Einstein Spaces as Attractors for Einstein's Equations

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M compact, connected, orientable
dimension $n \geq 2$

γ a Riemannian metric on M

(M, γ) is (negative) Einstein
if $Ric(\gamma) = \lambda \gamma$, $\lambda < 0$
λ constant

Rescale to arrange $R(\gamma) = -1$
$Ric(\gamma) = -\frac{1}{n}\gamma$

Vacuum Spacetime: $(M \times R^+, \eta)$
$$\eta = -dt \otimes dt + \frac{t^2}{n(n-1)} \gamma_{ij} dx^i \otimes dx^j$$

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(i) $Ric(\eta) = 0$ i.e., η is Ricci flat

(ii) If (M, γ) is hyperbolic then η is flat (always true in $n = 2, 3$ dimensions)

These are vacuum, $k = -1$ FRW solutions (compactified in space)

(iii) For $n > 3$ γ need not be hyperbolic in which case η need not be flat

Use mean curvature as time

$$\tau = tr_g K = -\frac{n}{t}$$

$$-\infty < \tau < 0 \iff 0 < t < \infty$$

$$g = \frac{t^2}{n(n-1)} \gamma, \quad K = \text{2nd fund. form}$$

$$\eta = -\left(\frac{n}{\tau^2}\right)^2 d\tau \otimes d\tau + \frac{n}{n-1} \frac{\gamma_{ij}}{\tau^2} dx^i \otimes dx^j$$

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1 Impose CMC slicing and spatial harmonic coordinate conditions on vacuum field equations. Resulting system is non-autonomous

2. Define new (rescaled) variables and reexpress system in autonomous form

3. Check that only fixed points are solutions of the above type

4. Use (higher order) energy arguments to study stability of these fixed points (non-linear stability for small perturbations)

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Another characterization of the fixed points

(i) vacuum solutions admitting a compact (orientable, connected) CMC slice

(ii) a proper homothety (one checks that $Y = \tau \frac{\partial}{\partial \tau}$ obeys $\mathcal{L}_Y \eta = -2\eta$)

Define usual ADM variables

$$K = -\frac{1}{\mu_g}\left(\pi - \frac{1}{(n-1)} g \operatorname{tr}_g \pi\right)$$

$$\mathcal{H}(g,\pi) = \frac{1}{\mu_g}\left(\operatorname{tr}_g(\pi \times \pi) - \frac{1}{(n-1)}(\operatorname{tr}_g \pi)^2\right) - \mu_g R(g)$$

$$\mathcal{J}(g,\pi) = -2\delta_g \cdot \pi$$

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$$\partial_t g = -2NK + \mathcal{L}_{\tilde{X}} g$$

$$\partial_t \pi = \text{long formula} + \mathcal{L}_{\tilde{X}} \pi$$

taking $t = \tau = \mathrm{tr}_g K$ forces

$$\frac{\partial \tau}{\partial t} = 1 = -\Delta_g N + \frac{N}{(\mu_g)^2}\left[\mathrm{tr}_g(\pi \times \pi) + \frac{1}{n(n-1)^2}(\mathrm{tr}_g \pi)^2\right]$$

Everything becomes non-autonomous — constraints, evolution equations, gauge conditions

Note physical dimensions

$[g] = (\text{length})^2$

$[\pi] = (\text{length})^{n-3}$

$[\tau] = (\text{length})^{-1}$

$[N] = (\text{length})^2$

$[X] = \text{length}$

$[K] = \text{length}$

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Define (dimensionless) rescaled variables

$\hat{g} = \tau^2 g$, $\quad \hat{N} = \tau^2 N$

$\hat{X} = \tau X$, $\quad \hat{\pi} = \tau^{n-3}\pi$

$\mu_{\hat{g}} = |\tau|^n \mu_g$, $\quad \tilde{\tau} = \frac{\tau}{\tau} = 1$

but $\frac{\partial}{\partial \tau} \longrightarrow \tau \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial T}$

where $T = -\ln(-\tau)$

$T \in (-\infty, \infty)$

Field equations now autonomous

$$-\frac{\partial \hat{g}}{\partial T} = 2\hat{g} + \mathcal{L}_{\hat{X}}\hat{g} - 2\hat{N}(-1)^n \hat{K}$$

$$-\frac{\partial \hat{\pi}}{\partial T} = \text{long formula}$$

$$1 = -\Delta_{\hat{g}}\hat{N} + \hat{N}\left[\frac{\mathrm{tr}_{\hat{g}}\, \hat{\pi}\times\tilde{\pi}}{(\mu_{\hat{g}})^2} + \frac{1}{n}\right]$$

similarly for $\{\hat{N}, \tilde{J}\}$

Look for fixed points: $\frac{\partial \hat{g}}{\partial T} = \frac{\partial \hat{\pi}}{\partial T} = 0$

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(i) contract $\frac{\partial \hat{g}}{\partial T} = 0$ equation with $\hat{\pi}$, integrate over M

$$0 = (-1)^n \int_M \mathrm{tr}_{\hat g}\hat{\pi} + \int_M \frac{\hat N}{\mu_{\hat g}}\left[\mathrm{tr}_{\hat g}(\hat\pi \times \hat\pi) - \frac{1}{(n-1)}(\mathrm{tr}_{\hat g}\hat\pi)^2\right]$$

(ii) Take $\hat g$ trace of same equation multiply by 2 and integrate

$$0 = (-1)^n \int_M \mathrm{tr}_{\hat g}\hat\pi - \int_M \frac{\hat N}{\mu_{\hat g}} \frac{(\mathrm{tr}_{\hat g}\hat\pi)^2}{n(n-1)}$$

(iii) $0 = \int_M \frac{\hat N}{\mu_{\hat g}} \mathrm{tr}_{\hat g}(\hat\pi^{tr} \times \hat\pi^{tr})$

but $\hat N > 0$ by maximum principle

$\therefore\ \hat\pi^{tr} = 0 \qquad \therefore\ \hat N = n$

(iv) $\frac{\partial \tilde g}{\partial T} = 0 \Rightarrow \mathcal{L}_{\tilde X}\tilde g = 0$

$\frac{\partial \hat\pi}{\partial T} = 0 \Rightarrow Ric(\hat g) = -\frac{(n-1)}{n^2}\tilde g$

(v) $\mathcal{L}_{\tilde X}\hat g = 0$ but $\hat g$ Einstein

$\Rightarrow \hat X = 0$.

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Conversely (Eardley, Isenberg, Marsden and VM '86) show that a homothetic Killing field, CMC slice and vacuum equations (in 3+1) reproduces the above solutions

argument also works in n+1 dimension

Linearization about a fixed point

$\delta g = h, \quad \delta\pi = p$

Can take $\delta N = \delta X = 0$ and require $\{h, p\}$ to be TT w.r.t. the Einstein metric γ

For $n \geq 4$ we have the Conformal (Weyl) tensor of γ, $R(\gamma) = -1$

$$\mathrm{Riem}(\gamma)_{kijl} = \mathrm{Conf}(\gamma)_{kijl} + \frac{R}{n(n-1)}(\gamma_{kj}\gamma_{il} - \gamma_{kl}\gamma_{ij})$$

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$$0 = -h^{TT}_{,zz} + \frac{(n-6)}{z} h^{TT}_{,z} + \frac{2(n-2)}{z^2} h^{TT} + \frac{n(n-1)}{z^2}\left[\Delta_\gamma h^{TT} - 2h^{TT}\cdot \mathrm{Conf}(\gamma)\right]$$

where $(h^{TT}\cdot \mathrm{Conf})_{kl} = h^{TT}_{rs}\, \mathrm{Conf}^{r\ s}_{\ kl}(\gamma)$

Separate variables via eigentensors of

$$\Delta_\gamma u^{TT} - 2u^{TT}\cdot \mathrm{Conf}(\gamma) = -\lambda u^{TT}$$

Eigensolutions behave like

$$h^{TT} \propto (-z)^{\alpha_\pm} u^{TT}$$

$$\alpha_\pm = \frac{n-5 \pm i\sqrt{4n(n-1)\lambda - (n^2-2n+9)}}{2}$$

$$= \frac{n-5 \pm i\sqrt{C(n,\lambda)}}{2}$$

if $C(n,\lambda) \neq 0$ i.e., $\alpha_+ \neq \alpha_-$

Otherwise get 2nd solution

$$h^{TT} \propto (-z)^{n-5/2} \ln(-z)\, u^{TT}$$

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Linearized Stability: decay of rescaled perturbations $z^2 h^{TT}$ as $T \to \infty \Leftrightarrow C(n,\lambda_{min}) > -(n-1)^2$

or $\lambda_{min} > \dfrac{2}{n(n-1)}$

A sufficient condition for this is the ("integration by parts identity) inequality

$$\mathrm{Conf}(\gamma)_{kijl} h^{TT\,ij} h^{TT\,kl} \geq \frac{-(n-2)}{n(n-1)} h^{TT}_{ij} h^{TT\,ij}$$

automatically holds if $\mathrm{Conf}(\gamma) = 0$ and $n \gtrless 2$ — identical to a "stability" criterion in Besse Section 12 H.

If $C(n,\lambda) > 0$ get decaying oscillations

$$h^{TT} \propto (-z)^{\frac{n-5}{2}} \cos\left[\frac{\sqrt{C(n,\lambda)}}{2} \ln(-z) + \mathring{\varphi}\right] u^{TT}$$

A model for the linearized problem — the damped oscillator

$$\frac{d^2X}{dT^2} + 2\alpha\frac{dX}{dT} + \beta^2 X = 0$$

$\beta^2 > \alpha^2 \Longleftrightarrow$ underdamped, universal decay rate

$$X = X_0 e^{-\alpha T}\cos\left(\sqrt{\beta^2-\alpha^2}\,T + \varphi_0\right)$$

Try an energy argument

$$H = \frac{1}{2}\left(\frac{dX}{dT}\right)^2 + \frac{1}{2}\beta^2 X^2$$

$$\frac{dH}{dT} = -2\alpha\left(\frac{dX}{dT}\right)^2 \qquad \text{decay but no rate}$$

Modify to $H + \Gamma$, $\Gamma = \alpha X \frac{dX}{dT}$

$$\frac{d}{dT}(H+\Gamma) = -2\alpha(H+\Gamma)$$

$$(H+\Gamma)(T) = \frac{1}{2}(\beta^2-\alpha^2)X_0^2 e^{-2\alpha T}$$

$$|\Gamma| \leq \frac{1}{2}\epsilon^2\left(\frac{dX}{dT}\right)^2 + \frac{1}{2}\frac{\alpha^2}{\epsilon^2}X^2 \qquad \text{choose } \epsilon < 1,\ \epsilon > \alpha^2/\beta^2$$

Now take $\beta^2 < \alpha^2 \qquad \beta \neq 0$

$$e^{i\omega T} \sim e^{-(\alpha \pm \sqrt{\alpha^2-\beta^2})T}$$

overdamped, slowest decay $\Leftrightarrow \alpha - \sqrt{\alpha^2-\beta^2}$

Take $H + \Gamma$, $\Gamma = \frac{\beta^2}{\alpha} X \frac{dX}{dT}$

$$\frac{d(H+\Gamma)}{dT} \leq -2\left(\alpha - \sqrt{\alpha^2-\beta^2}\right)(H+\Gamma)$$

$H+\Gamma$ still positive definite

Non-linear Forces (autonomous)

$$\frac{d^2X}{dT^2} + 2\alpha\frac{dX}{dT} + \beta^2 X = F\left(X, \frac{dX}{dT}\right)$$

Now get (underdamped case)

$$\frac{d}{dT}(H+\Gamma) = -2\alpha(H+\Gamma) + J\left(X, \frac{dX}{dT}\right)$$

$$\leq -2\alpha(H+\Gamma) + C(H+\Gamma)^{1+\delta}$$

$$= -2\alpha(H+\Gamma)\left[-1 + \frac{C}{2\alpha}(H+\Gamma)^{\delta}\right]$$

asymptotic decay at linearized rate

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$$\gamma_{mn,T}\Big|_{\text{linear}} = 2n\tilde{K}^{tr}_{mn}$$

$$\tilde{K}^{tr}_{mn,T}\Big|_{\text{almost linear}} = -(n-1)\tilde{K}^{tr}_{mn}$$

$$-\frac{n}{2}\Big\{-\gamma^{ij}\hat{\nabla}_i\hat{\nabla}_j(\gamma_{mn}-\hat{\gamma}_{mn}) - \frac{2}{n^2}(\gamma_{mn}-\hat{\gamma}_{mn})$$

$$+(\hat{C}^s{}_{mn}{}^r + \hat{C}^s{}_{nm}{}^r)(\gamma_{rs}-\hat{\gamma}_{rs})\Big\}$$

$$= -(n-1)\tilde{K}^{tr}_{mn} - \frac{n}{2}\mathcal{L}(\gamma-\hat{\gamma}_{mn})$$

where

$$(\mathcal{L}u)_{mn} = -\gamma^{ij}\hat{\nabla}_i\hat{\nabla}_j u_{mn} - \frac{2}{n^2}u_{mn}$$

$$+(\hat{C}^s{}_{mn}{}^r + \hat{C}^s{}_{nm}{}^r)u_{rs}$$

Spatial gauge condition (harmonic gauge)

$$\tilde{\nabla}_m(\mu_\gamma\gamma^{mn}) = -\mu_\gamma\gamma^{ij}(\Gamma^n_{ij}-\hat{\Gamma}^n_{ij}) = 0$$

Define:

$$\overset{(0)}{\mathcal{E}} := \int_M \mu_\gamma\Big\{\hat{\gamma}^{im}\hat{\gamma}^{jn}\tilde{K}^{tr}_{ij}\tilde{K}^{tr}_{mn} + \frac{1}{4}(\gamma_{ij}-\hat{\gamma}_{ij})(\mathcal{L}(\gamma-\hat{\gamma})_{kl})\,\hat{\gamma}^{ik}\hat{\gamma}^{jl}\Big\}$$

$$\overset{(0)}{\Gamma} := \int_M \mu_\gamma\tilde{K}^{tr}_{ij}(\gamma_{kl}-\hat{\gamma}_{kl})\hat{\gamma}^{ik}\hat{\gamma}^{jl}$$

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For case of universal decay rate we want the combinations

$$\overset{(0)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(0)}{\Gamma},\ \cdots\ \overset{(k)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(k)}{\Gamma}$$ where

$$\overset{(2k)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(2k)}{\Gamma} = n^{4k}\Big\{\int_M\mu_\gamma\hat{\gamma}^{ik}\hat{\gamma}^{jl}\Big[(\mathcal{L}^k\tilde{K}^{tr}_{ij})(\mathcal{L}^k\tilde{K}^{tr}_{kl})$$

$$+\frac{1}{4}\mathcal{L}^k(\gamma-\hat{\gamma})_{ij}\,\mathcal{L}^{k+1}(\gamma-\hat{\gamma})_{kl}\Big]$$

$$+\int_M\mu_\gamma\hat{\gamma}^{ik}\hat{\gamma}^{jl}\left(\frac{n-1}{2n}\right)\mathcal{L}^k\tilde{K}^{tr}_{ij}\,\mathcal{L}^k(\gamma-\hat{\gamma})_{kl}\Big\}$$

$$\overset{(2k+1)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(2k+1)}{\Gamma} = n^{4k+2}\Big\{\int_M\mu_\gamma\hat{\gamma}^{ik}\hat{\gamma}^{jl}\Big[\mathcal{L}^{k+1}\tilde{K}^{tr}_{ij}\,\mathcal{L}^{k+1}\tilde{K}^{tr}_{kl}$$

$$+\frac{1}{4}\mathcal{L}^{k+1}(\gamma-\hat{\gamma})_{ij}\,\mathcal{L}^{k+1}(\gamma-\hat{\gamma})_{kl}\Big]$$

$$+\int_M\mu_\gamma\hat{\gamma}^{ik}\hat{\gamma}^{jl}\left(\frac{n-1}{2n}\right)\mathcal{L}^{k+1}(\gamma-\hat{\gamma})_{ij}\,\mathcal{L}^k(\tilde{K}^{tr})_{kl}\Big\}$$

Can verify that

$$\left(\overset{(2k)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(2k)}{\Gamma}\right)_{,T}\Bigg|_{\text{lowest order only}} = -(n-1)\left(\overset{(2k)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(2k)}{\Gamma}\right)$$

$$\left(\overset{(2k+1)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(2k+1)}{\Gamma}\right)_{,T}\Bigg|_{\text{lowest order only}} = -(n-1)\left(\overset{(2k+1)}{\mathcal{E}} + \left(\frac{n-1}{2n}\right)\overset{(2k+1)}{\Gamma}\right)$$

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For anomalously slow decay rates (overdamped case) add further corrections

$$\overset{(2k)}{\Delta} = n^{4k}\left(-\frac{(n-1)}{2} + \frac{2n^2}{(n-1)}\hat{\lambda}_{min}\right) \times \int \mu_{\hat\gamma}\left[\frac{\hat{\gamma}^{ik}\hat{\gamma}^{jl}}{n}\mathcal{L}^k(\hat{K}^{tr}_{ij})\,\mathcal{L}^k(\gamma-\hat{\gamma})_{kl}\right]$$

$$\overset{(2k+1)}{\Delta} = n^{4k+2}\left(-\frac{(n-1)}{2} + \frac{2n^2}{(n-1)}\hat{\lambda}_{min}\right) \times \int \mu_{\hat\gamma}\left[\frac{\hat{\gamma}^{ik}\hat{\gamma}^{jl}}{n}\mathcal{L}^k(\hat{K}^{tr}_{kl})\,\mathcal{L}^{k+1}(\gamma-\hat{\gamma})_{ij}\right]$$

Need to verify that

1. corrected energies bound needed Sobolev norms (i.e., $s > \frac{n}{2}+1$ for "classical" results on local existence)
2. junk terms are bounded by a power > 1 of the corrected energies