# Dynamical Gauge Conditions for Numerical Relativity

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- The ‘gauge fields’ $N$ and $N^i$ may be chosen freely in general relativity, but poor choices can result in problems:
  - Coordinate shocks or singularities,
  - Ill posed evolution equations,
  - Inefficient and expensive evolution algorithms,
  - Complicated representations of simple geometries.
- We attempt to avoid these problems by choosing gauge fields:
  - which represent stationary geometries in a time independent way;
  - which satisfy linearly degenerate evolution equations to avoid shocks;
  - and which—together with the dynamical fields—are determined by a unified system of symmetric hyperbolic evolution equations.

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## Representing Stationary Spacetimes in a Time Independent Way:

- In a stationary spacetime we may always choose coordinates so that $\partial_t$ is the timelike Killing field. In these coordinates it follows that:

$$
\begin{aligned}
0 &= -\partial_t K = \nabla^i\nabla_i N - N K_{ij}K^{ij} - N^i\nabla_i K, \\
0 &= \partial_t(g^{jk}\Gamma^i_{jk}) = \partial_j\big[\Sigma^{ij} - \tfrac{1}{2}g^{ij}\Sigma\big] + \tfrac{1}{2}\big[\Sigma^{ij}\partial_j \log g + \Sigma\partial_j g^{ij}\big].
\end{aligned}
$$

  where $\Sigma^{ij} \equiv -\partial_t g^{ij} = \nabla^i N^j + \nabla^j N^i - 2NK^{ij}$.

- These conditions are elliptic equations for $N$ and $N^i$ (when $g_{ij}$ and $K_{ij}$ are considered fixed) which tend to avoid coordinate singularities and tend to represent time independent geometries in a time independent way. The main disadvantages at present are the lack of generic well-posedness theorems for coupled elliptic-hyperbolic systems, and the greater computational cost of solving the elliptic equations.
- Instead we attempt to convert these conditions to hyperbolic by adding appropriate time derivative terms, *e.g.*

$$
\begin{aligned}
\partial_t^2 N + \kappa N\partial_t N &= -\mu N^2\partial_t K, \\
\partial_t^2 N^i + \kappa N\partial_t N^i &= \mu N^2\partial_t(g^{jk}\Gamma^i_{jk}).
\end{aligned}
$$

## Choose Linearly Degenerate Gauge Evolution to Avoid Shock Formation:

- Consider a hyperbolic system of equations:

$$\partial_t u^\alpha + A^{k\alpha}{}_\beta(u)\partial_k u^\beta \quad = \quad F^\alpha(u).$$

- The characteristic speeds $v_I$ and characteristic fields $e^\alpha_I$ relative to the direction $n_k$ are defined as:

$$v_I e^\alpha_I = n_k A^{k\alpha}{}_\beta e^\beta_I.$$

- A system is called linearly degenerate if every characteristic speed is constant in the direction of its characteristic field:

$$0 = e^\alpha_I \frac{\partial v_I}{\partial u^\alpha}.$$

- The folklore is that smooth initial data do not form shocks when evolved with linearly degenerate evolution equations.

- The characteristic matrices $A^{k\alpha}{}_{\beta}$ for the Einstein evolution equations depend only on the metric variables: $N$, $N^i$, and $g_{ij}$. Thus the characteristic speeds depend only on these fields, and for many (typical) representations of the Einstein evolution equations they have the form:

$$v_I = N v_I^0 - n_k N^k,$$

where the $v_I^0$ are constants that depend on the particular representation.

- The only Einstein equation that involves explicit derivatives of the metric is
$$\partial_t g_{ij} - N^k \partial_k g_{ij} = 2 g_{k(i}\partial_{j)} N^k - 2 N K_{ij}.$$

- Thus the fixed densitized lapse and shift representations of the Einstein equations are linearly degenerate: the only characteristic fields having components in the $g_{ij}$ direction have characteristic speeds $-n_k N^k$.

- We can ensure that the addition of dynamical lapse and shift to the system does not spoil linear degeneracy by demanding that the principal parts of their equations have the forms

$$\partial_t N \simeq 0, \qquad\qquad \partial_t N^i \simeq 0.$$

## Constructing a Unified Symmetric Hyperbolic System of Evolution Equations:

- Start with the second order model gauge evolution equations:

$$\partial_t^2 N + \kappa N \partial_t N = -\mu N^2 \partial_t K,$$
$$\partial_t^2 N^i + \kappa N \partial_t N^i = \mu N^2 \partial_t (g^{jk}\Gamma^i_{jk}).$$

- Simplify the system by using first integral forms of these equations:

$$\begin{aligned} 0 &= \partial_t(N - N_0) - N^j \partial_j (N - N_0) + \kappa_L N(N - N_0) + \mu_L N^2 (K - K_0), \\ 0 &= \partial_t(N^i - N^i_0) - N^j \partial_j (N^i - N^i_0) + \kappa_S N(N^i - N^i_0) \\ &\quad -\mu_S N^2 (g^{jk}\Gamma^i_{jk} - g_0^{jk}\Gamma^i_{0jk}). \end{aligned}$$

- These gauge evolution equations are hyperbolic (by construction) when the other dynamical fields are considered fixed. But in reality we evolve the gauge fields at the same time as the other dynamical fields.
Is the full unified system of evolution equations really hyperbolic?

- Let $u^\alpha$ denote the complete collection of fields to be evolved:

$$u^\alpha = \{g_{ij}, N, N^i, K_{ij}, D_{kij} = \partial_k g_{ij}, T_i = \partial_i N, M_k{}^i = \partial_k N^i\}.$$

- The Einstein evolution equations combined with the dynamical gauge evolution equations described above can be written as a first order quasi-linear system in terms of these fields:

$$\partial_t u^\alpha + A^{k\alpha}{}_\beta(u)\partial_k u^\beta = F^\alpha(u).$$

- First order systems of this type are hyperbolic if there exists a positive definite metric $S_{\alpha\beta}$ on the space of fields, which symmetrizes the spatial derivative terms in the evolution equation:

$$S_{\alpha\gamma}A^{k\gamma}{}_\beta \equiv A^k_{\alpha\beta} = A^k_{\beta\alpha}.$$

- The problem is to construct a suitable $S_{\alpha\beta}$. Start by writing down the most general $S_{\alpha\beta}$ that depends only on $g_{ij}$:

$$dS^2 = S_{\alpha\beta}\, du^\alpha du^\beta = B_1 dK^2 + B_2\, g^{ik}g^{jl}dK_{ij}dK_{kl} + \cdots$$

- The bad news is that no choice of the 19 parameters that define $S_{\alpha\beta}$ can make the "standard" representation of the evolution equations symmetric, and hence hyperbolic.
- The evolution equations can be modified—leaving the equations as a first-order system but without changing any of the physical solutions—by adding multiples of the various constraints:

$$\begin{aligned}
\mathcal{C} &= R^{(3)} - K_{ij}K^{ij} + K^2, \\
\mathcal{C}^i &= \nabla_j K^j{}_i - \nabla_i K, \\
\mathcal{C}_{klij} &= \partial_{[k} D_{l]ij}, \\
\mathcal{C}_{ij} &= \partial_{[i} T_{j]}, \\
\mathcal{C}_{nk}{}^i &= \partial_{[n} M_{k]}{}^i.
\end{aligned}$$

  Proceeding in a systematic way, we add 15 different multiples of these constraints to the evolution equations.
- This more complicated system of equations with these 15 additional free parameters contains a 14 parameter family of systems with positive definite symmetrizers $S_{\alpha\beta}$.

In summary, we have constructed symmetric-hyperbolic representations of the Einstein equations coupled to fully dynamical gauge evolution equations. These systems have the following nice properties:

- The gauge evolution equations incorporated into these systems reduce in the time independent case to the necessary conditions on the lapse and shift in a stationary spacetime.
- These systems are linearly degenerate, a necessary condition to prevent the formation of gauge shocks.
- These systems have 14 parameters that can be adjusted to make all of the characteristic speeds less than or equal to the speed of light.
- These systems can also be extended in a straightforward way to include transformed independent dynamical fields. These extended systems have a total of 35 freely adjustable parameters that can be used to optimize the stability of numerical spacetime evolutions.