High-resolution finite volume methods for hyperbolic PDEs on manifolds

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Supported in part by NSF, DOE

Overview

- High-resolution methods for first-order hyperbolic systems
- Shock waves in nonlinear problems
- Heterogeneous media with discontinuous properties
- Godunov-type methods based on Riemann solvers
- Second-order correction terms with limiters to minimize dissipation and dispersion

Software

CLAWPACK (Conservation LAWs Package):

`http://www.amath.washington.edu/~claw`

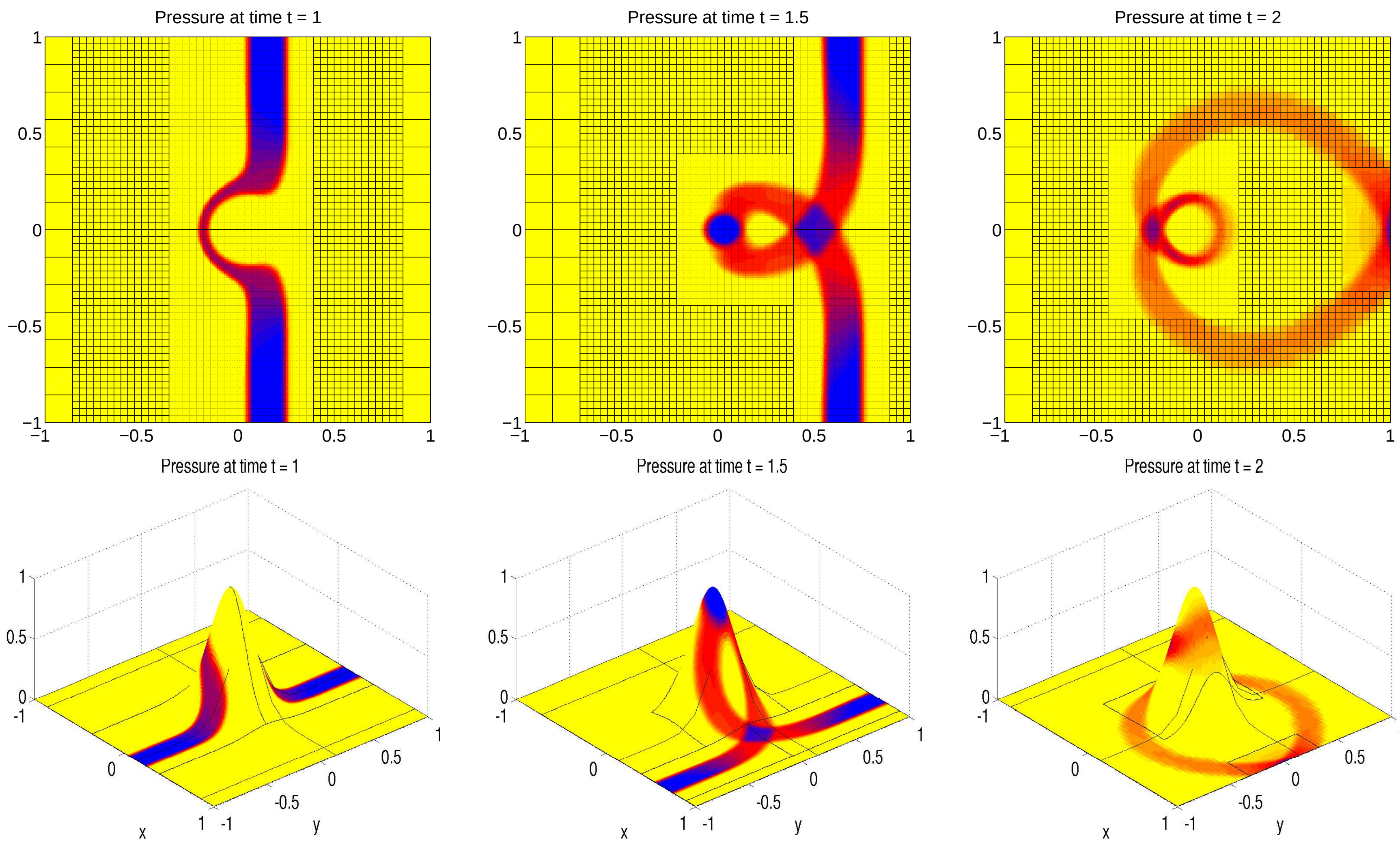
Includes a preliminary version of CLAWMAN for manifolds.

Developed by

- Derek Bale (relativistic flow)
- James Rossmanith (geophysical flow on the sphere)

Also includes AMRCLAW for adaptive mesh refinement on rectangular patches (with Marsha Berger).

AMR on a manifold



Outline

- Brief review of Godunov-type methods
- wave propagation approach
- f-wave approach for discontinous fluxes and source terms
- curvilinear grids in flat space
- manifolds:
  - parallel transport data to cell edges
  - Express in local orthonormal frame
  - Solve locally-flat Riemann problem
  - Metric terms and geometric source term naturally incorporated

References:

`http://www.amath.washington.edu/~rjl/publications` and `/students`

J. Pons, Font, Ibanez, Marti, Miralles, *General relativistic hydrodynamics with special relativistic Riemann solvers*, Astron. Astrophys. 339 (1998), 638-642

Finite-difference Methods

- Pointwise values $Q_i^n \approx q(x_i, t_n)$
- Approximate derivatives by finite differences
- Assumes smoothness

Finite-volume Methods

- Approximate cell averages: $Q_i^n \approx \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} q(x, t_n)\, dx$
- Integral form of conservation law,

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)\,dx \quad = \quad f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t))$$

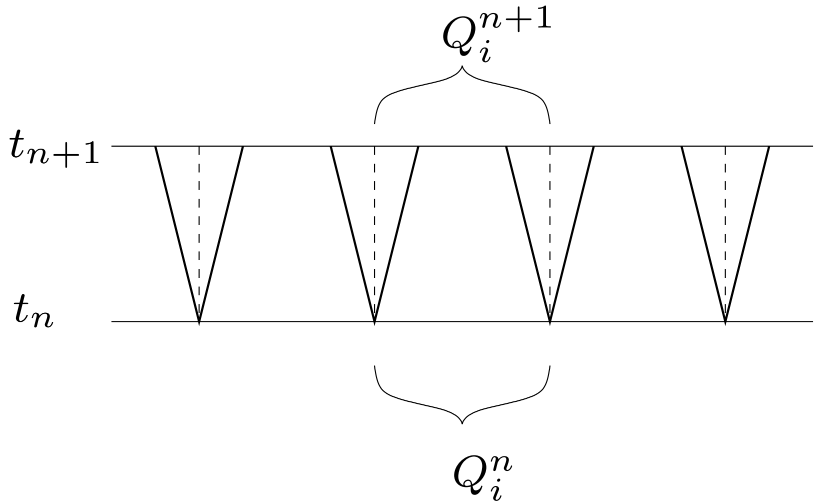
leads to conservation law $q_t + f_x = 0$ but also directly to numerical method.

Godunov's method

Q_i^n defines a piecewise constant function

$$\tilde{q}^n(x,t_n) = Q_i^n \quad \text{for } x_{i-1/2} < x < x_{i+1/2}$$

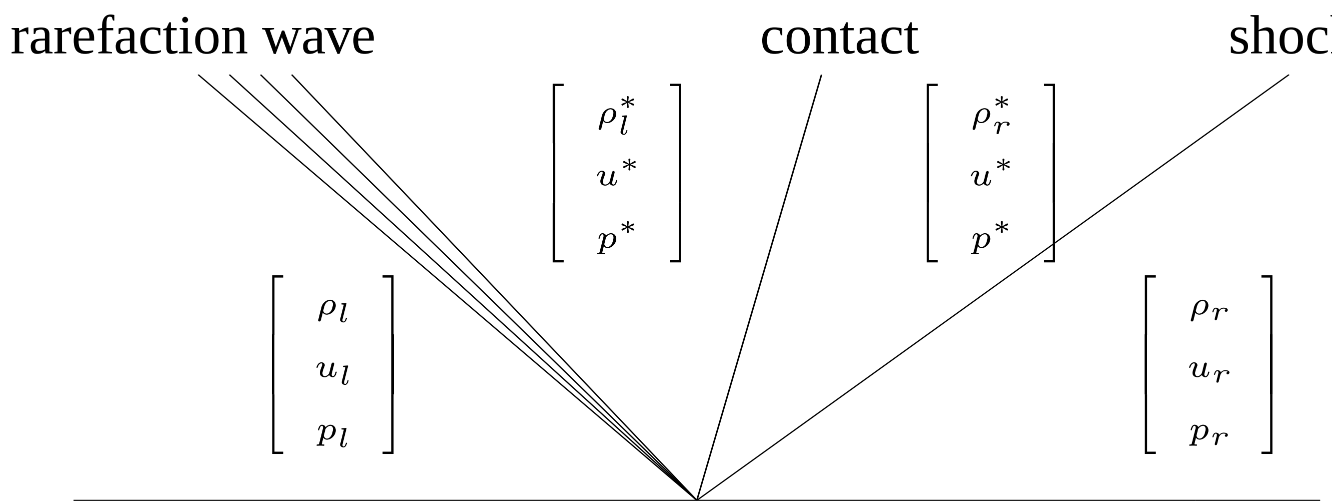
Discontinuities at cell interfaces $\implies$ Riemann problems.



$$\tilde{q}^n(x_{i-1/2},t) \equiv q^{\Downarrow}(Q_{i-1},Q_i) \quad \text{for } t > t_n.$$

$$F_{i-1/2}^n = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(q^{\Downarrow}(Q_{i-1}^n,Q_i^n))\,dt = f(q^{\Downarrow}(Q_{i-1}^n,Q_i^n)).$$

Riemann solution for the Euler equations



The Roe solver uses the solution to a linear system

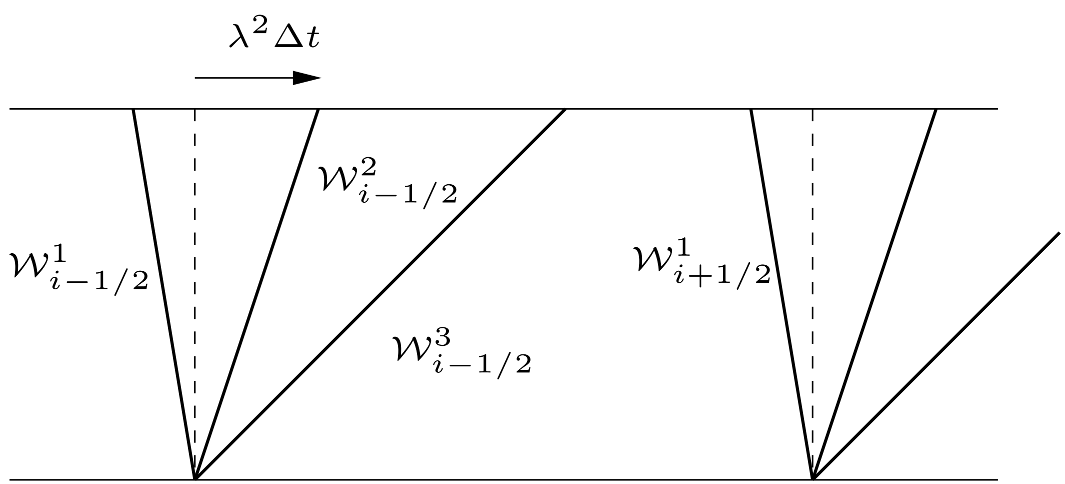
$$q_t + \hat{A}_{i-1/2} q_x = 0.$$

All waves are simply discontinuities.

Typically a fine approximation if jumps are approximately correct.

Wave-propagation viewpoint

For linear system $q_t + Aq_x = 0$, the Riemann solution consists of waves $\mathcal{W}^p$ propagating at constant speed λ^p.



$$Q_i - Q_{i-1} = \sum_{p=1}^{m} \alpha^p_{i-1/2} r^p \equiv \sum_{p=1}^{m} \mathcal{W}^p_{i-1/2}.$$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^2 \mathcal{W}^2_{i-1/2} + \lambda^3 \mathcal{W}^3_{i-1/2} + \lambda^1 \mathcal{W}^1_{i+1/2}\right].$$

High-resolution method for systems

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}\right] - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$$

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p=1}^{M_w}(s_{i-1/2}^p)^-\mathcal{W}_{i-1/2}^p,$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p=1}^{M_w}(s_{i-1/2}^p)^+\mathcal{W}_{i-1/2}^p,$$

Correction flux:

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{M_w}|s_{i-1/2}^p|\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right)\tilde{\mathcal{W}}_{i-1/2}^p$$

where $\tilde{\mathcal{W}}_{i-1/2}^p$ is a limited version of $\mathcal{W}_{i-1/2}^p$.

CLAWPACK

`http://www.amath.washington.edu/~claw/`

- Fortran codes with Matlab graphics routines.
- Many examples and applications to run or modify.
- 1d, 2d, and 3d.

User supplies:

- Riemann solver, splitting data into waves and fluctuations
  (Need not be in conservation form)
- Boundary condition routine to extend data to ghost cells
  Standard `bc1.f` routine includes many standard BC's
- Initial conditions — `qinit.f`

Some applications

- Gas dynamics, Euler equations
- Waves in heterogeneous / random media
- Acoustics, ultrasound, seismology
- Elasticity, plasticity, soil liquifaction
- Flow in porous media, groundwater contamination, oil recovery
- Geophysical flow on the sphere
- Shallow water equations, bottom topography, tsunami propagation
- Chemotaxis and pattern formation
- Traffic flow
- Crystal growth
- Multi-fluid, multi-phase flows, bubbly flow
- Streamfunction–vorticity form of incompressible flow
- Projection methods for incompressible flow
- Combustion, detonation waves
- Astrophysics: binary stars, planetary nebulae, jets
- Magnetohydrodynamics, shallow water MHD
- Relativistic flow, black hole accretion
- Numerical relativity — Einstein equations, gravity waves, cosmology

Spatially-varying flux functions

In one dimension:

$$q_t + f(q, x)_x = 0$$

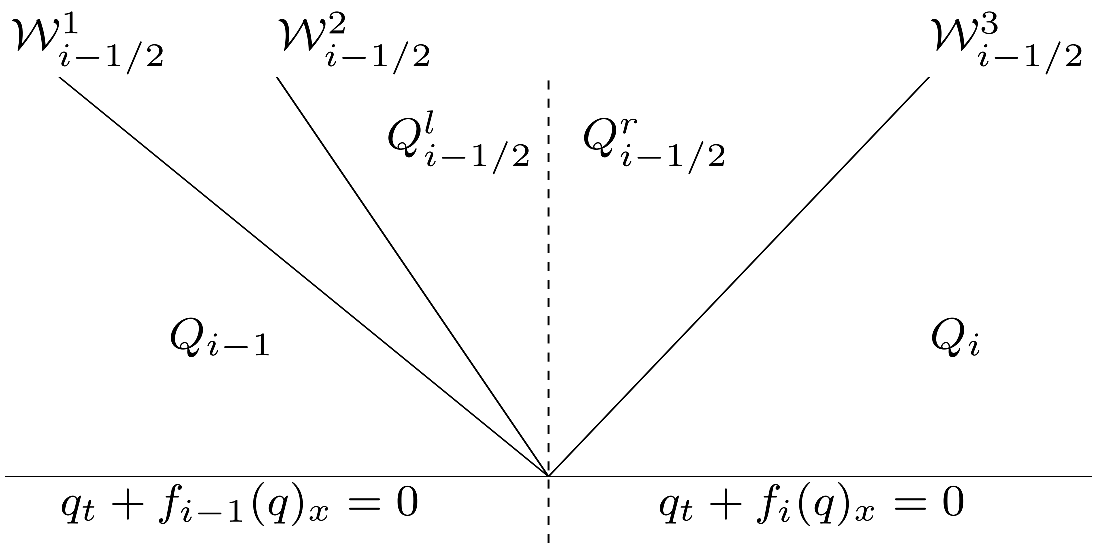
Examples:

- Nonlinear elasticity in heterogeneous materials
- Traffic flow on roads with varying conditions
- Flow through heterogeneous porous media
- Solving conservation laws on curved manifolds

Riemann problem for spatially-varying flux

$$q_t + f(q,x)_x = 0$$

Cell-centered discretization: Flux $f_i(q)$ defined in ith cell.



Flux-based wave decomposition:

$$f_i(Q_i) - f_{i-1}(Q_{i-1}) = \sum_{p=1}^{m} \beta^p_{i-1/2} r^p_{i-1/2} \equiv \sum_{p=1}^{m} \mathcal{Z}^p_{i-1/2}$$

Wave-propagation algorithm using f-waves

$$
\begin{aligned}
Q_i^{n+1} &= Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}] \\
&\quad - \frac{\Delta t}{\Delta x}[\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}]
\end{aligned}
$$

Standard version: $Q_i - Q_{i-1} = \sum_{p=1}^m \mathcal{W}_{i-1/2}^p$

$$
\begin{aligned}
\mathcal{A}^-\Delta Q_{i+1/2} &= \sum_{p=1}^m (s_{i+1/2}^p)^- \mathcal{W}_{i+1/2}^p \\
\mathcal{A}^+\Delta Q_{i-1/2} &= \sum_{p=1}^m (s_{i-1/2}^p)^+ \mathcal{W}_{i-1/2}^p \\
\tilde{F}_{i-1/2} &= \frac{1}{2}\sum_{p=1}^m |s_{i-1/2}^p| \left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right) \widetilde{\mathcal{W}}_{i-1/2}^p.
\end{aligned}
$$

Wave-propagation algorithm using f-waves

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}[\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}] - \frac{\Delta t}{\Delta x}[\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}]$$

Using f-waves: $f_i(Q_i) - f_{i-1}(Q_{i-1}) = \sum_{p=1}^m \mathcal{Z}_{i-1/2}^p$

$$\mathcal{A}^-\Delta Q_{i-1/2} = \sum_{p:s_{i-1/2}^p<0} \mathcal{Z}_{i-1/2}^p,$$

$$\mathcal{A}^+\Delta Q_{i-1/2} = \sum_{p:s_{i-1/2}^p>0} \mathcal{Z}_{i-1/2}^p,$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^m \text{sgn}(s_{i-1/2}^p)\left(1 - \frac{\Delta t}{\Delta x}|s_{i-1/2}^p|\right)\widetilde{\mathcal{Z}}_{i-1/2}^p$$

Source terms

$$q_t + f(q)_x = \psi(q)$$

Quasi-steady problems with near-cancellation:

$f(q)_x$ and ψ both large but $q_t \approx 0$.

Examples:

- Atmosphere or stellar dynamics with gravity balanced by hydrostatic pressure
- Shallow water equations in a lake over bottom topography

Source terms

$$q_t + f(q)_x = \psi(q)$$

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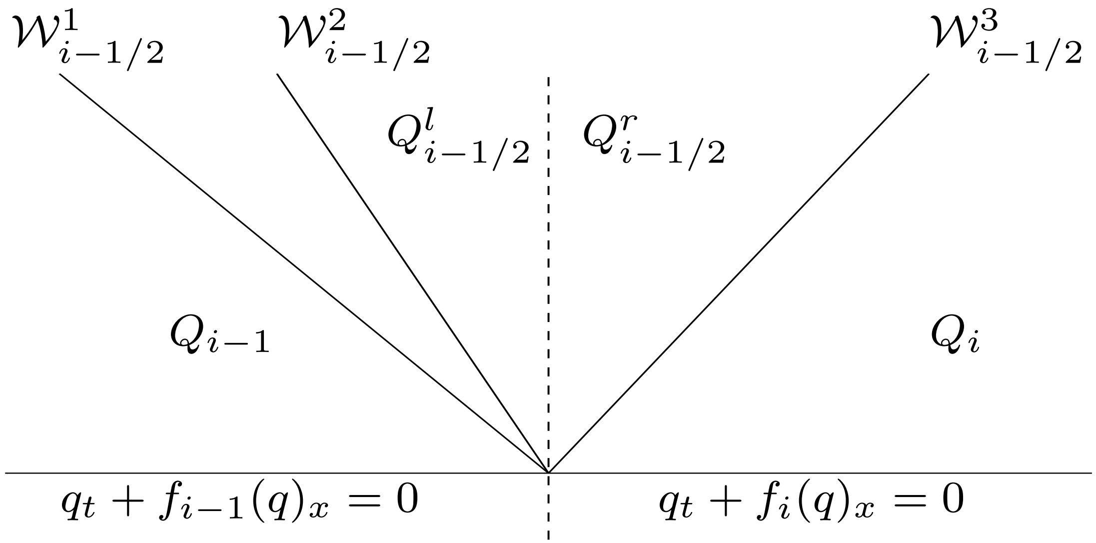
Fractional step method: Alternate between

1. $q_t + f(x)_x = 0$,
2. $q_t = \psi(q)$

Large motions induced in each step should cancel out, but won't numerically.

Riemann problem with a delta-function source term

$$q_t + f(q)_x = \Delta x \Psi_{i-1/2}\, \delta(x - x_{i-1/2})$$



Flux is no longer continuous:

$f(Q^r_{i-1/2}) - f(Q^l_{i-1/2}) = \Delta x \Psi_{i-1/2}$.

$$f_i(Q_i) - f_{i-1}(Q_{i-1}) - \Delta x \Psi_{i-1/2} = \sum_{p=1}^{m} \beta^p_{i-1/2} r^p_{i-1/2} \equiv \sum_{p=1}^{m} \mathcal{Z}^p_{i-1/2}$$

Multidimensional Hyperbolic Problems

Integral form of conservation law:

$$\frac{d}{dt}\iint_{\Omega} q(x,y,t)\,dx\,dy = -\int_{\partial\Omega} \vec{n}\cdot\vec{f}(q)\,ds.$$

If q is smooth then the divergence theorem gives

$$\frac{d}{dt}\iint_{\Omega} q(x,y,t)\,dx\,dy = -\iint_{\Omega} \vec{\nabla}\cdot\vec{f}(q)\,dx\,dy,$$

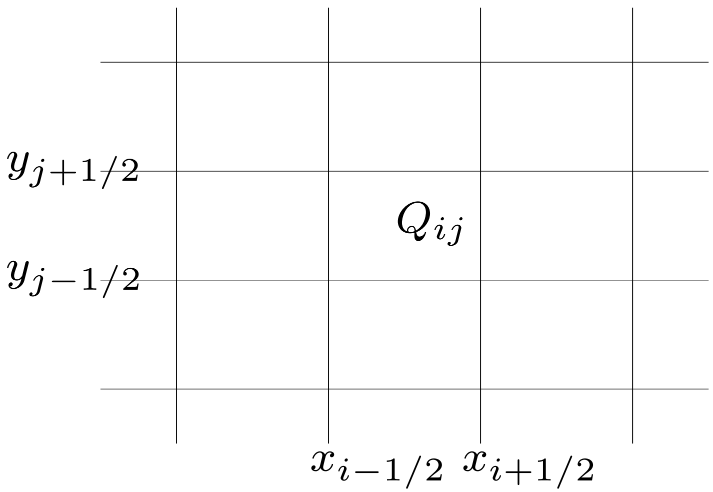
or

$$\iint_{\Omega} \left[q_t + \vec{\nabla}\cdot\vec{f}(q)\right]\,dx\,dy = 0.$$

True for all $\Omega \implies$

$$q_t + f(q)_x + g(q)_y = 0,$$

Finite volume method in 2D



$$Q_{ij}^n \approx \frac{1}{\Delta x \Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, y, t_n)\, dx$$

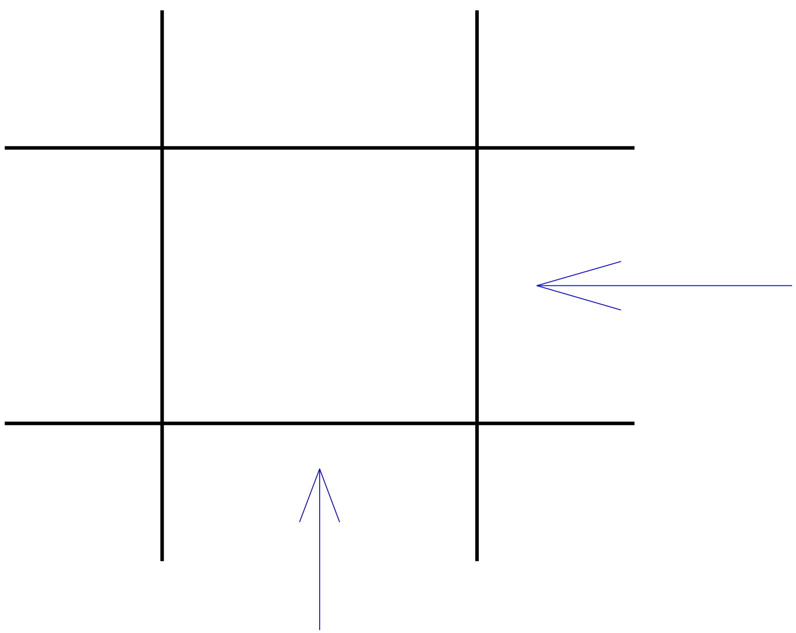
$$\frac{d}{dt} \iint_{\mathcal{C}_{ij}} q(x,y,t)\, dx\, dy = \int_{y_{j-1/2}}^{y_{j+1/2}} f(q(x_{i+1/2}, y, t)\, dy - \int_{y_{j-1/2}}^{y_{j+1/2}} f(q(x_{i-1/2}, y, t)\, dy$$
$$+ \int_{x_{i-1/2}}^{x_{i+1/2}} g(q(x, y_{j+1/2}, t)\, dx - \int_{x_{i-1/2}}^{x_{i+1/2}} g(q(x, y_{j-1/2}, t)\, dx.$$

Suggests the unsplit method

$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x}[F_{i+1/2,j}^n - F_{i-1/2,j}^n] - \frac{\Delta t}{\Delta y}[G_{i,j+1/2}^n - G_{i,j-1/2}^n],$$

Unsplit Godunov in 2D

At each cell edge, the flux is determined by solving a Riemann problem in the normal direction with data from the neighboring cells.



$$q_t + f(q)_x = 0$$

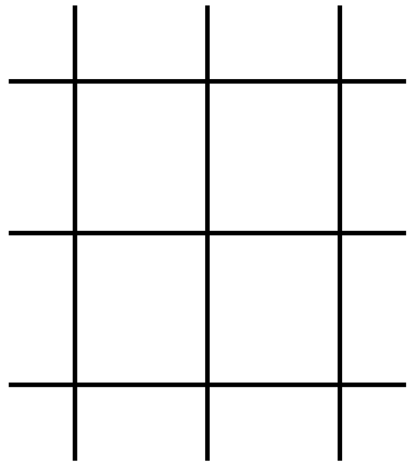
$$F^n_{i+1/2,j} = f(q^{\Downarrow}(Q^n_{ij}, Q^n_{i+1,j}))$$

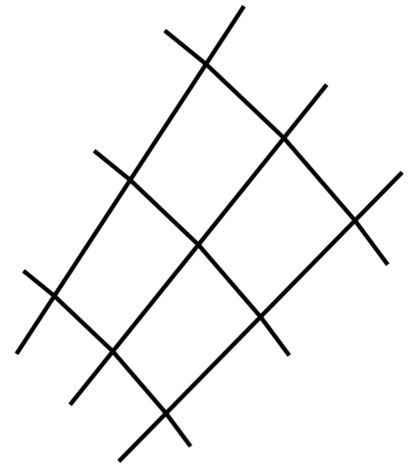
$$q_t + g(q)_y = 0$$

$$G^n_{i,j-1/2} = g(q^{\Downarrow}(Q^n_{i,j-1}, Q^n_{ij}))$$

Finite volume method on a curvilinear grid

(Flat space)





Two possible approaches:

1. Transform equations to computational space.
Discretize equations that include metric terms, source terms.

2. Update cell averages in physical space.
Solve 1d Riemann problems for physical equations in direction normal to cell edges to compute flux.

Example: Linear acoustics

Homogeneous medium with

density $\rho \equiv 1$, bulk modulus $K \equiv 1$, sound speed $c \equiv 1$,

$p =$ pressure, $u =$ velocity, $T = pI =$ stress tensor

$$\frac{\partial}{\partial t}\,p + \frac{1}{\sqrt{h}}\,\frac{\partial}{\partial x^k}\left(\sqrt{h}\,u^k\right) = 0$$

$$\frac{\partial}{\partial t}\,u^m + \frac{1}{\sqrt{h}}\,\frac{\partial}{\partial x^k}\left(\sqrt{h}\,T^{km}\right) = -\Gamma^m_{nk}T^{kn}\,.$$

One approach:

Discretize these equations directly in computational coordinates x^k.

$$\frac{\partial}{\partial t} p + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, u^k \right) = 0$$

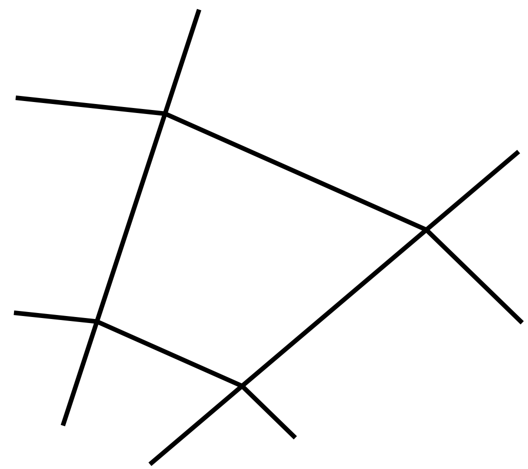
$$\frac{\partial}{\partial t} u^m + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, T^{km} \right) = -\Gamma^m_{nk} T^{kn} \, .$$

Note:

- Spatially varying flux functions
- Source term — conservative?

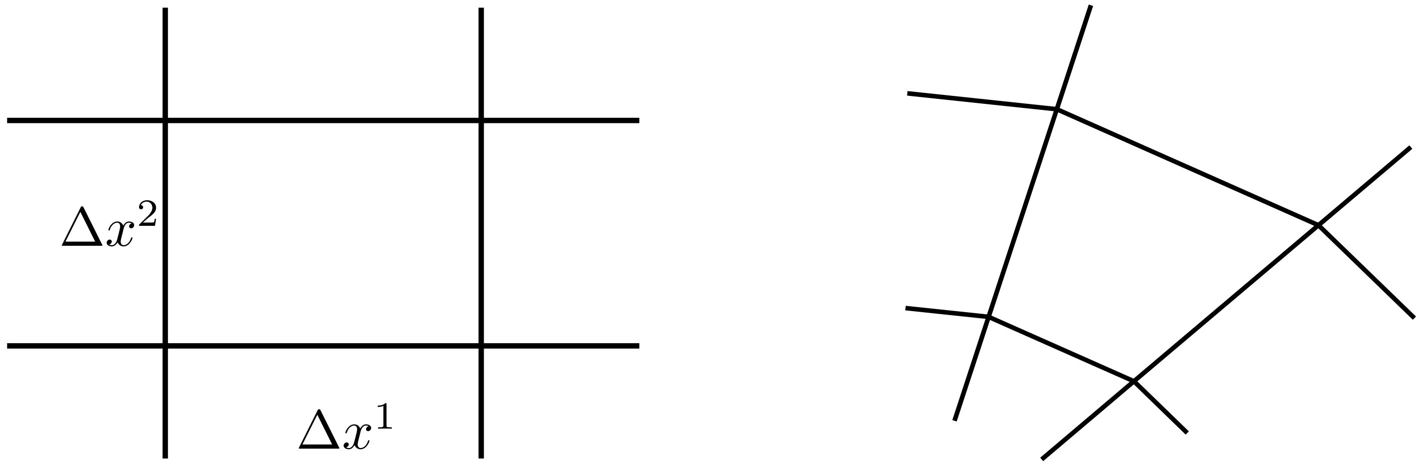
Better approach:

Update cell average of q over physical finite volume cell



- Store Cartesian velocity components u, v in each cell.
- At each cell edge, use data on each side to
  - compute normal velocities at edge,
  - solve 1d Riemann problem in normal direction,
  - scale resulting waves by length of side,
  - use to update cell average

Grid mapping:



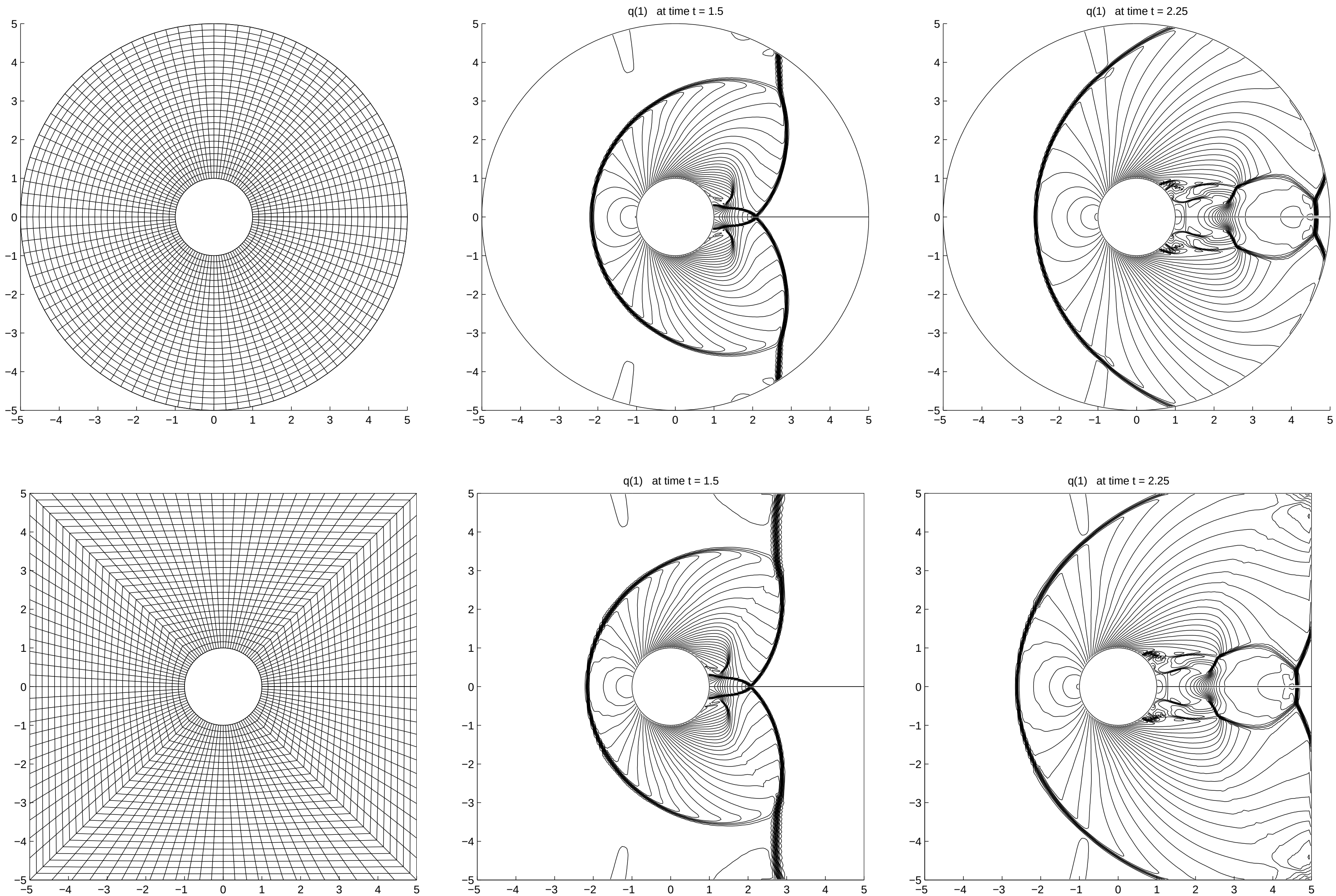
$$J = \begin{bmatrix} \partial x/\partial x^1 & \partial x/\partial x^2 \\ \partial y/\partial x^1 & \partial y/\partial x^2 \end{bmatrix}, \quad H = J^T J = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \quad h = \text{det}(H).$$

Lengths of sides in physical space $\approx h_{11}\Delta x^1,\ h_{22}\Delta x^2$,

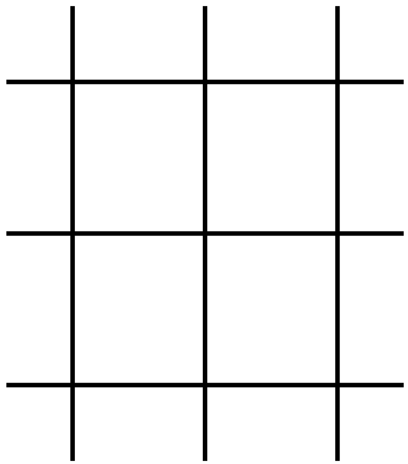
Area of cell $\approx \sqrt{h}\Delta x^1 \Delta x^2$.

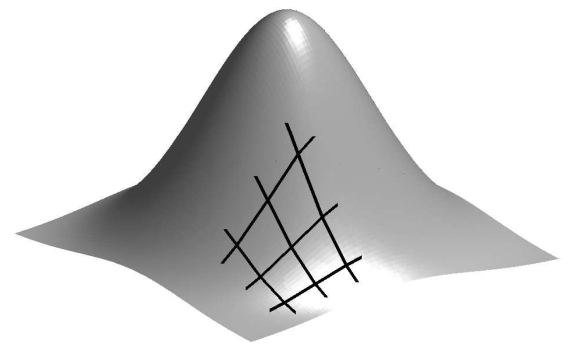
Flux differencing around boundary approximates covariant divergence of flux.

Shallow water flow into a cylinder



Acoustics on a manifold





$$\frac{\partial}{\partial t} p + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, u^k \right) = 0$$

$$\frac{\partial}{\partial t} u^m + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, T^{km} \right) = -\Gamma^m_{nk} T^{kn} \,.$$

Now we must represent velocities in "computational coordinates"

At each cell edge:

- Parallel transport cell-centered velocities to edge,
- Change coordinates to a local orthonormal frame at cell edge to obtain normal and tangential velocities,
- Solve 1d Riemann problem normal to cell edge (assuming locally flat)
- Scale resulting waves by length of side, transform back to cell-centered coordinates,
- Update cell averages.

CLAWMAN software

Currently only 2d.

Requires metric tensor H

- 2×2 matrix as function of x^1 and x^2,
- Used to compute scaling factors for edge lengths, cell areas,
- Used for orthonormalization at cell edges.

Christoffel symbols are needed for parallel transport

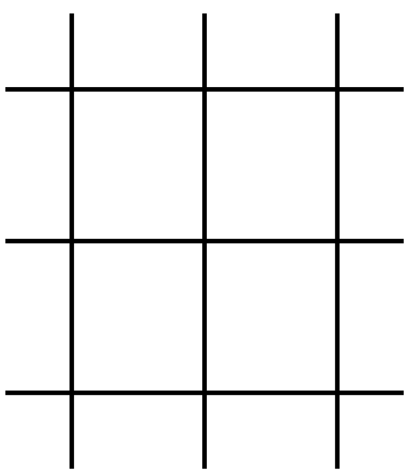
- Computed by finite differencing H.

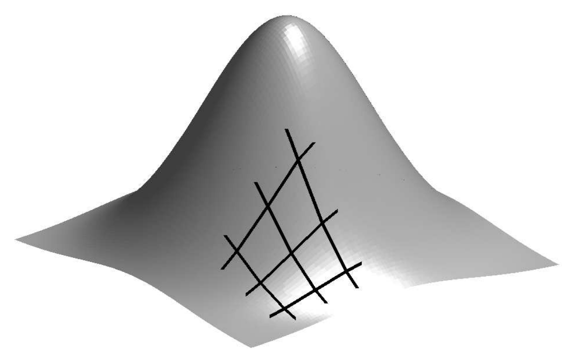
Parallel Transport

$$\frac{\partial}{\partial x^k} u^m + \Gamma^m_{nk} u^n = 0 \quad \text{or} \quad \frac{\partial}{\partial x^k} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} \Gamma^1_{1k} & \Gamma^1_{2k} \\ \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = 0.$$

Approximate using Taylor series:

$$u^m \left(x^k \pm \frac{1}{2} \Delta x^k \right) \approx u^m(x^k) \mp \frac{1}{2} \Delta x^k \Gamma^m_{nk} u^n(x^k).$$





Parallel Transport for acoustics

For acoustics with $q = (p, u^1, u^2)^T$, solve Riemann problem with

$$q^{\ell}_{i-1/2,j} = \left(I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix} \right) q_{i-1,j}$$

and

$$q^{r}_{i-1/2,j} = \left(I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix} \right) q_{i,j}$$

f-wave formulation

Split jump in fluxes ΔF into waves.

$$\Delta F^1 = \left(I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix}_{ij} \right) f^1_{ij} - \left(I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Gamma^1_{1k} & \Gamma^1_{2k} \\ 0 & \Gamma^2_{1k} & \Gamma^2_{2k} \end{bmatrix}_{i-1,j} \right) f^1_{i-1,j}$$

$$= f^1_{ij} - f^1_{i-1,j} - \Delta x^1 \psi^1_{i-1/2,j}$$

where

$$\psi^1_{i-1/2,j} = -\frac{1}{2} \left(\begin{bmatrix} 0 \\ \Gamma^1_{11} T^{11} + \Gamma^1_{12} T^{21} \\ \Gamma^2_{11} T^{11} + \Gamma^2_{12} T^{21} \end{bmatrix}_{ij} + \begin{bmatrix} 0 \\ \Gamma^1_{11} T^{11} + \Gamma^1_{12} T^{21} \\ \Gamma^2_{11} T^{11} + \Gamma^2_{12} T^{21} \end{bmatrix}_{i-1,j} \right)$$

This is $n = 1$ portion of the source term

$$\psi = -\Gamma^m_{nk} T^{kn}.$$

Acoustics on a manifold

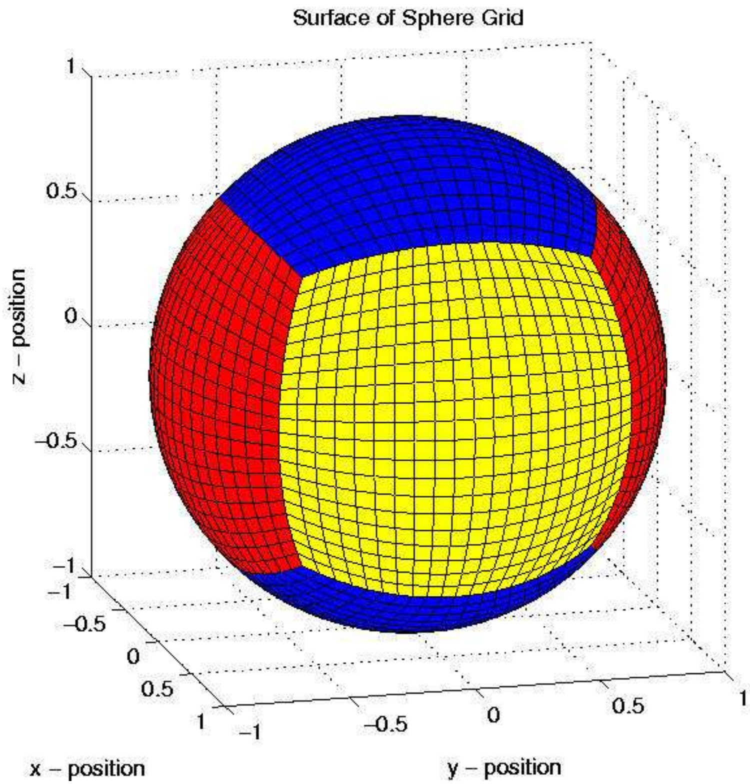
$$\frac{\partial}{\partial t} p + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, u^k \right) = 0$$

$$\frac{\partial}{\partial t} u^m + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^k} \left(\sqrt{h}\, T^{km} \right) = -\Gamma^m_{nk} T^{kn}\,.$$

With finite volume formulation,

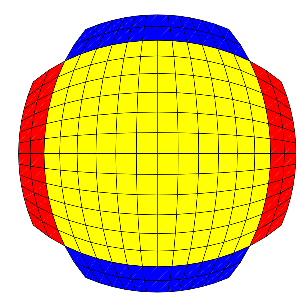
- Source term is automatically incorporated by parallel transport of fluxes,
- Covariant divergence is handled by use of edge lengths and cell volume,
- Parallel transport and orthonormalization allows use of standard flat-space Riemann solver at interface.

Cubed sphere grid



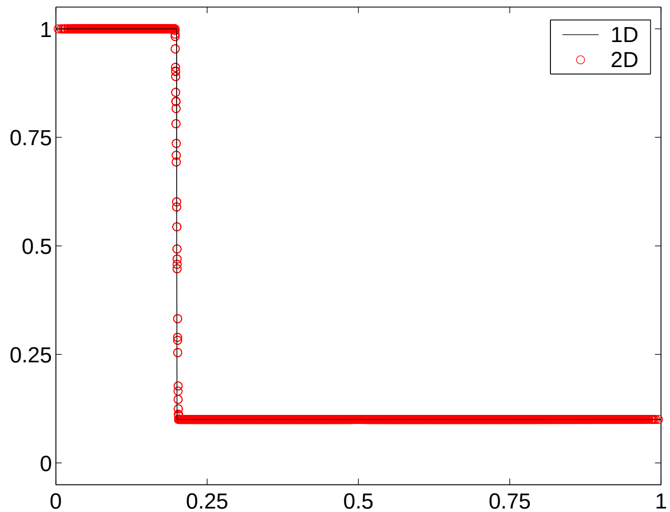
Six logically rectangular grids are patched together.

Data is transferred between patches using ghost cells



Shallow water on the sphere

$t = 0$

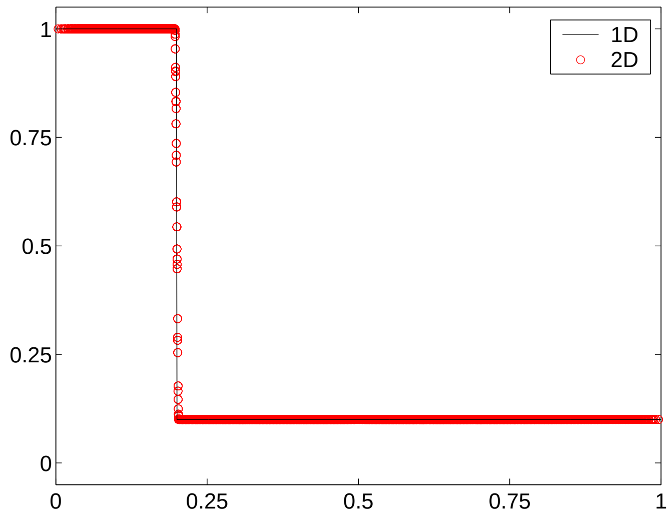


Scatterplot of depth vs. lattitude (from north to south pole)

Solid line is “exact” solution from 1D equation

Shallow water on the sphere

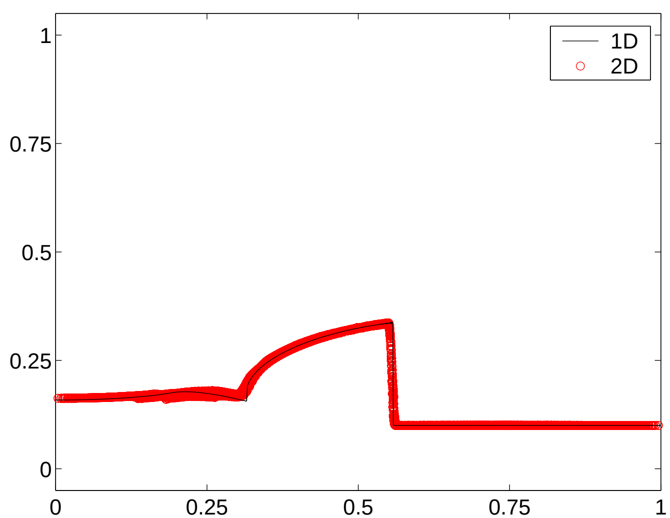
$t = 0$



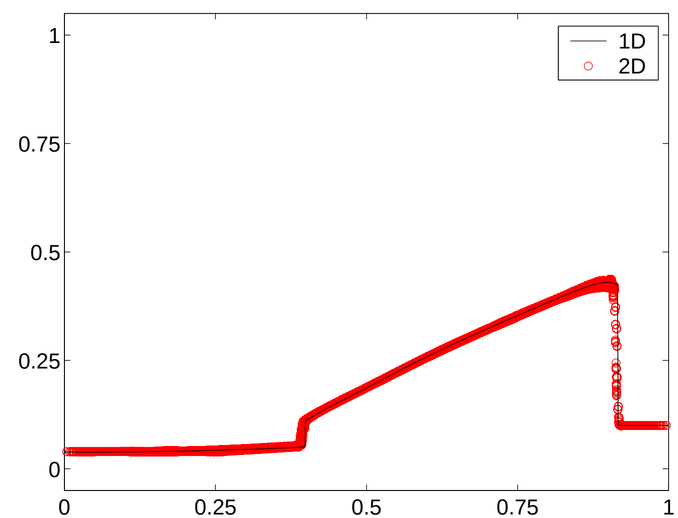
Scatterplot of depth vs. lattitude (from north to south pole)

Solid line is “exact” solution from 1D equation

$t = 1.25$



$t = 2.5$



Outline

- Brief review of Godunov-type methods
- wave propagation approach
- f-wave approach for discontinous fluxes and source terms
- curvilinear grids in flat space
- manifolds:
    - parallel transport data to cell edges
    - Express in local orthonormal frame
    - Solve locally-flat Riemann problem
    - Metric terms and geometric source term naturally incorporated

References:

`http://www.amath.washington.edu/~rjl/publications` and `/students`

J. Pons, Font, Ibanez, Marti, Miralles, *General relativistic hydrodynamics with special relativistic Riemann solvers*, Astron. Astrophys. 339 (1998), 638-642