# Well-posed formulation of some gravitational effective field theories

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Aron Kovacs and HSR 2003.04327, 2003.08398
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# Motivation

Detection of gravitational waves from BH (or NS) mergers is an opportunity to perform precision tests of GR in the *strong field* regime.

To do this we need theoretical predictions for how a deviation from GR would affect the gravitational waves emitted in a merger. Focus on BH/BH mergers so looking for deviations from GR in vacuum.

Two problems:

1. Could try to predict using a theory of modified gravity but which theory should we use?
2. To make predictions we need to perform numerical simulations. This requires that the theory admits a *well-posed initial value problem*, i.e., given suitable initial data there should exist a unique (up to diffeos) solution of the equations of motion that depends continuously on the initial data.

# Effective field theory

Provides a way of studying (small) deviations from GR that is agnostic about whatever "UV physics" causes this deviation.

Specify light fields and symmetries, then write down most general Lagrangian for these fields as an expansion in terms with increasing numbers of derivatives e.g. vacuum gravity:

$$\mathcal{L} = -2\Lambda + R + \alpha R^2 + \beta R_{ab} R^{ab} + \gamma L_{GB} + \dots$$

where $\alpha, \beta, \gamma \propto L^2$ for some UV length scale $L$ and $L_{GB} \propto R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2$

Expansion in increasing numbers of derivatives: makes sense provided higher derivative terms become successively less important. So need curvature to be small compared to $L^{-2}$. Call this the *weakly coupled* regime. Compatible with strong field BH dynamics provided BH large compared to $L$.

To be observable, need $L \sim \mathrm{km}$. Seems very unlikely from perspective of fundamental theory! Instead view this just as a framework for parameterising strong field tests of GR, analogous to the PPN formalism.

# Higher derivatives

If we truncate EFT at some number of derivatives then resulting equations involving higher than second derivatives of fields. Problematic because

- Well-posedness of initial value problem is determined mainly by the terms with the highest number of derivatives in the eqs of motion. They need to have "nice structure". But no reason for this structure to be present, and in EFT these terms should be the least important terms, not the most important!
- With higher order equations, need to specify more initial data: corresponds to additional (heavy) degrees of freedom that should not be present in EFT.

Fortunately, for several theories of interest we can write the *leading order* EFT corrections in a way that sidesteps these problems...

# Field redefinitions

In EFT one can perform field redefinitions to simplify action.

e.g. for vacuum gravity can simplify to

$$\mathcal{L} = -2\Lambda + R + \gamma L_{GB} + \dots$$

where $L_{GB} \propto R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$

$d > 4$: $L_{GB}$ gives leading (4-derivative) EFT corrections to GR and has *second order eqs of motion*

$d = 4$: $L_{GB}$ topological so no 4-derivative corrections to eqs of motion, leading EFT corrections start at 6 derivatives

# Scalar-tensor EFT

Light fields: metric plus scalar field. After field redefinitions, assuming a parity symmetry, Lagrangian can be written

$$\mathcal{L} = -V(\phi) + R + X + \alpha(\phi)X^2 + \beta(\phi)L_{\rm GB} + \dots$$

where $X = -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

Attractive features of this theory:

- Leading EFT corrections now start at 4 derivatives
- $L_{GB}$ can source scalar field: *guaranteed deviation from GR for vacuum BHs*
- If we neglect terms with more than 4 derivatives then equations of motion are second order

# Einstein-Maxwell EFT

Light fields: metric plus Maxwell field (the only observed fundamental fields for which the classical approximation is useful)

Assume parity symmetry. Leading EFT corrections have 4 derivatives and, using field redefinitions, these can be written in a way that gives second order equations of motion for $g_{\mu\nu}$, $A_\mu$

Field redefinitions can be used to write 4-derivative terms in a way that gives second order equations of motion for:

- Vacuum gravity (although trivial in 4d)
- Parity symmetric scalar-tensor
- Parity symmetric Einstein-Maxwell

Do these theories admit a well-posed initial value problem?

If 4-derivative terms become comparable to 2-derivative terms then well-posedness can fail Papallo & HSR 2017.

The best we can hope for is well-posedness at *weak coupling*. But that's all we need for EFT! However, even this is highly non-trivial.

# Strong hyperbolicity

A sufficient condition for a well-posed initial value problem is that the eq is *strongly hyperbolic*.

1st order linear constant coefficients system $\partial_t u = M^i \partial_i u + Nu$

$$u(t,x) \propto \int d\xi e^{i\xi_j x^j} e^{(iM^i\xi_i + N)t} \tilde{u}(0,\xi)$$

For convergence of integral demand $||e^{iM^i\xi_i t}|| \leq f(t)$ as $\xi \to \infty$. This implies that $M^i\xi_i$ must be *diagonalizable* with *real eigenvalues* (which fix phase velocities of modes). This is the definition of strong hyperbolicity, even when coefficients are not constant. (*Weakly* hyperbolic: real evals but not diagonalisable.)

Second order systems: reduce to first order and apply this definition. Nonlinear eqs: apply definition to linearisation around general background (weakly coupled in our case).

Strategy for proving well-posedness of a gravitational theory:

(1) Find a way of gauge-fixing to give strongly hyperbolic equations of motion: ensures well-posedness of initial value problem for *any* initial data, even data that violates constraints and gauge condition

(2) Show that if initial data satisfies constraints and gauge condition then, in the resulting solution, the gauge-fixing terms vanish and so one obtains a solution of the original (non-gauge-fixed) equations

Simplest gauge choice for conventional GR that gives strongly hyperbolic eqs is (generalised) harmonic gauge: gives $M^i\xi_i$ diagonalizable with real *but degenerate* eigenvalues (all modes have same phase velocity)

View weakly coupled 4-derivative theory as a small deformation of conventional GR: small deformation of $M^i\xi_i$

Generic deformation of a real matrix with degenerate eigenvalues is not diagonalizable!

In harmonic gauge, $M^i\xi_i$ is not diagonalizable in a *generic* weakly coupled background $\Rightarrow$ eqs only *weakly* hyperbolic Papallo & HSR 2017, Papallo 2017

Similar problem seen in other gauge-fixing approaches e.g. BSSN Kovacs 2019

Problem arises from mixing between two types of unphysical solutions of gauge-fixed eqs: "pure gauge" solutions and "gauge-condition violating" solutions; these travel at same speed as physical solutions

New idea: try to deform the harmonic gauge condition of conventional GR to give these modes different phase velocities (different eigenvalues); prevents mixing when we deform to a 4-derivative theory so should then retain diagonalizability of $M^i \xi_i$

# Vacuum GR in modified harmonic gauge

Introduce two auxiliary (inverse) Lorentzian metrics $\tilde{g}^{\mu\nu}$, $\hat{g}^{\mu\nu}$.
Gauge condition: $H^\mu = 0$ where

$$H^\mu \equiv \tilde{g}^{\nu\rho}\Gamma^\mu_{\nu\rho}$$

Gauge-fixed equation: $E^{\mu\nu} = 0$ where

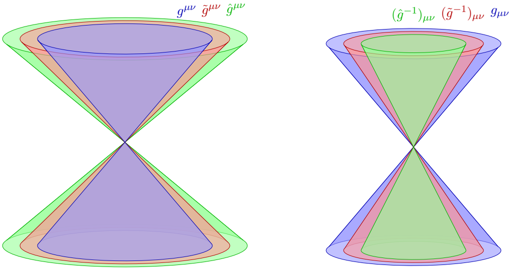
$$E^{\mu\nu} \equiv G^{\mu\nu} + \hat{P}_\alpha{}^{\beta\mu\nu}\partial_\beta H^\alpha \qquad \hat{P}_\alpha{}^{\beta\mu\nu} \equiv \delta_\alpha^{(\mu}\hat{g}^{\nu)\beta} - \frac{1}{2}\delta_\alpha^\beta\hat{g}^{\mu\nu}$$

Implies $\hat{\Box}H^\mu + \ldots = 0$

In linearised theory, gauge-condition violating solutions propagate along null cone of $\hat{g}^{\mu\nu}$ and pure gauge solutions propagate along null cone of $\tilde{g}^{\mu\nu}$. Physical solutions propagate along null cone of $g^{\mu\nu}$.

# The three metrics

We choose the unphysical metrics so that their null cones do not intersect each other or the null cone of the physical metric.




With this choice, can prove that vacuum GR is strongly hyperbolic in our modified harmonic gauge formulation.

Straightforward to include a minimally coupled (2-derivative) scalar field or Maxwell field (modified Lorenz gauge for $A_\mu$)

# Well-posedness of our EFTs

In the 2-derivative theory, $M^i \xi_i$ is diagonalisable with real evals.

The evals associated with pure gauge and gauge-condition violating modes are distinct from each other and from the evals associated with physical modes.

Using this, can show that $M^i \xi_i$ remains diagonalisable with real evals when we deform the theory to include 4-derivative terms, assuming weak coupling (i.e. a small deformation).

Hence, at weak coupling, our formulation gives strongly hyperbolic equations so the initial value problem is well posed in the 4-derivative theories I have described.

# Numerics

The first numerical simulations using our formulation have been performed East & Ripley 2020:

- Shift-symmetric theory Einstein-scalar-GB

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 + \lambda\phi L_{\rm GB}$$

- Dynamical scalarisation of rotating BHs
- Head-on collisions of BHs
- Inspiral and merger of BHs
- Typical values $\lambda M^{-2} \sim 0.01$ to $0.2$

# Generalisations

Our modified harmonic gauge formulation gives well-posed equations of motion for any weakly coupled Horndeski theory.

It also works for weakly coupled Lovelock theories such as Einstein-Gauss-Bonnet:

$$\mathcal{L} = R + \alpha L_{GB}$$

Opens possibility of studying effect of higher curvature corrections on dynamical processes in $d > 4$ gravity e.g. black string instability?

# Summary

Effective field theory is an attractive formalism for parameterising possible strong field deviations from GR.

For $d > 4$ vacuum gravity, $d = 4$ scalar-tensor theory, or $d = 4$ Einstein-Maxwell theory, the leading EFT corrections have 4 derivatives but second order eqs of motion.

For numerical simulations of BH mergers it is essential that a formulation of these equations is found that is strongly hyperbolic and hence admits a well-posed initial value problem.

We have found such a formulation, based on modified harmonic gauge. It is well-posed at weak coupling.

The first simulations of BH mergers have been performed using this formulation.