Numerical Relativity beyond General Relativity
status, challenges and new directions

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presentation with Luis Lehner

GREFT20: Probing Effective Theories of Gravity in Strong Fields and Cosmology
27 August 2020

DiRAC

1. Numerical (General) Relativity in a nutshell
2. Status: Numerical Relativity beyond the standard model
3. Interludium I: new directions and approaches – a discussion
4. Part II: Roadblocks and a potential way through (Luis Lehner)

Numerical (General) Relativity

$=$

Solve
$G_{\mu\nu} = 16\pi T_{\mu\nu}$

in $3+1$ dimensions
on High-Performance Computing Facilities

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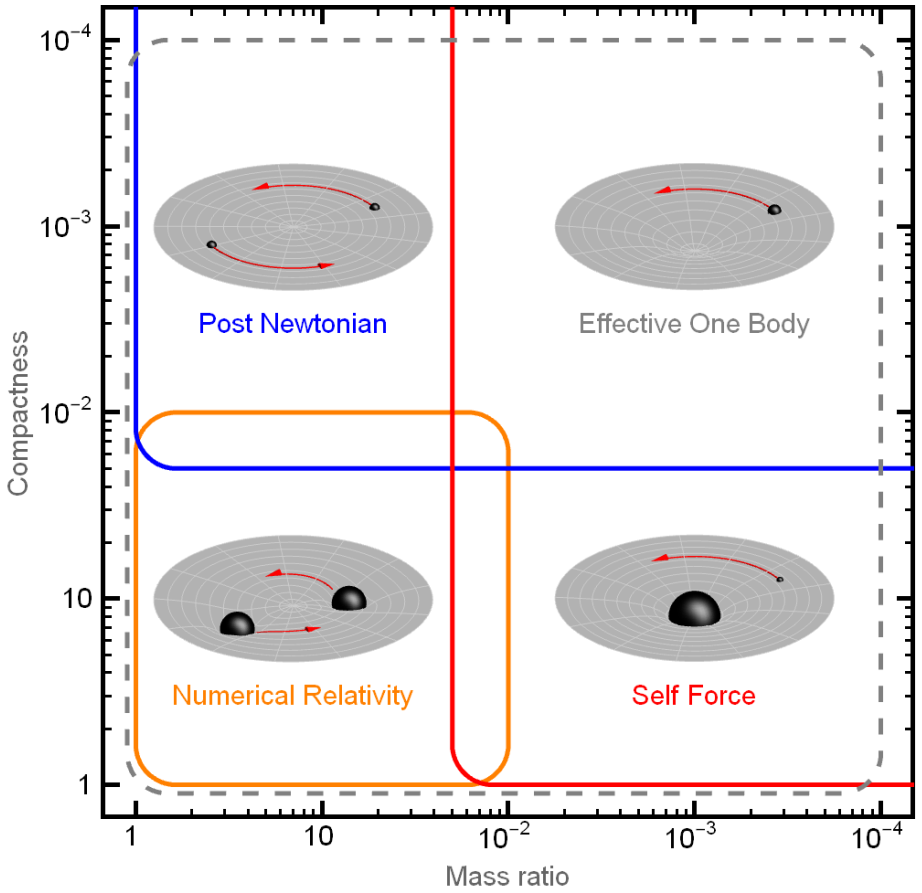
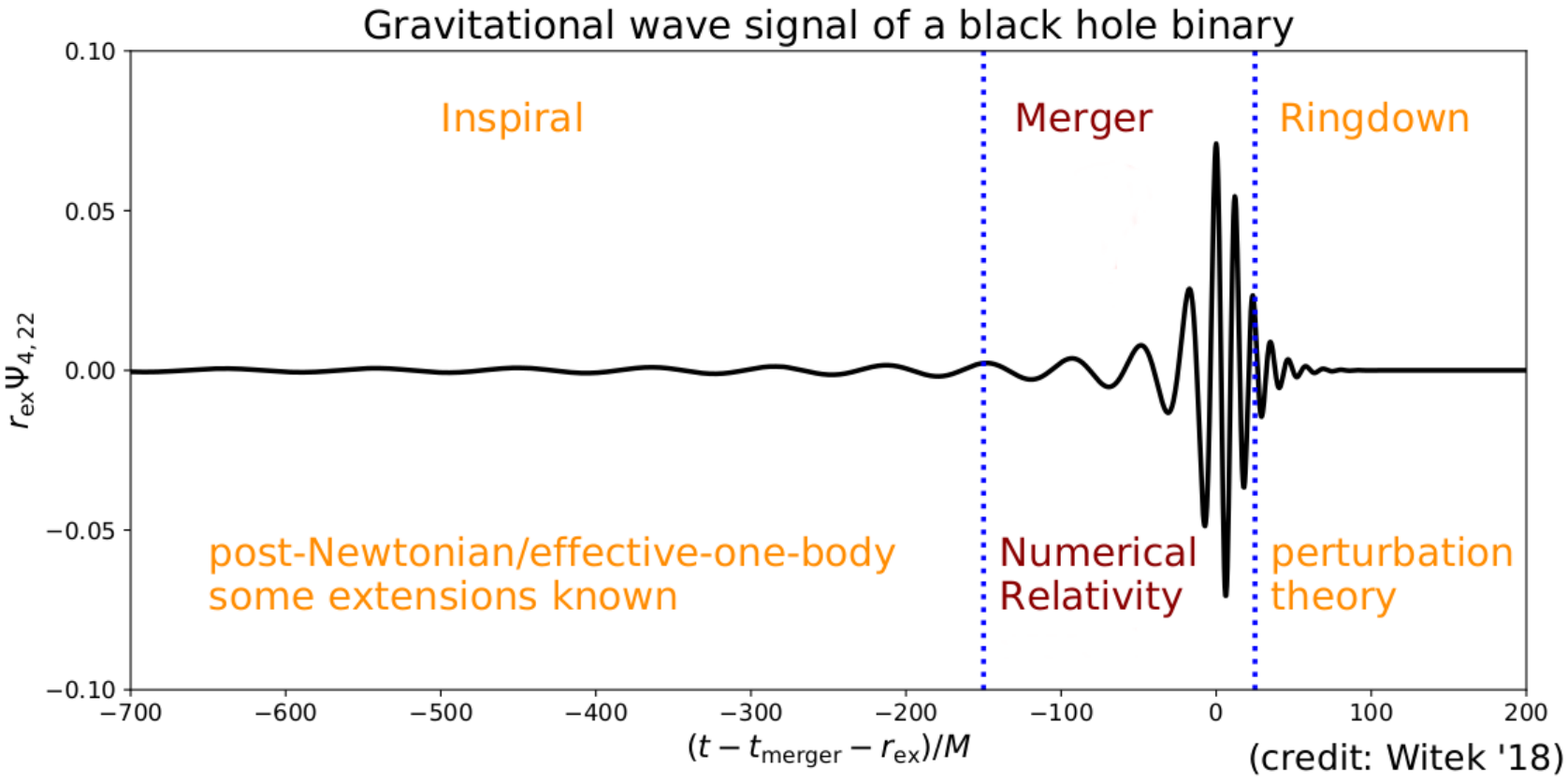
The realms of Numerical Relativity



(https://www.wikiwand.com/en/Two-body_problem_in_general_relativity)


Gravitational wave signal of a black hole binary
Inspiral
Merger
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Numerical
Relativity
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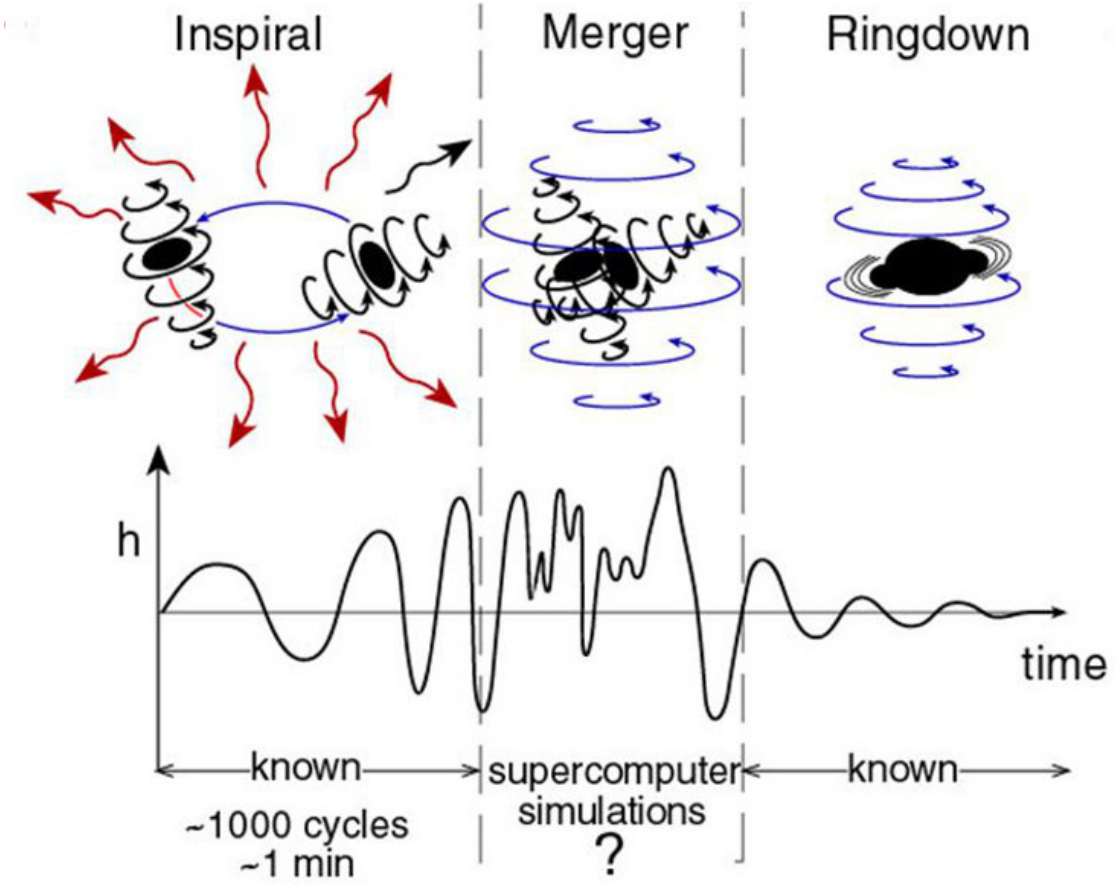
−100
0
100
200
$r_{ex}\Psi_{4,22}$
$(t - t_{merger} - r_{ex})/M$
(credit: Witek '18)

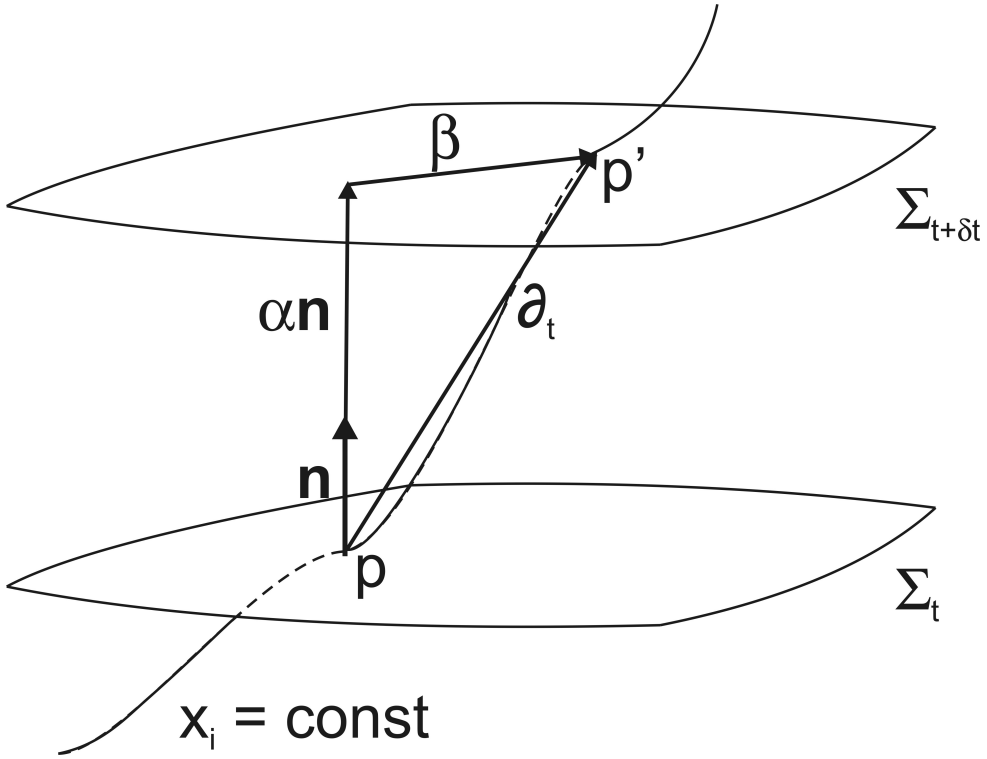


(credit: Thorne; pre-NR breakthrough by Pretorius '05)

Numerical General Relativity in a nutshell

- make space and time dependence explicit: foliation of spacetime $\mathcal{M} = \Sigma_t \times R$
- 4D metric $g_{\mu\nu} \longrightarrow$ 3D metric γ_{ij}, lapse α, shift β^i
- extrinsic curvature $K_{ij} = -\gamma^\mu{}_i \gamma^\nu{}_j \nabla_\mu n_\nu = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$

  $\Rightarrow$ kinematic evolution equation $\partial_t \gamma_{ij} \simeq -2\alpha K_{ij}$



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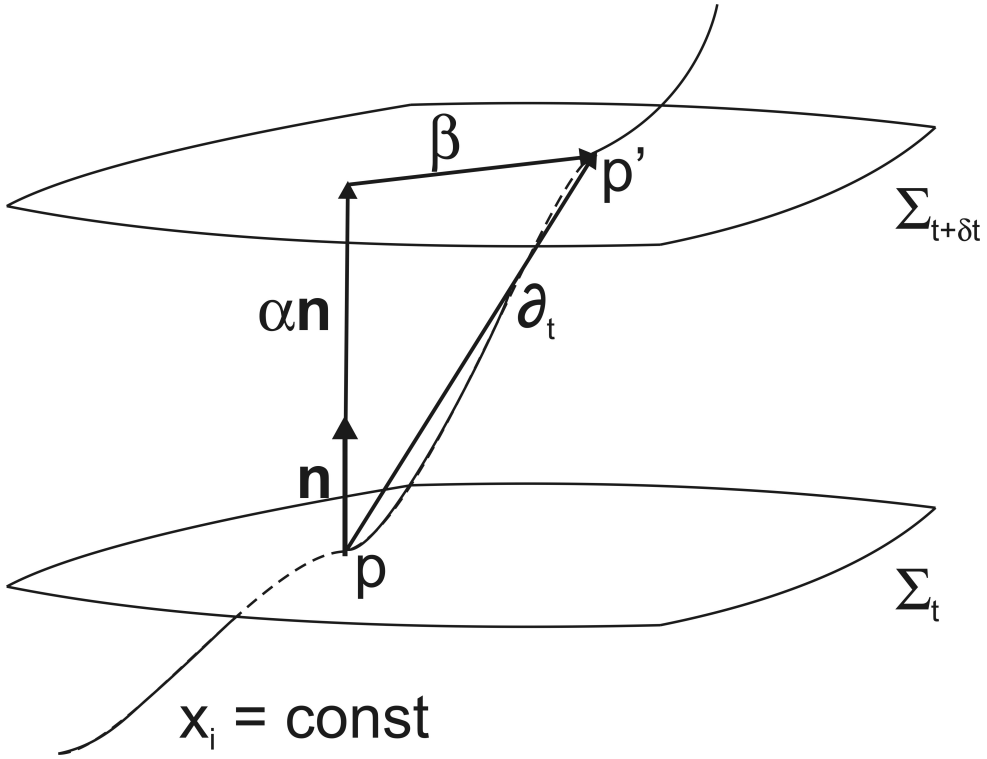

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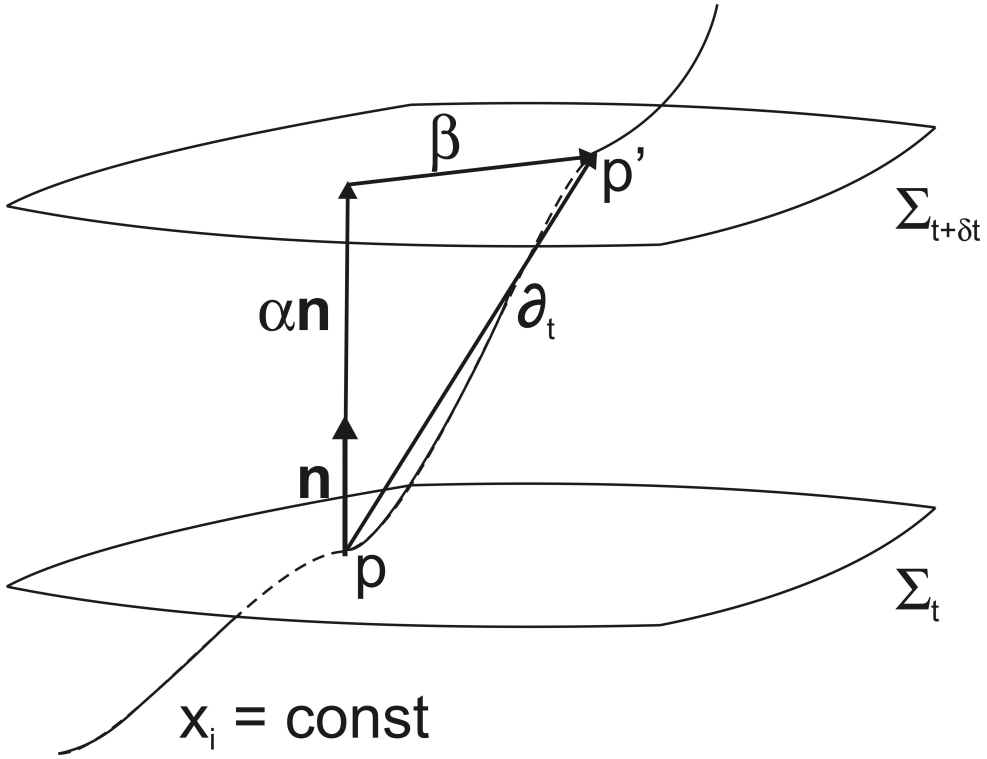
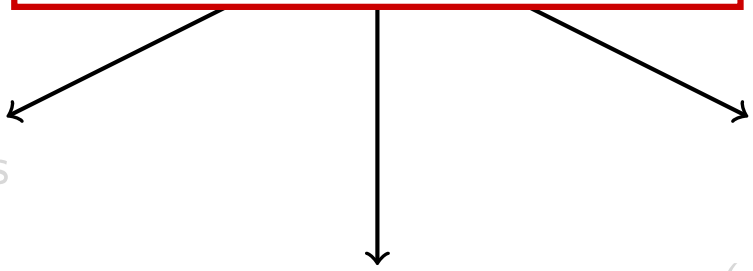


Numerical General Relativity in a nutshell

Dynamics: decompose field equations

$$\underbrace{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{geometry}} = \underbrace{16\pi T_{\mu\nu}}_{\text{matter}}$$



Constraint equations
- elliptic PDEs
- initial conditions

Evolution equations
- hyperbolic (time evolution) PDEs
- well-posed initial value formulation
- supplement with gauge conditions

Diagnostic tools
- Wave extraction
- (apparent) horizon finder

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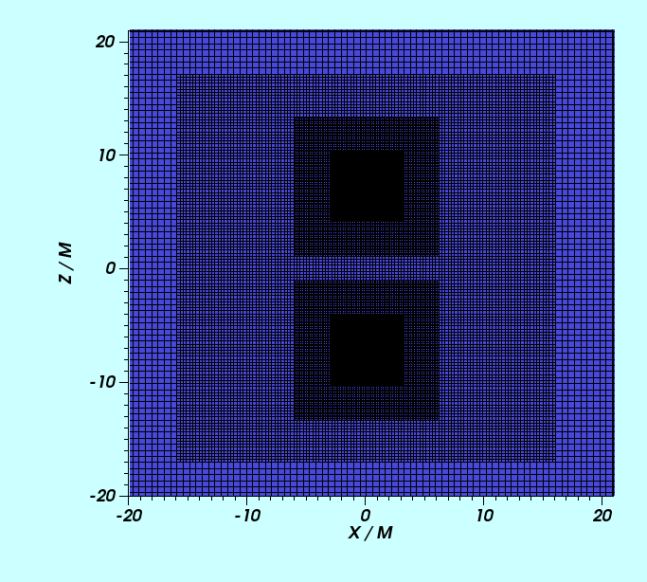
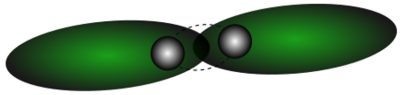


Technical implementation

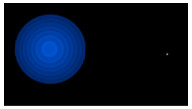
- numerical schemes
  (e.g. method of lines with finite differences and RK time integrator)
- adaptive mesh refinement
- high performance computing

Numerical Relativity beyond the standard model

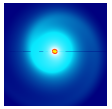
Black holes as cosmic particle detectors



Black hole mimickers boson stars



Testing the nonlinear regime of gravity



Numerical Relativity in modified gravity

$$
\begin{aligned}
\mathcal{L} = \quad & f_0(\phi)R - \omega(\phi)(\nabla\phi)^2 - V(\phi) + \mathcal{L}_{\rm M}[\Psi, A^2(\phi)g_{ab}] && \text{scalar-tensor theory} \\
& + f_1(\phi)\left(Riem^2 - 4R_{ab}R^{ab} + R^2\right) + f_2(\phi)\,{}^*R_{abcd}R^{abcd} && \text{quadratic gravity} \\
& + \ldots && \text{Horndeski} \\
& + \ldots && \text{Lorentz violation} \\
& + \ldots
\end{aligned}
$$

Numerical Relativity in modified gravity

black-hole formation

- Einstein-æther theory (Garfinkle et al '07)
- Gauss–Bonnet gravity (Benkel, Sotiriou, **HW** '16; Ripley & Pretorius '19, '20, Dima et al '20)
- Horndeski gravity (Ripley & Pretorius '19, Bernard et al '19, Figueras & França '20)
- (massive) scalar-tensor theory (Gerosa et al '16, Sperhake et al '17, Rosca-Mead et al '19, '20)

compact binaries

- scalar-tensor theory (Barausse et al '12, Shibata et al '13, Healy et al '11, Berti et al '13)
- Einstein-Maxwell-Dilaton models (Hirschmann et al '17)
- dynamical Chern-Simons gravity (Okounkova et al '17 – '19)
- scalar Gauss–Bonnet gravity (**HW**, L. Gualtieri, P. Pani, T. P. Sotiriou '18, Okounkova '20)

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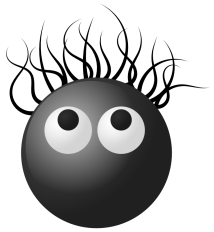
Example: NR in scalar Gauss–Bonnet gravity

Example: scalar Gauss-Bonnet gravity (sGB) (Kanti et al '95)

$$G_{\mu\nu} = \frac{1}{2} T^{\Phi}_{\mu\nu} - \frac{\alpha_{\rm GB}}{4} \mathcal{G}^{\rm GB}_{\mu\nu} \,, \qquad \Box\Phi = -\frac{\alpha_{\rm GB}}{8} f'(\Phi) \mathcal{R}_{\rm GB}$$

with $\mathcal{R}_{\rm GB} = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \sim (\partial\partial g_{\mu\nu})^2$

- low-energy limit of quantum gravity candidates
- black holes with scalar hair for $f(\Phi) \sim \Phi$
  (Kanti et al '95, Pani et al '09, '11, Stein et al '11, Sotiriou & Zhou '14, Ayzenberg & Yunes '14, Maselli et al '15, Benkel, Sotiriou, **HW** '16, '17; **HW**, Gualtieri, Pani, Sotiriou '19)
- black hole scalarization for $f(\Phi) \sim \Phi^2$
  (Silva et al '17; Doneva et al '17; Antoniou et al '17; Silva et al '18; Macedo et al '19; Doneva et al '19; Ripley & Pretorius '20; Dima et al '20)



Example: NR in scalar Gauss–Bonnet gravity

- well-posed initial value formulation necessary for numerical stability
  (for higher derivative theories see Choquet-Bruhat '88; Delsate, Hilditch, **HW** '14; Papallo & Reall '17; J. Cayuso et al '17; Allwright & Lehner '18; Kovacs & Reall '20; Kovacs '20; R. Cayuso & Lehner '20)
- full 3+1 formulation? double-valued Hamiltonian (Julié & Berti '20; **HW**, Gualtieri, Pani '20)
- expansion in coupling $\epsilon = \alpha_{\rm GB}/\ell^2 \ll 1$ (see Okounkova et al '17 for dCS)

$$\epsilon^0 : \; G^{(0)}_{ab} = \tfrac{1}{2} T^{(0)}_{ab} \qquad \Box^{(0)}\Phi^{(0)} = 0 \qquad \Rightarrow \; (g^{(0)}_{ab}, \Phi^{(0)}) = (g^{\rm GR}_{ab}, 0)$$

$$\epsilon^1 : \; G^{(1)}_{ab} = 0 \qquad \Box^{(0)}\Phi^{(1)} = -f' \mathcal{R}^{(0)}_{\rm GB} \qquad \Rightarrow \; (g^{(1)}_{ab}, \Phi^{(1)}) = (0, \Phi^{(1)})$$

$$\epsilon^2 : \; G^{(2)}_{ab} = \tfrac{1}{2} T_{\rm eff}(g^{(0)}_{ab}, \Phi^{(1)}) \qquad \Box^{(0)}\Phi^{(2)} = -\tfrac{M^2}{4} f'_{(1)} \mathcal{R}^{(0)}_{\rm GB}$$

Discussion appetizer (1 slide to go)

- + structure applicable to theories with EFT-type expansion
- + pathway towards "parametrized numerical relativity"
- - missing physics? nonlinear effects?
- - validity?

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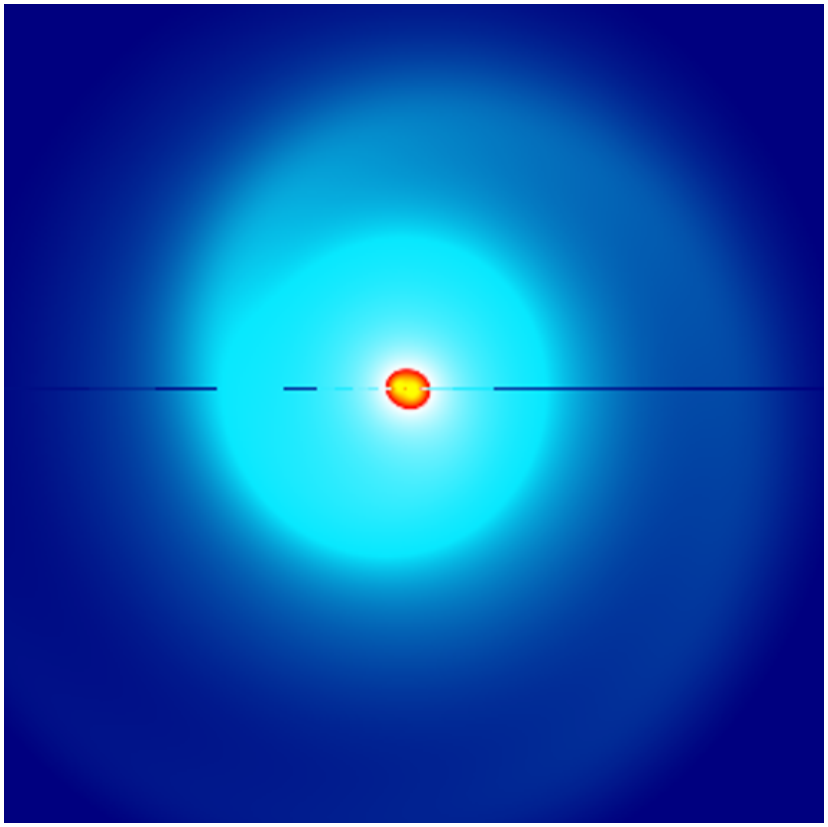
Proof-of-principle: evolution of hairy black holes



(**HW**, Gualtieri, Pani, Sotiriou '19)

Interludium I: Discussion

Guiding principle?

- Developing the numerical infrastructure takes 1-2 years per theory
- Covering parameter space takes years and large amount of resources!
  (black hole mass ratio, spins; coupling parameters; . . .)
- Which extensions of the standard model are scientifically most interesting?
  - physical motivation from cosmology / HEP?
  - existence, uniqueness and stability of solutions? Is flat space stable?
- Which extensions of the standard model are practical from NR perspective?
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Methodologies?

- "Parametrized" versus theory-specific numerical relativity?
- effective-field theory approach versus "full theory"?
  - EFT $\rightarrow$ hyperbolic equations with sources
  - What might we miss in EFT?
  - Nonlinear effects, treatment of "secular" effects, validity?
- well-posedness of specific theory?
  - Maths: well-posed initial value formulation
  - Physics: well-defined (theoretically well-posed) model
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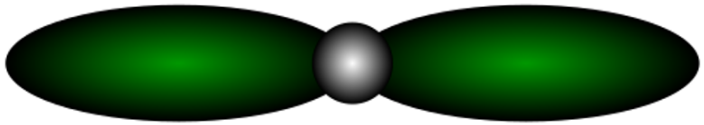
(Press & Teukolsky '72; Damour et al '76; Detweiler '80; Zouros & Eardley '79; Cardoso et al '05; Dolan '07; Rosa & Dolan '11; Pani et al '12; **HW** et al '12; Dolan '12; Shlapentokh-Rothman '14; Okawa, **HW** et al '14; Brito et al '15; Zilhao, **HW** et al '15; Moschidis '16; East '17, '18; Frolov et al '18; Dolan '18; Ficarra, Pani, **HW** '19, Baumann et al '18, '19, Herdeiro et al '19, Siemonsen & East '19, Creci, Vandoren, **HW** '20, ...)

- superradiant instability
  $\rightarrow$ formation of bosonic condensates



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BHs ($M_\odot$):	10^{10}	10^6	10^3	50	3
μ (eV):	10^{-21}	10^{-17}	10^{-14}	10^{-12}	10^{-11}

- axion-like particles as dark matter candidates, string axiverse
  (Peccei & Quinn '77, Arvanitaki & Dubovsky '10, '11, Kodama & Yoshino '11, Hui et al '16, Baumann et al '18, '19, ...)
- "hairy" black-holes (Herdeiro et al '14, Hui et al '19, Clough et al 19, ...)
- Any ultra-light bosonic field coupled to gravity
  $\Rightarrow$ black holes as probe for BSM particles complementary to traditional colliders

Black holes as cosmic particle detectors

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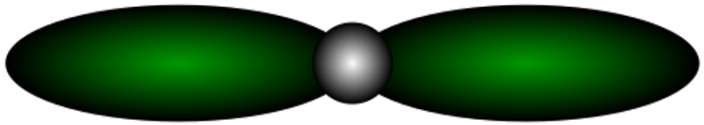

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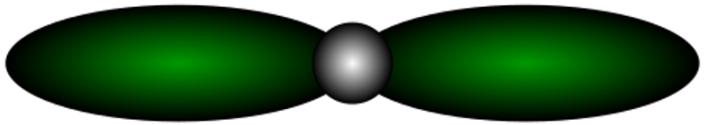

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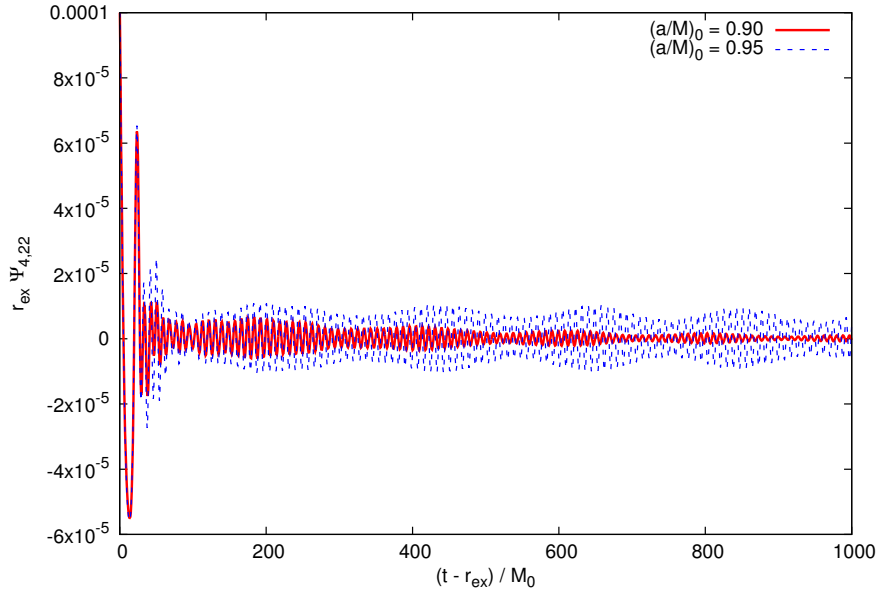
- "hairy" black-holes (Herdeiro et al '14, Hui et al '19, Clough et al 19, ...)
- Any ultra-light bosonic field coupled to gravity
  $\Rightarrow$ black holes as probe for BSM particles complementary to traditional colliders

Black holes as cosmic particle detectors

Observable signatures:

- gaps in spin–mass phase space of black hole population
  (Arvanitaki et al '09, '10; Pani et al '12; Brito et al '15-'20; Ficarra et al '18; . . .)
- black hole shadow (Herdeiro et al '19; Creci, Vandoren, **HW** '20,. . .)
- gravitational waves with $f_{22} \sim 20 \left[\frac{M}{M_\odot}\right]^{-1}$ kHz

  (Arvanitaki et al '14; Yoshino et al '13; Okawa, **HW** , Cardoso '14; Zilhão, **HW** '15, East et al '17-'20)

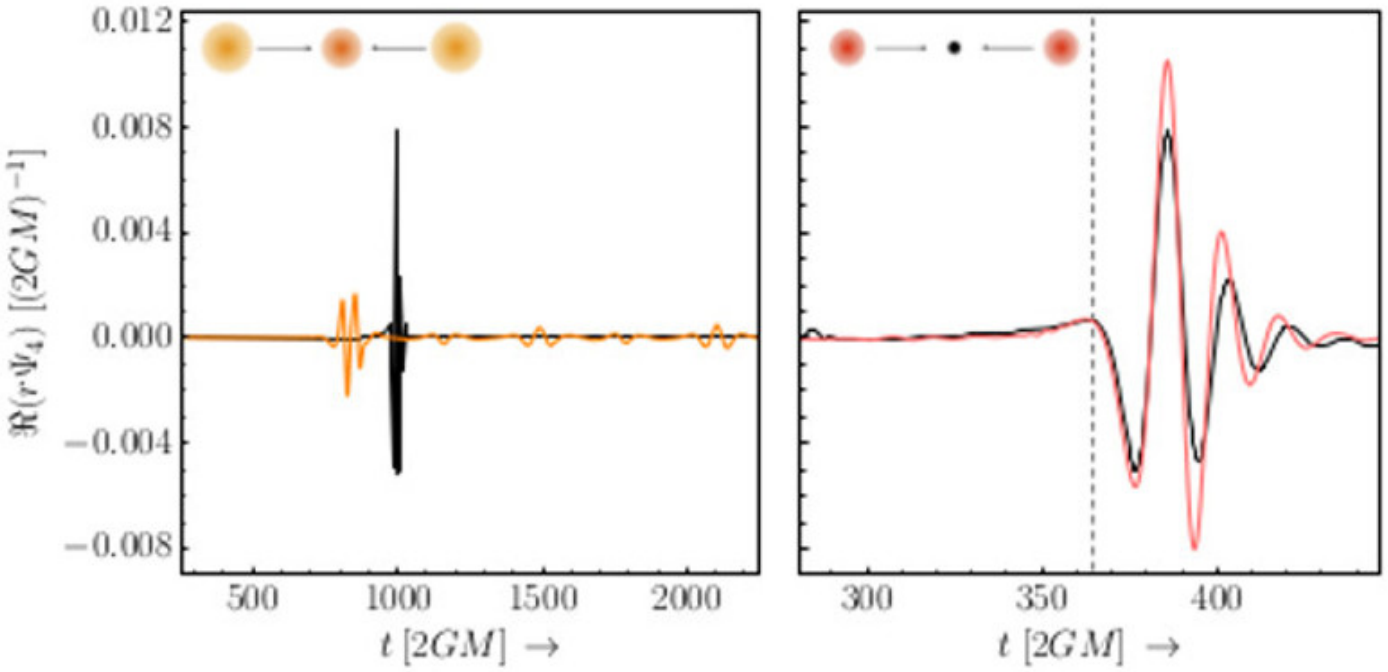


(**HW** & Zilhão; code: Einstein Toolkit & Canuda)

- Ongoing: BH binary evolution
  (Baumann et al '18 – '20, Wong et al '19, Hang & Zhang '19, Berti et al '19. . .)

Did we really detect the black holes of GR?

- (scalar) boson stars (Liebling & Palenzuela '12, Palenzuela et al '17, Helfer et al '18, Bezares et al '18, Alcubierre et al '19, ...),
- Proca stars (Sanchis-Gual et al '18)
- black holes with near-horizon fluctuations (Liebling et al '17)
- axion stars and black holes or neutron stars (Clough, Dietrich et al '18)



Example: boson stars (Helfer et al '18)

Extensions to GR: roadblocks and potential way through

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[based on results with: Juan Cayuso, Gwyneth Allwright, Raimon Luna, Ramiro Cayuso, Guillaume Dideron, *Nestor Ortiz & Laura Bernard*]

Preliminaries

- *Not a goal here*: to advocate for/or analyze a particular theory

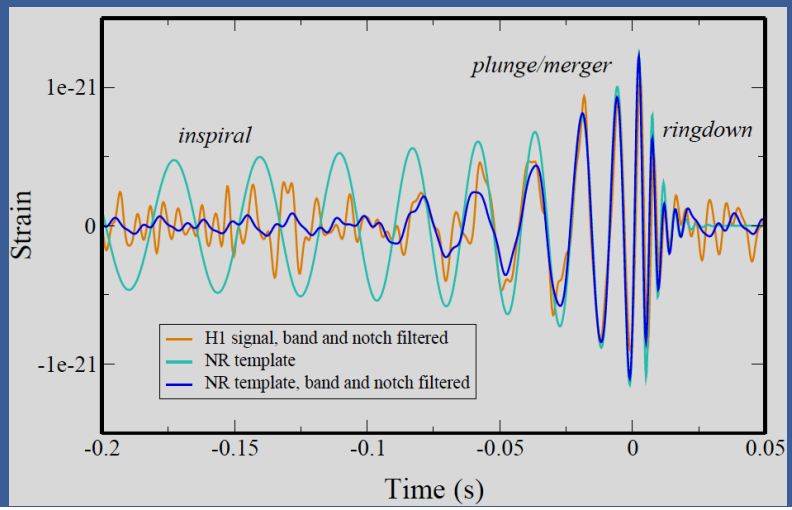
- *Goals:*
  - Discuss issues to bear in mind when considering any extension
  - Illustrate such issues do arise & their consequences
  - Discuss options to *plow* forward

Terminology

- A modified gravity theory called *interesting* : consistent with existing tests/experiments of general relativity in the weak-field, but may offer observable differences in the strong-field

- A modified gravity theory called *viable* if it possesses a well-posed initial value formulation that can be solved to make concrete predictions to confront with data

  - *i.e., looking for viable, interesting modifications to GR*



What should one consider modifying?

- modified coupling between matter/geometry
- modified geometry (LHS) [e.g. further degrees of freedom, relaxing symmetries…]
- modified matter (RHS): “exotic” alternatives to black holes/neutron stars
- We have strong anchors! GR/BHs/NSs → EFT approach, corrections to Einstein comes with further operators/higher derivatives and some *key* assumptions

Underappreciated (?) issues

- *'linearization stability'* : soln to linearized eqns <-> solution to full problem in linearized regime
  - e.g. take Navier Stokes → all modes decay (*no turbulence!*)

- *'regime of applicability'* : higher operators / corrections stay small.
  - But, we have both 'stability of Minkowski' and *singularity theorems* in GR already

- *Well posedness* is more than a requirement!

‘Over’ appreciated points

- 2^{nd} order equations → lack of Ostrogradski’s ghost, we’re good….

- Reduction of order strategy → treat corrections perturbatively (as in S-matrix calculations), we’re good….

Modified Coupling Between Geometry and Matter

- Prototypical example, scalar tensor theories

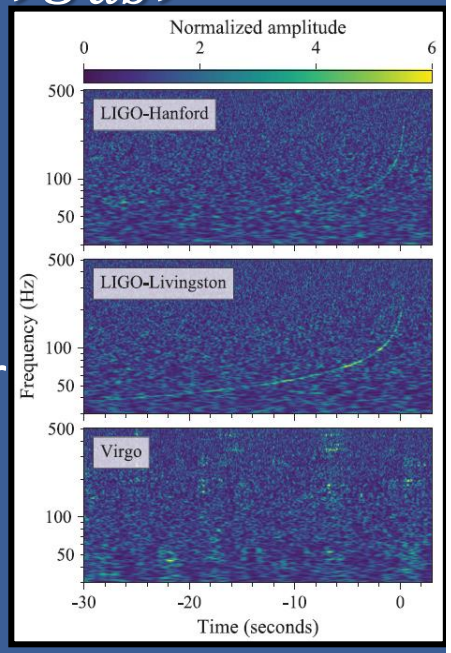
$$L \propto R(g_{ab}) + L_\theta(\theta, g_{ab}) + L_\varphi(\varphi, f(\varphi)g_{ab})$$

  - Einstein gravity with a scalar field θ, but other matter ϕ experiences "physical" geometry through a scalar field rescaled metric

- These are arguably the only class of modified gravity theories shown to be *interesting and viable*; however

  - with scalar tensor theories need "spontaneous scalarization" (Damour & Esposito Farese), or 'dynamical/induced scalarizaton' (Barause etal) to provide observationally interesting modifications, and also only relevant for binaries with neutron stars

  - with Einstein-Maxwell-Dilaton, need black holes with a significant charge [Hirschmann etal] , also [Julie] ….. Or use Jacobson's tonic ;-)



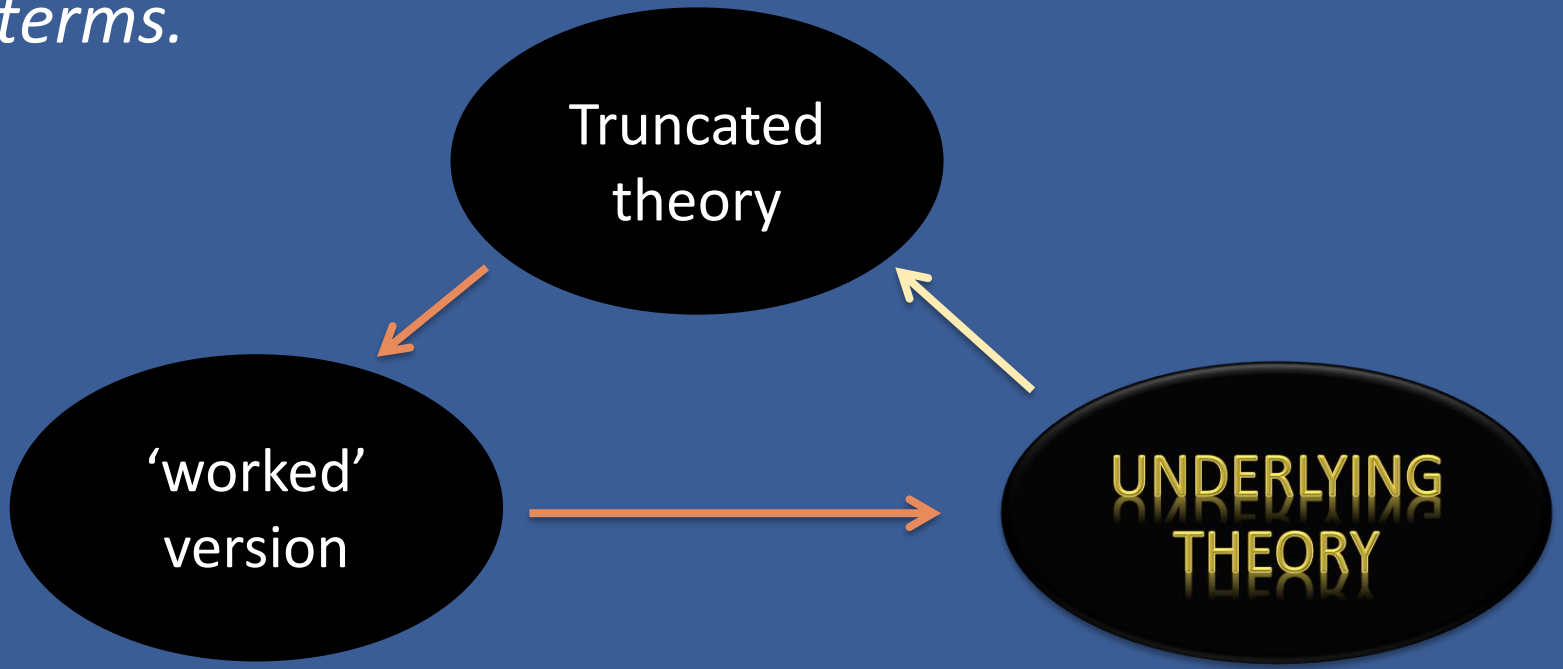
Horndenski: observations

- Ill posedness (wrt a specific frame) if $G_4=G_5=0$ (and nonlinearly even G_3) [Papallo-Real]
  - Specific frame? Keep G_4 → breakdown of hyperbolicity (depending on coupling and ID) (regardless of frame [Bernard-Luna-LL]. Also, for 'weak data' all likely good, *for the full nonlinear problem*
  - Different frame, and *for weak coupling*, strong hyperbolicity [Kovacs-Reall]
  - (related works by Garfinkle; Ripley-Pretorious; Figueras-Franca, Bezares etal)

- Caveat: even with coupling/ID ``under control''. Can we rely on conditions for strong hyperbolicity holding?
  - what happens for smooth data, travelling in the vicinity of strong focusing effects? If (when) characteristic cross what is the *extra condition*?
  - Wrt to initial conditions (e.g. in cosmology), what ensures 'past evolution' leads to such weak data?
  - These are tough qns to answer in general….[recall Minkowski vs sing form!]
  - What criteria guides 'where to truncate'? Doesn't this run in tension with the EFT philosophy? Eg: $L(x) = b X^2 + c X^3$ (why not $X^4 \cdots$?)

So, that's at 2nd order…..beyond?

- Effective Field Theories: no new degrees of freedom (as they are integrated out) beyond some scale. Effects enters through suitable expansions wrt a given scale.
  - This generically introduces *higher order derivative terms.*

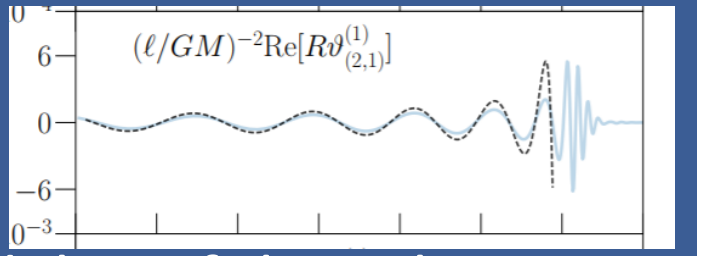


Strategies?

- 'Iteration/perturbation'

$$B(g^*) = 0 \;\rightarrow B(g) = S(g^*)$$

  - Rinse and repeat: *but during what time frame*?
  - Perturbative hierarchy [e.g. Okounkova-Stein +]



  - justified? can one guarantee the fidelity of the solution obtained? [recall Penco's comment, though smallness doesn't immidiately imply correctness as effects accumulate]

- 'Modification'

$$B(g) = F\,;F_t = -\lambda(F - S(g))$$

  - Modify system of equations to 'fix' problems [Cayuso+ '17]
  - *Introduces a new timescale* λ, can one guarantee the fidelity of the solution obtained?

- *Option 1:*

Take Navier-Stokes, solution of:

$$\partial_t \hat{v} + \hat{v}\partial\hat{v} = 0 \ (\sim \eta \partial^2 v)$$

displays turbulence for all wavelengths. What recovers the laminar regime? [short wavelength!]

$$\partial_t v + \hat{v}\partial v + v\partial\hat{v} = \eta\partial^2 v$$

Introduces a secular dependence, and if resummation is possible, it would reveal the laminar/turbulent regimes.

Option 2

- Modify the system of eqns, in an *ad-hoc* manner to control higher gradients and prevent wild runaway to the UV. E.g. Israel-Stewart formulation of viscous relativistic hydrodynamics:
- $T = T^{pf}$ + gradient terms
  - Define $\Pi =$ (shear/bulk)$_{ab}$ + Grad(shear/bulk..)$_{ab}$ as new and independent variable
  - Force an eqn on Π such that $\Pi \sim$ (shear/bulk)$_{ab}$ to leading order always

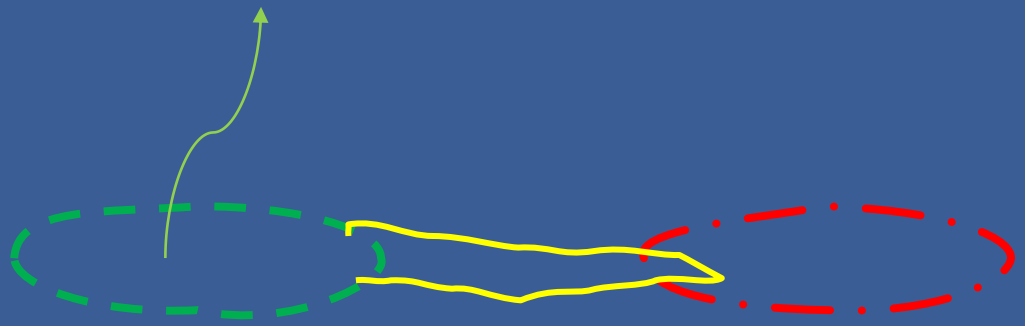
    $\tau \, \Pi_{,t} = - \Pi +$ (shear/bulk)$_{ab}$ .... [Geroch, details shouldn't matter]

So, mathematically now 'in check'. How about physically? Without a complete problem it is hard to tell

Higher than 2nd order 'observations'

- Explicit study, in the context of a complete theory, which when integrating out heavy fields → EFT with higher order derivs. [Burgess-Williams]. *For this model* [Allwright-LL]
- Reduction of order copes in weak coupling/smooth ID but fails otherwise (or gives solns with large errors wrt to 'true' solution)
  - 'Fixing' eqns keeps the solution in check and with rather small errors even for relatively large coupling
- Recently applied to $L \sim R + k\,(R_{abcd}\,R^{abcd})^2$ type theory [R. Cayuso-LL]
  - Role of corrections in QNMs
  - Growth (and temp decrease!) of Horizon area (due to effective NEC violation)
- *Some practical aspects*
  - Higher time derivatives? → 'reduction of order'/field redefinition [e.g. Trodden-Solomon]
  - Potential re-interpretation of 'fixing variables' in ``physical terms'' [discussions with Tolley]

- Why fix?
  - IVP & EFT arguments could be well defined wrt particular sets of ID!
  - Non-linearities require simulations (at least till we understand what to expect)



  - Genuinely non-linear 2nd order eqns & higher order derivs bring 'similar' issues to deal with

Does it work in 'Horndenski'?

$$\partial_t \phi = \partial_{xx}\phi + \lambda \partial_x \phi \partial_{xx}\phi \rightarrow k^2\phi + \lambda k^3 \phi^2$$

- 'EFT'

$$\partial_t \phi = \partial_{xx}\phi + \lambda \partial_{xxx}\phi \rightarrow k^2\phi + \lambda k^3 \phi$$

Final words

- 'fixing equations' gives a controlled way to plow ahead
  - Preliminary work indicates can also tame issues in Horndenski theories [Dideron]
  - Ongoing work to tie/classify potential departures through a membrane paradigm [R. Cayuso]

- Explicit examples of *potential* limitations of 'reduction of order' approach [warning: terminology varies!]

- GWaves + fluid-gravity correspondence gives hopes for d=4 gravity through an EFT.
  - If not, perhaps (?) employ a strategy for studying Quark-Gluon Plasma with Israel-Stewart.

THANKS!

Example!

- Burgess-Williams ’14

$$\frac{S}{v^2} = -\int d^4x \left[\frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}(1+\rho)^2\partial_\mu\theta\,\partial^\mu\theta + V(\rho)\right]$$

where

$$V(\rho) = \frac{M^2}{2}\left(\rho^2 + \rho^3 + \frac{1}{4}\rho^4\right).$$

Varying this action gives the classical equations of motion:

$$\Box\rho - (1+\rho)(\partial\theta)^2 - V'(\rho) = 0$$
$$\partial_\mu\left[(1+\rho)^2\partial^\mu\theta\right] = 0\,,$$

- Integrating out ϱ $\rightarrow$ $$-\int d^4x \left\{\frac{1}{2}(\partial\theta)^2 - \frac{1}{2M^2}(\partial\theta)^4 + \frac{2}{M^4}(\theta_{\mu\nu}\theta^{\mu\rho})\,(\partial_\rho\theta\,\partial^\nu\theta)\right\}$$

- and EOM

$$\begin{aligned} 0 \;=\; & -\Box\phi \\ & + \frac{1}{M^2}\left(4\phi^{,l}\phi^{,m}\phi_{,lm} + 2\phi_{,l}\phi^{,l}\Box\phi\right) \\ & + \frac{1}{M^4}\left(4\phi^{,l}\phi^{,m}(\Box\phi)_{,lm} + 4\phi^{,lm}\phi_{,lm}\Box\phi + 8\phi_{,m}\phi_{,lq}\phi^{,mlq} + 4\phi_{,q}(\Box\phi)\partial^q\Box\phi + 4\phi_{,q}\phi^{,mq}\partial_m\Box\phi\right) \end{aligned}$$

[large M cases]

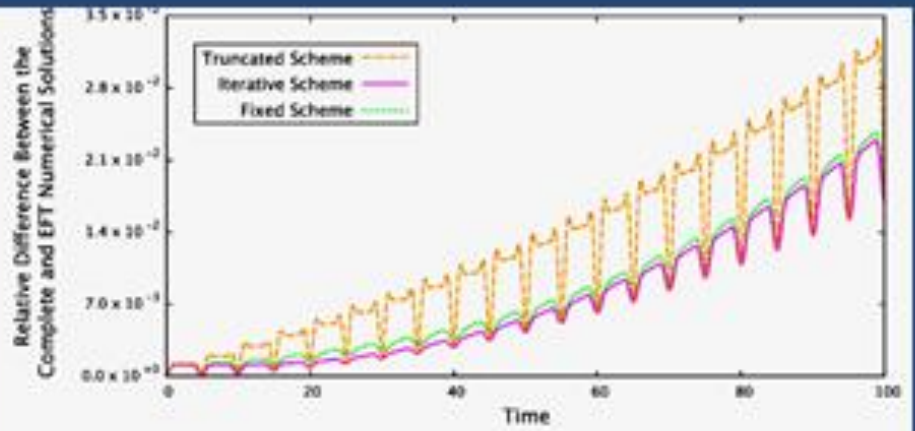


Figure 2: Plot of the relative difference between the UV-complete and EFT solutions for three different numerical techniques: the truncated, iterative and fixed schemes, using $M = 0.1$.

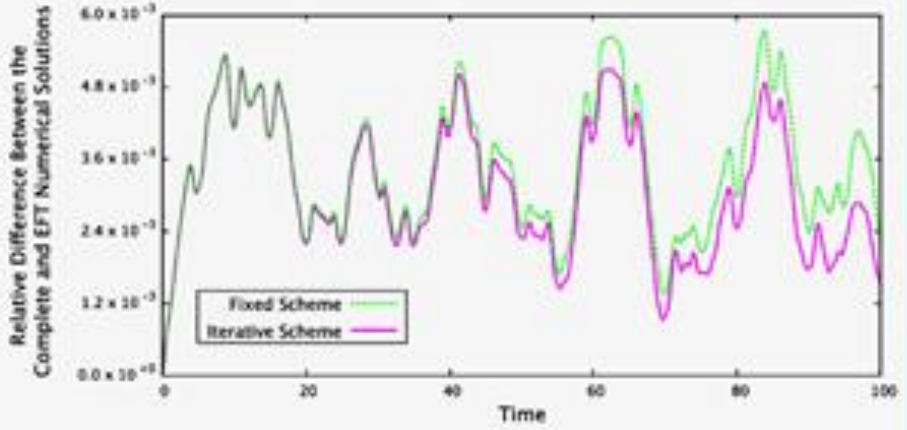


Figure 3: Plot of the relative difference between the UV-complete and EFT solutions for two different numerical techniques: the iterative and fixed schemes, using $M = 1$.

Small m case

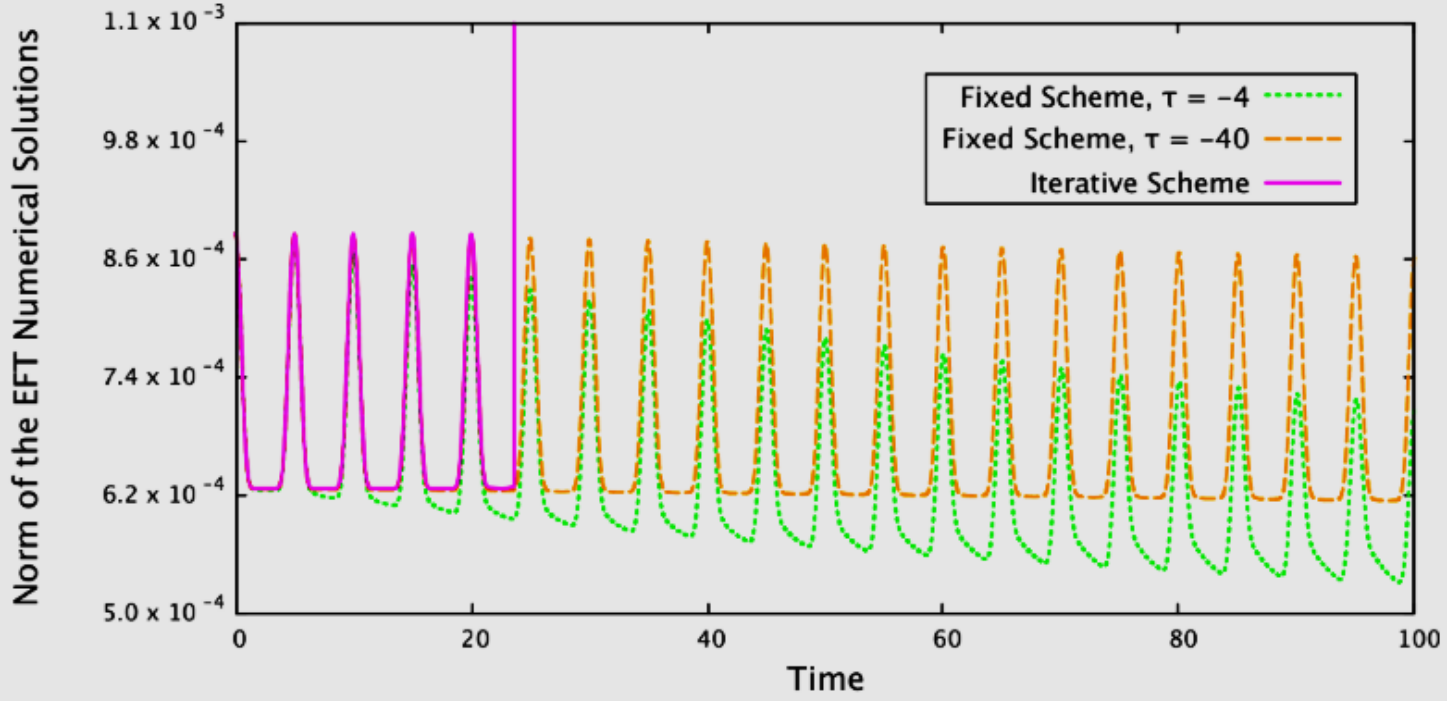


Figure 4: Plot of the L_2-norm of the EFT solutions for two numerical techniques: the iterative and fixed schemes, using $M = 0.01$.

Probing further, most general 2nd order theory

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(\Sigma_{i=1}^{5}\mathcal{L}_i) \tag{1}$$

where,

$$\mathcal{L}_1 = R + X - V(\phi)\,, \tag{2}$$
$$\mathcal{L}_2 = \mathcal{G}_2(\phi, X)\,, \tag{3}$$
$$\mathcal{L}_3 = \mathcal{G}_3(\phi, X)\Box\phi\,, \tag{4}$$
$$,\mathcal{L}_4 = \mathcal{G}_4(\phi, X)R + \partial_X\mathcal{G}_4(\phi, X)\delta^{ac}_{bd}\nabla_a\nabla^b\phi\nabla_c\nabla^d\phi\,, \tag{5}$$
$$\mathcal{L}_5 = \mathcal{G}_5(\phi, X)G_{ab}\nabla^a\nabla^b\phi - \frac{1}{6}\partial_X\mathcal{G}_5(\phi, X)\delta^{ace}_{bdf}\nabla_a\nabla^b\phi\nabla_c\nabla^d\phi\nabla_e\nabla^g\phi\,. \tag{6}$$

with $X = -1/2\nabla_a\phi\nabla^a\phi$, G_{ab} the Einstein tensor, $\mathcal{G}_i$ are functions of the scalars $\{\phi, X\}$, V is a potential and $\delta^{b_1..b_n}_{a_1..a_n}$ is the generalised Kronecker delta symbol.

Linearized study: Papallo-Real's, in a special frame, better be that G_4=G_5=0 or *problem will be ill-posed*

Counter argument: analysis relies in a particular gauge, could this be regarded as too restrictive?

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[(1+\mathcal{G}_4(\phi))R + X - V(\phi) + \mathcal{G}_2(\phi,X)\right] ,$$

- The ‘look’ of the theory can be simplified by a conformal transformation…

form $\tilde{g}_{ab} = \Omega^2 g_{ab}$ with $\Omega = \sqrt{1+\mathcal{G}_4(\phi)}$ allows one to rewrite the above action as,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left\{\tilde{R} + \frac{1}{(1+\mathcal{G}_4(\phi))^2}\left[\left(3[\mathcal{G}_4'(\phi)]^2 + 1 + \mathcal{G}_4(\phi)\right)\tilde{X} - V(\phi) + \mathcal{G}_2\left(\phi, (1+\mathcal{G}_4(\phi))\tilde{X}\right)\right]\right\} , \quad (8)$$

where $\tilde{X} = -1/2\tilde{\nabla}_c\phi\,\tilde{\nabla}^c\phi$. From this action, the equation of motions are

$$\tilde{G}_{ab} = \left[\frac{3[\mathcal{G}_4'(\phi)]^2 + 1 + \mathcal{G}_4(\phi)}{2(1+\mathcal{G}_4(\phi))^2}\tilde{X} + \frac{-V(\phi) + \mathcal{G}_2(\phi,X)}{2(1+\mathcal{G}_4(\phi))^2}\right]\tilde{g}_{ab} + \left[\frac{3[\mathcal{G}_4'(\phi)]^2}{2(1+\mathcal{G}_4(\phi))^2} + \frac{1+\partial_X\mathcal{G}_2(\phi,X)}{2(1+\mathcal{G}_4(\phi))}\right]\tilde{\nabla}_a\phi\tilde{\nabla}_b\phi , \quad (9)$$

$$\begin{aligned}\tilde{\Box}\phi = \frac{1}{3[\mathcal{G}_4'(\phi)]^2 + (1+\mathcal{G}_4(\phi))(1+\partial_X\mathcal{G}_2(\phi,X))}\Bigg\{ & V'(\phi) - \partial_\phi\mathcal{G}_2(\phi,X) - 2\,\mathcal{G}_4'(\phi)\frac{V(\phi)-\mathcal{G}_2(\phi,X)}{1+\mathcal{G}_4(\phi)} \\ & + \left[2(1+\mathcal{G}_4(\phi))\partial^2_{\phi X}\mathcal{G}_2(\phi,X) + \mathcal{G}_4'(\phi)\left(6\mathcal{G}_4''(\phi) - 1 - 3\partial_X\mathcal{G}_2(\phi,X) - 6\frac{[\mathcal{G}_4'(\phi)]^2}{1+\mathcal{G}_4(\phi)}\right)\right]\tilde{X} \\ & + 2\mathcal{G}_4'(\phi)(1+\mathcal{G}_4(\phi))\partial^2_{XX}\mathcal{G}_2(\phi,X)\tilde{X}^2 + (1+\mathcal{G}_4(\phi))^2\partial^2_{XX}\mathcal{G}_2(\phi,X)\tilde{\nabla}^a\phi\tilde{\nabla}^b\phi\tilde{\nabla}_a\tilde{\nabla}_b\phi\Bigg\} . \end{aligned} \quad (10)$$

[Bernard-Luna-LL]

- Or rather…

$$\left[\tilde{g}^{ab} - \frac{(1+\mathcal{G}_4(\phi))^2\partial^2_{XX}\mathcal{G}_2(\phi,X)}{3[\mathcal{G}_4'(\phi)]^2 + (1+\mathcal{G}_4(\phi))(1+\partial_X\mathcal{G}_2(\phi,X))}\tilde{\nabla}^a\phi\tilde{\nabla}^b\phi\right]\tilde{\nabla}_a\tilde{\nabla}_b\phi$$

$$= \frac{1}{3[\mathcal{G}_4'(\phi)]^2 + (1+\mathcal{G}_4(\phi))(1+\partial_X\mathcal{G}_2(\phi,X))}\left\{V'(\phi) - \partial_\phi\mathcal{G}_2(\phi,X) - 2\,\mathcal{G}_4'(\phi)\frac{V(\phi)-\mathcal{G}_2(\phi,X)}{1+\mathcal{G}_4(\phi)}\right.$$

$$+\left[2(1+\mathcal{G}_4(\phi))\partial^2_{\phi X}\mathcal{G}_2(\phi,X) + \mathcal{G}_4'(\phi)\left(6\mathcal{G}_4''(\phi) - 1 - 3\partial_X\mathcal{G}_2(\phi,X) - 6\frac{[\mathcal{G}_4'(\phi)]^2}{1+\mathcal{G}_4(\phi)}\right)\right]\tilde{X}$$

$$\left.+2\mathcal{G}_4'(\phi)(1+\mathcal{G}_4(\phi))\partial^2_{XX}\mathcal{G}_2(\phi,X)\tilde{X}^2\right\}.$$

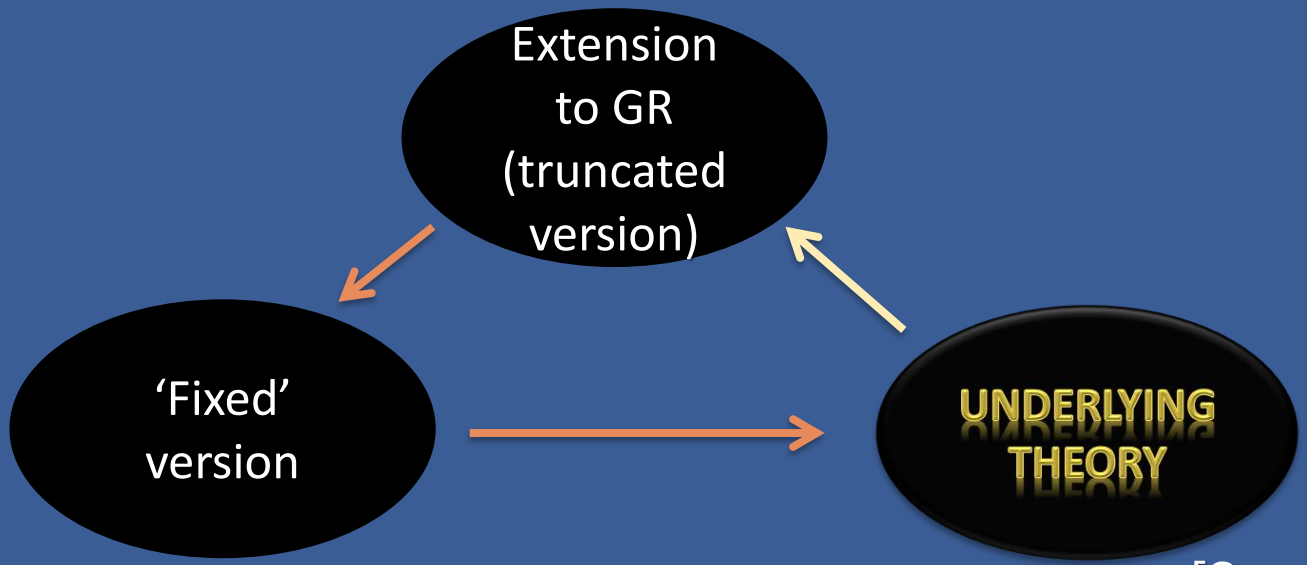
- Thus, the field propagates according to a different metric that depends on its gradients
  - For ‘weak data’ the eqn satisfies the ‘null condition’ (Klainerman) which together with Straus’ condition in V → global solutions
  - If the $v_a = X\,\phi_{,\alpha}$ is ‘twist free’, a field redefinition takes it to a ‘standard’ wave equation wrt to conformal metric

Beyond these cases, can we expect good behavior?

- If shocks arise → uniqueness is lost -> thus well posedness
- The character of the equation changes before!
- Issues are *independent* of the gauge adopted for Ees
- What would happen in a ‘generic+relevant’ case for GWs? For Cosmo?

- *Option 1: delicate (& uncertain?). Option 2: fine but can it be justified? One could argue yes in 3+1 dimensions but not above*
  - (drawing from LIGO/VIRGO, fluid-gravity correspondence and specific '2nd order' BH perturbation calculations) → all methods are intrinsically relying on this!

There seems to be a way to avoid 'not going to non-linear-land' with (many) GR alternatives and face upcoming data



[Cayuso,Ortiz,LL '17]

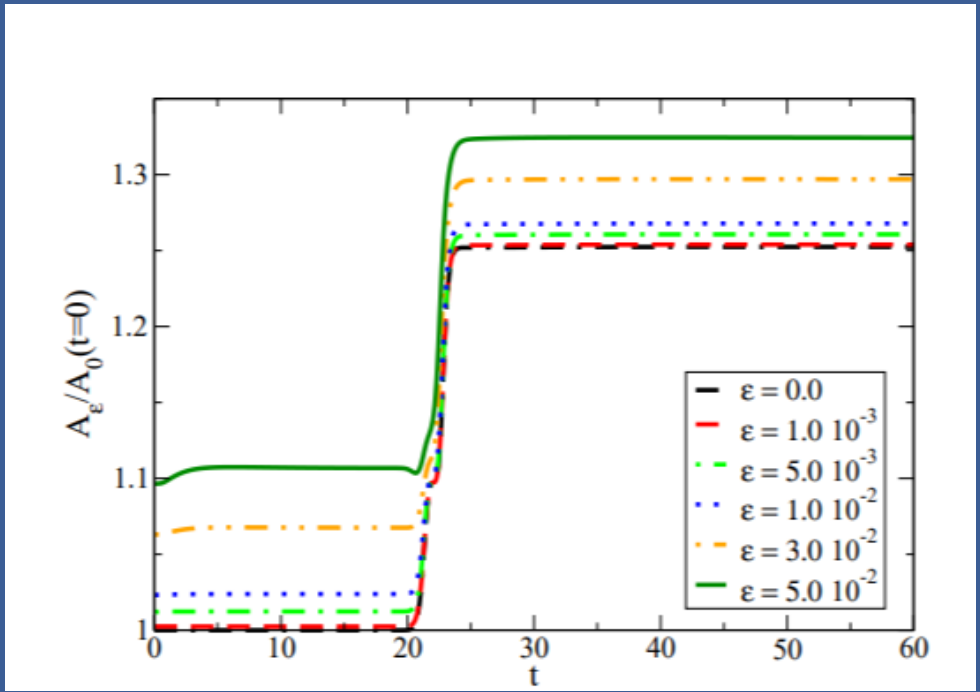
‘Gravity application’

$$S_{eff} = \int d^4x\sqrt{-g}2M_{pl}^2\left(R - \frac{\mathcal{C}^2}{\Lambda^6} - \frac{\widetilde{\mathcal{C}}^2}{\widetilde{\Lambda}^6} - \frac{\mathcal{C}\widetilde{\mathcal{C}}}{\Lambda_-^6} + \dots\right), \tag{1}$$

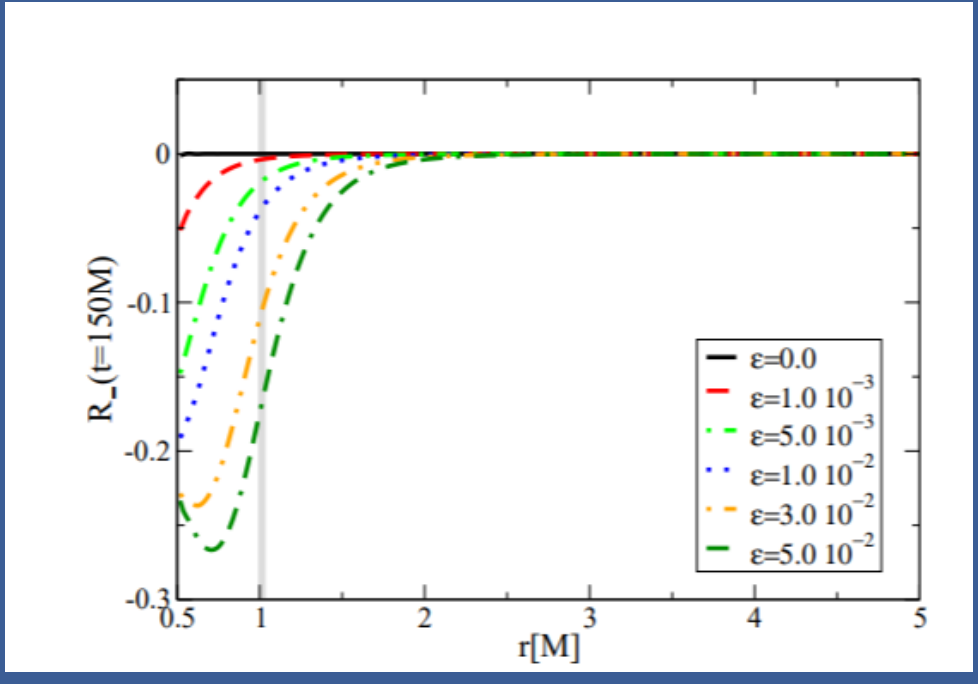
where $\mathcal{C} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and $\widetilde{\mathcal{C}} \equiv R_{\alpha\beta\gamma\delta}\widetilde{R}^{\alpha\beta\gamma\delta}$, with $\widetilde{R}^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu\gamma\delta}$, and the $+\dots$ correspond to sub-leading contributions.

$$\begin{aligned} G_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{1}{\Lambda^6}\Big(&- 8\mathcal{C}\, R_\mu{}^\alpha R_{\nu\alpha} + 8\mathcal{C}\, R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ &+ 4\mathcal{C}\, R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{C}^2 - 4\mathcal{C}\nabla_\mu\nabla_\nu R \\ &- 32R^{\beta\gamma\sigma\delta}\nabla_{(\mu}R_{\nu)}{}^\alpha\nabla_\alpha R_{\beta\gamma\sigma\delta} + 8\mathcal{C}\nabla_\alpha\nabla^\alpha R_{\mu\nu} \\ &+ 32R^{\beta\gamma\sigma\delta}\nabla_\alpha R_{\beta\gamma\sigma\delta}\nabla^\alpha R_{\mu\nu} + 8R_\mu{}^\alpha{}_\nu{}^\beta\nabla_\beta\nabla_\alpha\mathcal{C}\Big), \end{aligned} \tag{2}$$

$$\nabla^\mu T_{\mu\nu} = 0. \tag{3}$$

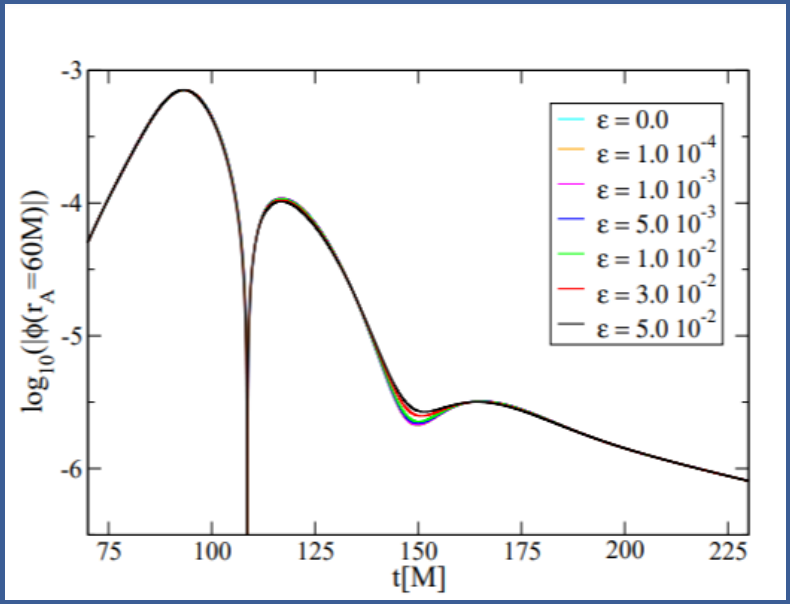


- Horizon growth



- Null convergence condition violation

Quasi normal modes of SF



$$\omega^R = \omega^R_{GR}(1 - 0.54\epsilon + 0.77\epsilon^2),$$
$$\omega^I = \omega^I_{GR}(1 + 0.45\epsilon - 1.33\epsilon^2).$$

Further freedom!

$$\theta \to \theta + \frac{2}{M^4} \left(\partial_\mu \partial_\nu \theta\right) \left(\partial^\mu \theta\right) \left(\partial^\nu \theta\right) . \tag{36}$$

We can then rewrite the EFT action given by eq. (28) as

$$\frac{S}{v^2} \simeq -\int_{\mathbb{R}^4} \mathrm{d}^4 x \left\{ \frac{1}{2} \partial_\nu \theta(x) \cdot \partial^\nu \theta(x) - \frac{1}{2M^2} [\partial_\nu \theta(x) \cdot \partial^\nu \theta(x)]^2 \right.$$
$$\left. + \frac{2}{M^4} \left[\partial_\mu \partial_\nu \theta(x) \cdot \partial^\sigma \partial^\mu \theta(x) \cdot \partial_\sigma \theta(x) \cdot \partial^\nu \theta(x) + \partial_\sigma \theta(x) \cdot \partial^\sigma \left[\partial_\mu \partial_\nu \theta(x) \cdot \partial^\mu \theta(x) \cdot \partial^\nu \theta(x) \right] \right] \right\} . \tag{37}$$

By integrating the last term by parts, we obtain

$$\frac{S}{v^2} \simeq -\int_{\mathbb{R}^4} \mathrm{d}^4 x \left\{ \frac{1}{2} \partial_\nu \theta(x) \cdot \partial^\nu \theta(x) - \frac{1}{2M^2} [\partial_\nu \theta(x) \cdot \partial^\nu \theta(x)]^2 \right.$$
$$\left. + \frac{2}{M^4} \partial_\mu \partial_\nu \theta(x) \cdot \left[\partial^\sigma \partial^\mu \theta(x) \cdot \partial_\sigma \theta(x) \cdot \partial^\nu \theta(x) - \Box \theta(x) \cdot \partial^\mu \theta(x) \cdot \partial^\nu \theta(x) \right] \right\} . \tag{38}$$

↓ [Solomon-Trodden ’18]

$$- \Box \theta + \frac{2}{M^2} \left[\left(\partial_\nu \theta\right) \left(\partial^\nu \theta\right) \Box \theta + 2 \left(\partial_\mu \partial_\nu \theta\right) \left(\partial^\nu \theta\right) \left(\partial^\mu \theta\right) \right]$$
$$+ \frac{2}{M^4} \left[3 \, \Box \theta \left(\partial_\mu \partial_\nu \theta\right) \left(\partial^\mu \partial^\nu \theta\right) - \left(\Box \theta\right)^3 - 2 \left(\partial_\mu \partial_\nu \theta\right) \left(\partial_\sigma \partial^\mu \theta\right) \left(\partial^\sigma \partial^\nu \theta\right) \right] = 0 .$$