From Plasticity to a Renormalisation Group

RC Ball	University of Warwick
R Blumenfeld *	‘’

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SF Edwards	Cambridge
D Grinev *	‘’
R Farr	Unilever
C Thornton	Aston
J Melrose	Unilever

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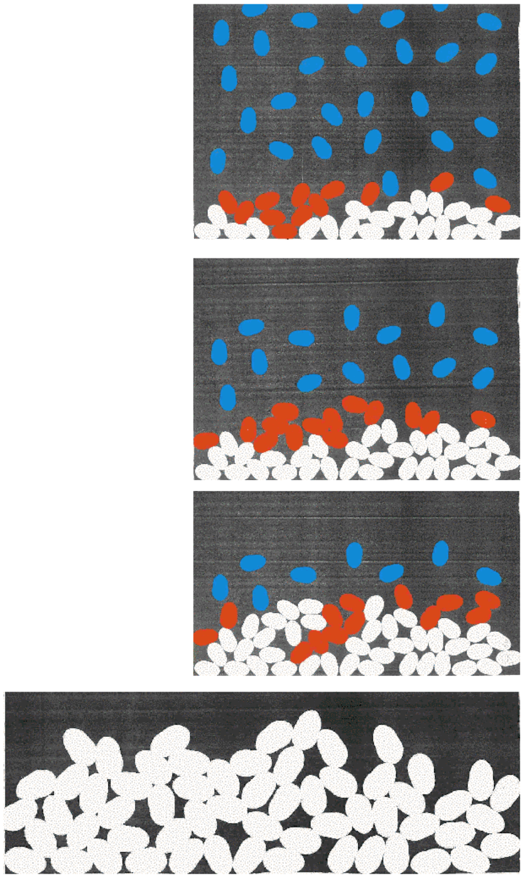
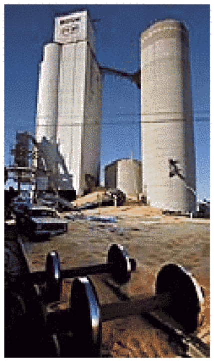
Outline of talk

- Marginally Rigid State
  - Experimental Evidence
  - Stress Transmission
    ab initio vs FPA
  - RG & generalised statics
- Yield Equations & Grain Rolling
- Rough vs Smooth grain response

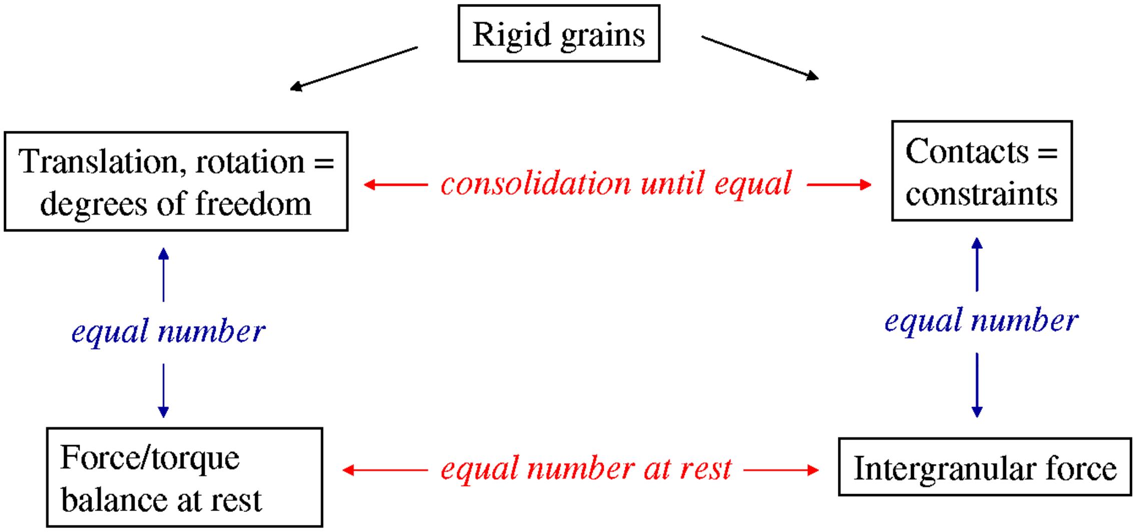


Stress in Granular Materials

- Soil Mechanics
  - design of foundations
  - dams, embankments
  - landslides
- Materials handling,
  - silos
- Counterintuitive Physics
  - load distribution
  - under pile



Marginally Rigid State



Rigid grains

Translation, rotation = degrees of freedom

Contacts = constraints

consolidation until equal

equal number

equal number

Force/torque balance at rest

Intergranular force

equal number at rest

•Static equilibrium determines all the intergranular forces

and hence the transmission of stress

Test of Marginal Rigidity: Critical Coordination Number

d=2

per grain 3 d.o.f

per contact 2 constraints, with friction => $z_c= 3$

checked by experiment

no friction => $z_c = 6$ *topological maximum*

“ & discs => $z_c = 4$ both *match sequential (disordered) packing*

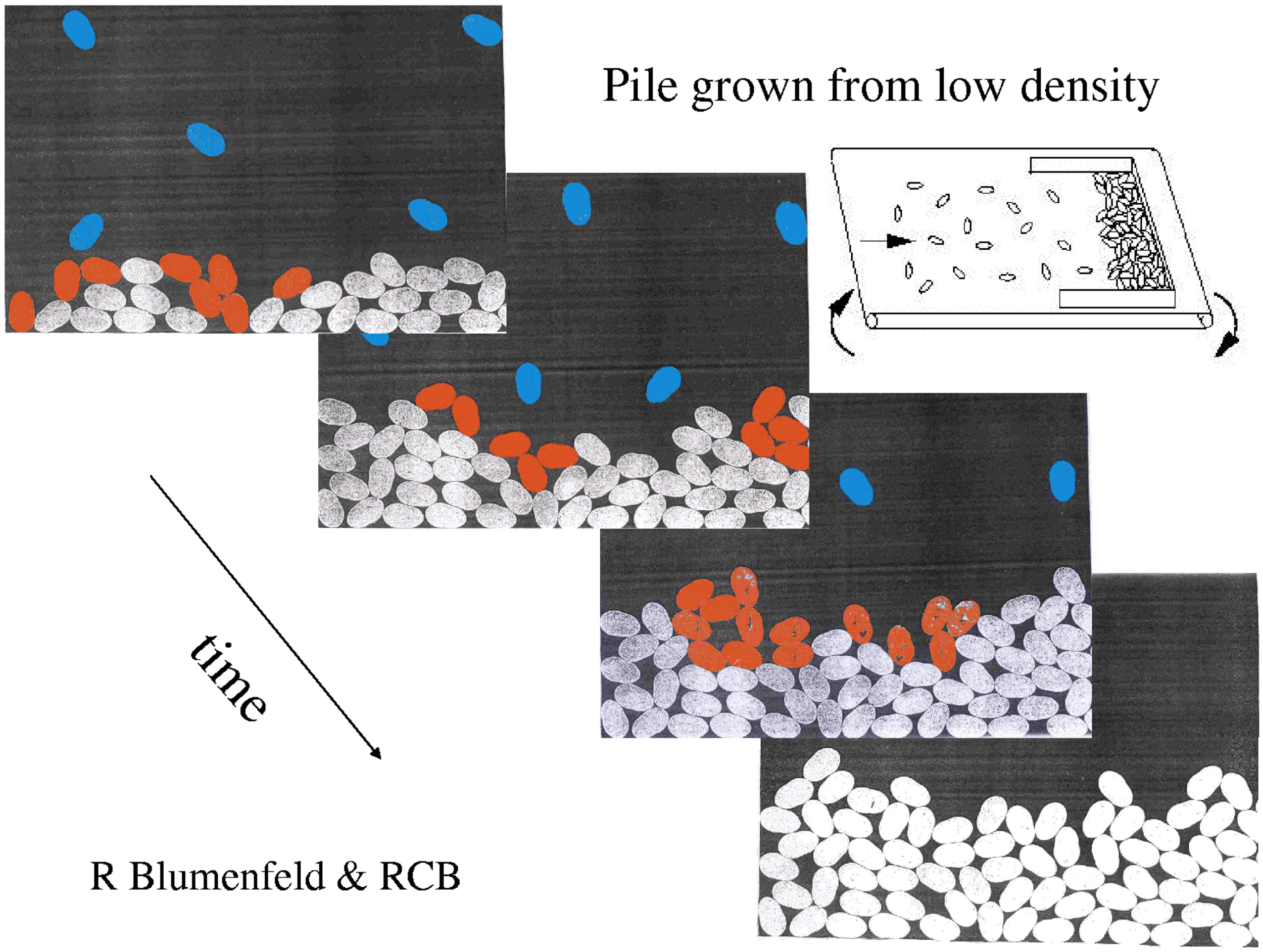
d=3

friction => $z_c = 4$

no friction => $z_c = 12$ *hard to exceed (but possible: Donev et al)*

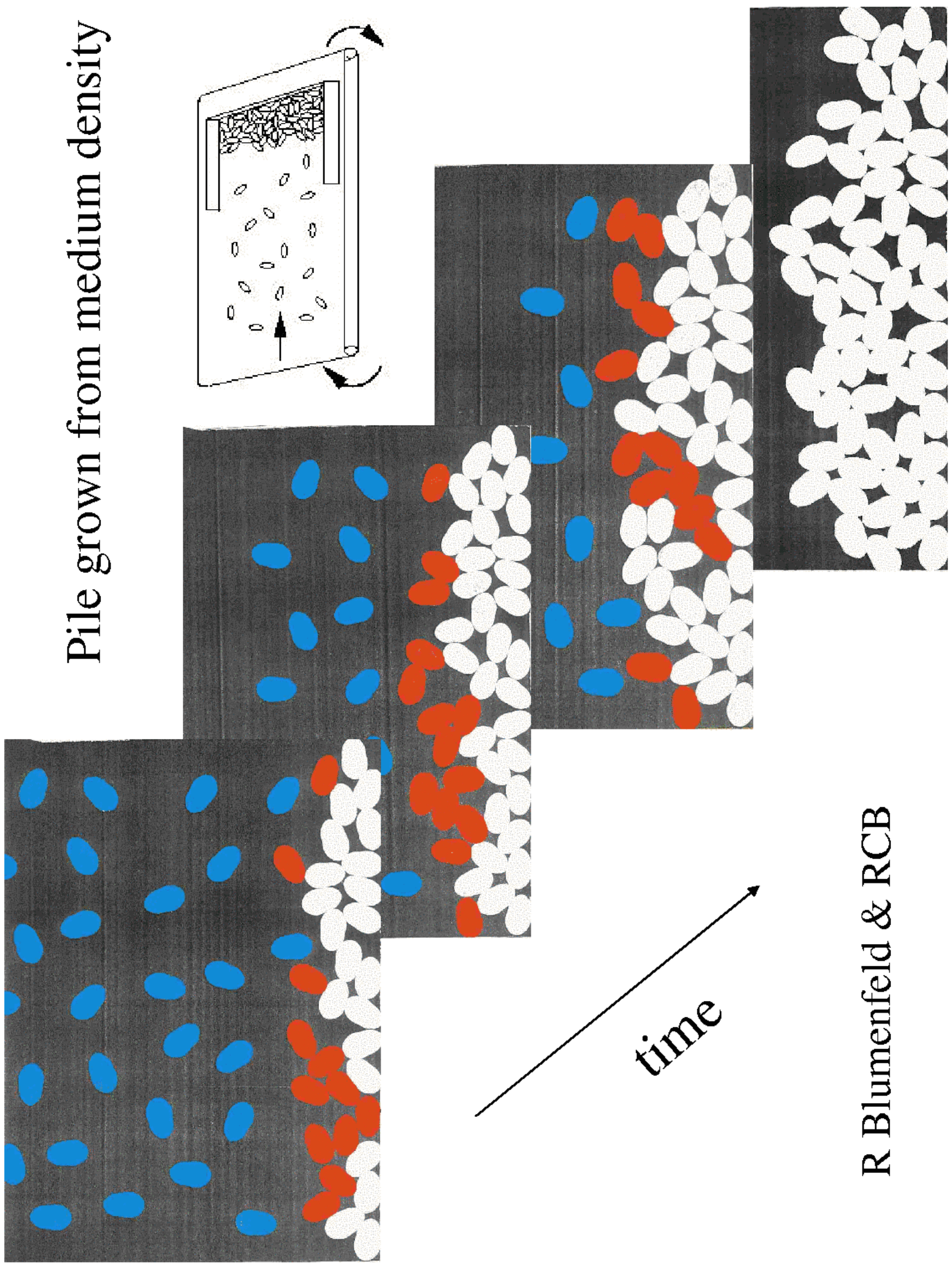
“ & discs => $z_c = 6$ *both match sequential (disordered) packing*

Pile grown from low density



R Blumenfeld & RCB

Pile grown from medium density



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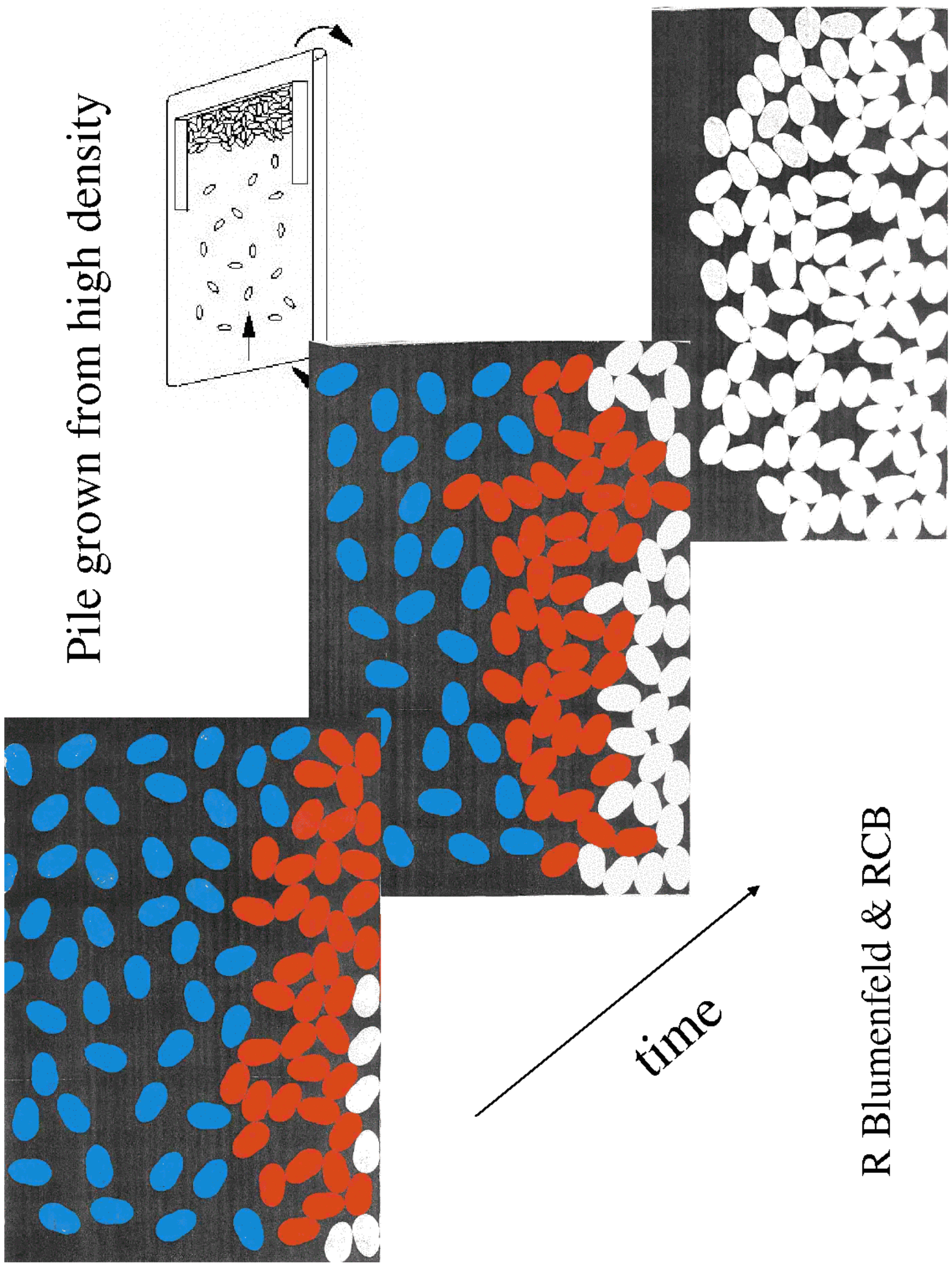
Pile grown from high density



R Blumenfeld & RCB

Final density
vs initial density

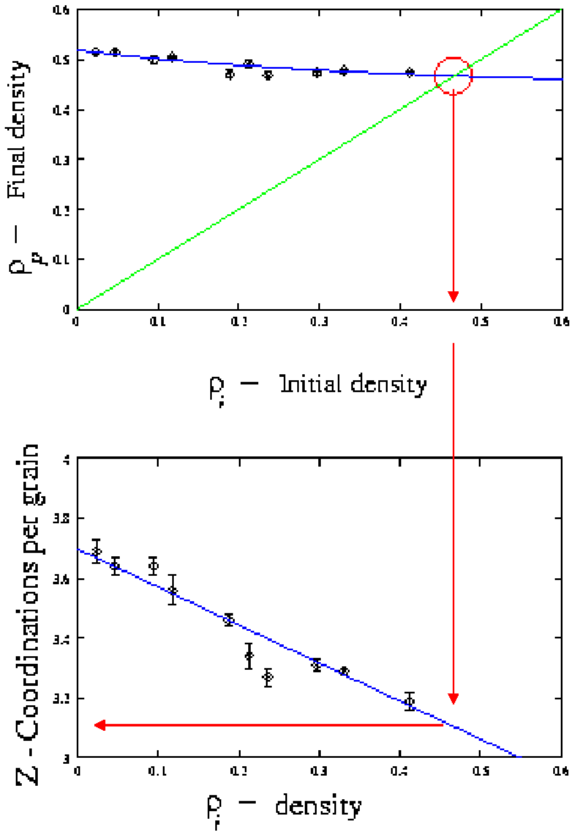
limiting value $\rho_c = 0.47$

Experimental results

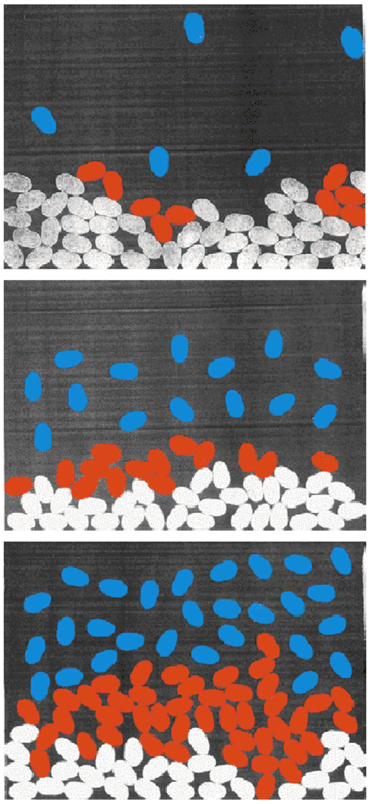
Coordination number

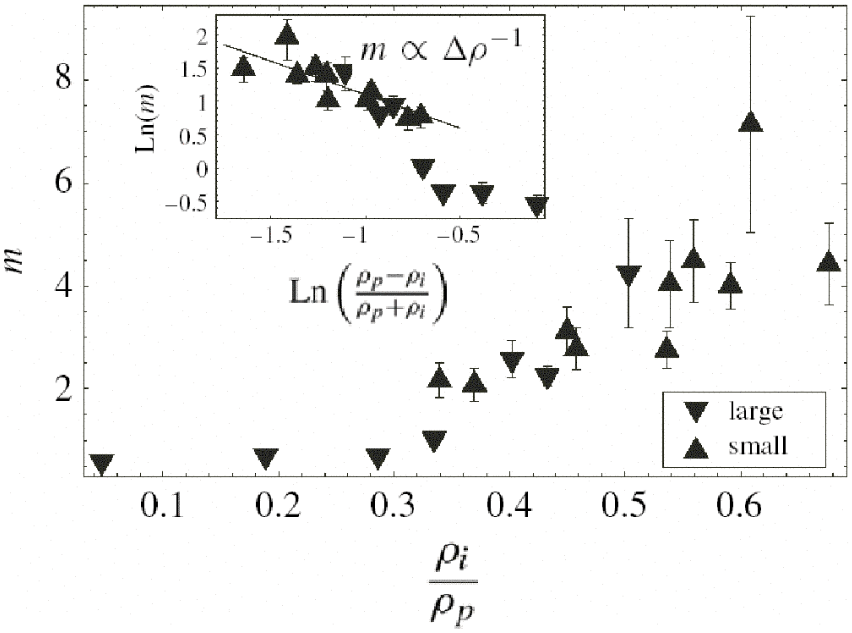
limiting value $z_c = 3.1$

RB & RB



The Consolidation Front

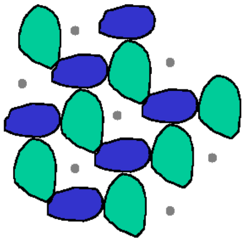




Stress transmission calculations

$$\boxed{p_{ij\,kl}\sigma_{kl}=0} \qquad (*)$$

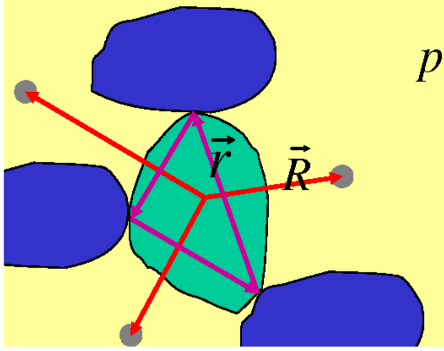
antisymmetric symmetric indices



Simplest periodic lattice at $z_{c:}$

2 grains per unit cell =>
2 conditions balancing torque:

$$\sigma_{kl}=\sigma_{lk} \quad \text{and} \quad (*) \quad \text{above}$$



$$p_{ij\,kl}=\varepsilon_{ij}\ \varepsilon_{kp}\sum_{\text{round grain}}\left(r_pR_q+R_pr_q\right)\ \varepsilon_{ql}$$

$d=2$

- Generalises off-lattice
- Spatial average is zero
- Anti-correlation between neighbours

Local linear constitutive equations

$$p_{ijkl}\sigma_{kl}=0$$

"OSL" - *Cates et al* *"Isostatic"* *"Null-stress"*

FPA: $p_{ijkk}=0$ for all ij

$$\vec{\vec{\sigma}}=\sigma_{aa}\vec{a}\vec{a}+\sigma_{bb}\vec{b}\vec{b}+\sigma_{cc}\vec{c}\vec{c}\ ,\quad \vec{a},\vec{b},\vec{c}\ \text{ set by sample history}$$

d=2: $p_{kl}\sigma_{kl}=0$ degenerate case

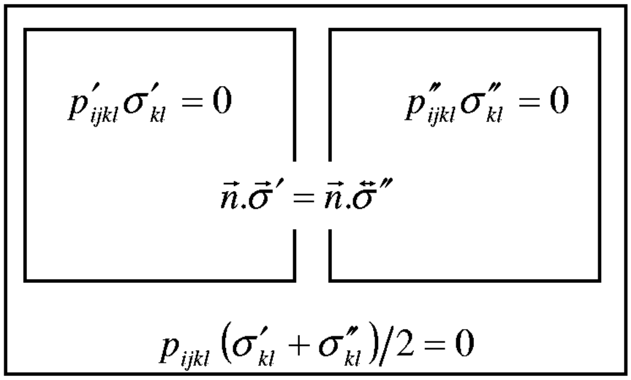
Cates et al, Edwards & Grinev ....

$$\updownarrow$$

$$\sigma_{ij}=f_1\tau_{1ij}+f_2\tau_{2ij}$$ *Tkachenko & Witten*

Renormalisation of Static Equations

Simple
model



$$p'_{ijkl}\sigma'_{kl} = 0 \qquad p''_{ijkl}\sigma''_{kl} = 0$$

$$\vec{n}.\vec{\sigma}' = \vec{n}.\vec{\sigma}''$$

$$p_{ijkl}\left(\sigma'_{kl} + \sigma''_{kl}\right)/2 = 0$$

•simple solution in *d=2* : $p_{ij} = B\left(p'^{-1}_{tt}\, p'_{ij} + p''^{-1}_{tt}\, p''_{ij}\right)$ *t* => transverse to *n*

(left, antisymmetric indices are redundant in *d=2*)

•generalises to all *d* but *B*, p_{tt}^{-1} not scalar; right indices of *p'*, *p"* preserved.

Fixed Points

1. *Constant Fixed Points: p=any constant*

•Sensible for smooth grains.

•Significance unclear for rough grains:
average(*p*) is zero at grain level, but RG transformation not linear.

2. *Pressureless fixed points*

•Where (one of) the $p_{ij..}$ constrains the stress to be traceless,
clearly transitive under the stress combination.

•Linearly stable to fluctuations in *d=2*:
analytic calculations
seen numerically by R Farr

•Pressureless stress incompatible with cohesionless granular matter.

3. *Fixed Principle Axes fixed points*

•All *p* traceless wrt rh indices, i.e. p_{ijkk}=0 for all *i,j* ;
clearly transitive under renormalisation of *p*

•Random axes fixed point neutrally stable wrt small isotropic parts
analytic calculations in *d=2*
numerical indications in *d=3*

Higher derivative cases

$$p_{ijkl}\,\sigma_{kl} + q_{ijklm}\,\partial_m\,\sigma_{kl} + r_{ijklmn}\,\partial_m\partial_n\,\sigma_{kl} + \ldots = 0$$

Rough Grains :

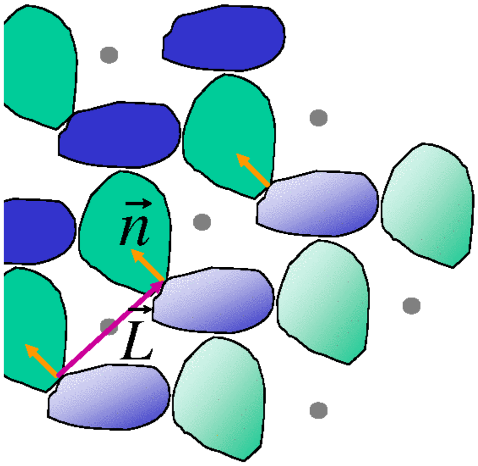
periodic, aspherical: $p \neq 0$ (but staggered)

periodic, spherical: $p = 0, q \neq 0$

statistically isoptropic: $p = 0, q = 0, r \neq 0$

Smooth Grains :

periodic, aspherical: $p \neq 0$ (no stagger).



Yield Equations

Condition for sliding at plane:

$$\left|\vec{n}\times\vec{\sigma}\times\vec{L}\right| \;>\; \mu\left(-\vec{n}.\vec{\sigma}\times\vec{L}\right)$$

|shear force| coeff friction normal force

$$Y(\vec{\sigma}) = \prod_{\text{sliding planes}}\left(\left|\vec{n}\times\vec{\sigma}\times\vec{L}\right| + \mu\,\vec{n}.\vec{\sigma}\times\vec{L}\right) > 0 \quad \text{yielding}$$

$$\boxed{Y(\vec{\sigma}) = 0 \quad \text{creeping yield => plastic flow,} \quad \text{direction} = \quad g(\vec{\sigma})}$$

$Y(\vec{\sigma}) < 0$ unyielding

Plastic Flow

conventional plasticity grain rolling

$$\partial_k u_l + \partial_l u_k \quad = \quad A(\vec{x},t)\, g_{kl}(\vec{\sigma}) \quad + \quad p^T{}_{kl\,ij}(\vec{x})\ \ \omega_{ij}(\vec{x},t)$$

local sliding rate

new (?) coefficients

grain angular velocity

- supplemented by:

plastic yield condition $Y(\vec{\sigma}) = 0$

force balance $\vec{\nabla}.\vec{\sigma} + \vec{f} = 0$

'constitutive equation' $p_{ij\,kl}\sigma_{kl} = 0$

- grain rolling is required for the equations to close correctly
- Gallilean invariance requires average of $p^T = 0$

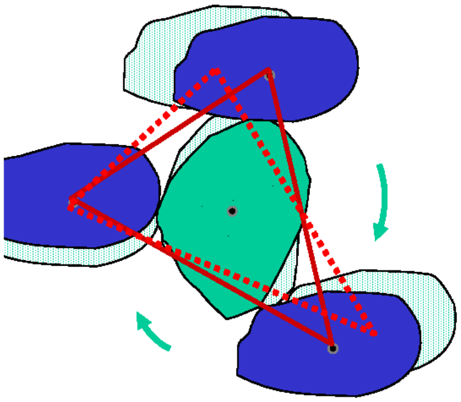
σ_{kl}

A

$\vec{u}$

ω_{ij}

Grain Rolling



Rotate central grain & translate neighbours, maintaining rolling contact

Explicit calculation of shear rate in triangle:

$$\partial_k u_l + \partial_l u_k = \frac{1}{\text{area}} \int (u_k dS_l + u_l dS_k) == \omega_{ij}\, p_{ij\,kl}$$

i.e. p^T really is the transpose of p, (for d=2, periodic lattice)

Dissipation rate & Interpretation

$$2\dot{\Delta} = (\partial_k u_l + \partial_l u_k)\sigma_{kl} = A\ \underset{\underset{\text{sliding}}{\uparrow}}{g_{kl}}(\vec{\sigma})\sigma_{kl} + \omega_{ij}\underset{\underset{\text{rolling}}{\uparrow}}{p_{ij\,kl}}\sigma_{kl}$$

- The force conjugate to rolling is $p_{ij\,kl}\sigma_{kl}$

where *p* can quite generally be defined from the plastic flow equation.

- Rolling is not (directly) disspative => $p_{ij\,kl}\sigma_{kl} = 0$

hence this force must be balanced - explaining the ‘constitutive eqn’.

Cf Tkachenko & Witten PRE 62 2510 (2000)

Microscopic views of Rolling/Sliding

What happens when we load a sample, incompatibly with $p_{ijkl}\sigma_{kl} = 0$?

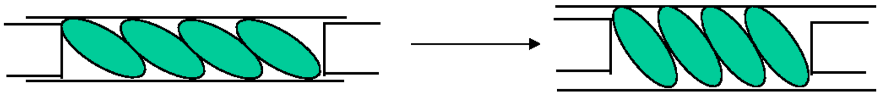
Change in geometry -> local changes in *p*
-> changes in σ **which could be much less local**
- what do these look like?

SUGGESTION:
Smooth grains: imposed deformations penetrate bulk, so the bulk deforms

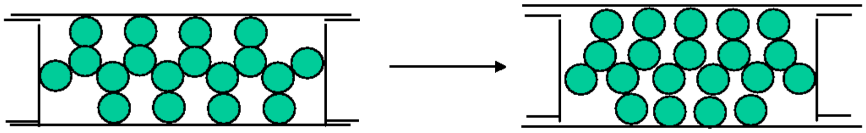
Rough grains: any strain disproportionately concentrated at walls
-> wall rearrangement
-> loads transmitted to bulk are compatible with $p_{ijkl}\sigma_{kl} = 0$

Simple Cases

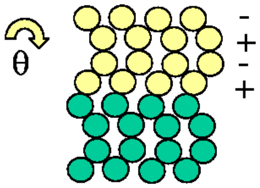
Smooth grains



Rough grains



Grain rotation -> overshoot of affine at walls



Local shear $\gamma \sim a \nabla\theta$

wall shear $\gamma_{wall} \sim \theta$

Wall shears >> bulk shears for rough grains
-> rearrangement preferentially at walls

Summary and Outlook

- Direct experimental evidence for marginally rigid state
  d=2 yes, d=3 outstanding
- Theory of stress transmission …. empirical theory
  non-lattice calculations required in d=3

- RG indicates FPA may be stable
- RG in d=3; systemmatic/improvements ?
- Gradient terms and order of equtions ?

- New hydrodynamics: Yield = plasticity + rolling
  interpretation of dilatancy: shear => roll => dilate
- "missing" static equations relate to Rolling Force
- Closure: evolution of P_{ijkl} under shear ?

- are rough and smooth systems fundamentally different ?

- Impact engineering soil mechanics?

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