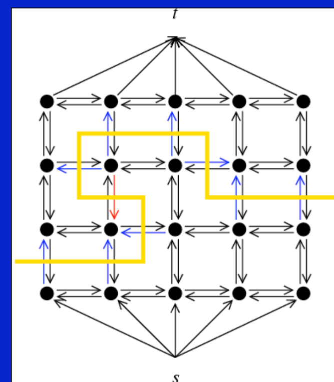


$S_i = -1$

Min-Cut-Max-Flow Problem

network G(V,A), arcs (Bonds) $(i,j)\in A$ have capacity $u_{ij}>0$,
flow $0<=n_{ij}<=u_{ij}$ fulfills mass balance constraint

$$\sum_{\{j|(j,i)\in A\}} n_{ji} - \sum_{\{j|(i,j)\in A\}} n_{ij} = \begin{cases} v & \text{for } i=s \\ -v & \text{for } i=t \\ 0 & \text{else} \end{cases}$$



Find the maximum flow n^* with value v from s to t

residual network G(n) with residual capacities

$$r_{ij} = u_{ij} - n_{ij} + n_{ji}$$

n^* maximum flow $\Leftrightarrow$ no directed path $s\rightarrow t$ in $G(n^*)$

s-t cut [S,S‘] is a partition of V in two disjoint sets with $s\in S$, $t\in S'=V\backslash S$,
$(S,S') = \{(i,j)\in A | i\in S, j\in S'\}$; capacity of the s-t-cut $v[S,S'] = \sum_{(i,j)\in(S,S')} u_{ij}$

Min-Cut-Max-Flow-Theorem: $\max_{\{n\}} v = \min_{[S,S']} v[S,S']$ and $r_{ij}^*=0$ along (S,S‘)

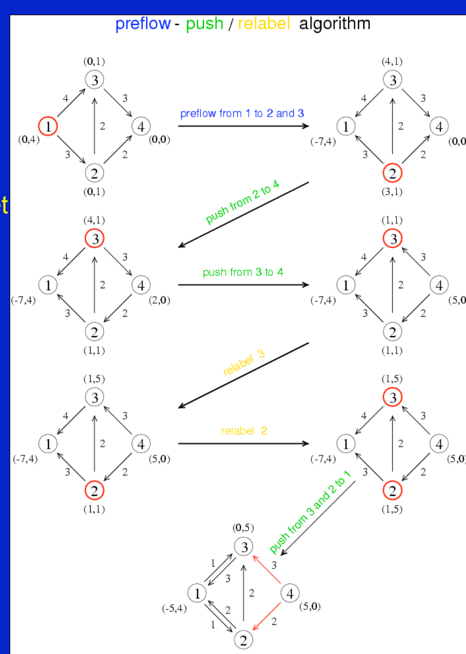
Preflow-Push Algorithm

Strategy:
1) flood the network from source
2) propagate the flooding toward the target
3) push excess flow back towards source

excess flow $e(i) = \sum_{\{j|(j,i)\in A\}} n_{ji} - \sum_{\{j|(i,j)\in A\}} n_{ij}$

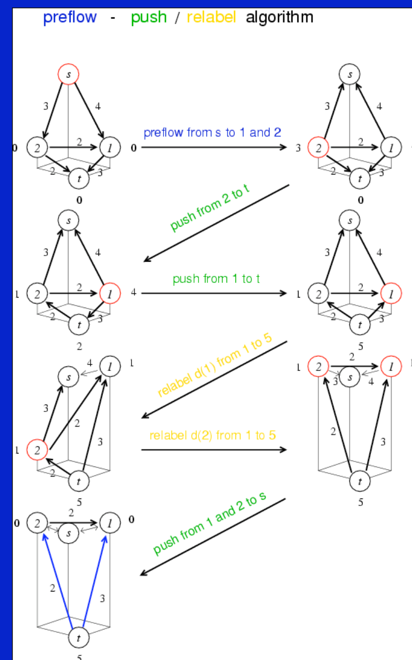
distance function d(i) (w.r.t. target)

begin
d(i) = exact distance from target, d(s)=|V|
for all $(s,j)\in A$: $n_{sj}=u_{sj}$;
choose $i\in V\backslash\{s,t\}$ with e(i)>0
if there is $(i,j)\in A$ with d(i)=d(j)+1:
push $\delta=\min\{e(i),r_{ij}\}$ flow units from i to j
else
relabel $d(i)\leftarrow \min_j\{d(j)+1|(i,j)\in A, r_{ij}>0\}$;
end



Preflow-Push-Algorithm (2)

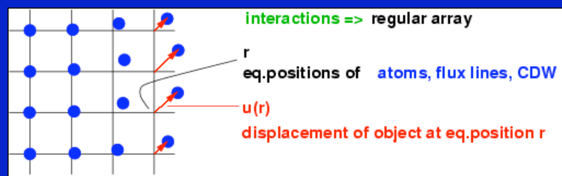
Interprete
distance function (or labels) as
height of the nodes.



Problems that can be mapped on min-cut / max-flow

- Interfaces / wetting in random media
- Random field Ising model (in any dimension)
- Periodic media (flux lines, CDW, etc.) in disordered environments
- Elastic manifolds with periodic potential and disorder

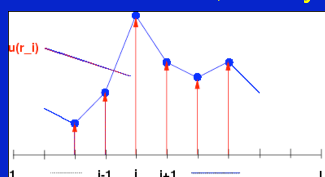
Example: Elastic manifolds / Elastic media



Expansion of potential energy around equilibrium positions: $H_{elast} = \int d^D r\, (\nabla u)^2 \xrightarrow{lattice} \sum_{(ij)} [u(r_i) - u(r_j)]^2$

Effects of impurities, i.e. disorder: lattice distortions: $H_{rand} = \int d^D r\, V[r,u(r)] \xrightarrow{lattice} \sum_{(ij)} V[r_i,u(r_i)]$

Manifolds: D=1 KPZ, in any D ↔ interface in D+1 RBIFM



$w^2 = \sum_i (u_i - \langle u \rangle)^2 \sim L^{\zeta}$

ζ=2/3, 0.41, 0.22 in D=2,3,4

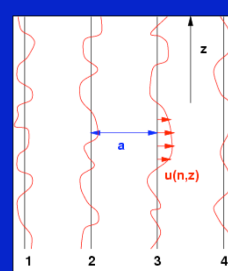
Periodic elastic media

Symmetry: H{u}=H{u+a·n}, e.g.: flux lines

Random pinning potential: first harmonics

$V[r,u(r)] = \cos(2\pi u(r)/a - \phi(r))$

$\phi \in [0,2\pi[$, random



$H = \int d^D r\, (\nabla u)^2 + \lambda \cos(u(r) - \phi(r)) \xrightarrow{lattice} \sum_{(ij)} (u_i - u_j)^2 + \lambda \sum_i \cos(u_i - \phi_i)$

Flory argument: Roughness $\langle\langle u^2 \rangle\rangle \sim \ln L/L_{\xi}$

in 2d RG: $\langle\langle u^2 \rangle\rangle \sim (\ln L/L_{\xi})^2$

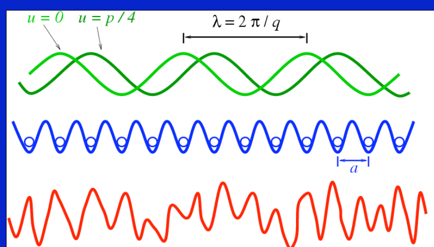
Periodic elastic media + periodic potential

$$H = \int d^D r\, (\nabla u)^2 - v\cos(p\cdot u(r)) + \eta(r)\cos(u(r) - \theta(r))$$

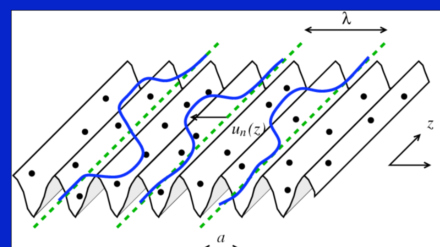
$$<\eta(r)>=0 \quad <\eta(r)\eta(r')>=\Delta\delta(r-r') \quad <e^{i\theta(r)}>=0 \quad <e^{i\theta(r)}e^{-i\theta(r')}>=\delta(r-r')$$

2 periodicities with ratio p

T=0 roughening transition via □



Charge density wave system



Flux line system

Periodic elastic medium + periodic potential (2)

Mapping to an RBIFM interface problem

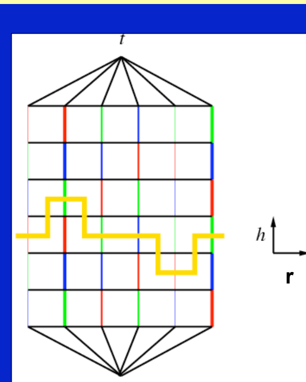
Discrete interface hamiltonian:

$$H = \sum_{(ij)} (h_i - h_j)^2 - \sum_i \eta_i \cos(2\pi h_i / p - \theta_i)$$

Ising model:

$$H = -\sum_{(ij)} J_{ij} S_i S_j$$

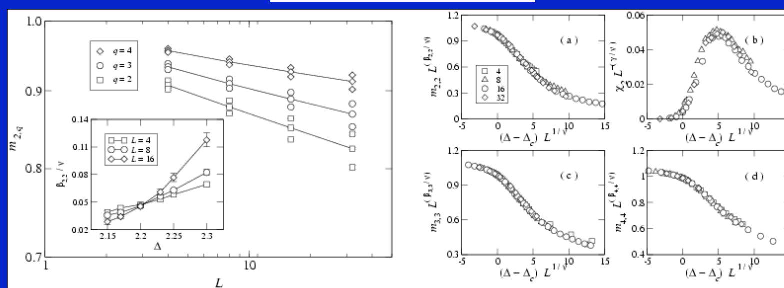
$$J_{h-direction} = \eta(r)\cos(2\pi h / p - \theta(r))$$
$$J_{r-direction} = const. \;\forall\; (r,h)$$



Periodic elastic medium + periodic potential (3)

The roughening transition in 3d:

Order parameter: $m_{p,q}(L,\Delta) = |\langle e^{2\pi i \phi/q} \rangle|$



p	Δ_c	$\beta_{p,2}/\nu$	$\beta_{p,3}/\nu$	$\beta_{p,4}/\nu$	ν
2	2.20	0.046	0.034	0.022	1.25
3	2.48	0.049	0.037	0.024	1.29
4	2.95	0.044	0.033	0.022	1.28

Further applications of combinatorial optimization methods in Stat-Phys.

- Flux lines with hard core interactions
- Vortex glass with strong screening
- Interfaces, elastic manifolds, periodic media
- Wetting phenomena in random systems
- Random field Ising systems
- Spin glasses (2d polynomial, d>2 NP complete)
- Statistical physics of complexity (K-Sat, vertex cover)
- Random bond Potts model at T_c in the limit $q \to \infty$
- ...

Further reading:

H. Rieger:
Ground state properties of frustrated systems,
Advances in computer simulations, Lexture Notes in Physics 501
(ed. J. Kertesz, I. Kondor), Springer Verlag, 1998

M. Alava, P. Duxbury, C. Moukarzel and H. Rieger:
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A. Hartmann and H. Rieger,
Optimization in Physics,
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