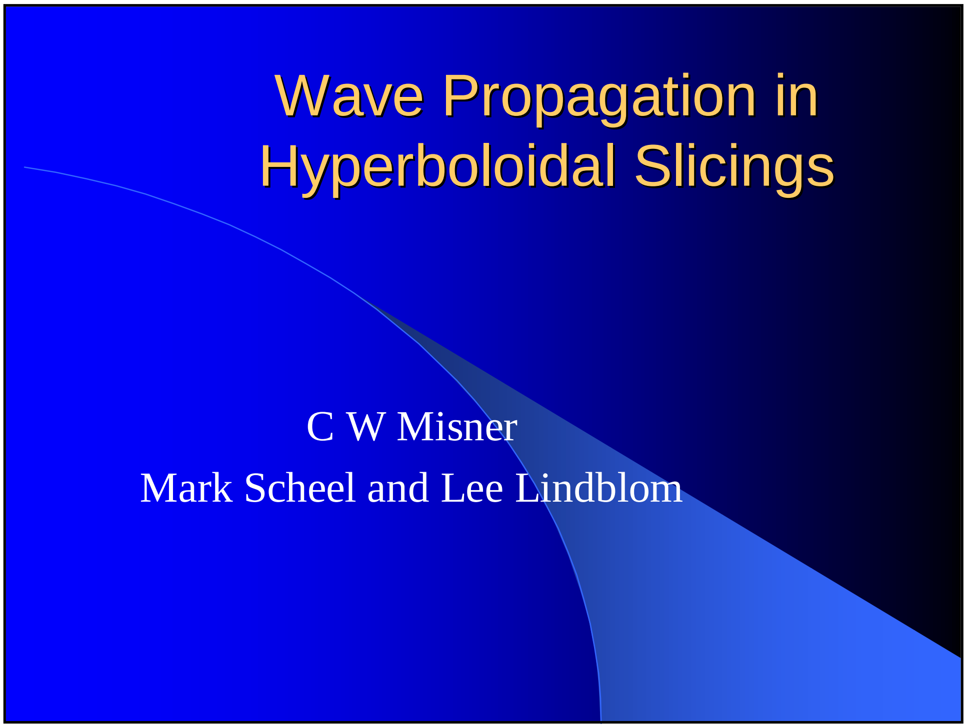
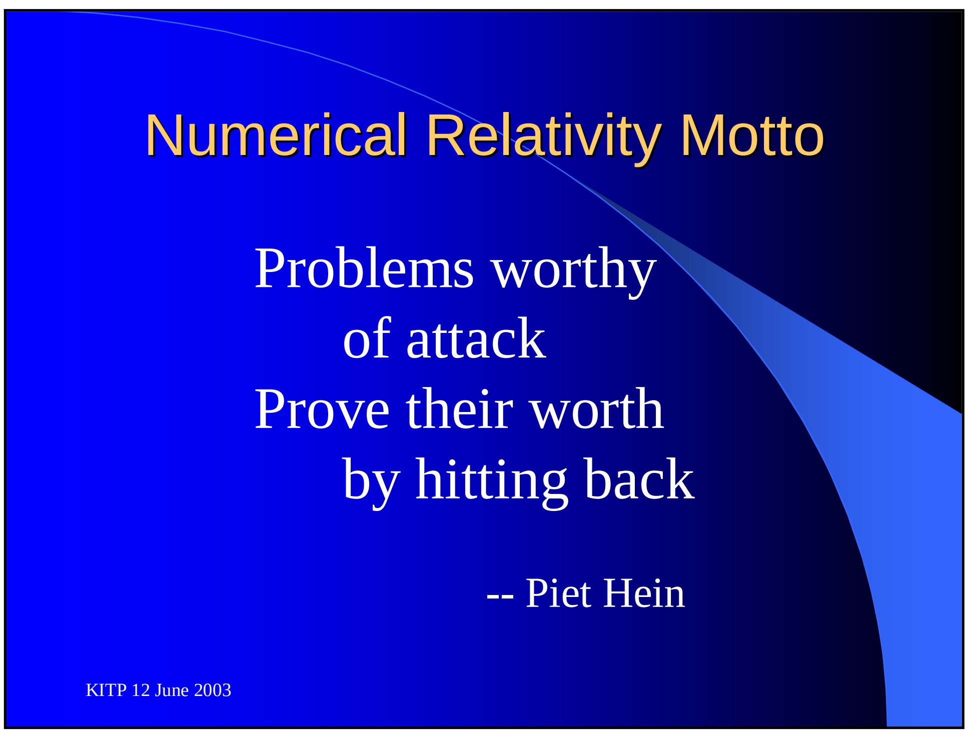
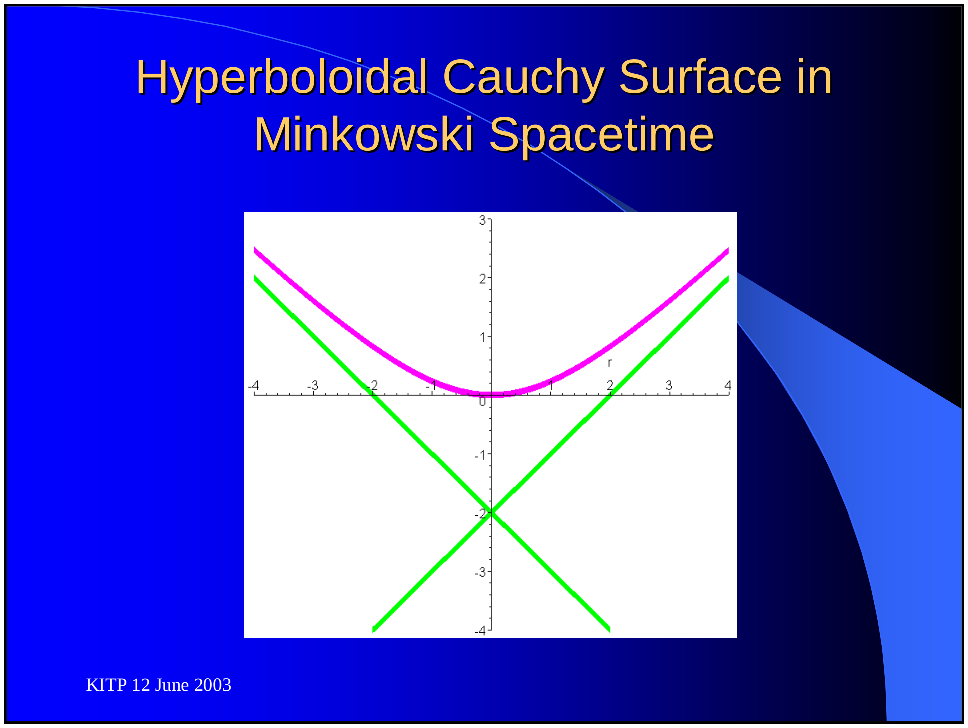

Wave Propagation in
Hyperboloidal Slicings
C W Misner
Mark Scheel and Lee Lindblom


Numerical Relativity Motto
Problems worthy
of attack
Prove their worth
by hitting back
-- Piet Hein
KITP 12 June 2003


Hyperboloidal Cauchy Surface in Minkowski Spacetime
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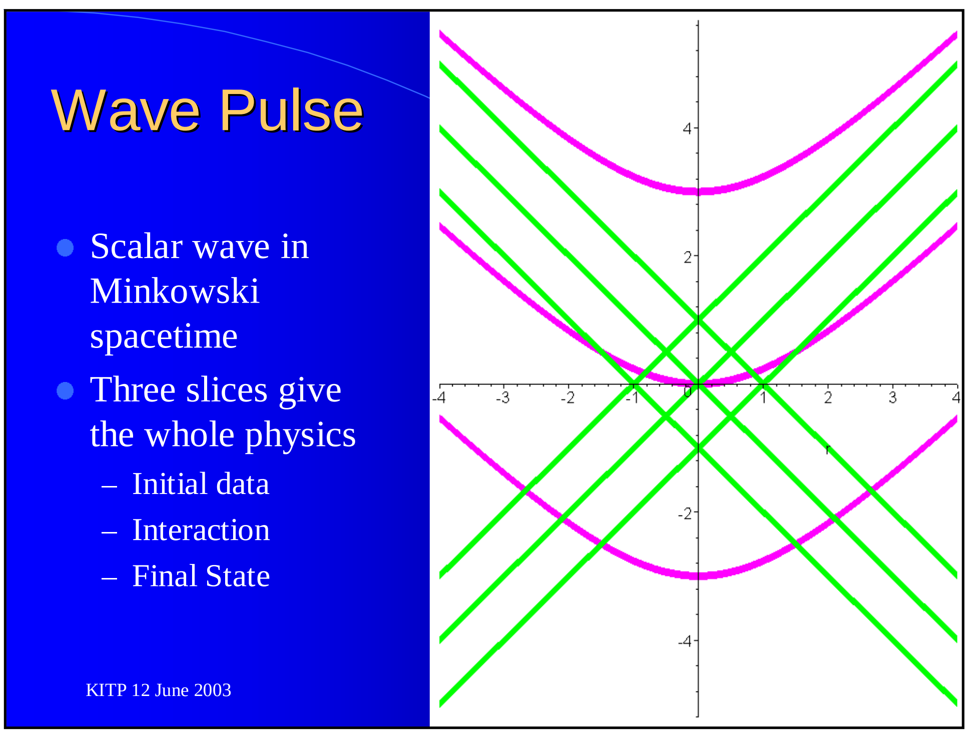
Wave Pulse
Scalar wave in Minkowski spacetime
Three slices give the whole physics
– Initial data
– Interaction
– Final State
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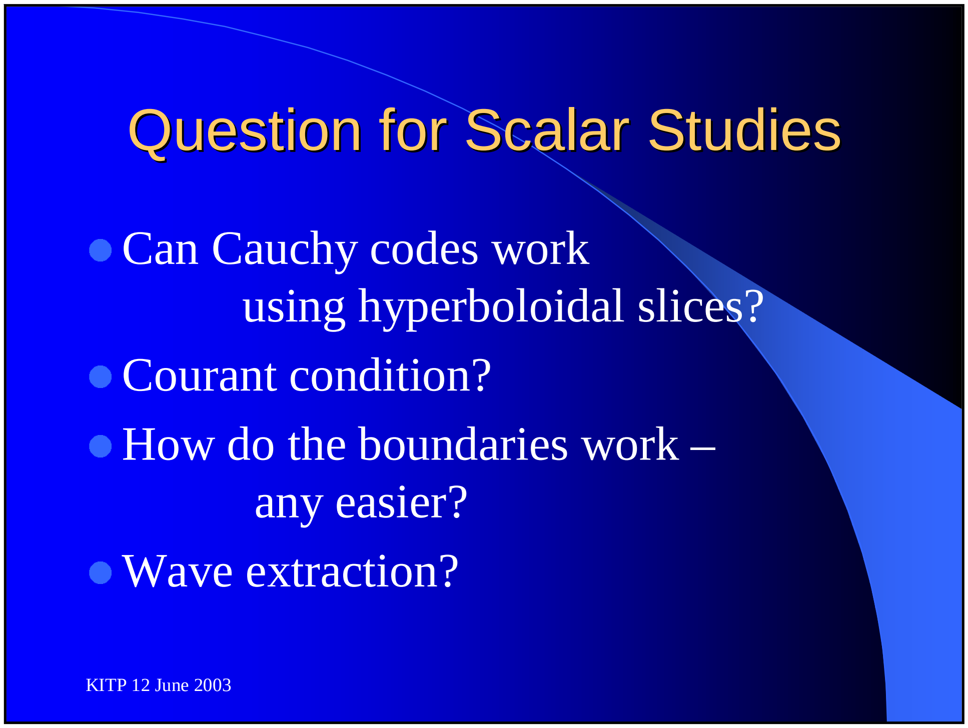
Question for Scalar Studies
Can Cauchy codes work using hyperboloidal slices?
Courant condition?
How do the boundaries work – any easier?
Wave extraction?
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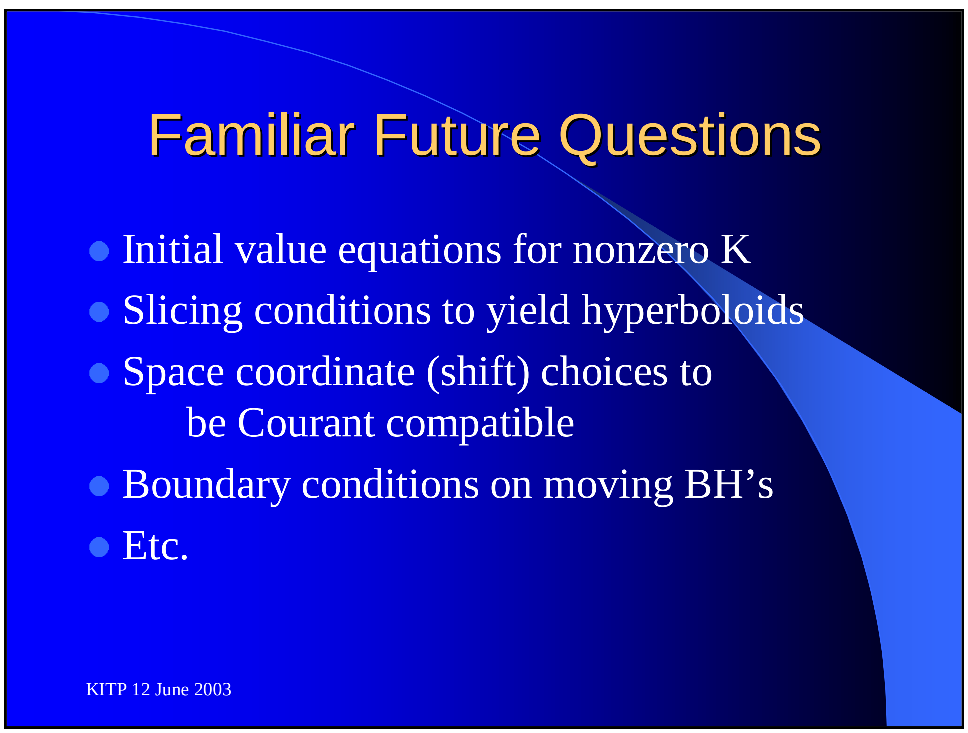
Familiar Future Questions
Initial value equations for nonzero K
Slicing conditions to yield hyperboloids
Space coordinate (shift) choices to be Courant compatible
Boundary conditions on moving BH's
Etc.
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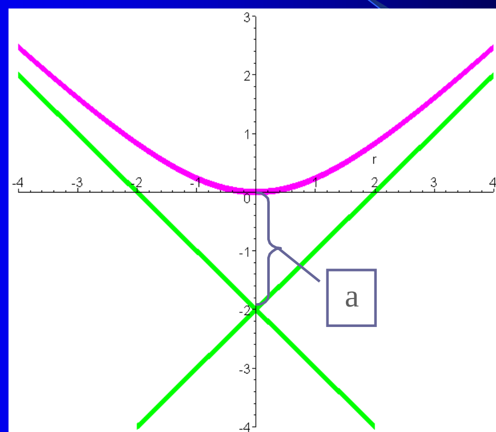
First Example: Minkowski Spacetime



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Change the Time Coordinate

$$T = u - a + \sqrt{a^2 + R^2}$$

a = ‘*Scri*-delay’ -- events at $(R,u)=(0,0)$ influence *Scri+* no earlier than $u=a$

$$ds^2 = -\left(1 + \frac{R^2}{a^2}\right) du^2 + \left(\frac{a\,dR}{\sqrt{a^2 + R^2}} - \frac{R\,du}{a}\right)^2 + R^2\, d\Omega^2$$

Check the coordinate velocity of light: dR/du when ds=0.

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What About the Courant Condition?

$$c\_out = \left(\frac{R}{a} + \sqrt{1 + \frac{R^2}{a^2}} \right) \sqrt{1 + \frac{R^2}{a^2}}$$

$$c\_in = \left(\frac{R}{a} - \sqrt{1 + \frac{R^2}{a^2}} \right) \sqrt{1 + \frac{R^2}{a^2}}$$

Trouble from c_out when r gets large: $\Delta u < \frac{\Delta R}{c}$

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Redshifted Outgoing Waves

$$\Phi(R, T) = \frac{A(T-R) - A(T+R)}{R}$$

$$T - R = u - a + \frac{a^2}{2R} - \frac{a^4}{8R^3} \quad \cdots$$

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Analytic Mesh Refinement

$$\frac{R}{a} = \frac{r}{1 - \frac{r^2}{4}}$$

Dimensionless radial coordinate compactifies the slice: Scri+ at r=2.

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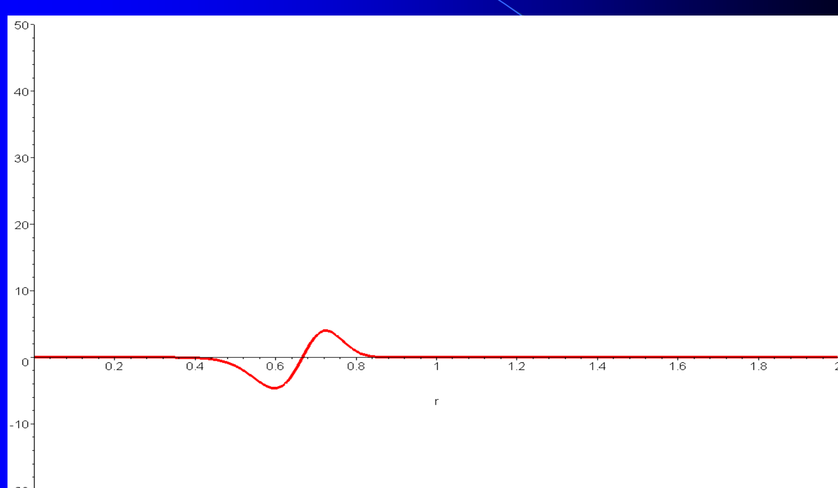
Analytic Solution



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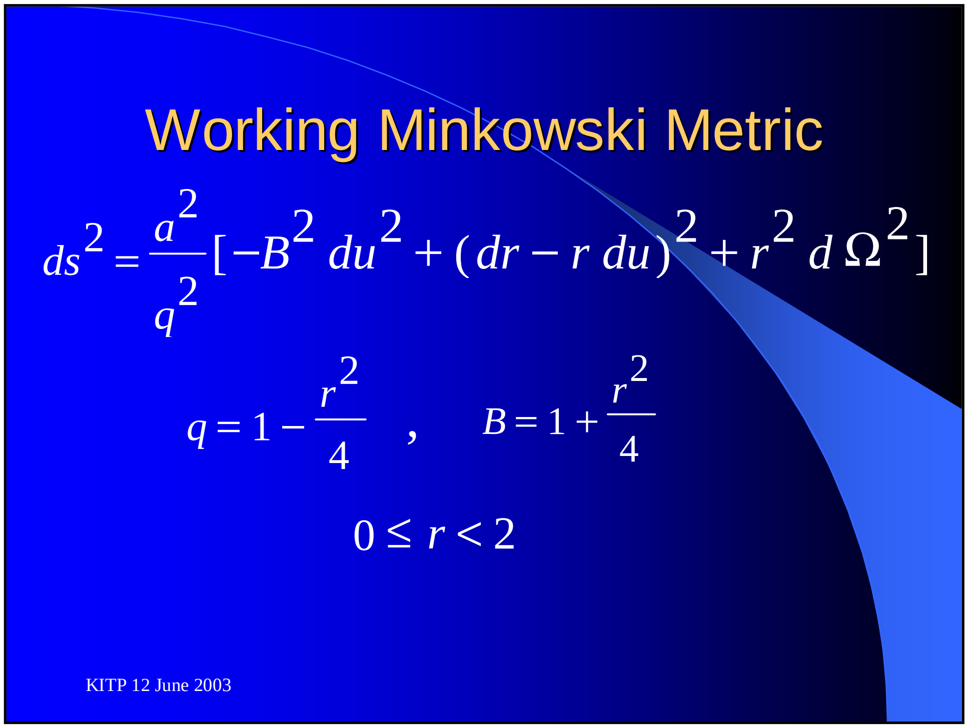
Working Minkowski Metric
$$ds^2 = \frac{a^2}{q^2}[-B^2\, du^2 + (dr - r\, du)^2 + r^2\, d\Omega^2]$$
$$q = 1 - \frac{r^2}{4} \quad , \quad B = 1 + \frac{r^2}{4}$$
$$0 \le r < 2$$
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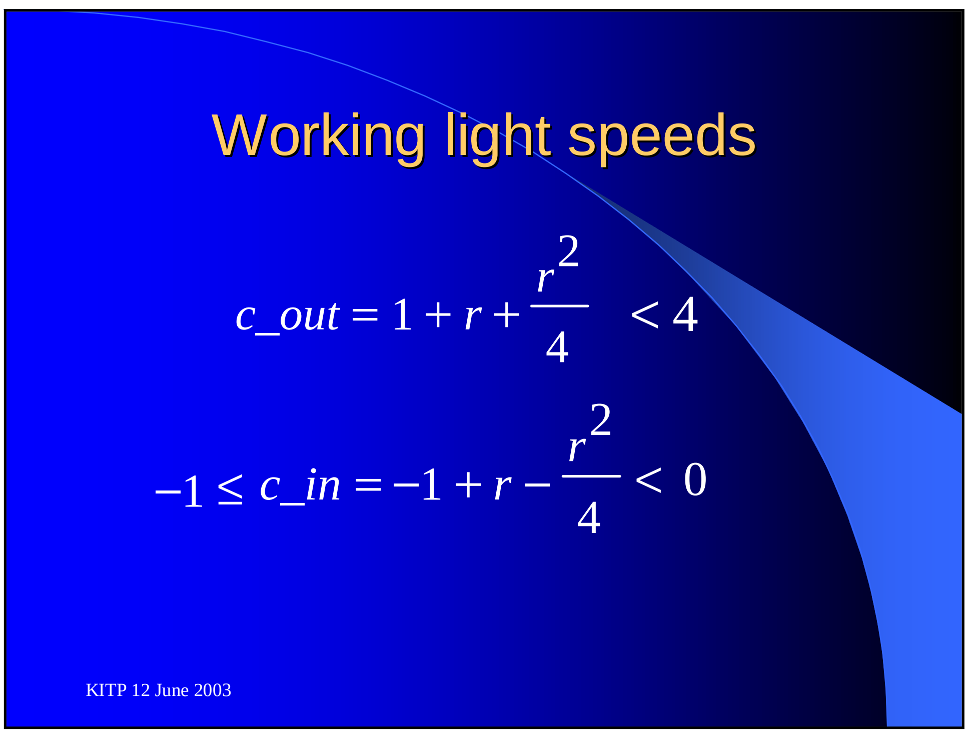
Working light speeds
$$c\_out = 1 + r + \frac{r^2}{4} < 4$$
$$-1 \le c\_in = -1 + r - \frac{r^2}{4} < 0$$
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Try Schwarzschild background

- Slicings from Malec and O`Murchadha's catalogue with K=3/a .
- The simplest don't cross the BH horizon (white hole r=2m instead); choose a time independent set that do.
- Still need to understand scalar wave initial conditions.

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M-O`M Schwarzschild

$$ds^2 = -B^2\, du^2 + \frac{(dR + \beta\, du)^2}{B^2} + R^2\, d\,\Omega^2$$

$$B = \sqrt{1 - \frac{2\,m}{R} + \left(\frac{R}{a} - \frac{\kappa\, a^2}{R^2}\right)^2}$$

$$\beta = -\mathrm{B}\left(\frac{R}{a} - \frac{\kappa\, a^2}{R^2}\right)$$

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Light speeds

$$c_{out} = B^2 - \beta \quad , \quad c_{in} = -B^2 - \beta$$

At R=2m one finds $B^2 = |\beta|$ so that $c_{out} = 0$ only if β is positive. This requires that

$$\left(\frac{2\,m}{a}\right)^3 < \kappa$$

Then light going outward at R=2m can have constant R, zero speed, so BH.

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Extrinsic Curvature (Schw)

$$K^a{}_b = \begin{bmatrix} -\frac{1}{a} - \frac{2\,a^2\,\kappa}{R^3} & 0 & 0 \\ 0 & -\frac{1}{a} + \frac{a^2\,\kappa}{R^3} & 0 \\ 0 & 0 & -\frac{1}{a} + \frac{a^2\,\kappa}{R^3} \end{bmatrix}$$

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3-curvature of slice

$$R^a{}_b = \begin{bmatrix} -\frac{2}{a^2} - \frac{2(m + a\kappa)}{R^3} + \frac{4a^4\kappa^2}{R^6} & 0 & 0 \\ 0 & -\frac{2}{a^2} + \frac{m + a\kappa}{R^3} + \frac{a^4\kappa^2}{R^6} & 0 \\ 0 & 0 & -\frac{2}{a^2} + \frac{m + a\kappa}{R^3} + \frac{a^4\kappa^2}{R^6} \end{bmatrix}$$

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Bring Scri to r=2

$$\frac{R}{a} = \frac{r}{1 - \frac{r^2}{4}} = \frac{r}{q}$$

$$q = 1 - \frac{r^2}{4}$$

Same transformation as in flat spacetime

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Compactified M-O`M

$$ds^2 = \frac{a^2}{q^2}\,[-B^2\,du^2 + f^2\,(dr + \beta\,du)^2 + r^2\,d\,\Omega^2]$$

$$f^2 = \frac{\left(1 + \frac{r^2}{4}\right)^2}{B^2} \quad , \quad \beta = -\left(r - \frac{\kappa\,q^3}{r^2}\right)\frac{1}{f}$$

$$B = \sqrt{q^2\left(1 - \frac{2\,m\,q}{a\,r}\right) + \left(r - \frac{\kappa\,q^3}{r^2}\right)^2}$$

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Nearly Minkowski at Scri+

Note that m and κ only appear in the combinations $m\,q^3$ and $\kappa\,q^3$. Since q ~ (2-r) for r->2 (*Scri*+) their effects die like $q^3 \sim \left(\frac{2\,a}{R}\right)^3$ toward Scri+. In particular c_out -> 4 and c_in -> 0 independent of m and κ.

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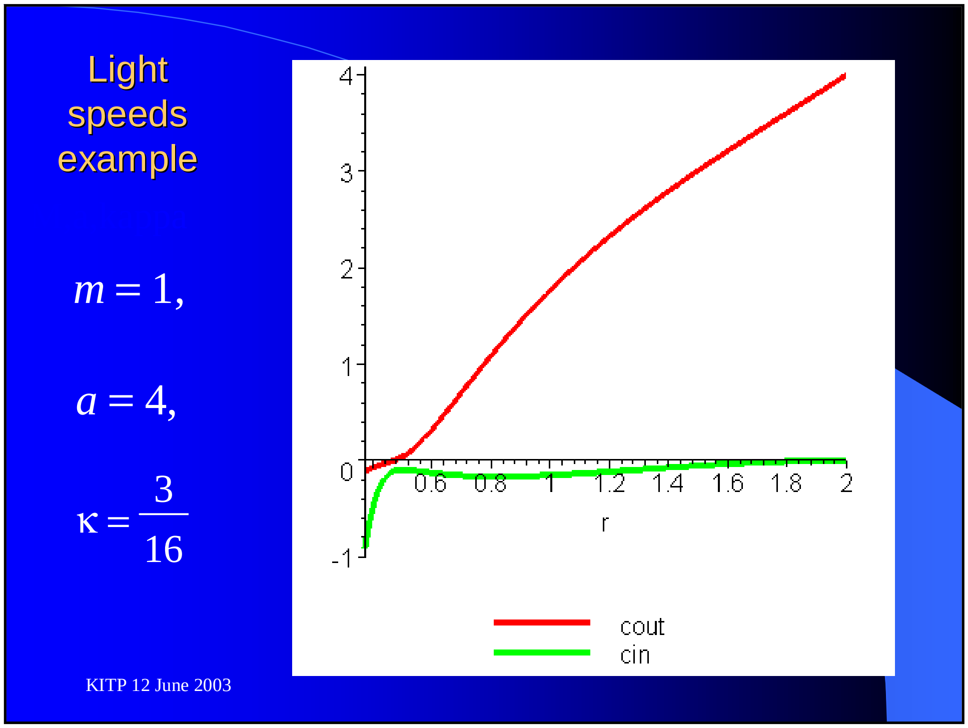
Light speeds example
m = 1,
a = 4,
κ = 3/16
4
3
2
1
0
-1
0.6 0.8 1 1.2 1.4 1.6 1.8 2
r
cout
cin
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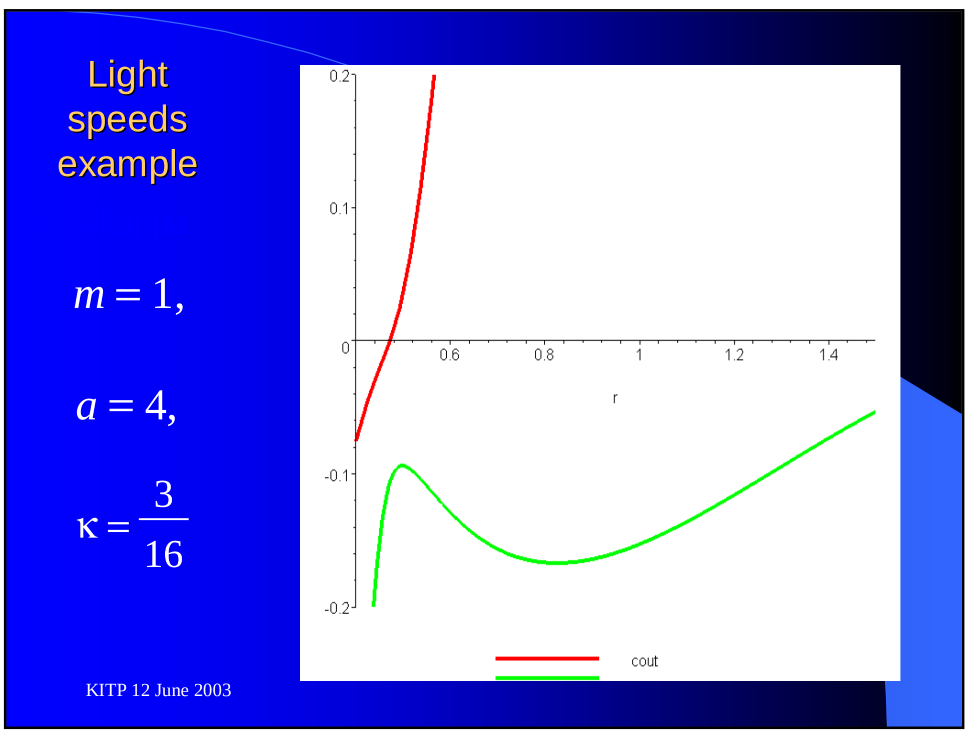
Light speeds example
m = 1,
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0.2
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-0.1
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0.6 0.8 1 1.2 1.4
r
cout
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