Elliptic-hyperbolic systems and the Einstein Equations.

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$(\bar{M}, \bar{g})$ 3+1 Lorentz $-+++$

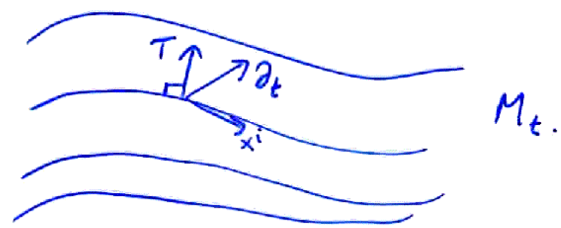
$\bar{\nabla}, \bar{R}$ $\quad i,j,k = 1,2,3 \quad \alpha,\beta,\gamma = 0,1,2,3,4.$

$t : \bar{M} \to \mathbb{R}$ time function

$\langle \bar{\nabla} t, \bar{\nabla} t \rangle < 0$

$\{M_t\}$. foliation by levelsets of t.

$T = N^{-1}(\partial_t - X).$



$$\bar{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$$

$$K_{ij} = -\tfrac{1}{2} \mathcal{L}_T \bar{g}_{ij} = -\langle \bar{\nabla}_i T, \partial_{x^j} \rangle$$

Adapted frame : $(e_\alpha) = (T, e_i)$.

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Background and motivation

3+1 GR - want to understand global structure of Maximal Globally Hyperbolic Vacuum spacetimes

- Cosmic Censorship ⟺ generic MGHV is C^2-inextendible
- Asymptotic behavior near (cosmological) singularity: generically spacelike oscillating singularity (BKL)

  AVTD, quiescent for models with symmetry or stiff fluid.
- Asymptotic behavior in expanding direction
- Global uniqueness a'la Choquet-Bruhat, Geroch. What regularity is needed?
- Approach these problems via CMC-conjecture (spatially cpt. case).

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Results:

1) Elliptic-hyperbolic formulation of Einstein Eqs (CMCSH gauge) w/p in H^s, $s > n/2+1$

   LA, Moncrief AHP vol 4, '03 gr-qc/0110111.

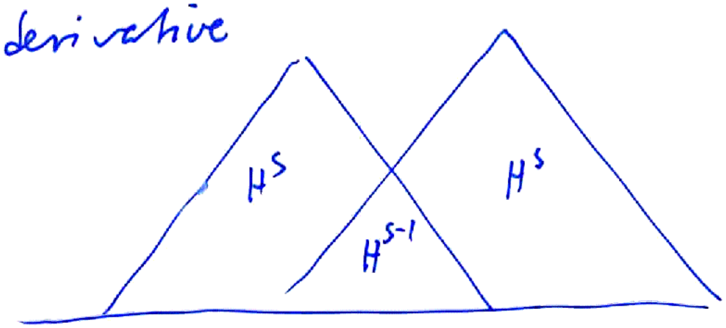
2) Future complete 3+1 vacuum spacetimes (small data global existence) [gr-qc/0303045. LA, Moncrief.].

3) CMCSH gauge version of Einstein Eq w/p in $H^{2+\varepsilon}$ in 3+1 d.

   [LA, Klainerman, Rodnianski, under preparation.]

(4)

- hyperbolic gauge cond. expected to be OK locally, not globally. For global results, need elliptic gauge like CMC.
- For global uniqueness, need elliptic gauge to gain 1 extra derivative



loose one derivative in coord. transformation $\Rightarrow$ need $H^{3+\varepsilon}$ w. current results.

Longterm goal: H^2 – this requires bilinear estimates.

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Einstein Eqs in Constant Mean Curvature, Spatial Hyperbolic gauge

$\bar{M} = \mathbb{R} \times M$ 3+1 spacetime, metric $\bar{g}_{\alpha\beta}$

$\bar{R}_{\alpha\beta}, \bar{\nabla}$, assume M compact, hyperbolic type.

Fix $\hat{g}$: C^∞ negative curvature background metric on M.

$\bar{R}_{\alpha\beta} = 0 \iff$

$$(\mathcal{E}) \quad \begin{cases} \partial_t g_{ij} = -2N K_{ij} + \mathcal{L}_X g_{ij} \\ \partial_t K_{ij} = -D_i D_j N + N(R_{ij} + \operatorname{tr}K\, K_{ij} - 2K_{im}K^m{}_j) + \mathcal{L}_X K_{ij} \end{cases}$$

$$(\mathcal{C}) \quad \begin{cases} R - |K|^2 + (\operatorname{tr}K)^2 = 0 \\ D_i \operatorname{tr}K - D^j K_{ij} = 0 \end{cases}$$

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Let $V^k = g^{mn}(\Gamma_{mn}^k - \hat{\Gamma}_{mn}^k)$

$-V^k$ = tension field of $Id : (M,g) \to (M,\hat{g})$

CMCSH gauge cond.

$$(\mathcal{G}) \quad \begin{cases} \mathrm{tr} K = t + t_0 \\ V^k = 0 \end{cases}$$

Note: 1) May consider inhomogeneous gauge cond of the form

$$\begin{cases} \mathcal{L}_{(\partial_t - X)} \mathrm{tr} K = f^0 \\ \mathcal{L}_{(\partial_t - X)} V^k = f^k \end{cases}$$

f^α given spacetime fcns.

$f^\alpha = (1,0) \Leftrightarrow (\mathcal{G})$

2) For AF case may use

$$\begin{cases} \mathrm{tr} K = 0 \\ V^k = 0 \\ N \to 1, X \to 0 \end{cases}$$

6a

$$\hat{\nabla}_m Y_n = \partial_m Y_n - \hat{\Gamma}_{mn}^k Y_k$$

$$\hat{\Delta}_g h_{ij} = \frac{1}{\mu_g} \hat{\nabla}_m (g^{mn} \mu_g \hat{\nabla}_n h_{ij})$$

$$\mu_g = \sqrt{\det g}$$

$$R_{ij} = -\tfrac{1}{2} \hat{\Delta}_g g_{ij} + S_{ij}[g, \partial g] + \delta_{ij}$$

where $\delta_{ij} = \frac{1}{2}(\nabla_i V_j + \nabla_j V_i)$

S_{ij} quadratic in ∂g.

$g_{ij} \mapsto R_{ij} - \delta_{ij}$ is quasilinear elliptic

(6b)

time differentiate (G):

$$(\mathcal{D}) \begin{cases} -\Delta N + |K|^2 N = 1 \\ \hat{\Delta}_g X^k + g^{mn} \hat{R}^k{}_{mln} X^l + V^l \hat{\nabla}_l X^k = \\ = \left(-2NK^{mn} + 2\nabla^m X^n\right)\left(\Gamma^k_{mn} - \hat{\Gamma}^k_{mn}\right) \\ + 2\nabla^m N K_m{}^k - \nabla^k N K_m{}^m \end{cases}$$

here $\hat{R}^k{}_{mln}$ is Riemann tensor of $(M, \hat{g})$.

Modified Evolution problem for CMCSH Einstein Eqs.

$$(\mathcal{E}') \begin{cases} \partial_t g_{ij} = -2N K_{ij} + \mathcal{L}_X g_{ij} \\ \partial_t K_{ij} = -\nabla_i \nabla_j N + N\left(R_{ij} + \operatorname{tr}K K_{ij} - 2K_{im}K^m{}_j - \delta_{ij}\right) \\ \qquad + \mathcal{L}_X K_{ij} \end{cases}$$

Fact: $(\mathcal{E}') + (\mathcal{D})$ is elliptic-hyperbolic.

Propagation of Constraints and Gauge (8)

Let
$$\begin{cases} A = \operatorname{tr}K - t - t_0 \\ V^k = g^{mn}\left(\Gamma^k_{mn} - \hat{\Gamma}^k_{mn}\right) \\ F = R + (\operatorname{tr}K)^2 - |K|^2 - \nabla_i V^i \\ D_i = \nabla_i \operatorname{tr}K - 2\nabla^m K_{mi} \end{cases}$$

let $\partial_0 = \partial_t - X$.

$$\begin{cases} \partial_0 A \cong NF \\ \partial_0 F \simeq N\Delta A \\ \partial_0 V^i \simeq N D^i \\ \partial_0 D^i \simeq N P_0 V^i \end{cases} \quad + \text{ lower order.}$$

where $P_0 V^i = \hat{\Delta}_g V^i + g^{mn} \hat{R}^i{}_{mln} V^l$

let $\mathcal{E} = \frac{1}{2}\int |A|^2 + |DA|^2 + |F|^2$
$\qquad + \frac{1}{2}\int |V|^2 + |\hat{D}V|^2 + |D|^2$

then $|\partial_t \mathcal{E}| \le C\mathcal{E}$ so $\mathcal{E} \equiv 0$ if $\mathcal{E}(0) = 0$.

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Thm: [LA, Moncrief, AHP 4, 2003]

The CMCSH Cauchy problem (E') + (D) is locally w/p in $H^{n/2+1+\varepsilon}$, $\varepsilon > 0$

n=3

Thm [LA, Klainerman, Rodnianski]

locally w/p in $H^{2+\varepsilon}$

[under preparation. Pf uses Strichartz estimates:

$$\Box \phi = 0$$

$$\Rightarrow \|\partial\phi\|_{L^p_t L^\infty_x} \le C T^\delta \left(\|\partial\phi[0]\|_{H^{s-1}} \right)$$].

9a

Strichartz:

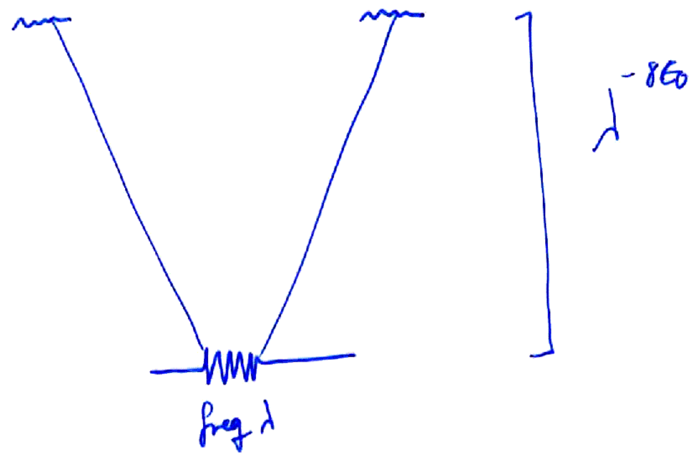
$\Box_g \phi = 0$

Expansion of null cones ⇒ decay of waves

⇒ $L^p_t L^\infty_x$ estimate for $\partial\phi$

via Littlewood–Paley.

Decay holds in rough spacetime satisfying (approximate) Einstein Eq, after frequency localization and rescaling.



Covariant wave equation for the metric. (10)

Let $u_{ij} = g_{ij}$, $v_{ij} = -2K_{ij}$

think of u, v as sections of bdle $Q \to M$, with covariant derivative D, defined by $D_k h_{ij} = \hat{\nabla}_k h_{ij}$.

Extend to $\mathbb{Q} \to \overline{M}$ with cov. der $\mathbb{D}_\alpha$, defined by

$$\begin{cases} \mathbb{D}_0 u_{ij} = \partial_t u_{ij} \\ \mathbb{D}_k u_{ij} = D_k u_{ij} \end{cases}$$

Let $L h = \bar{g}^{\alpha\beta} \mathbb{D}_\alpha \mathbb{D}_\beta h$.

(10a)

$(\mathcal{E}') \Leftrightarrow$

$$\begin{cases} \partial_t u_{ij} = N v_{ij} + \hat{\nabla}_X u_{ij} + \mathcal{F}_{1ij} \\ \partial_t v_{ij} = N \hat{\Delta}_g u_{ij} + \hat{\nabla}_X v_{ij} + \mathcal{F}_{2ij} \end{cases}$$

where

$$\mathcal{F}_{1ij} = u_{\ell j} \hat{\nabla}_i X^\ell + u_{i\ell} \hat{\nabla}_j X^\ell$$

$$\mathcal{F}_{2ij} = F_{ij} + v_{\ell j} \hat{\nabla}_i X^\ell + v_{i\ell} \hat{\nabla}_j X^\ell$$

$$F_{ij} = F_{ij}[u, \partial u, v, \partial N, \partial^2 N]$$

Then $(\mathcal{E}')$ can be written in the form

$$L u = G$$

$$G = -N^{-1}\mathcal{F}_2 - T^\alpha \mathbb{D}_\alpha (N^{-1}\mathcal{F}_1) + N^{-1} D^m N D_m u + \operatorname{tr} K \, T^\alpha \mathbb{D}_\alpha u$$

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- To Do:
  - AF version. $\hat{g}$ may be chosen as Euclidean or Schwarzschild for example.
  - inhomogenous gauge version (prescribed mean curvature)
  - conformally split version.
  - boundary value problem.

Expect: elliptic-hyperbolic formulation of E.E. → w/p completely gauge fixed initial-boundary value problem, for ex. using Dirichlet boundary cond. for (N, X)