On strong hyperbolicity

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OUTLINE

- Introduction, definitions, examples.
- Covariant Definitions
- Causality
- Applications
  - ADM-BSSN hyperbolicity
  - Subsidiary system hyperbolicity

First Order Constant Coefficient Systems

$$\partial_t u^\alpha = A^{\alpha i}{}_\beta \nabla_i u^\beta + B^\alpha{}_\beta u^\beta$$

- Question: When is the above system well posed in the L^2 sense?

$$||u^\alpha(t)||_{L^2} \leq C(t)||u^\alpha(0)||_{L^2}$$

- Answer: It is well posed if and only if for all co-vectors ω_i, the matrix $A^{\alpha i}{}_\beta \omega_i$ has only real eigen-values and a complete set of eigen-vectors.

First Order Constant Coefficient Systems II

$$\partial_t \hat{u}^\alpha = i A^{\alpha i}{}_\beta \omega_i \hat{u}^\beta$$

$$\hat{u}^\alpha(t) = (e^{iA^i \omega_i t})^\alpha{}_\beta \hat{u}^\beta(0)$$

$$||u^\alpha(t)||_{L^2} = ||\hat{u}^\alpha(t)||_{L^2} \leq C(t)||\hat{u}^\alpha(0)||_{L^2} \leq C(t)||u^\alpha(0)||_{L^2}$$

$$C(t) = \sup_{\tau \in [0,t]} \sup_{\omega_a} ||(e^{iA^a \omega_a \tau})^\alpha{}_\beta||$$

First Order Quasi-linear Systems I

$$\partial_t u^\alpha = A^{\alpha i}{}_\beta(u,x,t)\nabla_i u^\beta + B^\alpha(u,x,t)$$

- The above system is well posed (w.r.t. a Sobolev Norm) in a neighborhood of u_0^α if and only if for all u^α close enough to u_0^α, all co-vectors ω_i and all points, the matrix $A^{\alpha i}{}_\beta(u,x,t)\omega_i$ has only real eigen-values and a complete set of eigen-vectors. Plus some "technical" condition
- We call such systems **strongly hyperbolic**.

First Order Quasi-linear Systems II

- If a system is strongly hyperbolic then there exists a positive definite bilinear form (a metric) $H_{\alpha\beta} = H_{\alpha\beta}(u, x, t, \omega_a)$ **uniformly bounded by above and away from zero in** ω_a such that:

$$H_{\alpha\gamma} A^{\gamma a}{}_{\beta} \omega_a$$

is also symmetric. [Kreiss Matrix Theorem]

Technical condition requires H to be smooth also on ω_i

- If there exists a $H_{\alpha\beta}$ independent of ω_a we say that the system is **symmetric hyperbolic**
- If strong hyperbolicity fails it is easy to construct a sequence solutions whose initial data has norm one but whose norm at any future time tends to infinity. Non-linear behavior can not cure this.

Examples:

Example 1: Maxwell equations:

$$W_{ij} := \partial_i A_j$$

$$\begin{aligned} \partial_t E_i &= \partial^j W_{ji} - \partial^j W_{ji} - \alpha(\partial^j W_{ij} - \partial_i W_j{}^j) \\ \partial_t W_{ij} &= \partial_i E_j - \frac{1}{2}\beta e_{ij}\partial^k E_k \end{aligned}$$

This system is symmetric hyperbolic for $\alpha < 0$ and $\beta < -\frac{2}{3}$ (most general symmetrizer built out of the 3-metric). But strongly hyperbolic for all (α, β) such that $\alpha\beta > 0$.

Examples:

Example 2:

Consider the matrices,

$$\mathbf{A^0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \mathbf{A^1} = \begin{pmatrix} -2 & 10s_1 & 0 \\ 0 & 1 & -2s_2 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A^2} = \begin{pmatrix} 0 & s_1 & 0 \\ 0 & \frac{1}{2} & 7s_2 \\ 0 & 0 & 1 \end{pmatrix}$$

There is no positive definite $h_{\alpha\beta}$ which would symmetrize $A^i\omega_i$ for arbitrary ω_i. Nevertheless $(A^0)^{-1}(A^1 + \lambda A^2)$ is diagonalizable.

$$(\mathbf{A^0})^{-1}(\mathbf{A^1} + \lambda\mathbf{A^2}) = \begin{pmatrix} -2 & (10+\lambda)s_1 & 0 \\ 0 & \frac{1}{4}(2+\lambda) & \frac{1}{2}(-2+7)s_2 \\ 0 & 0 & 2\lambda \end{pmatrix}$$

Covariant definitions I:

$$A^{\alpha a}{}_{\beta}(u,p)\nabla_a u^{\beta} = J_{\alpha}(u,p)$$

The sum of two symmetric matrices is symmetric

Defi nition: The above system is symmetric hyperbolic if there exists $h_{\alpha\beta}(u,p)$ such that:

- $h_{\alpha\beta}(u,p)A^{\beta a}{}_{\gamma}(u,p)$ is symmetric.
- for some n_a, $h_{\alpha\beta}(u,p)A^{\beta a}{}_{\gamma}(u,p)n_a$ is positive defi nite.
- One can defi ne an energy vector:

$$E^a := h_{\alpha\beta}(u,p)A^{\beta a}{}_{\gamma}(u,p)\delta u^{\alpha}\delta u^{\gamma}$$

$E^a n_a \geq 0$

- If n_a is as above, then $n_a + \varepsilon w_a$ is also as above, for ε small enough.

Covariant definitions II:

$$A^{\alpha a}{}_{\beta}(u,p)\nabla_a u^{\beta} = J_{\alpha}(u,p)$$

The sum of two diagonalizable matrices is not necessarily diagonalizable

Definition A: The above system is strongly hyperbolic if there exists n_a such that:

- $A^{\alpha a}{}_{\beta} n_a$ is invertible, and
- for each loop $\kappa(\lambda)_a = \lambda n_a + \omega_a$ $\lambda \in [-\infty, \infty]$ where ω_a is not proportional to n_a, $dim(span\{\cup_{\lambda \in R} Kern\{A^{a\alpha}{}_{\beta}\kappa_a(\lambda)\}\}) = dim\{\text{manifold of fields}\}$

Definition B: The above system is strongly hyperbolic if there exists n_a such that for each co-vector ω_a there exists $h_{\alpha\beta}(u,p,\omega)$ satisfying:

- $h_{\alpha\beta}(u,p,\omega)A^{\beta a}{}_{\gamma}(u,p)\omega_a$ is symmetric.
- $h_{\alpha\beta}(u,p,\omega)A^{\beta a}{}_{\gamma}(u,p)n_a$ is symmetric and positive definite.
- If n_a is as above then $n_a + \varepsilon\omega_a$ is also as above for ε small enough.

First Order Pseudo-Differential Systems

$$\partial_t u^\alpha = P^\alpha{}_\beta(u,x,t,D)u^\beta := \int p(u,x,t,\omega_i)\alpha_\beta e^{i\omega_i x^i}\hat{u}^\beta d\Omega$$

- The above system is said to be pseudo-differential of first order if the following limit exists,

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}p(u,x,t,\lambda\omega_i)\alpha_\beta := p_1(u,x,t,\omega_i)\alpha_\beta$$

- If furthermore $ip_1(u,x,t,\omega_i)\alpha_\beta$ has only real eigenvalues and a complete set of eigen-vectors we say the systems is **strongly hyperbolic**.
- Strongly hyperbolic pseudo-differential operators plus technical condition are well posed. [Taylor, Kreiss-Ortiz-R]

Causality

- We consider the domain of dependence of the linearized equation at a given background u_0^α.
- The domain of dependence of a region Σ_0 of a Cauchy surface is given by the maximal foliation of such region produced by hypersurfaces whose normal is such that:

$$E^a(\delta u)n_a \geq 0 \;\; \forall \delta u \leftrightarrow \det(A^{\alpha a}{}_\beta n_a) \neq 0$$

- Surfaces with normal such that the determinant vanishes are called characteristic surfaces.
- For each co-vector k_a which is a characteristic there is a perturbation which in the high frequency limit moves along the integral lines of $V^a = \frac{\partial \det(A^c k_c)}{\partial k_a}$ at points where $\det(A^c k_c) = 0$ $(V^a k_a = 0)$.
- Question: what happens in the case of strongly hyperbolic systems?

Holmgren's Theorem

- Given an analytic coeffi cient equation system (not necessarily hyperbolic!)

$$\partial_t u^\alpha = A^{\alpha i}{}_\beta(x,t)\nabla_i u^\beta + B^\alpha{}_\beta(x,t)u^\beta$$

  and assuming the solution vanishes in a hypersurface Σ_0 then the solution, if suffi ciently smooth, vanishes in a whole neighborhood of it, given by the maximal foliations such that their normals do not become characteristics.

- Generalizable to the case of non-analytic coeffi cients for strng-hyp. systems.
  - Extend the space-time to R^n or T^n.
  - Approximate the system by an analytic sequence of strng-hyp. systems.
  - Use Holmgren's theorem on each one of them to conclude that the one parameter family of solutions so generated vanishes in some region Ω_n.
  - Use continuous dependence of solutions of strongly hyperbolic systems to show that the limiting solution would also vanish in a limiting set Ω.

Summary

- Strongly hyperbolic differential (and pseudo-differential) systems are well posed.
- There are global energy norms (pseudo-differential operators).
- There are covariant definitions. And open set of "space-like" hyper-surfaces.
- Strongly hyperbolic differential (and pseudo-differential) systems have finite propagation speeds. With domain of dependence given by their characteristic fields.
- Symmetric hyperbolic energy $\leftrightarrow$ Summation by parts in finite differences
- Strongly hyperbolic pseudo-energy $\leftrightarrow$ Pseudo-spectral methods.

Applications:

- ADM-BSSN first-second order systems [Frittelli-R, Sarbach-Calabrese-Pullin-Tiglio, Kreiss-Ortiz, Nagy-Ortiz-R]
- Constraint propagation. [Hyperbolicity properties of subsidiary systems of constraints.]

ADM equations I

$$G_{ab} = 0 \quad \Rightarrow \begin{cases} \mathcal{L}_n h_{ab} = -2k_{ab}, \\ \mathcal{L}_n k_{ab} = {}^{(3)}R_{ab} - 2k_a{}^c k_{bc} + k_{ab} k_c{}^c - \frac{D_a D_b N}{N}, \\ {}^{(3)}R + (k_c{}^c)^2 - k_{ab} k^{ab} = 0, \\ D^b k_{ba} - D_a k = 0, \end{cases}$$

ADM equations II

$$\mathcal{L}_{(t-\beta)} h_{ij} = -2N k_{ij}$$

$$\mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} \left[-\partial_k \partial_l h_{ij} - \partial_i \partial_j h_{kl} + 2\partial_k \partial_{(i} h_{j)l} \right] + B_{ij}$$

where

$$B_{ij} := N \left[\gamma_{ikl} \gamma_j{}^{kl} - \gamma_{ij}{}^k \gamma_{kl}{}^l - 2k_i{}^l k_{jl} + k_{ij} k_l{}^l - A_{ij} \right],$$

$$\gamma_{ij}{}^k := \frac{1}{2} h^{kl} (2\partial_{(i} h_{j)k} - \partial_k h_{ij}),$$

$$A_{ij} := a_i a_j - \gamma_{ij}{}^k a_k - 2\gamma_{ikl} \gamma_j{}^{(kl)} + \partial_i [(\partial_j N)/N],$$

$$a_i := (\partial_i N)/N.$$

ADM equations III

Hyperbolicity analysis: 1) consider only the principal part, 2) freeze coefficients, 3) substitute all derivatives by Fourier transforms ($\partial_k h_{ij} \to i\omega_k \hat{h}_{ik}$), and 4) define $\hat{\ell}_{ij} = i\omega\hat{h}_{ij}$. [Kreiss, Ortiz][Taylor]

The associated first order system is then

$$\begin{aligned} \partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2N\hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right], \\ \partial_t \hat{k}_{ij} &\hat{=} i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right] \end{aligned}$$

with $\tilde{\omega}_i = \omega_i / \omega$.

Result:

- ADM equations are only weakly hyperbolic (3 eigenvectors missing).

ADM equations IV

$N = h^b Q$ (h = determinant of h_{ij})

The associated first order system is then

$$\partial_t \hat{\ell}_{ij} \doteq i\omega \left[-2N\hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right],$$

$$\partial_t \hat{k}_{ij} \doteq i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + (1+b)\tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]$$

Result:

- Modified ADM equations for $b > 0$ still weakly hyperbolic (2 eigenvectors missing).
- Adding Hamiltonian constraint does not change hyperbolicity, but does change characteristics.

BSSN equations I

$$f^k = h^{ij}\gamma_{ij}{}^k + dh^{kl}\gamma_{lm}{}^m = h^{kl}(h^{ij}\partial_i h_{jl} + \partial_l \ln h)$$

$$\begin{aligned}
\mathcal{L}_{(t-\beta)} h_{ij} &= -2N k_{ij} \\
\mathcal{L}_{(t-\beta)} k_{ij} &= \frac{N}{2} h^{kl} \left[-\partial_k\partial_l h_{ij} - b\,\partial_i\partial_j h_{kl}\right] + N\partial_{(i} f_{j)} + \mathcal{B}_{ij} \\
\mathcal{L}_{(t-\beta)} f_i &= N[-(2-c)D^k k_{ki} + (1-c)D_i k_k{}^k] + \mathcal{C}_i
\end{aligned}$$

BSSN equations II

Hyperbolicity Analysis:

$$\begin{aligned}
\partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2\alpha \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right] \\
\partial_t \hat{k}_{ij} &\hat{=} i\omega \left[\frac{\alpha}{2} \left(-\hat{\ell}_{ij} - b\, \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} + 2\tilde{\omega}_{(i} \hat{f}_{j)} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right] \\
\partial_t \hat{f}_i &\hat{=} i\omega \left[\alpha \left((-2+c) \hat{k}_{ik} \tilde{\omega}^k + (1-c) \tilde{\omega}_i h^{kl} \hat{k}_{kl} \right) + \tilde{\omega}_k \beta^k \hat{f}_i \right]
\end{aligned}$$

Result: [Nagy-Ortiz-R]

- Modifi ed BSSN equations for $b > 0$ $c > 0$ strongly hyperbolic.
- Eigenvalues: $(0, \pm 1, \pm\sqrt{b}, \pm\sqrt{c/2})$

Constraint Propagation

Evolution System:

$$\partial_t u^\alpha = A(u,t,x)^{\alpha a}{}_\beta \partial_a u^\beta + B(u,t,x)^\alpha,$$

Constraints:

$$C^A = K(u,t,x)^{Aa}{}_\beta \partial_a u^\beta + L(u,t,x)^A,$$

Integrability condition (subsidiary system):

$$\partial_t C^A = S(u,t,x)^{Aa}{}_B \partial_a C^B + R(u,\partial u,t,x)^A{}_B C^B,$$

- Want to study what can we say about the properties of the subsidiary system from what we know from the evolution system.

Constraint Propagation II

- Problem: In general $S(u,t,x)^{Aa}{}_B$ is not unique if the constraint themselves satisfy certain identities.

For instance, if there is an $L_A(\omega)$ such that:

$$L_A(\omega)K^{An}{}_\alpha \omega_n = 0$$

we could add to $S(u,t,x)^{Aa}{}_B$

$$M^{Aa}L_B$$

- With this addition there are easy examples where one can get any sort of badly posed systems!

Constraint Propagation III

- Assume: For any ω_i, $K^{An}{}_{\alpha}\omega_n$ is surjective.
- In general this is not satisfied, but in examples of interest one finds subset of constraints which do satisfy it. [Maxwell, EC].

Constraint Propagation IV

Integrability condition implies:

$$K^{A(a}{}_{\alpha} A^{|\alpha|b)}{}_{\beta} - S^{A(a}{}_{B} K^{|B|b)}{}_{\beta} = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a. If (σ, u^α) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_{\beta}\omega_a$ then $(\sigma, v^A = K^{Aa}{}_{\alpha}\omega_a u^\alpha)$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_{B}\omega_a$.

Constraint Propagation IV

Integrability condition implies:

$$K^{Aa}{}_{\alpha} A^{\alpha b}{}_{\beta} \omega_a \omega_b - S^{Aa}{}_{B} K^{Bb}{}_{\beta} \omega_a \omega_b = 0$$

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Constraint Propagation IV

Integrability condition implies:

$$K^{Aa}{}_{\alpha} A^{\alpha b}{}_{\beta} \omega_a \omega_b - S^{Aa}{}_{B} K^{Bb}{}_{\beta} \omega_a \omega_b = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a. If (σ, u^{α}) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_{\beta}\omega_a$ then $(\sigma, v^A = K^{Aa}{}_{\alpha}\omega_a u^{\alpha})$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_{B}\omega_a$.
- **Corollary 1:** If the evolution system is strongly hyperbolic then so is the subsidiary system. [It does not work symmetric $\rightarrow$ symmetric].
- **Corollary 2:** The characteristics of the subsidiary system are a subset of the characteristics of the evolution system. The domain of dependence of the subsidiary system is at least as large as the domain of dependence of the evolution system.