# *Overlapping grids and moving boundaries in numerical relativity*

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# *Introduction*

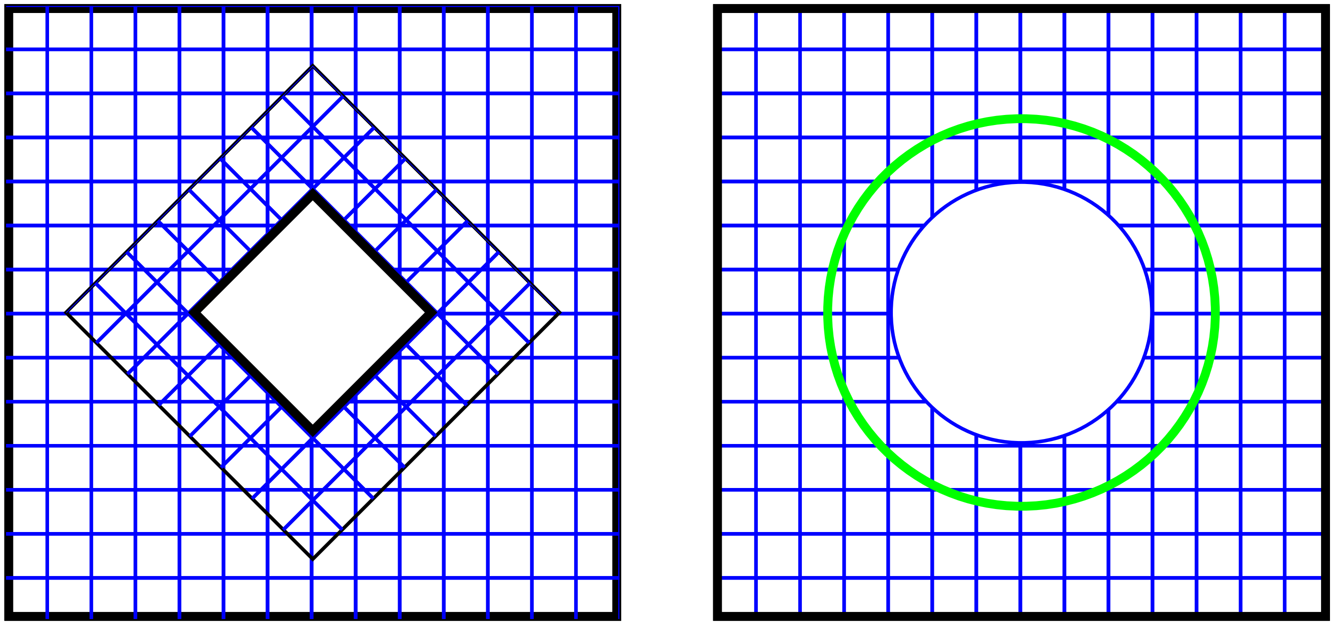
- Preliminary report on work with Gioel Calabrese
- Motivation: Black hole excision
  - Well-posed IBVP at the continuum.
  - Symmetry adapted grid boundaries.
  - Moving boundaries with overlapping grids
- Axisymmetric scalar field as a model problem
- Stable, energy conserving discrete system for axisymmetric wave equation
- Towards moving black-hole excision boundaries

# *Guiding philosophy ("the faith")*

- Well-posed IBVP at the continuum
  - Symmetric hyperbolic system of equations
  - Smooth initial data
  - Maximally dissipative boundary conditions specify incoming modes
- Convergent discrete systems using the energy method
  - Conserved energy for semi-discrete system
  - Boundary conditions using projection operators
  - Stable time discretization

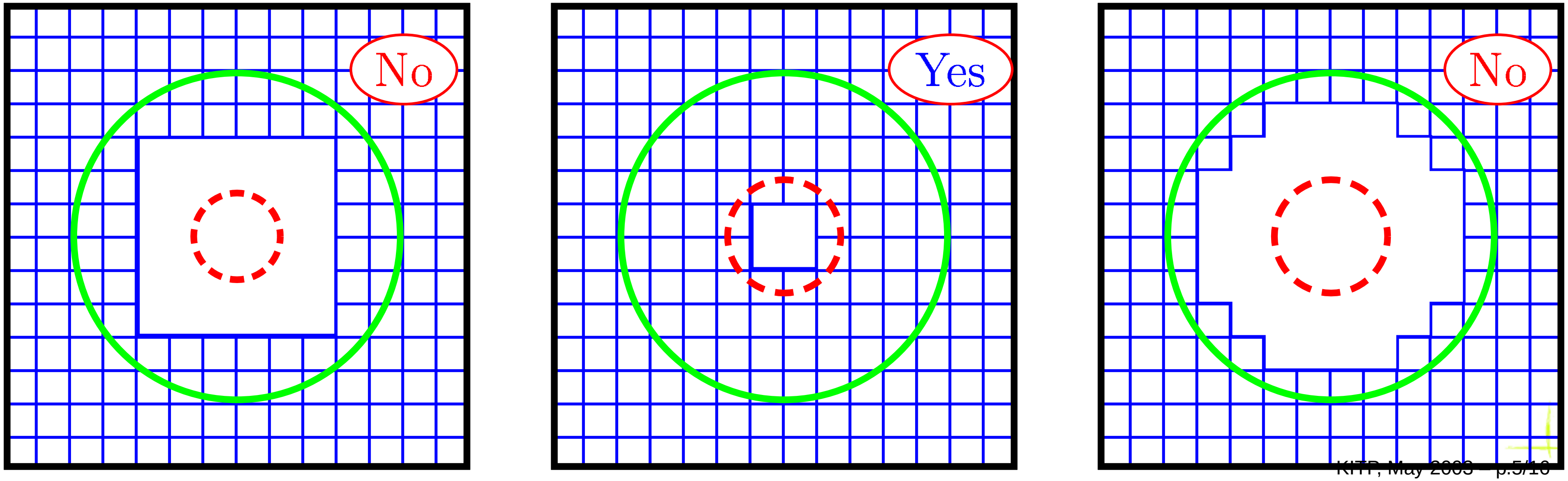
# Motivation—boundaries

- Physical boundary conditions often specify quantities normal and transverse to the boundary
- Physical boundaries of nontrivial systems often do not align with simple grid geometries



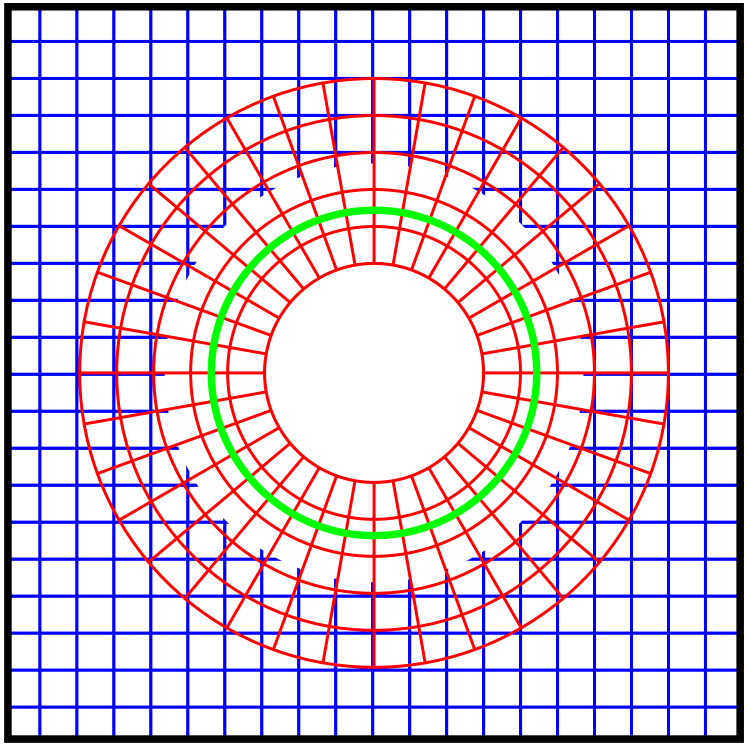
# *Simple, 3-D black hole excision*

- Avoid the black hole singularity with an inner boundary inside the horizon. Require outflow inner boundary.
- For Schwarzschild, $\beta^i \pm \alpha\sqrt{g^{ii}}$ are outgoing over a cube of length $2L$ only for $L < 2M/(3\sqrt{3}) \approx 0.38M$
- No known cubic excision for Kerr with well-posed IBVP.



# ***Improved excision boundaries***

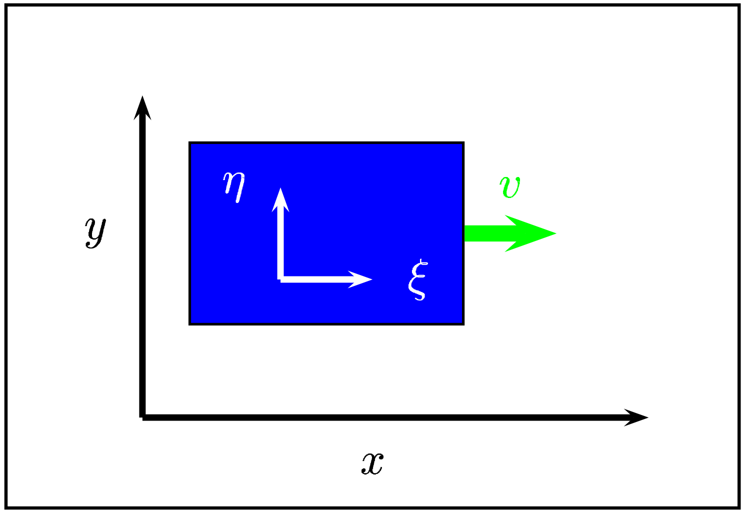
- Multiple coordinate mappings
- Unstructured grids
- Rectangular grids with nontrivial boundaries
- Overlapping grids
  - Grids adapt to symmetry
  - Grids are logically cartesian for simple computing
  - Individual grids may move to track boundaries



# Moving boundaries: toy problem

- The 1-D wave equation

(1) $$\partial_t^2 \phi = \partial_x^2 \phi$$




- Transform to coordinates ($\xi$,$\eta$)

(2) $$\tau = t$$
(3) $$\xi = x - vt$$
(4) $$T \equiv \partial_\tau \phi - v \partial_\xi \phi$$

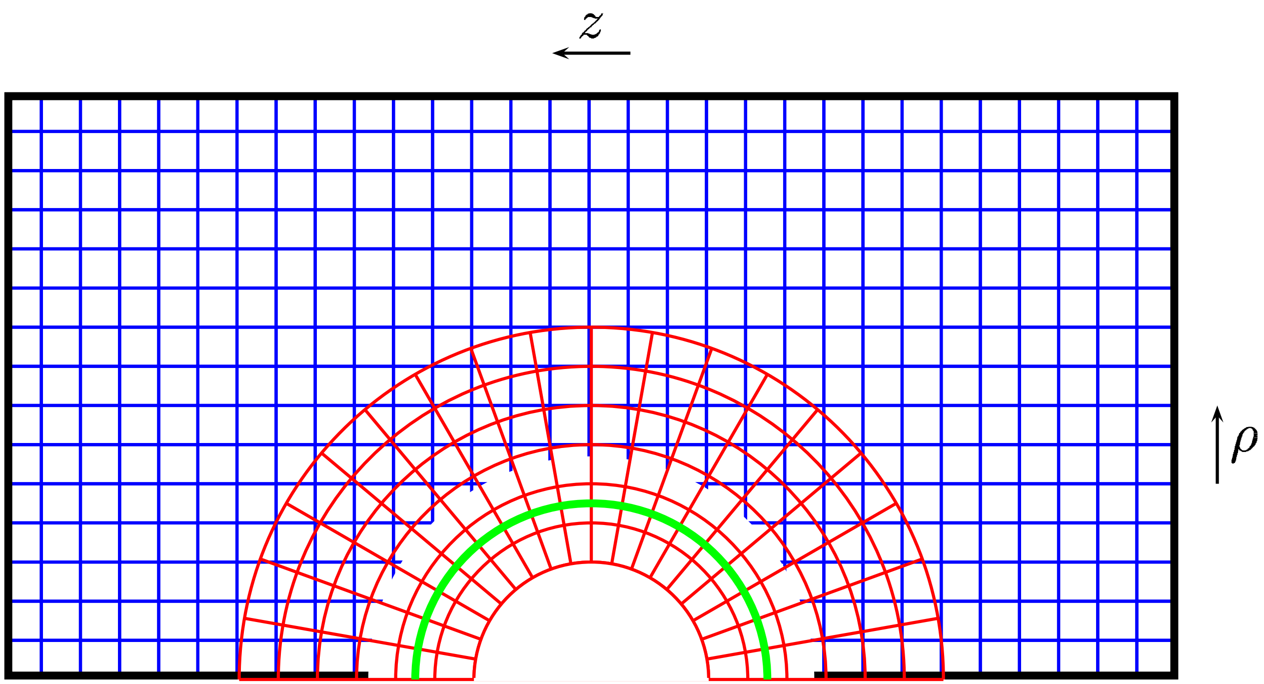
- The wave equation becomes

(5) $$\partial_\tau T = \partial_\xi \Xi + v \partial_\xi T$$
(6) $$\partial_\tau \Xi = \partial_\xi T + v \partial_\xi \Xi$$

(7) $$\lambda_\pm = \pm 1 + v$$
(8) $$|v| < 1$$

# Model problem

- Klein-Gordon scalar field, $\nabla_\mu \nabla^\mu \phi = 0$.
- Boosted Schwarzschild background geometry.
- Axial symmetry.



# Axisymmetric wave equation

- First order form

(9) $$T \equiv \partial_t \phi \qquad P \equiv \partial_r \phi \qquad Z \equiv \partial_z \phi$$

(10) $$\partial_t T = \frac{1}{\rho}\partial_\rho(\rho P) + \partial_z Z, \quad \partial_t P = \partial_\rho T, \quad \partial_t Z = \partial_z T$$

- Conserved energy

(11) $$E = \frac{1}{2}\int \mathrm{d}z \int \mathrm{d}\rho\, \rho \left(T^2 + P^2 + Z^2\right)$$

- Maximally dissipative boundary conditions

(12) $$w^{(+1;n)} = S w^{(-1;n)} + g^{(n)}$$

# Semi-discrete system I

- Discrete system

$$(13)\qquad \partial_t T_{ij} = \begin{cases} 2D_+^{(\rho)} R_{0j} + D^{(z)} Z_{0j}, & i = 0 \\ \frac{1}{\rho_i} D^{(\rho)} (\rho R)_{ij} + D^{(z)} Z_{ij}, & i \geq 1 \end{cases}$$

$$(14)\qquad \partial_t R_{ij} = D^{(\rho)} T_{ij}, \quad i \geq 1$$

$$(15)\qquad \partial_t Z_{ij} = D^{(z)} T_{ij}, \quad i \geq 0\ ,$$

- Discrete boundary conditions
  - Boundary conditions projected onto the RHS
  - Consistency conditions at grid corners

# Semi-discrete system II

- Energy for the semi-discrete system

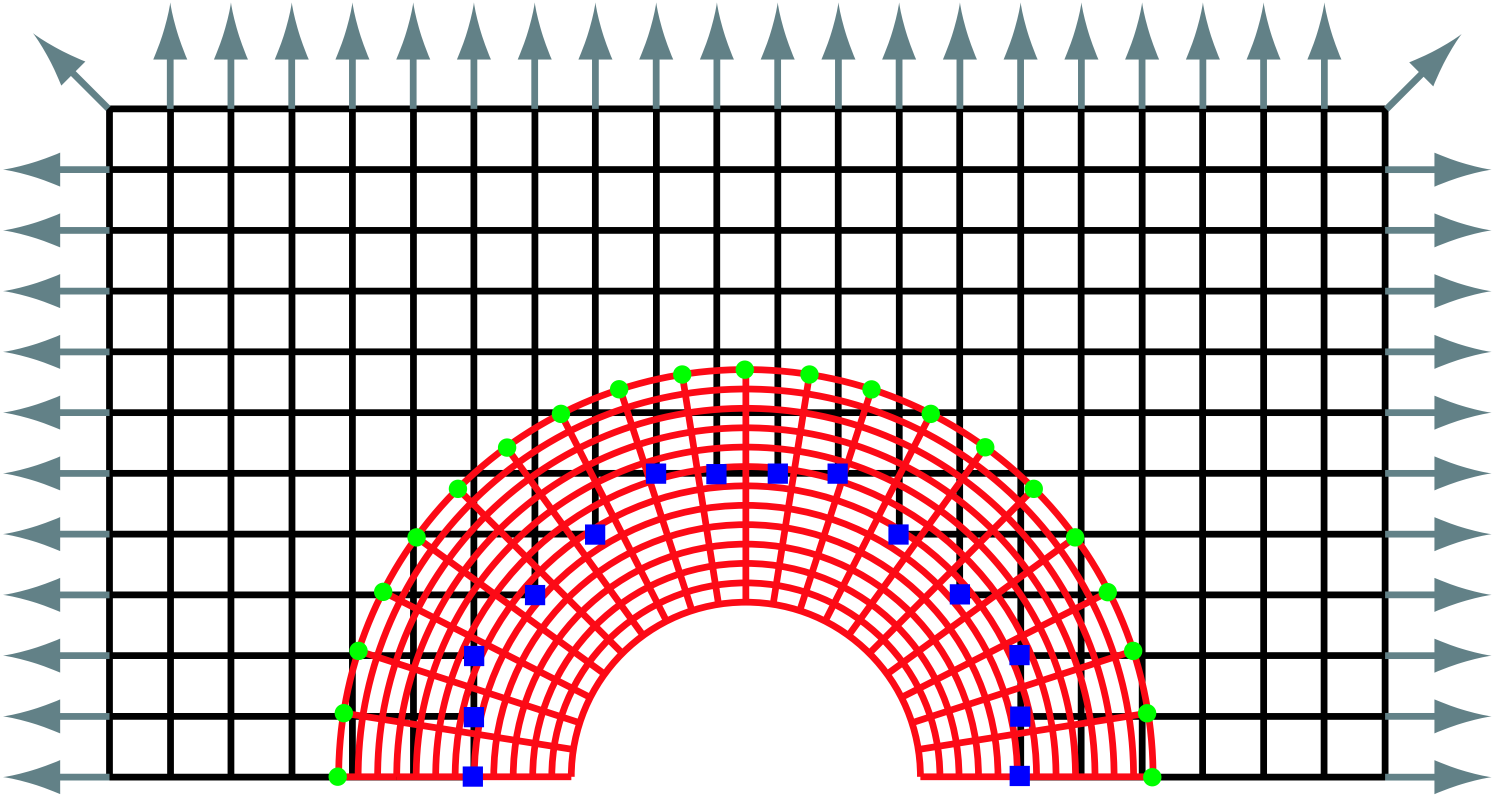
$$E = \frac{1}{2}\sum_{j=0}^{N_z}\left[\sum_{i=1}^{N_\rho}\left(T_{ij}^2 + R_{ij}^2 + Z_{ij}^2\right)\rho_i\sigma_i\triangle\rho + \frac{1}{4}\left(T_{0j}^2 + Z_{0j}^2\right)\triangle\rho^2\right]\sigma_j\triangle z$$

- The semi-discrete energy is conserved

$$\partial_t E = \sum_{j=0}^{N_z} T_{N_\rho j}\rho_{N_\rho}R_{N_\rho j}\sigma_j\triangle z + \sum_{i=1}^{N_\rho}(T_{iN_z}Z_{iN_z} - T_{i0}Z_{i0})\rho_i\sigma_i\triangle\rho$$
$$+\frac{1}{4}(T_{0N_z}Z_{0N_z} - T_{00}Z_{00})\triangle\rho^2$$

- Similar results for wave equation in spherical coordinates.

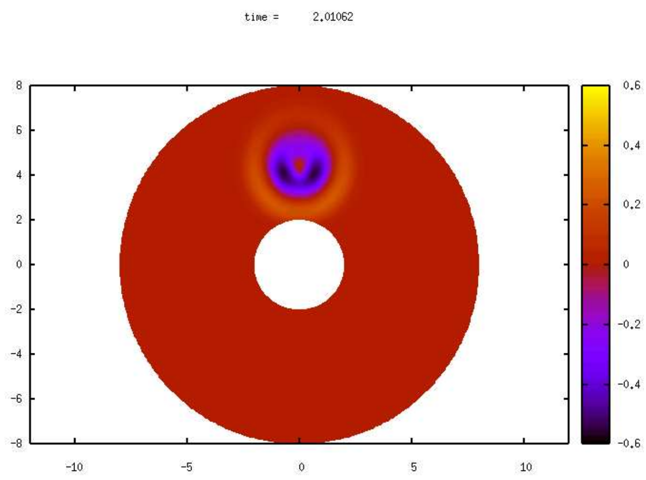
# *Semi-discrete system III*

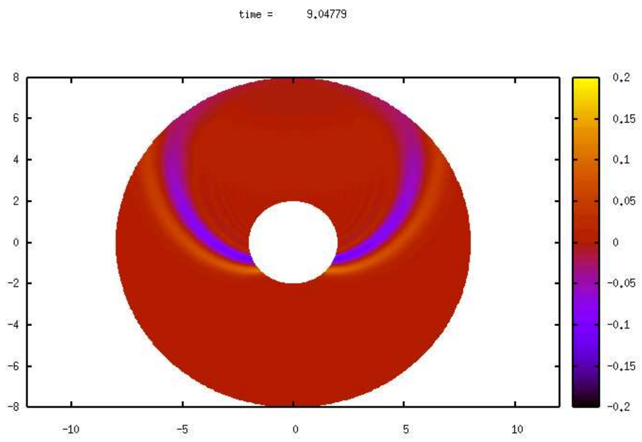


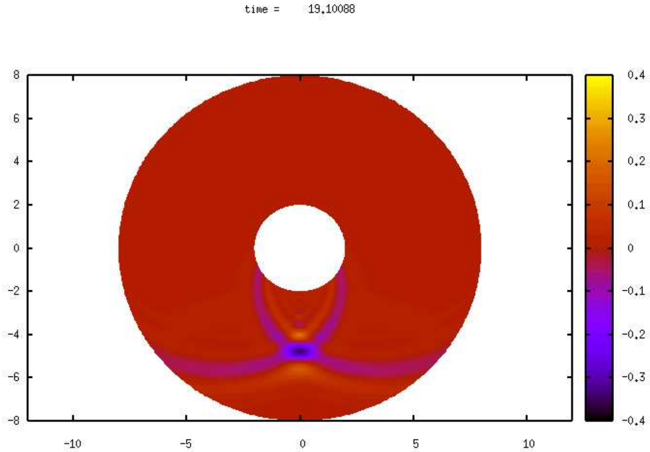
# *Excision of a boosted black hole*

- Cylindrical coordinate patch
  - Coordinates fixed in space
  - Known outer boundary conditions assumed ($P_{\rm max}$, $Z_{\rm min}$, $Z_{\rm max}$)
  - Black hole moves along the axis, and excised
  - Inner excision boundary covered by spherical patch
- Spherical coordinate patch
  - Tracks the black hole
  - $R_{\rm max}$ boundary covered by cylindrical grid
  - $\Theta_{\rm min}$ and $\Theta_{\rm max}$ have symmetry conditions
  - $R_{\rm min}$ must be an outflow boundary.

# Wave on Kerr-Schild background

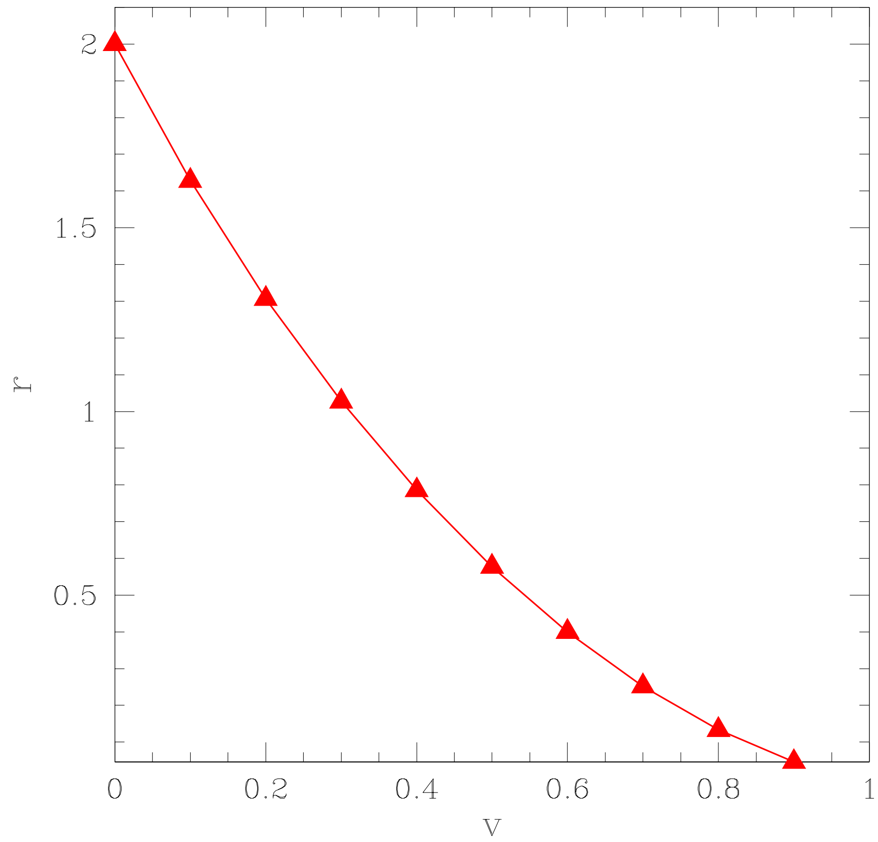


# *Excision radii for Schwarzschild*

- Spherical coordinates are *not,* of course, adapted to boosted black hole horizons.



# Conclusion

- Significant benefits to using symmetry adapted coordinates for black hole excision
  - Excise further away from the singularity
  - Required for Kerr
- Advantages of overlapping grids
  - Individual coordinate patches adapt to local symmetry
  - Natural fixed mesh refinement
  - Simple (cartesian) data structures
  - Easy to parallelize
  - Works with AMR (Overture code at LLNL)
  - Natural moving boundaries