Asymptotic simplicity:

completing the picture

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(For ref's cf. H.F. gr-qc/0304003)

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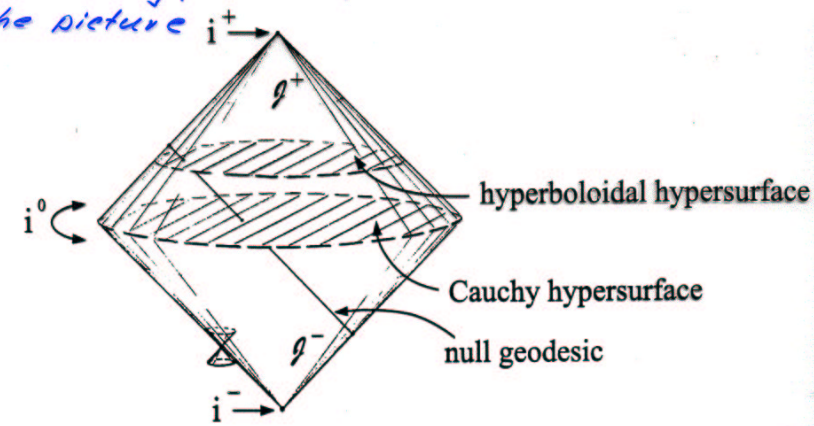
- Completing, not complete!
- the picture

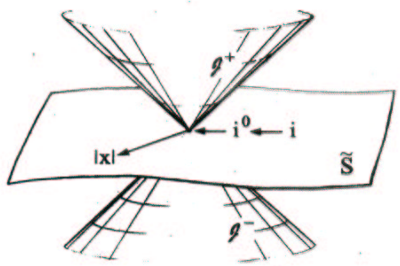


- "... non-trivial existence questionable ..." ?
- sharp analysis provides:
  - help with interpretation, concepts, ...
  - insight into equations, gauge cond's, ...
  - support for numerical analysis
- P. Chruściel, E. Delay CQG 19 (2002) L71 : $\exists$ non-triv. as. simple solutions to $R_{\mu\nu} = 0$
- 'How many' exist? $\exists$ characterisation in terms of Cauchy data?
- Need complete information on an arbitrary nbhd of space-like infty i^0!
- Also for generalizations!

- Conformal data singular at space-like infty i
- remove singularity (time symmetric case) by

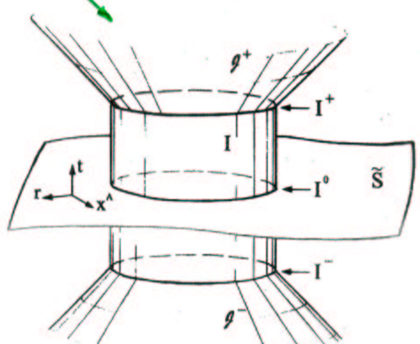
BLOW-UPs $i \longrightarrow I^0 \simeq S^2, \quad i^0 \longrightarrow I \simeq]-1,1[\times S^2$

Seen from inside:



not imposed!

consequence of conf. geometry and field equ's



- in suitable gauge of Θ, x^μ, e_k, f:
  data smooth on $S = \tilde{S} \cup I^0$
  equ.'s symmetric hyperbolic near S

- Unexpected. Specific for Einstein equ.'s. Most concise realization of the problem

- Reduced equations for $u = (v, \varphi)$ where
  $v = (e^\mu{}_k, \hat{\Gamma}_i{}^j{}_k, \hat{R}_{jk}), \ \varphi = \Theta^{-1} C^i{}_{jkl}$:

$$\begin{cases} \partial_t v = F(v, \varphi) \\ A^t \partial_t \varphi + A^r \partial_r \varphi + A^B \partial_{x^B} \varphi = C \varphi \end{cases}$$

- sol's exist, are smooth up to I near S
- cylinder I very special: $A^r|_I = 0$
  $\rightsquigarrow \exists$ inner equ.'s on I for $u^p \equiv \partial_r{}^p u|_I$
  $\rightsquigarrow$ calc. of u^p reduced to expressions

$$y(t) = X(t) X^{-1}(0) y_0 + X(t) \int_0^t X^{-1}(t') b(t') dt'$$

$y \in \mathbb{C}^2$, $b = b[u^0, \dots, u^{p-1}]$, $X \in M_{2\times 2}$ 'known'

- 'critical sets' $I^\pm$ very special: $\det(A^t)|_{I^\pm} = 0$
  $\rightsquigarrow$ break-down of hyperbolicity at $I^\pm$

in general: $u^p \sim (1-t)^k \log^j (1-t)$ as $t \to 1$

(u^p has polyhomogeneous expansion at $I^\pm$)

There are 2 sources of log-singularities:

- Some $X(t)$ behave as $(1-t)^k \log(1-t)$ as $t \to 1$
  in that case $b \equiv 0$, regularity depends on y_0

$\rightsquigarrow$ no log-terms occur from this if the data $(\bar{S}, h)$ satisfy the 'regularity condition'

(*) $R(D_{a_p} \dots D_{a_1} B_{ab})(i) = 0, \; 0 \le p \le p_*$ with $p_* = \infty$

(non-trivial after linearization)

- in the other cases $X(t) = Y(t)\,S(t)$ smooth with

$\det(Y(t)) \neq 0$, $|t| \le 1$, $S(t) = \begin{pmatrix} (1-t)^{p-2} & 0 \\ 0 & (1-t)^{p-2} \end{pmatrix}$ $p \ge 2$

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the regularity depends on $b = b[u^0, \dots, u^{p-1}]$

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the integrals can, in principle, be calculated recursively

(these log-terms disappear under linearization)

Implications for null infinity

- Linearize (with $\mathscr{J} = \{t = \pm 1, r > 0\}$) at Minkowski space

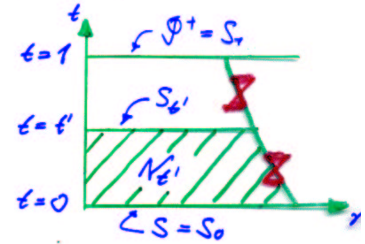
- Bianchi equ's then imply

$$
\begin{aligned}
(1+t)\,\partial_t \varphi_0 - r\,\partial_r \varphi_0 \; + X_+ \varphi_1 \qquad\qquad\quad &= -2\varphi_0 \\
2\,\partial_t \varphi_1 \; + X_+ \varphi_2 + X_- \varphi_0 &= -2\varphi_1 \\
2\,\partial_t \varphi_2 \; + X_+ \varphi_3 + X_- \varphi_1 &= 0 \\
2\,\partial_t \varphi_3 \; + X_+ \varphi_4 + X_- \varphi_2 &= 2\varphi_3 \\
(1-t)\,\partial_t \varphi_4 + r\,\partial_r \varphi \qquad\quad + X_- \varphi_3 &= 2\varphi_4
\end{aligned}
$$

- Standard energy estimates

$$\|\varphi\|_{L^2(S_{t'})} \le \frac{C}{(1-t')^{5/2}} \|\varphi\|_{L^2(S_0)}$$

useless as $t' \to 1$.



- Useful energy estimates

$\|\partial_r^p \varphi\|_{H_4^m(N_{t'})} \le C \|\partial_r^p \varphi\|_{H_4^m(S_0)}$, $p \ge m+2$, $0 \le t' < 1$, $C = $ const.

$\rightsquigarrow$ $\partial_r^p \varphi \in C^j(N_1)$, $p \ge j+6$. Two integrations give:

$$\varphi = \sum_{p'=0}^{p-1} \frac{1}{p'!} \varphi^{p'} r^{p'} + \hat{\varphi}, \; p \ge j+6 \text{ with } \hat{\varphi} \in C^j(N_1).$$

Solutions C^∞ if lin. reg. cond. hold with $p_* = \infty$ otherwise $\exists$ log-terms on $\mathscr{J}$ (expected but not obvious!)

- The log-terms are determined uniquely by the data $(\tilde{S}, \tilde{h})$, they are not generated by the setting

- the nature of the degeneracy at $I^{\pm}$ pinpoints the origin of the non-smoothness on $\mathscr{I}$

- the transition: linear → non-linear will not improve the situation, we can expect log-terms in general

- main tasks

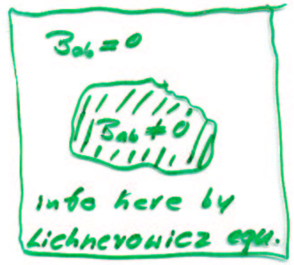
i) analyse the behaviour of u^p on $\bar{I} = I \cup I^- \cup I^+$

derive nec. & suff. regularity conditions

ii) analyse the consequences of the behaviour of the u^p near $I^{\pm}$ for the smoothness of $\mathscr{I}^{\pm}$ (generalize the linear estimates...)

- The reg. cond. (∗) is nec. + suff. for $p = 0, 1, 2, 3$.

- J. Valiente Kroon gr-qc/0211024 uses algebraic computer program studies data which are conformally flat near i:



⇝ u^4 analytic on $\bar{I}$

⇝ $u^5 = u^5_* + c^5 (1-t)^{k_5} \log(1-t)$ as $t \to 1$

u^5_* analytic on $\bar{I}$, $c^5 \sim$ NP const.

require $c^5 = 0$

⇝ $u^6 = u^6_* + c^6 (1-t)^{k_6} \log(1-t)$ as $t \to 1$

require $c^6 = 0$ & axial symmetry

⇝ $u^7 = u^7_* + c^7 (1-t)^{k_7} \log(1-t)$ as $t \to 1$

etc. etc.

- global conditions? things have changed!

- sequence of conditions suggests: regularity at all orders ⇒ asympt. Schwarzschild whence Schwarzschild near i (only static sol. in class!)

- $m = 0$: the reg. cond. (*) are nec. + suff. for the smoothness of $\mathcal{J}$
- static data with $m \neq 0$ satisfy the reg cond. (*) at all orders.
- conformal flatness, $m \neq 0$, and regularity seem to imply asymptotic staticity.
- $m \neq 0$: reg. cond. (*) is <u>not</u> sufficient for the smoothness of $\mathcal{J}$

- what is the role of the mass here?
- does asymptotic staticity play a particular role here?
- is the setting with the cylinder at space-like infinity smooth at $I^{\pm}$ in the static case?

static sol's : $\tilde{g} = v^2 dt^2 + \tilde{h}_{ab} dx^a dx^b$; $v(x^c)$, $h_{ab}(x^c)$

static equ's : $R_{ab}[\tilde{h}] = \frac{1}{v} \tilde{D}_a \tilde{D}_b v$, $\Delta_{\tilde{h}} v = 0$

For static solutions with $m \neq 0$ everything is real analytic (in a suitable sense) in a nbhd of $\bar{I} = I \cup I^- \cup I^+$.

Requires static field equ's in detail!

with: $v_0 = \partial_t$, $v_1 = r\,\partial_r$, $v_B = \partial_{x^B}$
$\alpha^0 = dt$, $\alpha^1 = \frac{1}{r} dr$, $\alpha^B = dx^B$ we have

$g = g_{ik}\,\alpha^i \alpha^k$, $\nabla_{v_i} v_k = \gamma_i{}^j{}_k\, v_j$ with

- g_{ik} analytic & Lorentzian near $\bar{I}$
- $\gamma_i{}^j{}_k$ analytic near $\bar{I}$
- all tensor fields derived from g are analytic in the frame above

$\rightsquigarrow$ u^p, $p = 0, \ldots,$ are analytic on $\bar{I}$

$\rightsquigarrow$ for data which are asympt. static up to order k the u^p are analytic on $\bar{I}$ for $p \leq p_*(k)$

(cf. P. Chrusciel, E. Delay gr-qc/0301073 for such data)

(m = 0)-reg. cond. $\longrightarrow$ ($m \neq 0$)-reg. cond. $\longrightarrow$ asympt. static

$\uparrow$

where precisely ?