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Hydrodynamics for a Granular Fluid

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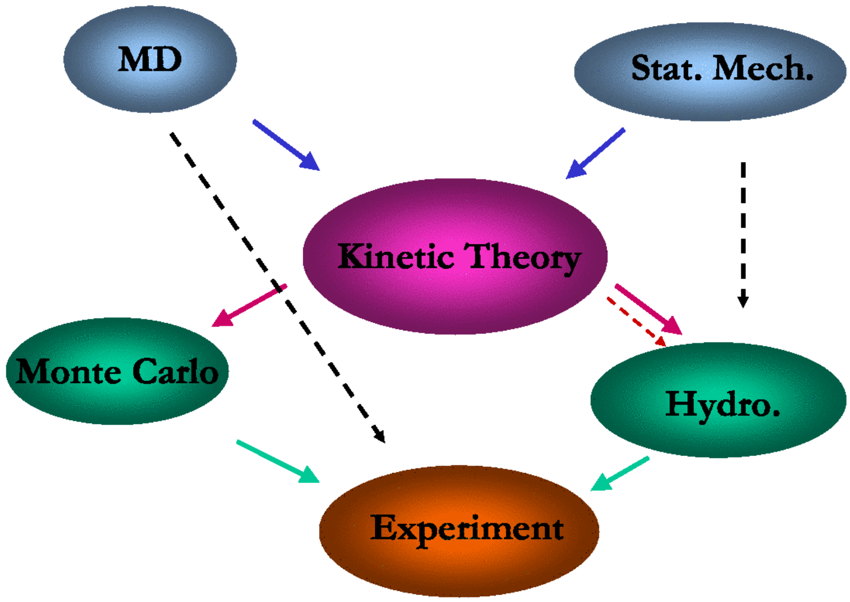
Overview

- What is a hydrodynamic description?
- Boltzmann equation and origin of hydrodynamics.
- Normal solution, a constructive example (CE).
- Hydrodynamics in spectrum of collision operator
- Exact results from kinetic models (X).



Origins of hydrodynamics: Boltzmann Kinetic Theory

$$\left(\frac{\partial}{\partial t} + \mathrm{v}\cdot\nabla\right) f(\mathbf{r},\mathrm{v},t) = J\left[\mathbf{r},\mathrm{v}|f(t)\right],$$

$$\begin{pmatrix} n(\mathbf{r},t) \\ n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) \\ \frac{3}{2}n(\mathbf{r},t)T(\mathbf{r},t) \end{pmatrix} = \int d\mathbf{v} \begin{pmatrix} 1 \\ \mathrm{v} \\ \frac{1}{2}m\left(\mathbf{v}-\mathbf{u}\right)^2 \end{pmatrix} f(\mathbf{r},\mathbf{v},t).$$

exact

$$\partial_t n + \nabla\cdot(n\mathbf{u}) = 0,$$

$$\left(\partial_t + \mathbf{u}\cdot\nabla\right) u_i + (mn)^{-1}\nabla_j P_{ij} = 0,$$

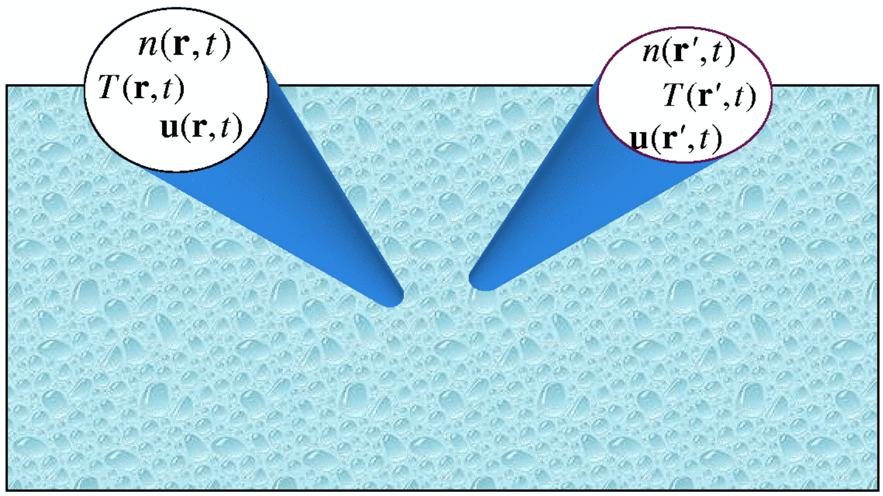
$$\left(\partial_t + \mathbf{u}\cdot\nabla + \zeta\right) T + \frac{2}{3n}\left(P_{ij}\nabla_j u_i + \nabla\cdot\mathrm{q}\right) = 0.$$

$$P_{ij}(\mathbf{r},t) = \int d\mathbf{v}\, m(v_i - u_i)(v_j - u_j) f(\mathbf{r},\mathbf{v},t),$$

$$\mathbf{q}(\mathbf{r},t) = \int d\mathrm{v} \frac{m}{2}(\mathrm{v}-\mathbf{u})^2(\mathrm{v}-\mathbf{u}) f(\mathbf{r},\mathrm{v},t).$$

$$\zeta(\mathbf{r},t) = \frac{(1-\alpha^2)m\pi\sigma^2}{24n(\mathbf{r},t)T(\mathbf{r},t)} \int d\mathrm{v} \int d\mathrm{v}_1\, g^3\, f(\mathbf{r},\mathbf{v},t,)f(\mathbf{r},\mathbf{v}_1,t).$$

Normal solution



$$f = f\left(\mathbf{r},\mathbf{v}\middle|n(t),\mathbf{u}(t),T(t)\right)$$

Normal solution

$$f(\mathbf{r},\mathbf{v},t) = f\left[\mathbf{r},\mathbf{v},t \middle| \{y_\alpha\}\right],$$

such that

$$\begin{pmatrix}\partial_t \\ \nabla\end{pmatrix} f\left[\mathbf{r},\mathbf{v},t \middle| \{y_\alpha\}\right] = \int dt' \int d\mathbf{r}' \frac{\delta f(\mathbf{r},\mathbf{v},t|\{y_\alpha\})}{\delta y_\beta(\mathbf{r}',t')} \begin{pmatrix}\partial_{t'} \\ \nabla'\end{pmatrix} y_\beta(\mathbf{r}',t').$$

↓

$$\zeta = \zeta\left[\mathbf{r},t \mid \{y_\alpha\}\right], \quad P_{ij} = P_{ij}\left[\mathbf{r},t \mid \{y_\alpha\}\right], \quad \mathbf{q} = \mathbf{q}\left[\mathbf{r},t \mid \{y_\alpha\}\right].$$

↓

still exact

$$\partial_t y_\alpha(\mathbf{r},t) = N_\alpha\left[\mathbf{r},t \mid \{y_\beta\}\right],$$

Hydrodynamics !

***Key concept*: rapid relaxation to normal form implies hydrodynamics; same as for normal gases.**

Navier-Stokes Hydrodynamics - I

Special case: states with *small* gradients

$$f = f^{(0)}\left(\mathbf{v},\{y(\mathbf{r},t)\}\right)\left[1 + X_\beta\left(\mathbf{v},\{y(\mathbf{r},t)\}\right)\nabla y_\beta(\mathbf{r},t) + \cdots\right]$$

$$P_{ij}(\mathbf{r},t|f) \to -\eta(T,n,\alpha)\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right)$$

$$\mathbf{q}(\mathbf{r},t|f) \to -\kappa(T,n,\alpha)\nabla T - \mu(T,n,\alpha)\nabla n$$

Explicit form of solution: technical details (Chapman-Enskog)

Another perspective on hydrodynamics spectrum of excitations (“hydrodynamic modes”)

Stronger Test for Hydrodynamics – Linear Response

Uniform solution: $n_0 = \text{constant}, \quad u_i = 0, \quad (\partial_t + \xi_0(T_0))T_0 = 0$

Dimensionless variables, Fourier transform

$$\delta\tilde{y}^*(\mathbf{k}, s) = \int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} \delta n^* \\ \delta T^* \\ \delta \mathbf{u}^* \end{pmatrix} \qquad \delta n^* = \frac{\delta n}{n_0} \qquad \delta T^* = \frac{\delta T}{T_0} \qquad \delta u_i^* = \frac{\delta u_i}{\sqrt{2T_0/m}}$$

$$ds = dt \frac{1}{\ell}\sqrt{\frac{T_0(t)}{m}}$$

Linearize *phenomenological* Navier-Stokes about uniform solution

$$\partial_s \delta\tilde{y}_\alpha(\mathbf{k}, t) + M_{\alpha\beta}(k)\delta\tilde{y}_\beta(\mathbf{k}, t) = 0$$

Hydrodynamic Modes

$$\partial_s \delta\tilde{y}_\alpha + M_{\alpha\beta}(k)\delta\tilde{y}_\beta = 0$$

Solution:

$$\delta\tilde{y}_\alpha = \sum_{n=1}^{5} A_\alpha^{(n)} e^{-\lambda_n(k)s}$$

Hydrodynamic modes

$$M_{\alpha\beta}(k)\psi_\beta^{(n)} = \lambda_n(k)\psi_\alpha^{(n)}$$

$$\lambda(k) = \left(0,\ \frac{1}{2}\zeta,\ -\frac{1}{2}\zeta,\ -\frac{1}{2}\zeta,\ -\frac{1}{2}\zeta\right) + \text{order } k^2$$

***Problem*:** **Calculate $\delta\tilde{y}_\alpha$ directly from Boltzmann equation and look for hydrodynamic modes**

$$f(\mathbf{r},\mathbf{v},t=0) = f_{hcs}(v,0)[1+\phi_\alpha(\mathbf{v})\delta y_\alpha(r,0)]$$

$$f(\mathbf{r},\mathbf{v},t) = f_{hcs}(v,t)[1+\phi(\mathbf{r},\mathbf{v},t)]$$

Linearized Boltzmann equation (dimensionless, Fourier transform)

$$(\partial_s + i\mathbf{k}\cdot\mathbf{v} + \Lambda)\phi = 0$$

$$\Lambda X = -J\left[f_{hcs}^*, f_{hcs}^* X\right] - J\left[f_{hcs}^* X, f_{hcs}^*\right] + \underline{\frac{1}{2}\xi^*\nabla_{v^*}\left(v^* f_{hcs}^* X\right)}$$

Calculate hydrodynamic fields

$$\delta y_\alpha(\mathbf{r},t) = \int d\mathbf{v}\psi_\alpha(\mathbf{v})\left(f(\mathbf{r},\mathbf{v},t) - f_{hcs}(\mathbf{v},t)\right)$$

$$\delta\tilde{y}_\mu(\mathbf{k},s) = R_{\mu\nu}(k,s)\,\delta\tilde{y}_\nu(\mathbf{k},0)$$

$$R_{\mu\nu}^*(\mathbf{k},s) = \int d\mathbf{v}^* f_{hcs}^* \psi_\mu^* e^{-(i\mathbf{k}\cdot\mathbf{v}+\Lambda)s}\phi_\nu^*$$

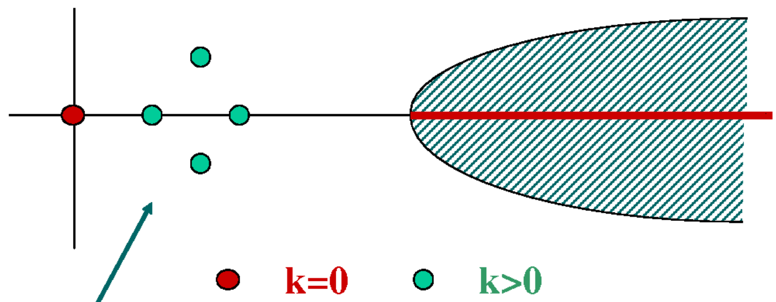
Spectrum of $i\mathbf{k}\cdot\mathbf{v}+\Lambda$ **?**

$$(i\mathbf{k}\cdot\mathbf{v}+\Lambda)\chi^{(n)} = \lambda_n(k)\chi^{(n)}$$

Hydrodynamic modes ??

Slowest modes ??

Normal (elastic) gas



Hydrodynamic modes (Carleman, Weyl, Grad, McClennan, …)

$$\frac{\partial}{\partial y_\alpha} J[f_M, f_M] = 0 \quad \Rightarrow \quad \Lambda \chi_\alpha^{(0)} = 0 \,, \qquad \chi_\alpha^{(0)} \propto \frac{\partial}{\partial y_\alpha} \ln f_M$$

What about granular gases ?

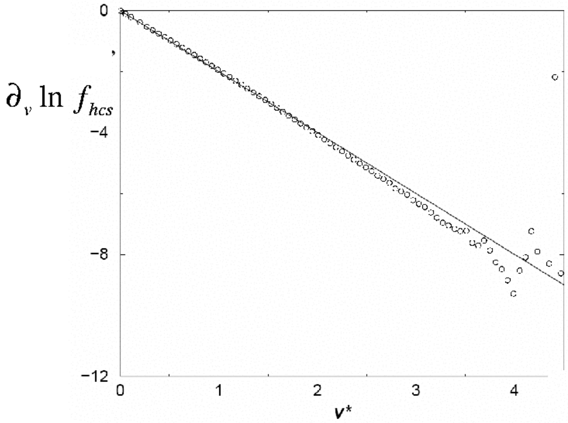
(same idea)

$$\frac{\partial}{\partial y_\alpha}\left(J[f_{hcs}, f_{hcs}] - \frac{1}{2}\zeta \nabla_v \cdot \left(v f_{hcs} \right) \right) = 0$$

HCS equation

$$\Rightarrow \quad \Lambda \chi_\alpha^{(0)} = \lambda_\alpha^{(0)} \chi_\alpha^{(0)} \,, \qquad \chi_\alpha^{(0)} \propto \frac{\partial}{\partial y_\alpha} \ln f_{hcs}$$

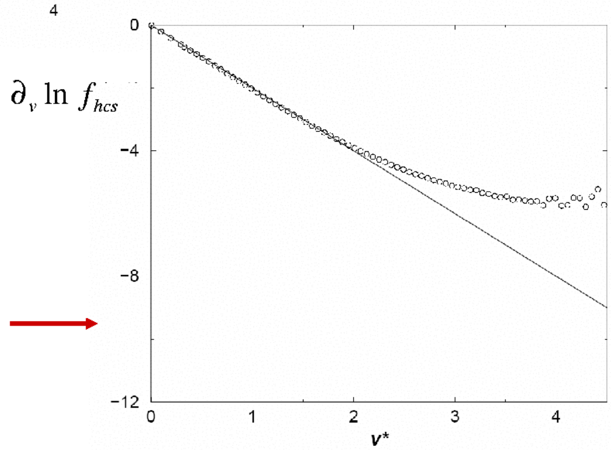
$$\lambda_n^{(0)} = \left(0, \ \frac{1}{2}\zeta^*, \ -\frac{1}{2}\zeta^*, \ -\frac{1}{2}\zeta^*, \ -\frac{1}{2}\zeta^* \right)$$ (existence of hydrodynamic excitations)



DSMC (circles)

Gaussian (solid)

$\alpha = 0.95$



DSMC (circles)

Gaussian (solid)

$\alpha = 0.6$

Navier-Stokes Hydrodynamics - II

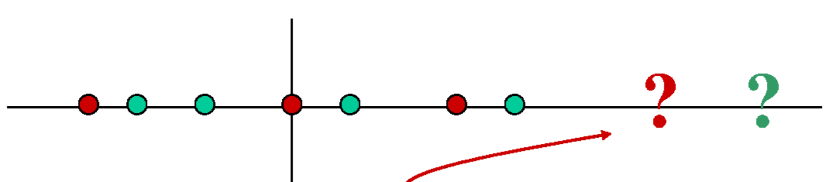
Perturbation theory

$$(i\mathbf{k} \cdot \mathbf{v} + \Lambda)\chi_n = \lambda^{(n)}(k)\chi_n$$

$$\lambda_n(k) = \lambda_n^{(0)} + \lambda_n^{(1)} ik + \lambda_n^{(2)} k^2 + \ldots$$

(Brey / Dufty - PRE 2003,2005)

Same as Chapman-Enskog at order k^2



(Results from kinetic models)

Mixtures: (similar)

Dense Fluids: Enskog Equation

$$\left(i\mathbf{k}\cdot\mathbf{v}B(k)+\Lambda(k)\right)\chi_n=\lambda^{(n)}(k)\chi_n$$

Mean field operator

$\Lambda(0)$ same as Boltzmann

HCS equation same as Boltzmann

$$\frac{\partial}{\partial y_\alpha}\left(J[f_{hcs},f_{hcs}]-\frac{1}{2}\zeta\nabla_v\cdot(vf_{hcs})\right)=0$$

→ $\lambda^{(n)}(0)$ and $\chi_n(0)$ same as Boltzmann

Summary

Granular gases are different, but:

- Concept of normal solution unchanged
- Hydrodynamic excitations in linear kinetic theory
- Hydrodynamics beyond Navier-Stokes may be required (e.g., steady shear flow)

Informal Discussion – Kinetic Theory

(today, 1:30)

Some possible topics:

- Validity and context of the Boltzmann equation
- Validity and context of the Enskog equation
- Representation of binary collisions
- Representation of boundary conditions
- Santa Barbara restaurants