Interdisciplinary viewpoints of resonance and its role in cochlear mechanics

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FIELDS

THE FIELDS INSTITUTE
JUNE 16, 2017

Workshop on the

Mathematics of Hearing

The ear is a remarkable detector, encoding sound pressure into neural signals that carry myriad pieces of information about the world around us to the brain. It is also highly selective, decomposing sound into constituent frequency components by virtue of acting as a hydrodynamic Fourier analyzer. Interestingly, not only does the ear respond to sound, but emits it as well, a facet that has revolutionized pediatric audiology.

Yet much still remains unknown about this system we all have a pair of built in to our heads. This one-day workshop aims to bring together a variety of researchers from different backgrounds to explore, from both mathematical and biological viewpoints, a wide range of topics related to the ear.

INVITED SPEAKERS

Bard Ermentrout (University of Pittsburgh) – nonlinear dynamics
Andre Longtin (University of Ottawa) – neural coding
Laura Miller (University of North Carolina) – fluid dynamics
Christopher Shera (University of Southern California) – inverse problems
Sarah Verhulst (Ghent University) – cochlear neurobiology and psychoacoustical modeling
George Zweig (MIT) – cochlear modeling

ORGANIZING COMMITTEE: CHRISTOPHER BERGEVIN (YORK UNIVERSITY) AND SUNIL PURIA (HARVARD MEDICAL SCHOOL)

For registration and more information, please visit:
www.fields.utoronto.ca/activities/16-17/mathhearing

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MOH

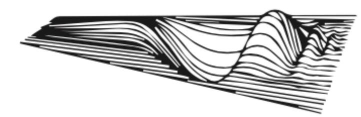
13th Mechanics of Hearing conference

- June 19-24, 2017
- Brock University (St. Catherines, Ontario, Canada)

Niagara region

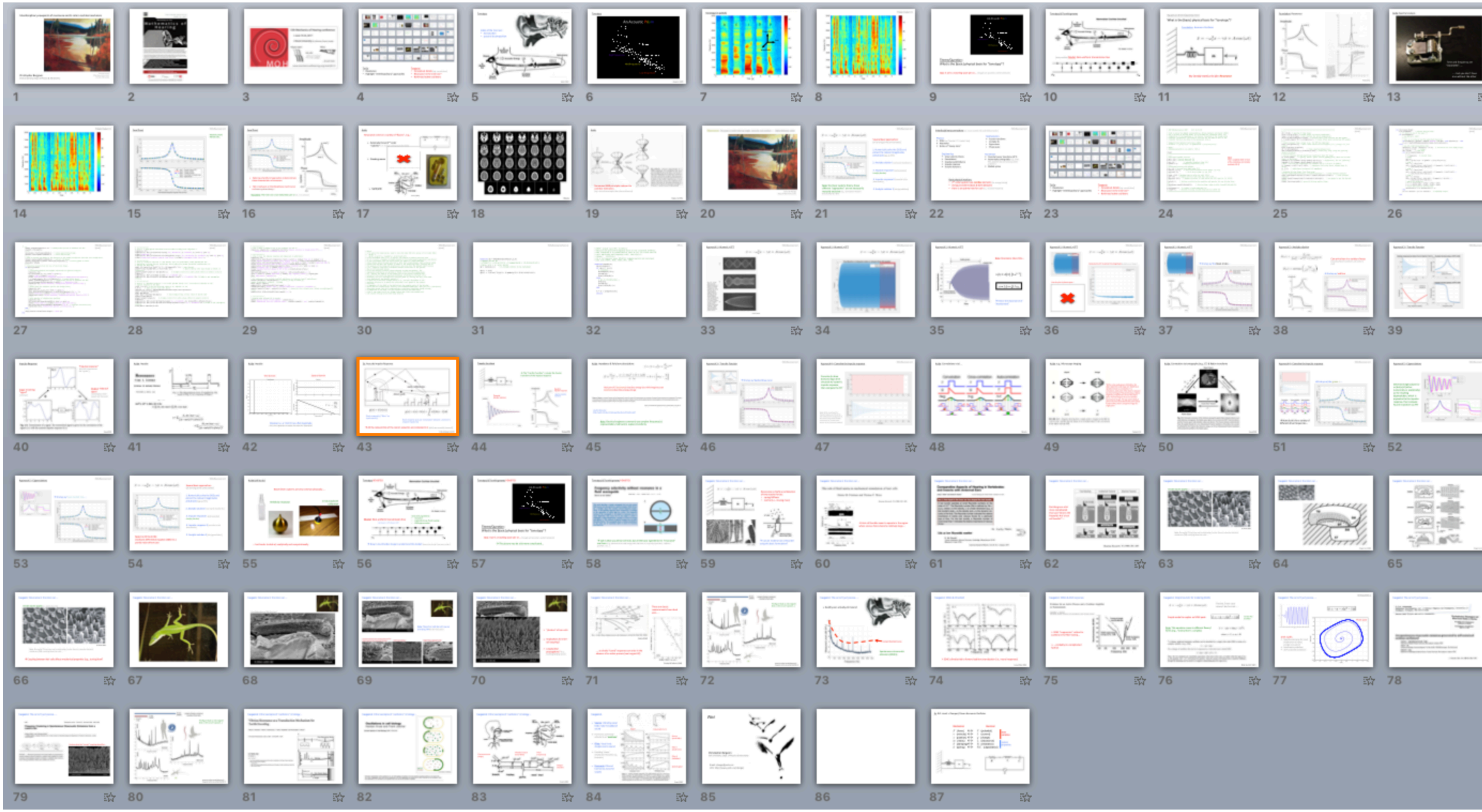
Hair cells

Waves



... & much more

www.mechanicsofhearing.org/moh2017/



<u>To Do</u>

- Resonance
- Highlight "interdisciplinary" approaches

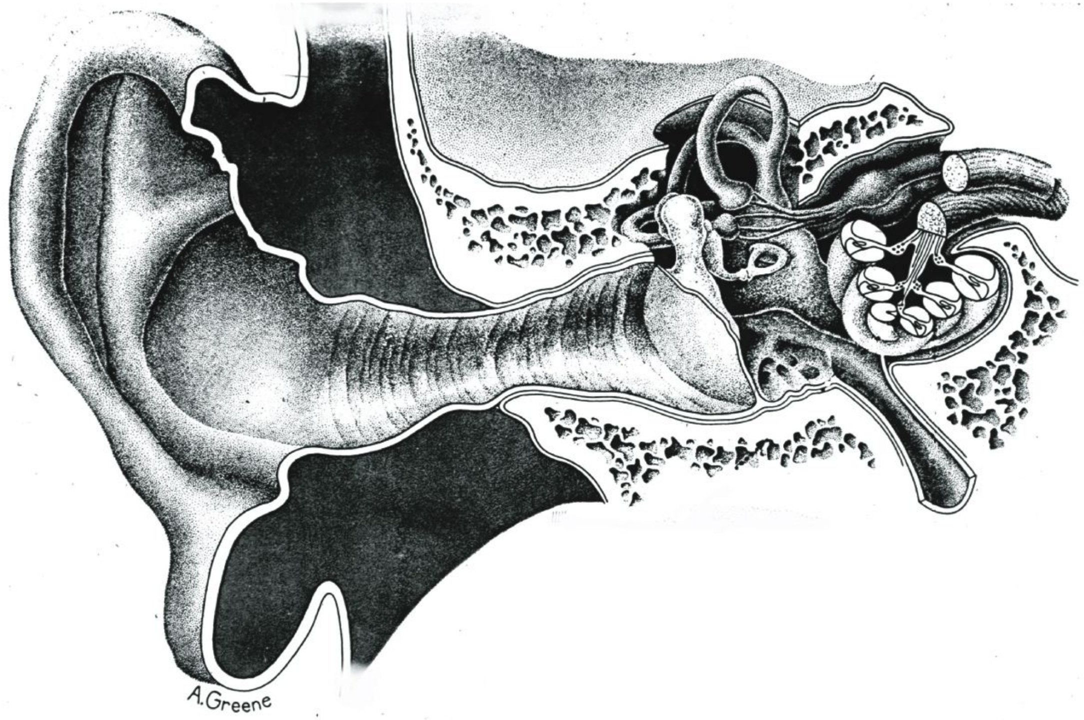
<u>Tangents</u>

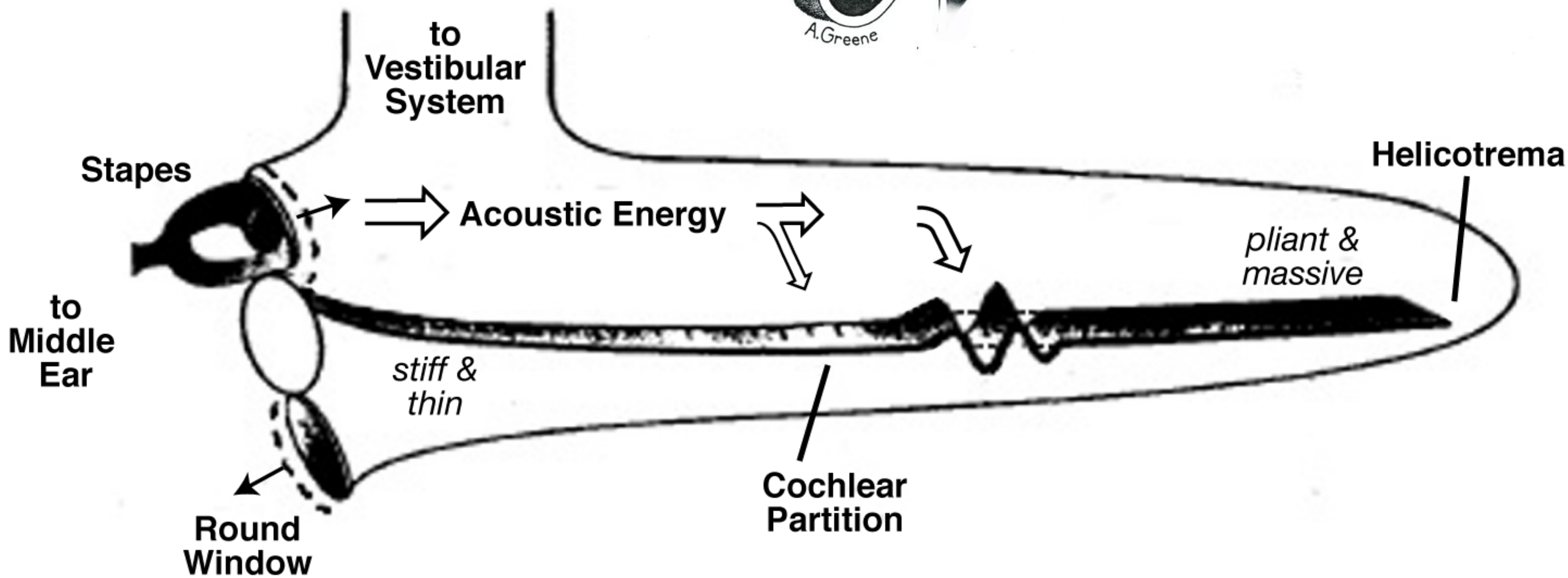
- Conceptual details (e.g., convolutions)
- Resonance in the inner ear?
- Nonlinear/active oscillators
- Other examples in biology

<u>Tonotopy</u>

Job(s) of the inner ear:

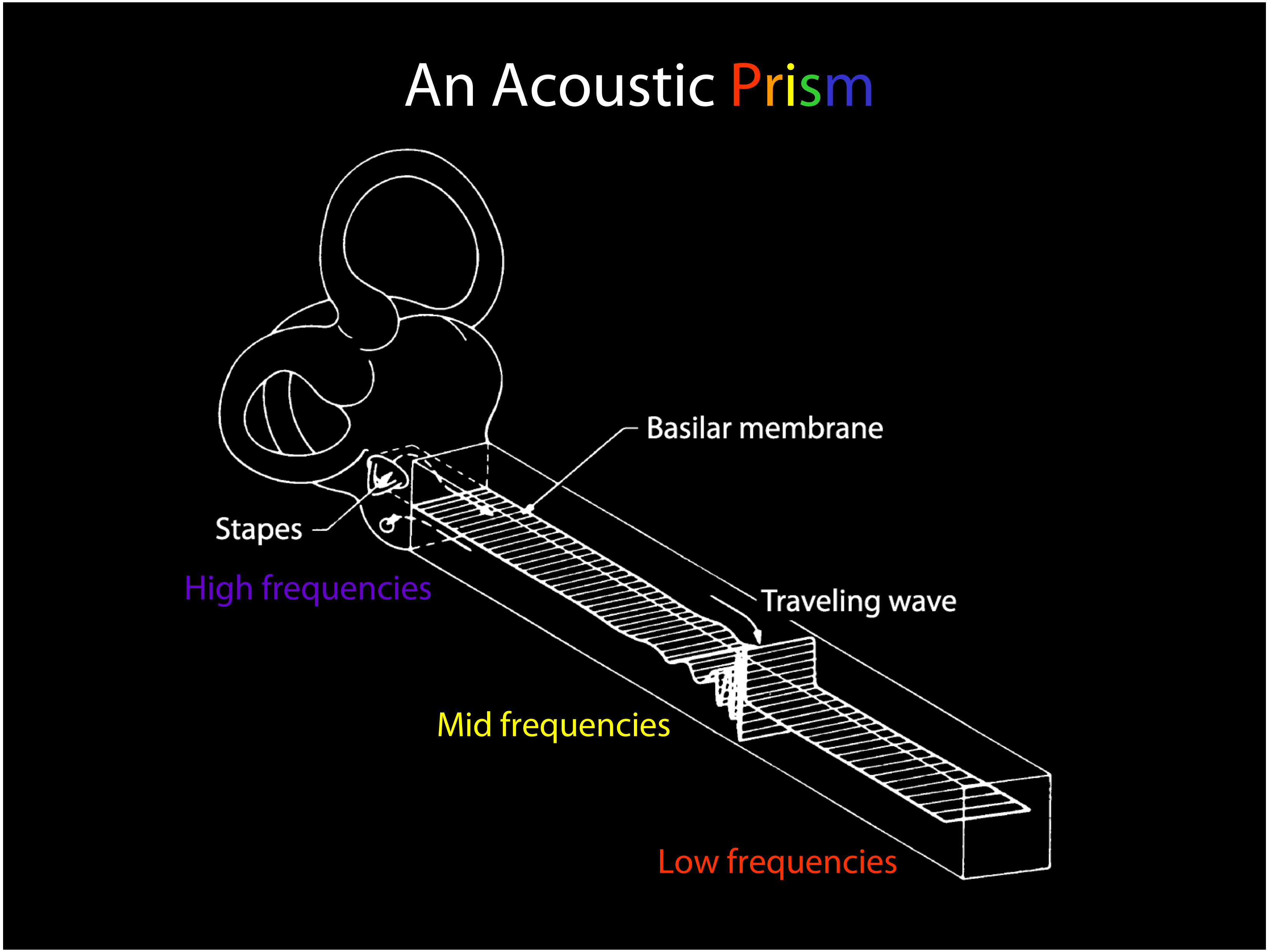
- transduction
- spectral decomposition





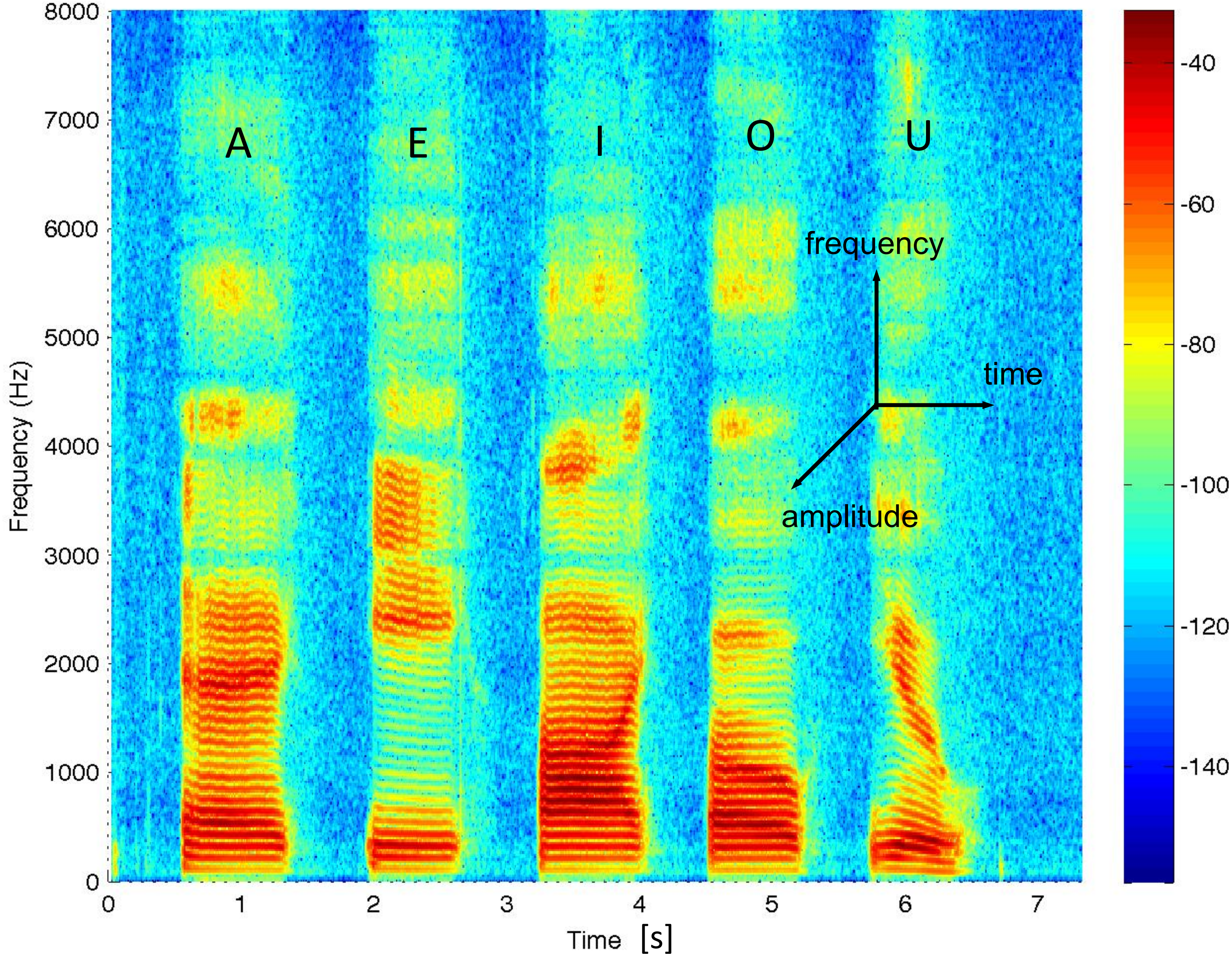
Geisler (1996)

Tonotopy

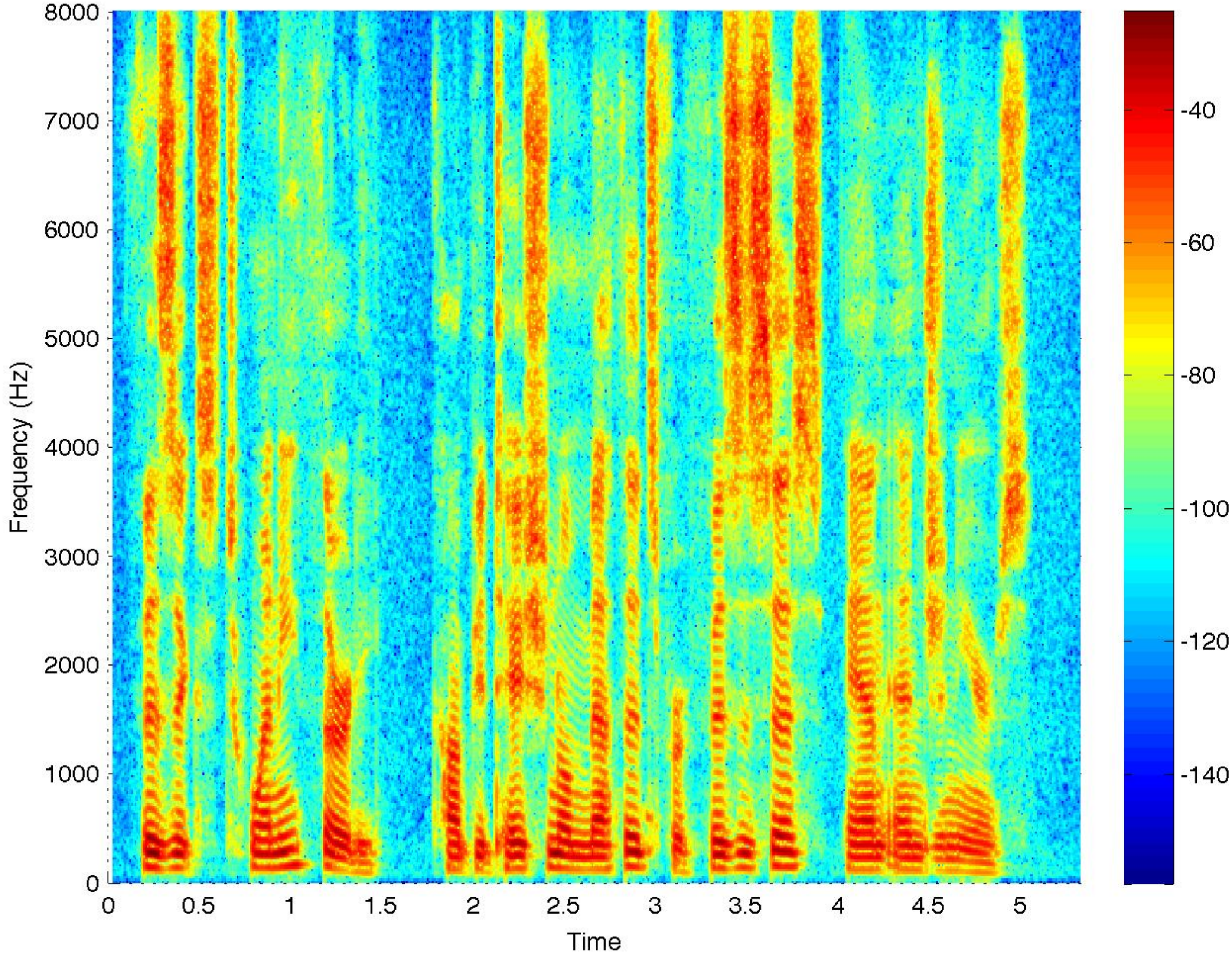


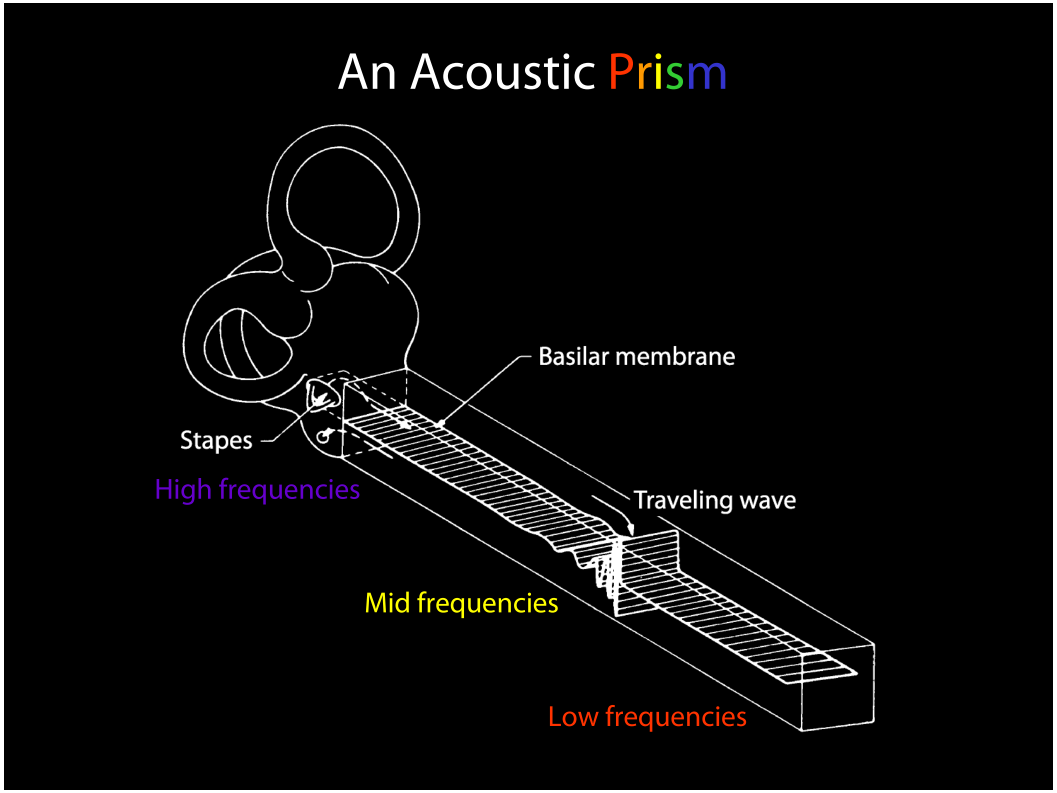
Zweig et al. (1976)

Tonotopy (re speech)



EXspectrogram.m

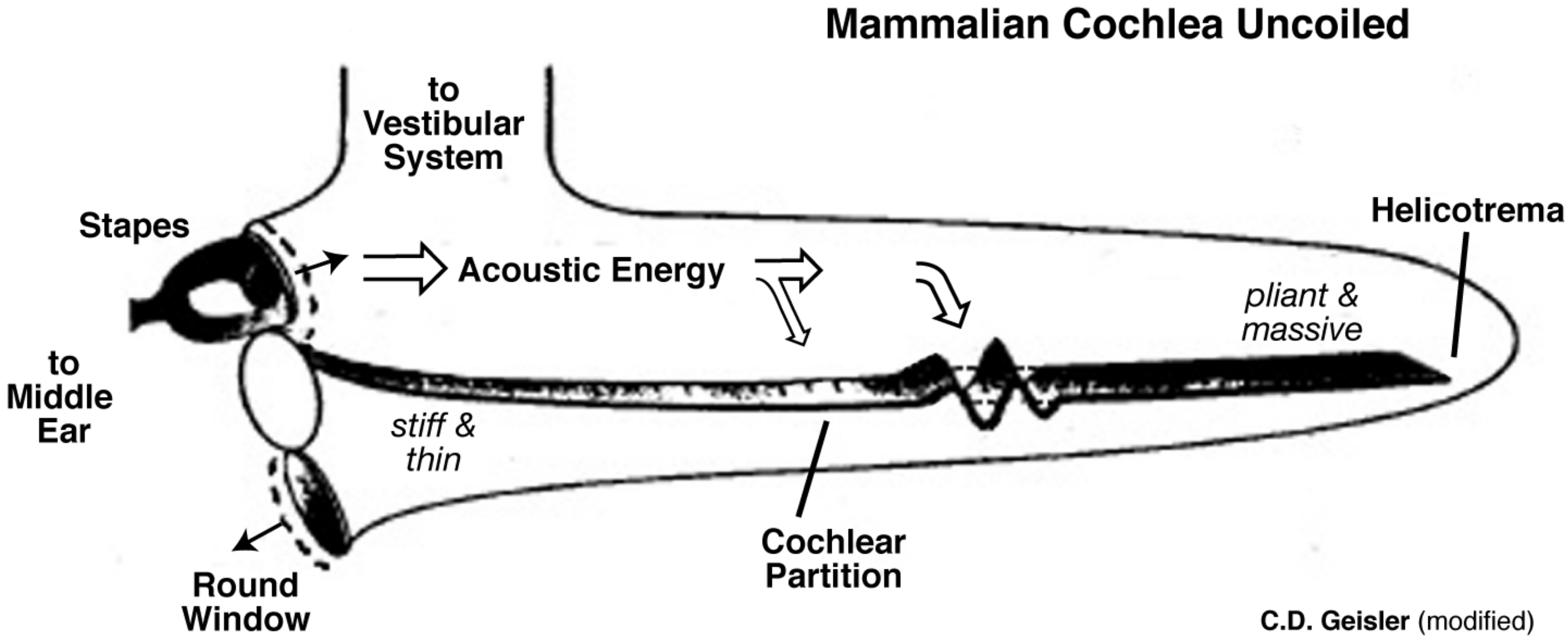




Theme/Question:
What is the (basic) physical basis for "tonotopy"?

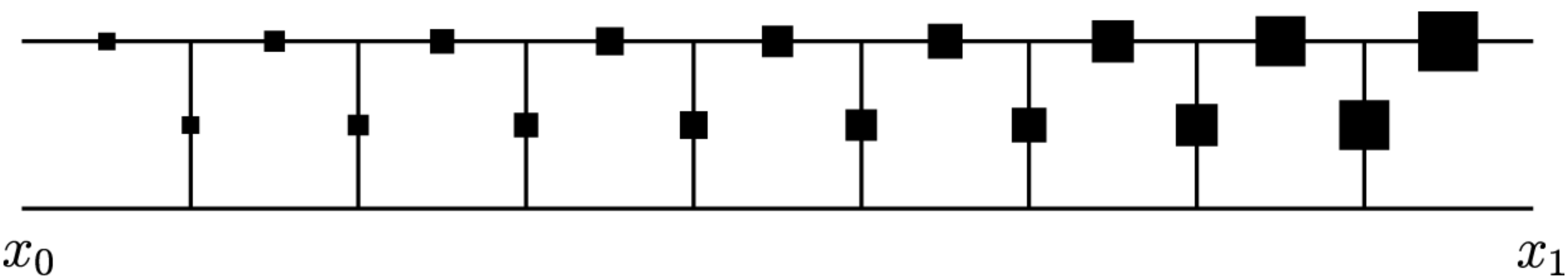
Hint: It ain't a traveling wave per se.... (though such provides a useful framework)

Tonotopy & Traveling waves



C.D. Geisler (modified)

(one possible) Model: Non-uniform transmission line

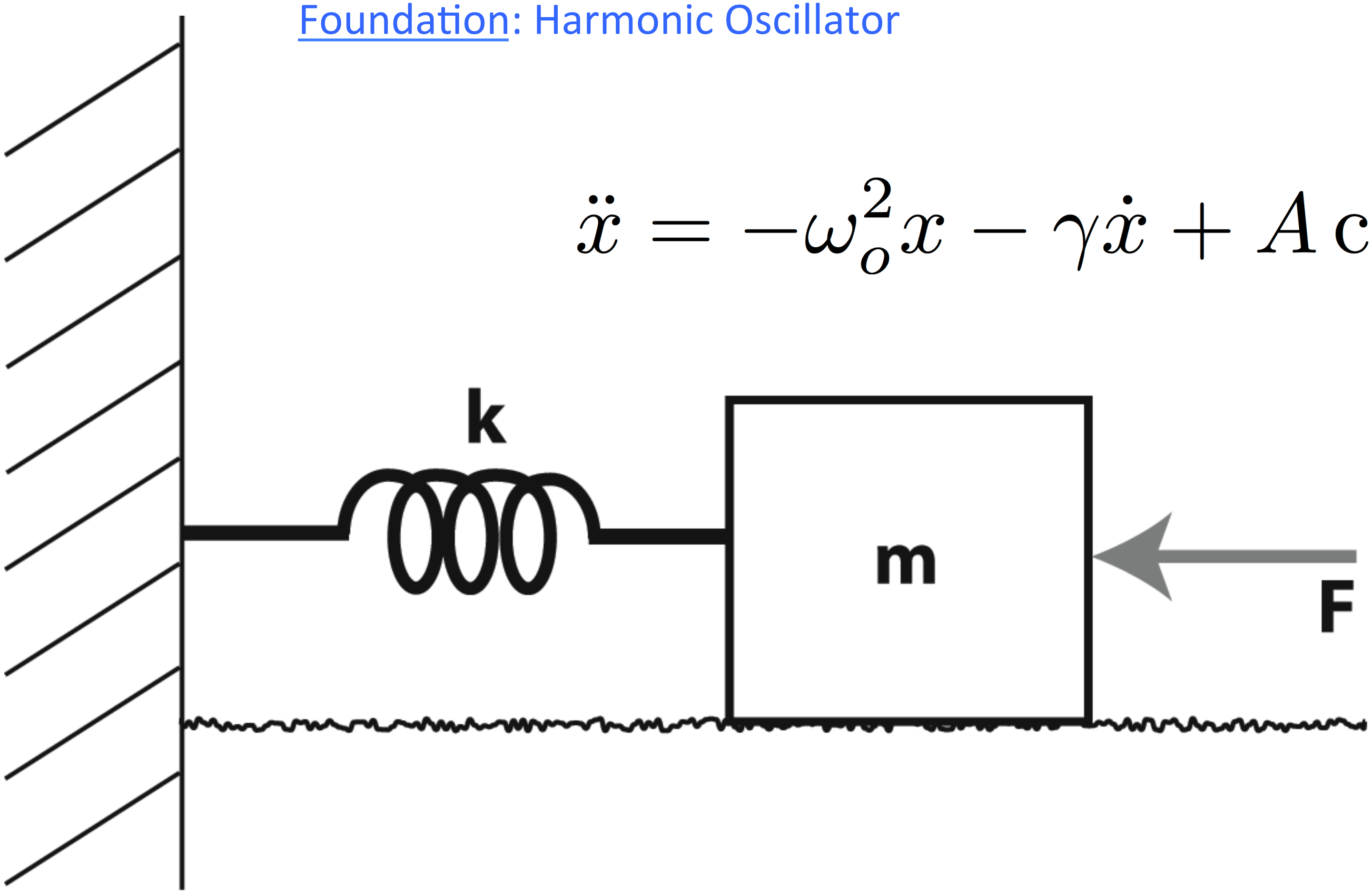


Big picture theme/question here:

What is the (basic) physical basis for “tonotopy”?

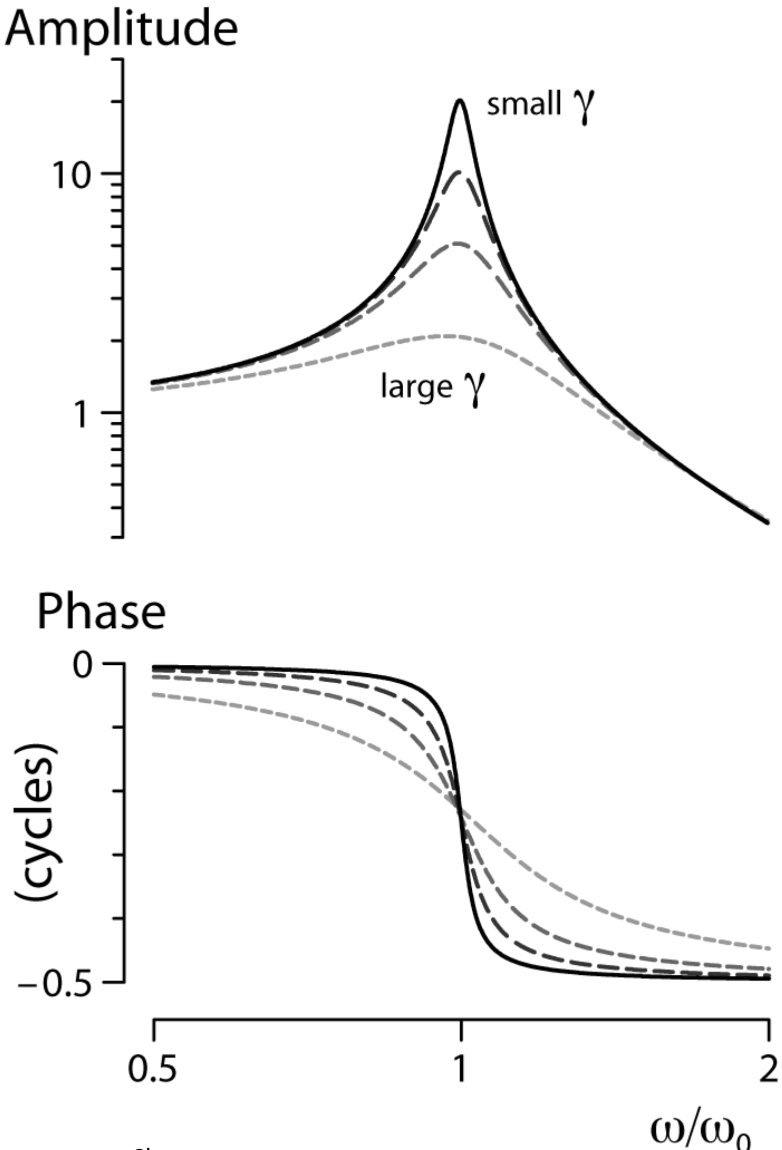
Foundation: Harmonic Oscillator

$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$



Key (steady-state) principle: ***Resonance***

Foundation: Resonance



Shera

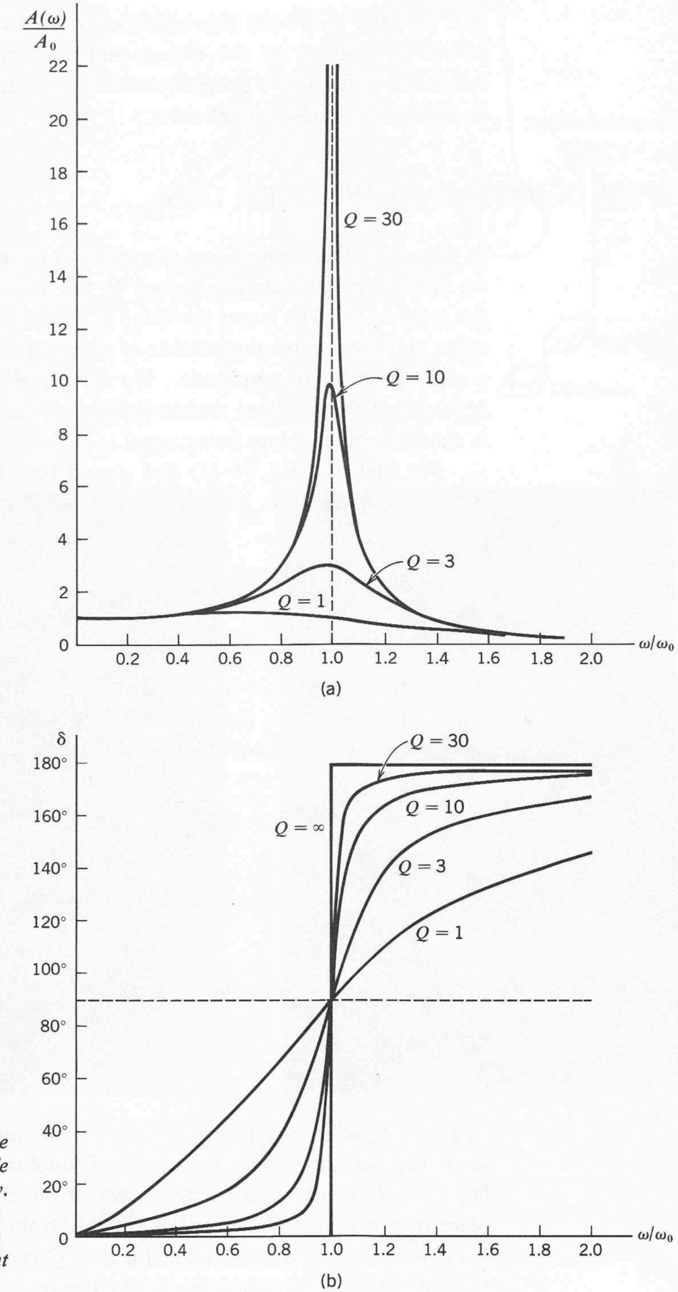
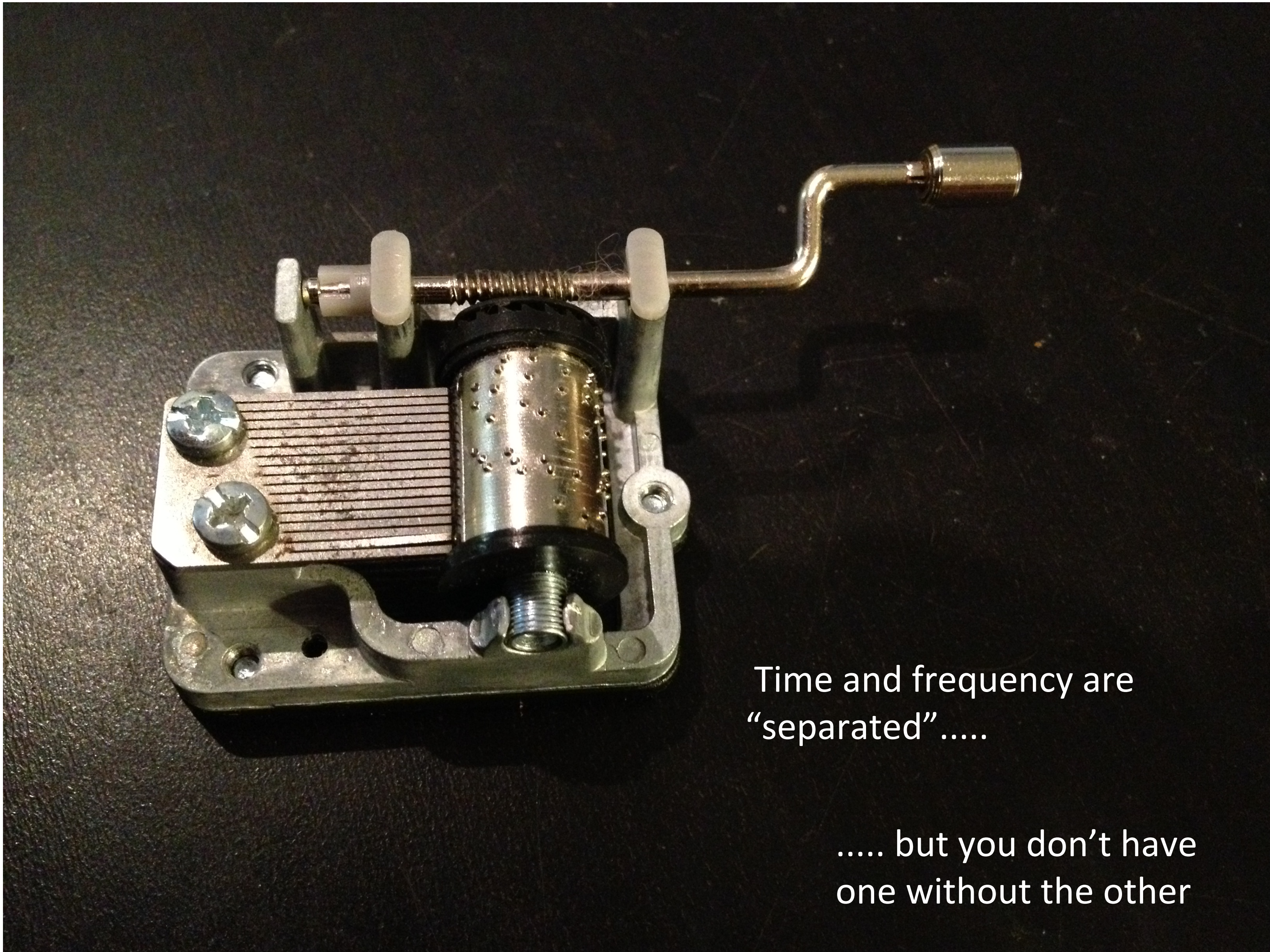
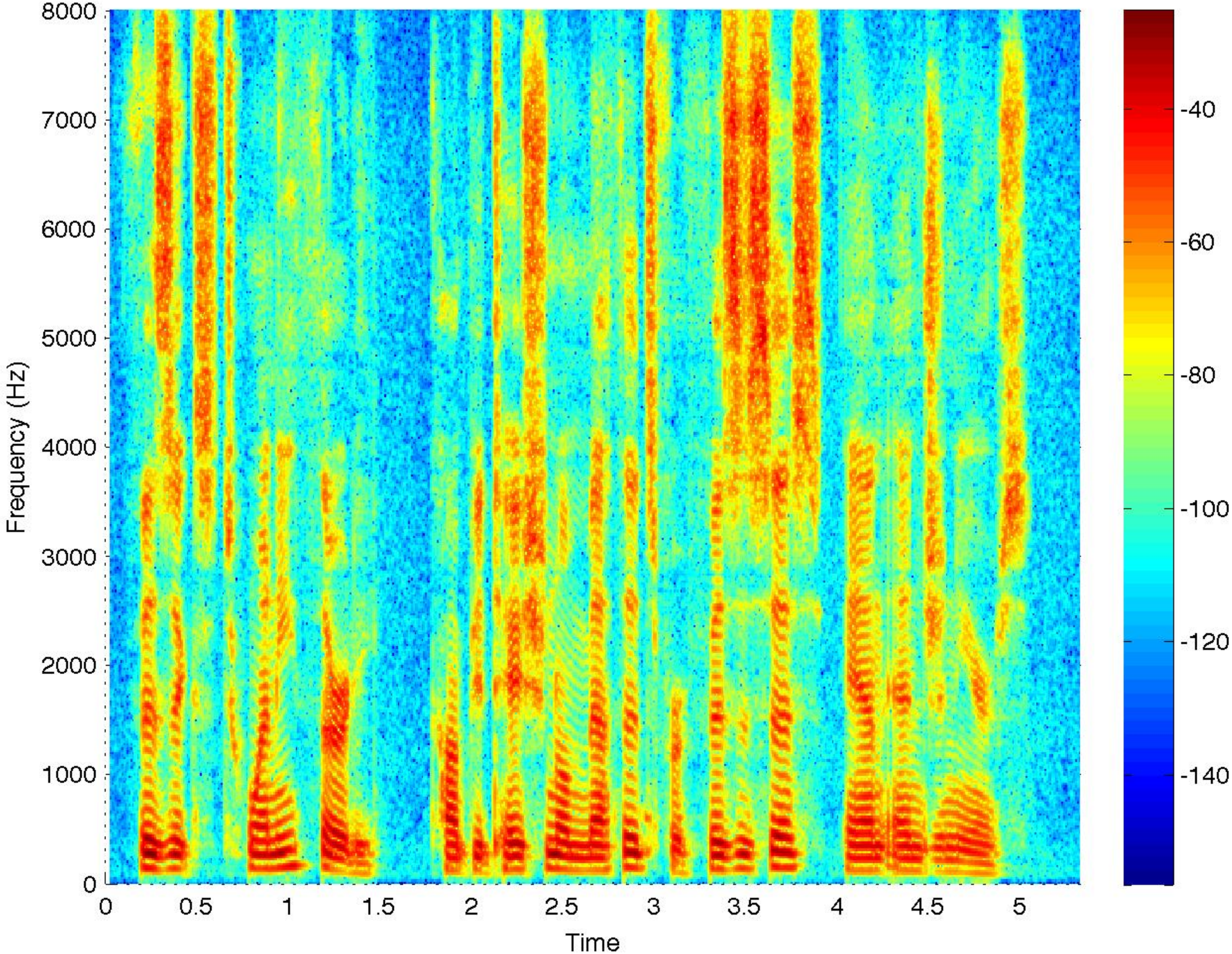


Fig. 4–9 (*a*) *Amplitude as function of driving frequency for different values of Q, assuming driving force of constant magnitude but variable frequency.* (*b*) *Phase difference δ as function of driving frequency for different values of Q.*

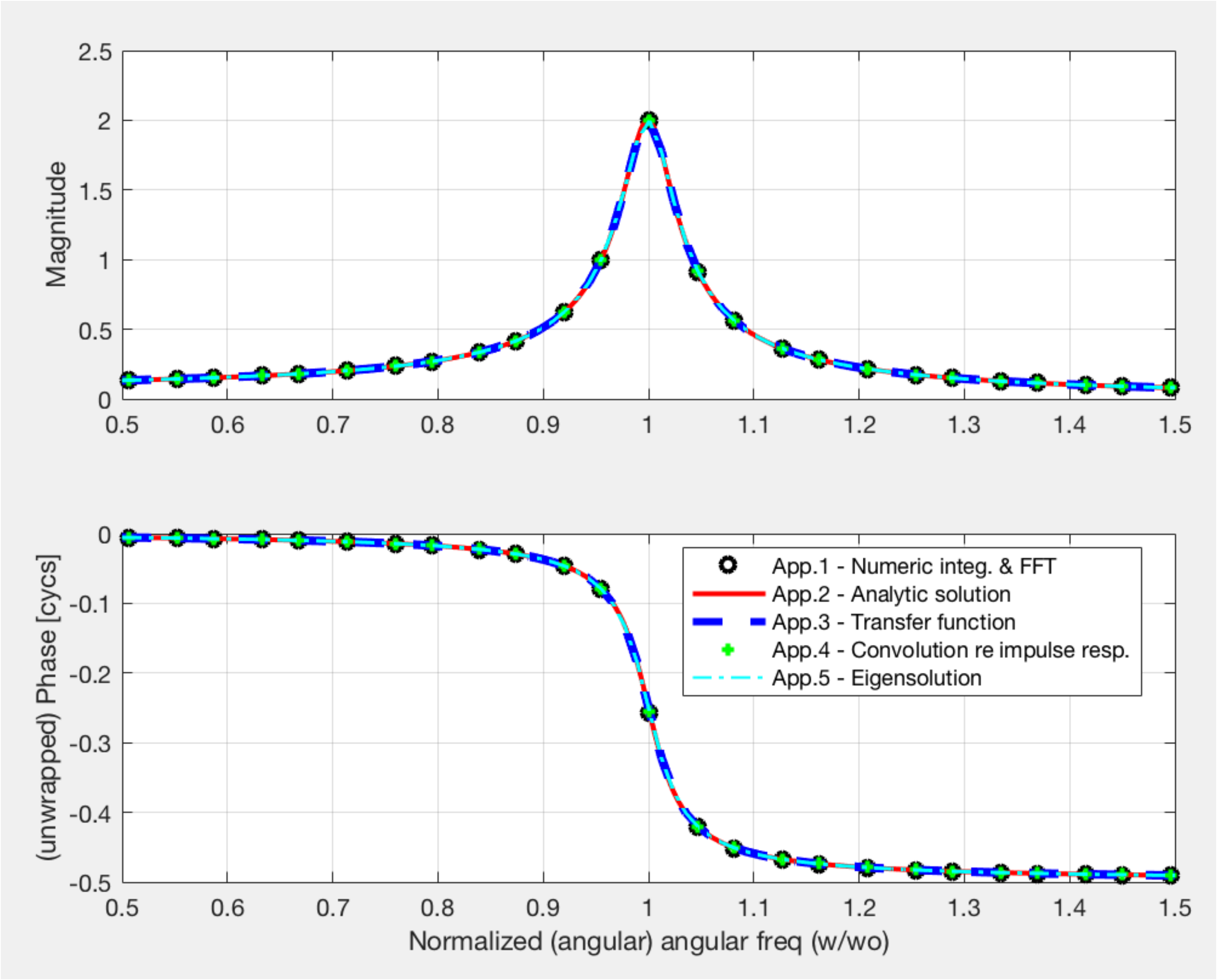
French (1971)

Aside: Spectral analysis



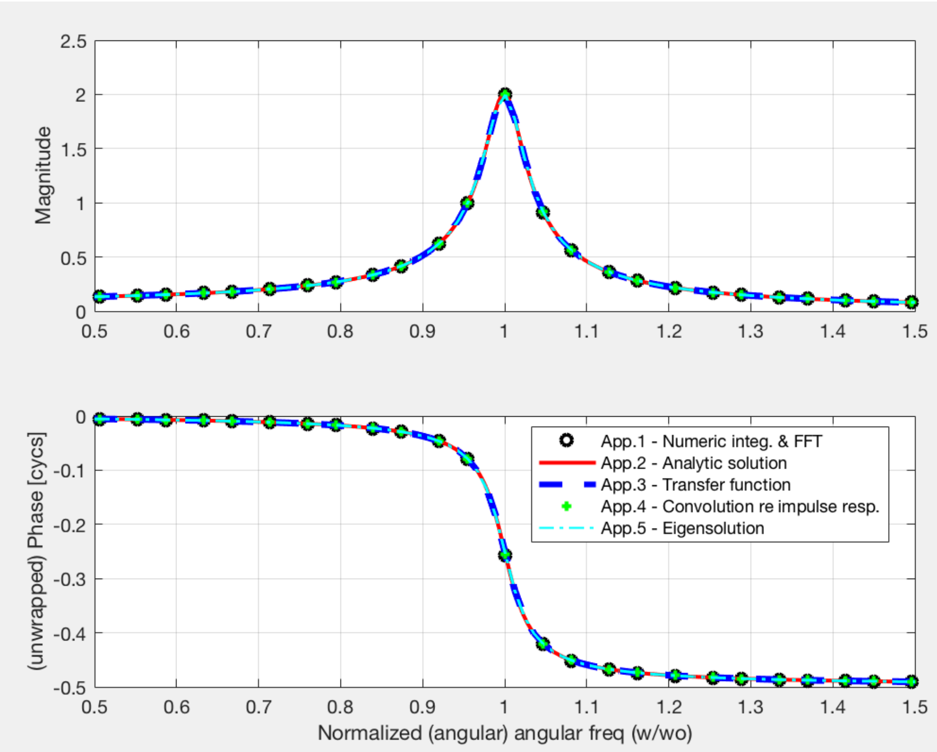

8000
7000
6000
5000
4000
3000
2000
1000
0
Frequency (Hz)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time
-40
-60
-80
-100
-120
-140

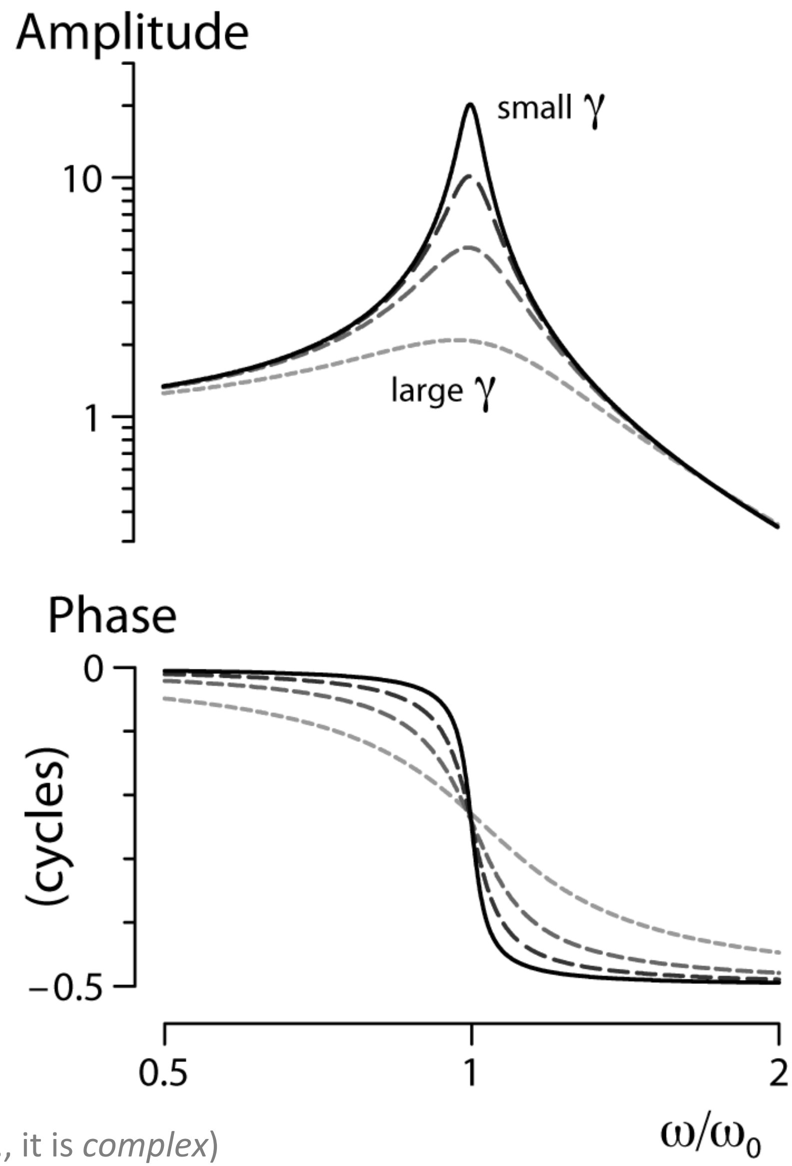
Focal Point



Relatively simple Matlab code...

Focal Point





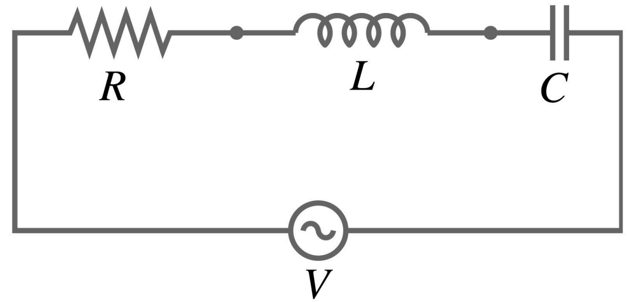
- Exploring a handful of approaches to demonstrate these characteristics of resonance
- Talk is really just an interdisciplinary crash course on *linear systems theory*...

Nota bene: The inner ear is not really linear per se (i.e., it is *complex*)

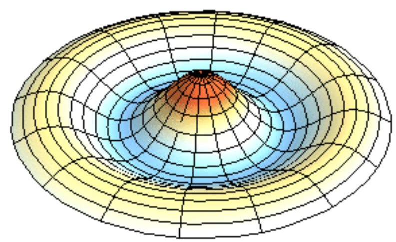
Aside

Resonance comes in a variety of “flavors”, e.g.,:

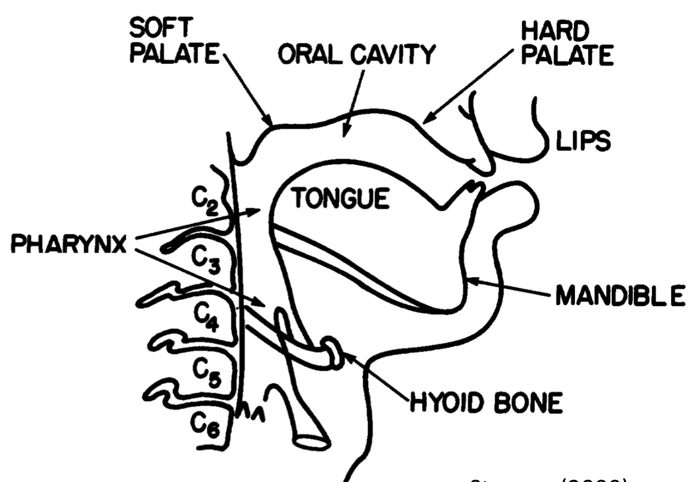
- Externally forced 2nd order “systems” (i.e., energy is being input into them)



- Standing waves

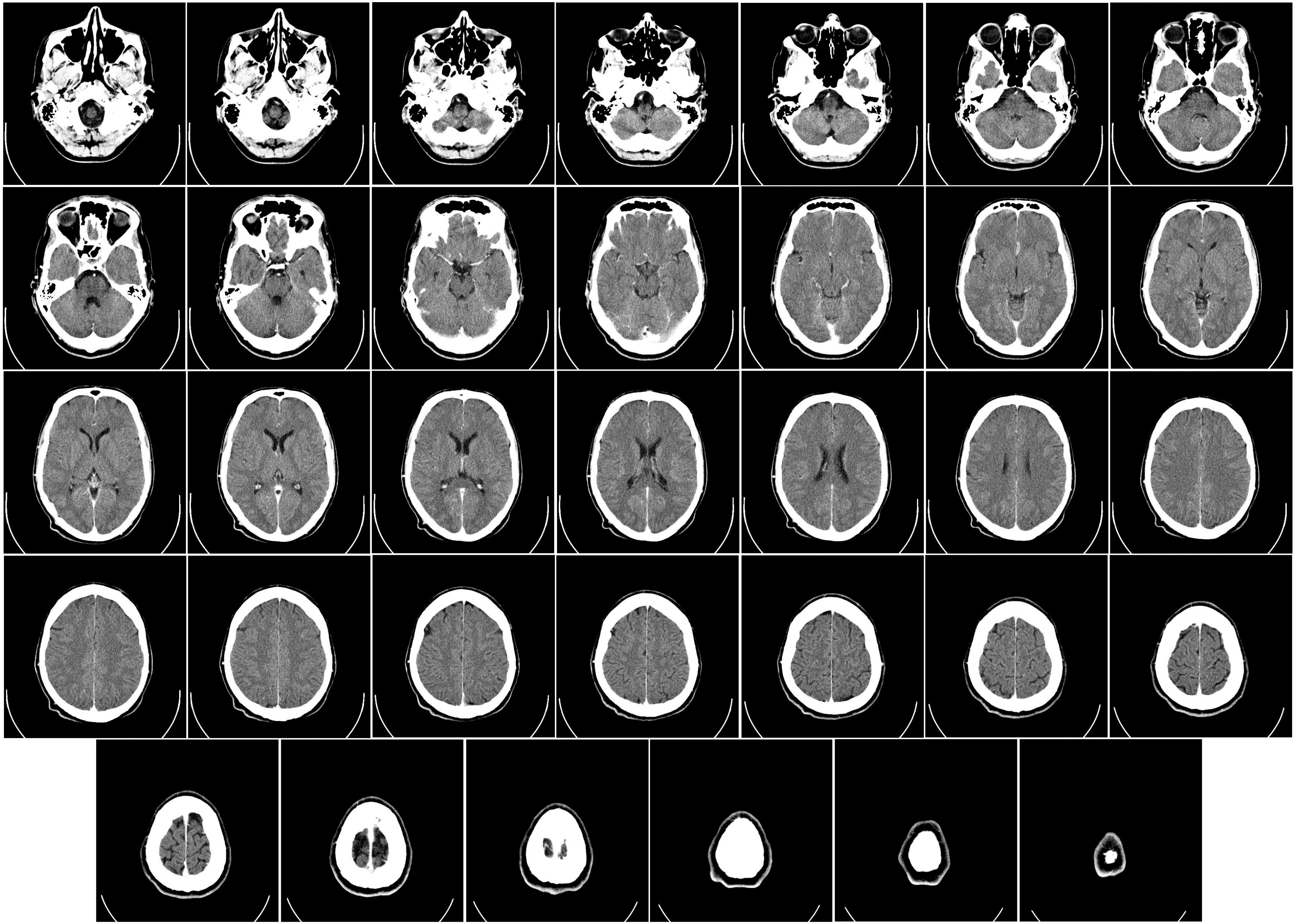


http://www.acs.psu.edu/drussell/demos/membranecircle/circle.html



Stevens (2000)

- NMR/MRI



Wikipedia

Aside

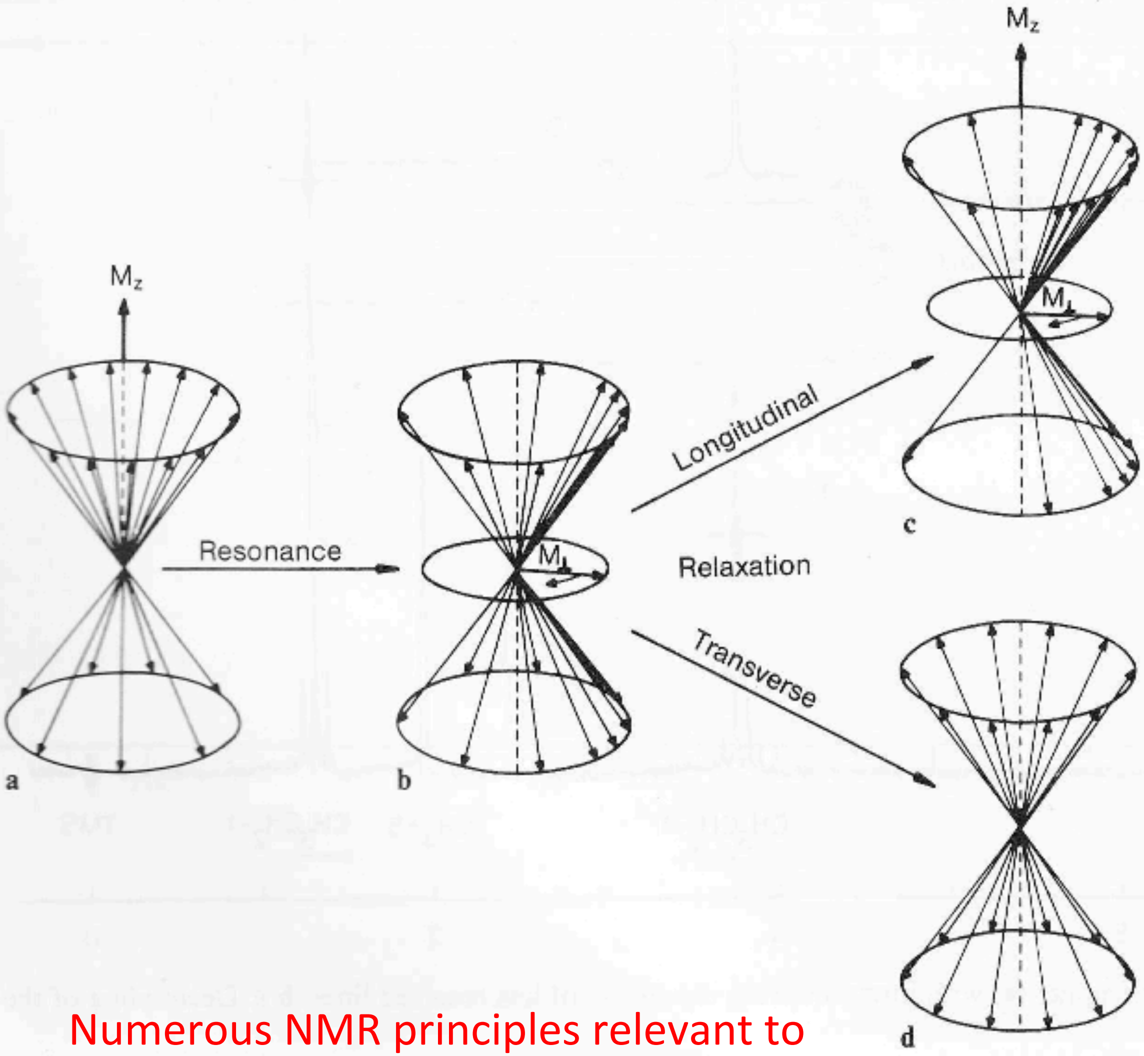


Fig. 3.147a–d. Classical representation of the NMR experiment. **a** In equilibrium the nuclear spins are distributed in the states α and β according to the Boltzmann distribution. **b** At resonance and with a sufficiently strong RF field, the populations of α and β are equalized and the spins precess in phase at the Larmor frequency ω_L. **c** Longitudinal relaxation restores the equilibrium distribution of the spins. **d** The phase coherence of the spins is lost by transverse relaxation. In reality the processes **c** and **d** proceed simultaneously

Numerous NMR principles relevant to cochlear mechanics

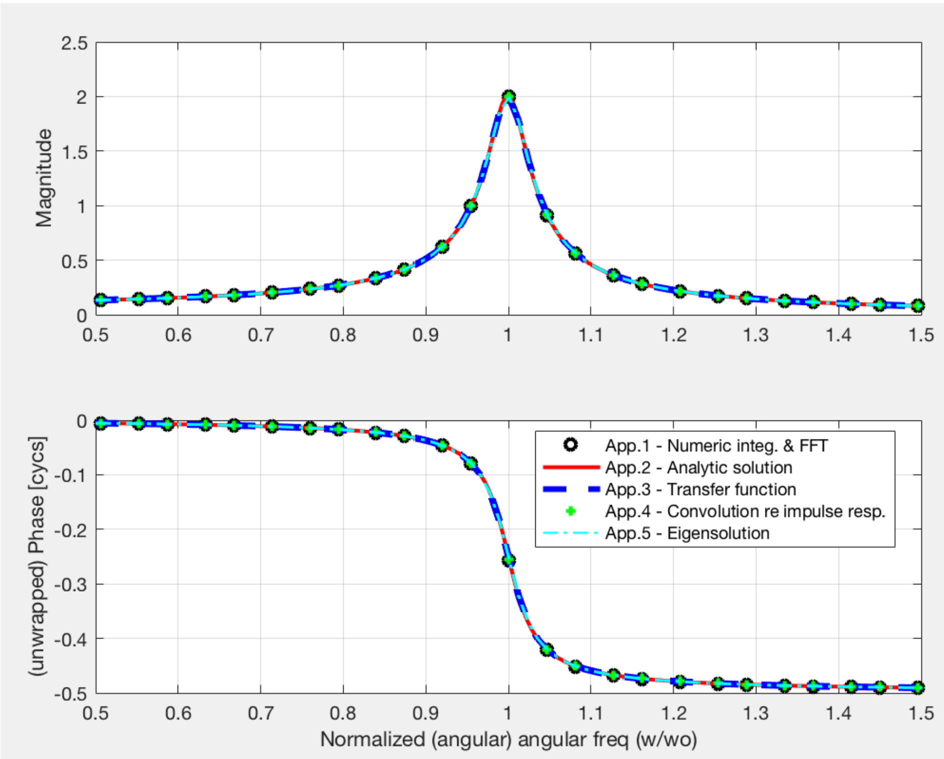
(e.g., micro/macroscopic states, phase coherence)

Hoppe et al (1983)

Resonance: The power to evoke enduring images, memories, and emotions — - *Oxford dictionary, online*

Sonia de Pasqua
http://www.d-sho.com/

$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$



Several basic approaches:
(all arriving at the same answer)

1. Numerically solve the ODEs and extract the relevant magnitudes and phases (via an FFT)

2. Analytic solution I (via Fourier transforms)

3. Impulse response I (and associated *transfer function*)

4. Impulse response II (convolve in the time domain)

5. Analytic solution II (via eigensolutions)

Note: Via linear systems theory, these different "approaches" are not necessarily mutually exclusive (e.g., convolution theorem directly links #s 3 & 4)

Interdisciplinary connections (i.e., basic concepts this code demonstrates)

Physics

- ODEs (e.g., Newton's 2nd, Hooke's Law)
- Resonance
- Notion of "steady-state"

Mathematics

- Fourier transforms
- Complex #s
- Eigenvalues
- Phase space

Engineering

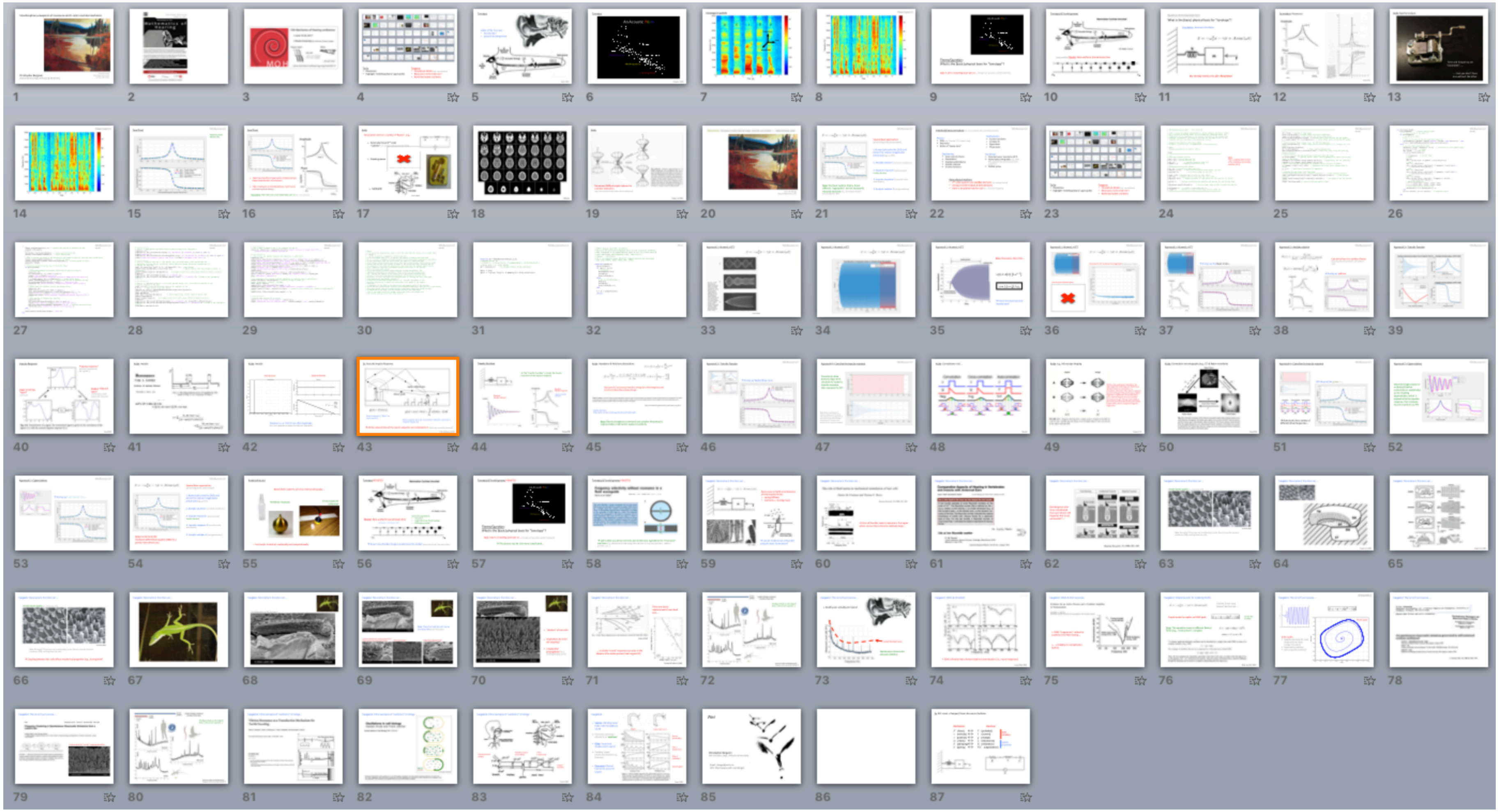
- Linear systems theory
- Convolutions
- Impedance/Admittance
- Impulse response
- Transfer functions

Numerical

- Discrete Fourier transforms (FFT)
- Numerically solving ODEs (e.g., Euler, RK4, adaptive step-size and associated problems)
- Matlab syntax

Basic physical intuition:

- 2nd order system: two ***reactive*** elements (i.e., energy-storing)
- energy transferred back & forth between
- there is an optimal rate for such (i.e., resonant frequency)



<u>To Do</u>

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<u>Tangents</u>

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```
% ### EXhoResonance.m ###     2017.02.04 CB

% Code to solve the damped (sinusoidally-) driven harmonic oscillator (DDHO)
% for a variety of driving freqs. so to buildup the "resonance curve" via
% computation of the mag/phase of the Fourier transform of the steady-state
% response. Furthermore, the analytic solution for the DDHO as well as the
% transfer function are shown to be equivalent (Fig.1)

% Damped driven Harmonic Oscillator (DDHO)
% d^2xdt^2 = -((P.wo)^2)*x - P.gamma*dx/dt + (P.A)*sin(P.w*t)

% Reqs:
% EXhoResonanceFunc.m (re ode45), rfft.m

clear;
% ==================================================================
% ---
% Oscillator params. and ICs
P.p0 = 0.0; % Initial position {0}
P.v0 = 0.0; % Initial velocity {0}
P.wo= 10;            % resonant (angular) freq {10}
P.gamma= 0.5;      % damping coefficient {0.5}
% ---
% Sinusoidal driving term params.
P.A = 10; % Driving force amplitude {10}
P.wDrive= [5 15];    % start and end angular drive freqs. {[5 15]}
P.wDriveN= 25;          % # of drive freqs. to run {25}
% ---
P.tmax = 200; % Maximum time to solve [s; arb] {200}
P.SR= 150;     % sample rate for time step [Hz; arb] {150}
P.Npoints= 8192;    % Number of points in time series for FFT, must be 2^n {8192}
% ---
P.plotN= 1;     % boolean re plotting the waveform and spectra for one driving freq.
{1}
P.plotNnum= round(P.wDriveN/2);     % driving freq. index to plot {round(P.wDriveN/2)}
% ---
P.solveType= 1;   % 0-ode45, 1-hard-coded RK4 {1}
P.stepF= 0;     % boolean re using a fixed step-size for ode45 {0}
% ==================================================================
```

Note:
This is a slightly older version of the code (does not include methods 4 & 5)

```
% ===============================================================
% ---
dt= 1/P.SR;  % spacing of time steps
init0 = [P.p0 P.v0]'; % Column vector of initial conditions.
tspan = [0:dt:P.tmax]; % time interval for entire computation
tW=[0:1/P.SR:(P.Npoints-1)/P.SR];  % (shorter/later) time interval for FFT window
L = length(tspan); TW = L-(P.Npoints-1);  % create offset point extracting FFT window
% ---
% create relevant freq. arrays (e.g., for FFT bin labeling)
freq= [0:P.Npoints/2];  % Note: these values are not angular (i.e., [freq]= 1/s, not
rads/s)
freq= P.SR*freq./P.Npoints;
df = P.SR/P.Npoints;            % freq. spacing between bins
wDT= linspace(P.wDrive(1),P.wDrive(2),500); % create ang. freq. array for plotting
analytic solution
% ++++++++++++++
% various relevant derived quantities (used post- main for loop)
Q= P.wo/P.gamma;    % "quality factor" (Note: tau=1/P.gamma=Q/P.wo, where tau is time
const. of build-up)
lambdaP= 0.5*(-P.gamma+ sqrt(P.gamma^2-4*P.wo^2));  % Eigenvalues, for x=0 (undriven)
lambdaM= 0.5*(-P.gamma- sqrt(P.gamma^2-4*P.wo^2));
% Note - Can also get eigenvalues via command: eig([0 1;-P.wo^2 -P.gamma])
Z= P.gamma+ i*(wDT- P.wo^2./wDT);   % impedance (see notes above; assumes mass is
unity)
Y= 1./Z;    % admittance (reciprocal of impedance)
% ---
% grabbing driving freqs. from freq array
%%indx= find(freq>=P.fDrive(1) & freq<=P.fDrive(2));    % find relevant indicies
indx= find(freq>=P.wDrive(1)/(2*pi) & freq<=P.wDrive(2)/(2*pi));    % find relevant
indicies
indxB= round(linspace(indx(1),indx(end),P.wDriveN));    % one means to get the desired
subset
freqD= 2*pi*freq(indxB); % array of driving angular freqs
```

```
for mm=1:numel(freqD)
    P.w= freqD(mm);     % extract driving freq.
     % *** Solve in one of two ways ***
    if P.solveType==0
        % use Matlab's ode45
        % ---
        % tell it to actually use the specified step-size
        if(P.stepF==1), options = odeset('MaxStep',1/P.SR);  else options=[];   end
        [t,y] = ode45(@EXhoResonanceFunc,tspan,init0,options,P);
    else
        % use 4th order Runge-Kutta code
        xPoints(1) = P.p0;  vPoints(1) = P.v0;   % initialize ICs into dummy arrays
        x= P.p0; v= P.v0;   % kludge
        dt= 1/P.SR;     % time step
        for nn=1:(length(tspan)-1)
            % ---
            t = tspan(nn);   % Current time.
            % step1
            xk1= v;
            vk1= -((P.wo)^2)*x - P.gamma*v + (P.A)*sin(P.w*t);
            % step 2
            xk2 = v + (dt/2)*vk1;
            vk2= -((P.wo)^2)*(x + (dt/2)*xk1) - P.gamma*(v + (dt/2)*vk1)...
                + (P.A)*sin(P.w*(t+(dt/2)));
            % step 3
            xk3 = v + (dt/2)*vk2;
            vk3= -((P.wo)^2)*(x + (dt/2)*xk2) - P.gamma*(v + (dt/2)*vk2)...
                + (P.A)*sin(P.w*(t+(dt/2)));
            % step 4
            xk4 = v + dt*vk3;
            vk4= -((P.wo)^2)*(x + (dt)*xk3) - P.gamma*(v + dt*vk3)...
                + (P.A)*sin(P.w*(t+(dt/2)));
            % apply RK4 weighting
            x = x + (dt/6)*(xk1 + 2*xk2 + 2*xk3 + xk4);
            v = v + (dt/6)*(vk1 + 2*vk2 + 2*vk3 + vk4);
            % store away position and velocity
            xPoints(nn+1) = x; vPoints(nn+1) = v;
        end
        y(:,1)= xPoints'; y(:,2)= vPoints';   % repackage output

    end
```

```
    % ---
        ySPEC= y(TW:TW+P.Npoints-1,1);  % steady-state portion of waveform for FFT
        sigSPEC= rfft(ySPEC);
        % ---
        wDrive(mm)= 2*pi*freq(indxB(mm));   % store away driving freqs.
        mag(mm)= abs(sigSPEC(indxB(mm)));   % store away SS mag.
        % ---
        % need to correct the phase re the duration of the window allowed for settling into steady-state
        tPhase= angle(sigSPEC(indxB(mm)));  % extract the phase
        tPhase= angle(exp(i*(tPhase- wDrive(mm)*tspan(TW))));  % correct phase re onset
        phase(mm)= tPhase;
        % ---
        % visualize relevant bits for one of the drive freqs.
        if mm==P.plotNnum
            % ---
            % integrated waveform and segment extracted for spectral analysis
            figure(2); clf;
            h1= plot(tspan,y(:,1)); hold on; grid on;
            xlabel('Time'); ylabel('Position');
            title('Time Waveform of integrated solution to damped driven HO equation')
            L = length(tspan); TW = L-(P.Npoints-1);  % create offset point
            ySPEC= y(TW:TW+P.Npoints-1,1);  % steady-state portion of waveform for FFT
            h2= plot(tspan(TW:TW+P.Npoints-1),ySPEC,'r.','MarkerSize',3);
            legend([h1 h2],'Entire waveform','Steady-state portion (used for FFT)')
            % ---
            % phase space for waveform (entire and steady-state)
            figure(3); clf;
            hPS1= plot(y(:,1),y(:,2)); hold on; grid on;
            hPS2= plot(y(TW:TW+P.Npoints-1,1),y(TW:TW+P.Npoints-1,2),'r.-');
            xlabel('Position'); ylabel('Velocity'); title('Phase plane');
            legend([hPS1 hPS2],'Entire waveform','Steady-state portion (used for FFT)')
            % ---
            % plot spectra of steady-state waveform
            figure(4); clf;
            hS1= plot(2*pi*freq,db(sigSPEC)); hold on; grid on;
            xlabel('Freq [rads/s]'); ylabel('Spectral amplitude [dB]');
            hS2= plot(2*pi*freq(indxB(mm)),db(mag(mm)),'rs');   % indicate extracted freq.
            legend([hS1 hS2],'Steady-state spectra','Driving freq.');
        end
        % ---
        disp([num2str(100*mm/numel(freqD)),'% done']);
    end
```

```
% ++++++++++++++
% [Fig.1] ** Mags/phases extracted from the numeric steady-state responses **
figure(1); clf;
subplot(211); hh1= plot(wDrive/P.wo,mag,'ko','MarkerSize',6,'LineWidth',2); hold on; grid on;
ylabel('Magnitude');
subplot(212); hh2= plot(wDrive/P.wo,unwrap(phase)/(2*pi),'ko','MarkerSize',6,'LineWidth',2); hold on; grid on;
xlabel('Normalized (angular) angular freq (w/wo)'); ylabel('(unwrapped) Phase [cycs]');

% ++++++++++++++
% [Fig.1]** Analytic solution ** (see French, 1971; as noted above, these expressions are
% equivalent to using Fourier transforms, which implicitly assume sinusoidal steady-state, to
% solving the main ODE)
magT= P.A./sqrt((P.wo^2-wDT.^2).^2 + ((P.gamma*wDT).^2));  % mag (theory)
phaseT= atan((P.gamma*wDT)./(-P.wo^2+wDT.^2));             % phase (theory; note sign change in denom. re
convention)
phaseT= phaseT+ phase(1)+ abs(phaseT(1));    % (kludge) correct for (arb?) phase offset in numeric solution
figure(1);
subplot(211); hh3= plot(wDT/P.wo,magT,'r-','LineWidth',2);
subplot(212); hh4= plot(wDT/P.wo,unwrap(2*phaseT)/(4*pi),'r-','LineWidth',2); % kludge to get unwrapping
working

% ++++++++++++++
% [Fig.1] ** "Transfer function" ** re linear systems theory (i.e., the Fourier transform of the
% impulse response of the DHO)
init0 = [0 10]'; % set ICs such that there is an "impulse" at t=0
P.w= 0;     % make sure to "turn off" drive
options= []; [t,yI] = ode45(@EXhoResonanceFunc,tspan,init0,options,P);
specI= rfft(yI(1:P.Npoints));
magI= abs(specI);
magI= magI* (max(mag)/max(magI));       % scale impulse mag. re max. value of driven case
phaseI= angle(specI);
phaseI= phaseI+ phase(1);    % (kludge) correct for (arb?) phase offset in numeric solution
figure(1);
subplot(211); hh5= plot(2*pi*freq/P.wo,magI,'b--','LineWidth',2); xlim([wDT(1) wDT(end)]/P.wo);
subplot(212); hh6= plot(2*pi*freq/P.wo,unwrap(2*phaseI)/(4*pi),'b--','LineWidth',2); % kludge to get unwrapping
working
xlim([wDT(1) wDT(end)]/P.wo);
```

```
% ++++++++++++++
% [Fig.1] Make a legend to put it all together (re Fig.1)
figure(1); subplot(211); legend([hh1 hh3 hh5],'Numeric solution re steady-state FFT',...
    'Analytic solution','Transfer function');

% ++++++++++++++
% [Fig.5] Plot the impulse response and comparison to admittance
figure(5); clf;
subplot(221); plot(tW,yI(1:P.Npoints)); hold on; grid on; xlabel('Time [s]'); ylabel('x');
title('Impulse response (no drive; P.w=0, P.p0=0, P.v0=10'); xlim([0 tW(round(numel(tW)/3))]);
subplot(222); hI2= plot(freq,db(specI),'LineWidth',2); grid on; hold on; ylabel('Amplitude [dB]');
title('Transfer function (mag. of FFT of IR)'); xlim(P.wDrive/(2*pi));
subplot(224); hI3= plot(freq,angle(specI)/(2*pi),'LineWidth',2);  grid on; hold on;
xlabel('Frequency [Hz]'); ylabel('Phase [cycles]');
title('Transfer function (phase of FFT of IR)'); xlim(P.wDrive/(2*pi));
subplot(223);
hZa= plot(wDT,abs(Y),'k-'); grid on; hold on;  hZb= plot(wDT,abs(Z),'r.');
grid on; hold on; ylabel('Amplitude'); xlabel('Ang. requency [rad/s]'); legend([hZa
hZb],'admittance','impedance');
% ---
% for reference, also include (scaled) admittance to indicate (near?) equivalence
offset= max(db(Y))- max(db(specI)); % scaling (in dB) to match up
%magY= fliplr(db(Y)- offset);  % kludge
magY= (db(Y)- offset);
subplot(222); hI2b= plot(wDT/(2*pi),magY,'r--');
legend([hI2 hI2b],'Transf. func.','(scaled) Admittance','Location','SouthWest');
angleY= angle(Y)/(2*pi) - angle(Y(1))/(2*pi);   % there will be a slight vert. ofseet re angle(specI)/
(2*pi)
subplot(224); hI3b= plot(wDT/(2*pi),angleY,'r--');


% ++++++++++++++
% display some relevant #s to screen
disp(['Quality factor (P.wo/P.gamma)= ',num2str(Q)]);
disp(['Eigenvalues (for x=0, undriven case): ',num2str(lambdaP),' and ',num2str(lambdaM)]);
```

```
% Notes
% o To solve this numerically, need to turn 2nd order ODE into series of 1st order ODEs:
%   dx/dt= y
%   dy/dt= -P.wo^2*x - P.gamma*y + (P.A)*sin(P.w*t)
% o For autonomous case (i.e., no drive), can rewrite in matrix form such that
% A= [0 1;-P.wo^2 -P.gamma]; straight-forward to find associated eigenvalues (see below)
% o via P.solveType, user can solve either via ode45 or a hard-coded RK4
% (both should yield the same solution!); Note that (surprisingly) ode45 actually seems
% slower than the RK4 (the slowest is ode45 w/ the fixed step-size), possibly due to
% the passing of the large-ish structure P; also note that the default ode45 routine
% (i.e., adaptive step-size) introduces harmonic disortions in the spectra
% due to its nonlinear nature
% o For the analytic solution (below manifest as magT and phaseT), the
% expression used below, as derived in French (1971) for the steady-state,
% is   exactly the same as if one simply put in the Fourier transform and
% solved for the resulting magnitude and phase [confirmed on the back of an
% envelope; let x(t)= X(w)exp(i*w*t) and plug back in, solving for X(w); note then
% that magT=abs(X) and phaseT= angle(X)]
% o There are a few minor kludges below [e.g., vertical adjustment of the
% analytic solution so to match the (arbitrary?) ref. phase of the numeric
% solution)
% o Impedance (Z) for DDHO is (by definition) the complex ratio of the driving
% force and the (steady-state) velocity (see 4080W2016L10REF.pdf). Real part of Z (resistance)
% describes energy loss while imaginary part (reactance) describes energy storage
% o Comparison of the mags. for the transfer function and admittance
% (Fig.5, top right) are a bit kludgy (unsure why fliplr was needed) and
% off (worser overlap as you move away from wo)
```

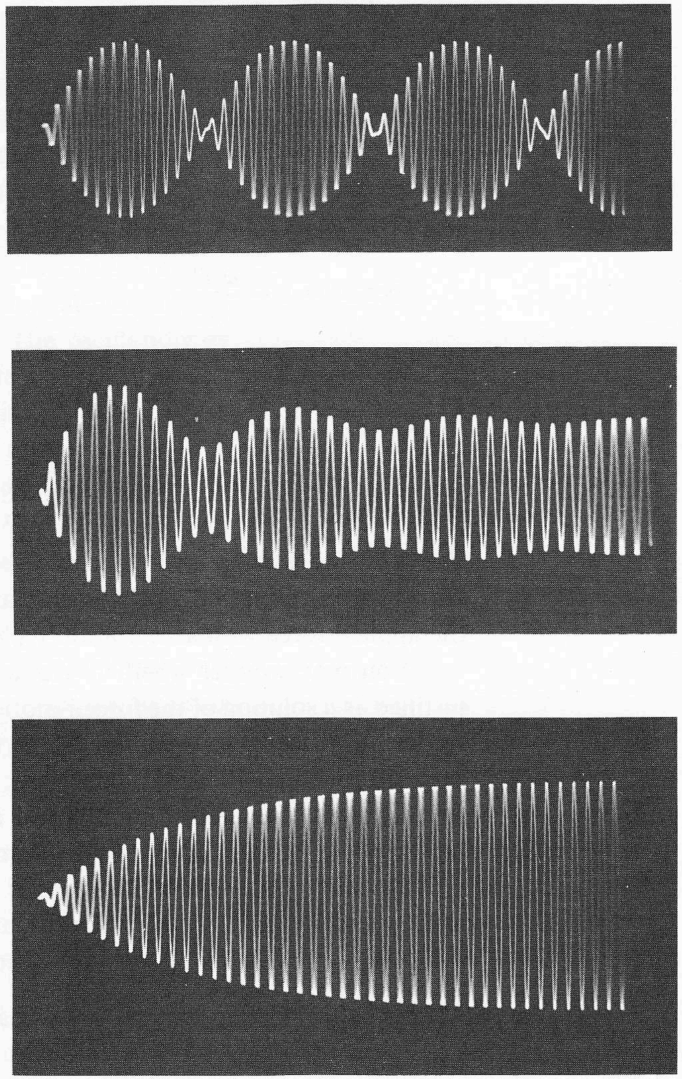
```
function dy = EXhoResonanceFunc(t,y,P)
% Damped driven HO
% d^2xdt^2 = -((P.wo)^2)*x - P.gamma*dx/dt + (P.A)*sin(P.w*t)
% Note: y(1) = x, y(2)= dx/dt
dy = zeros(2,1);    % A column vector to be returned

dy(1) = y(2);
dy(2) = -((P.wo)^2)*y(1)- P.gamma*y(2)+ (P.A)*sin(P.w*t);
```

```
% RFFT: scaled real FFT, X=rfft(x)
% Returns the positive-frequency half of the transform X=FFT(x).
% The transform X is normalized so that if {x} is a sine wave of
% unit amplitude and frequency n*df, then X[n]=1.
% Usage:    X=rfft(x)
% If x is N points long, NF=N/2+1 complex points are returned.
% See also IRFFT, FAST, FSST, FFT, IFFT,

function X=rfft(x)
  [m,n]=size(x);
  if (m==1 | n==1)
    % original...
    N=length(x)/2+1;
    xc=fft(x);
    X=xc(1:fix(N));
  else
    % do it column-wise...
    N=m/2+1;
    xc=fft(x);
    X=xc(1:fix(N),:);
  end

  X = X / (length(x)/2);
  return
```

EXhoResonance.m

Approach 1 – Numeric + FFT

$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$



(a)

(b)

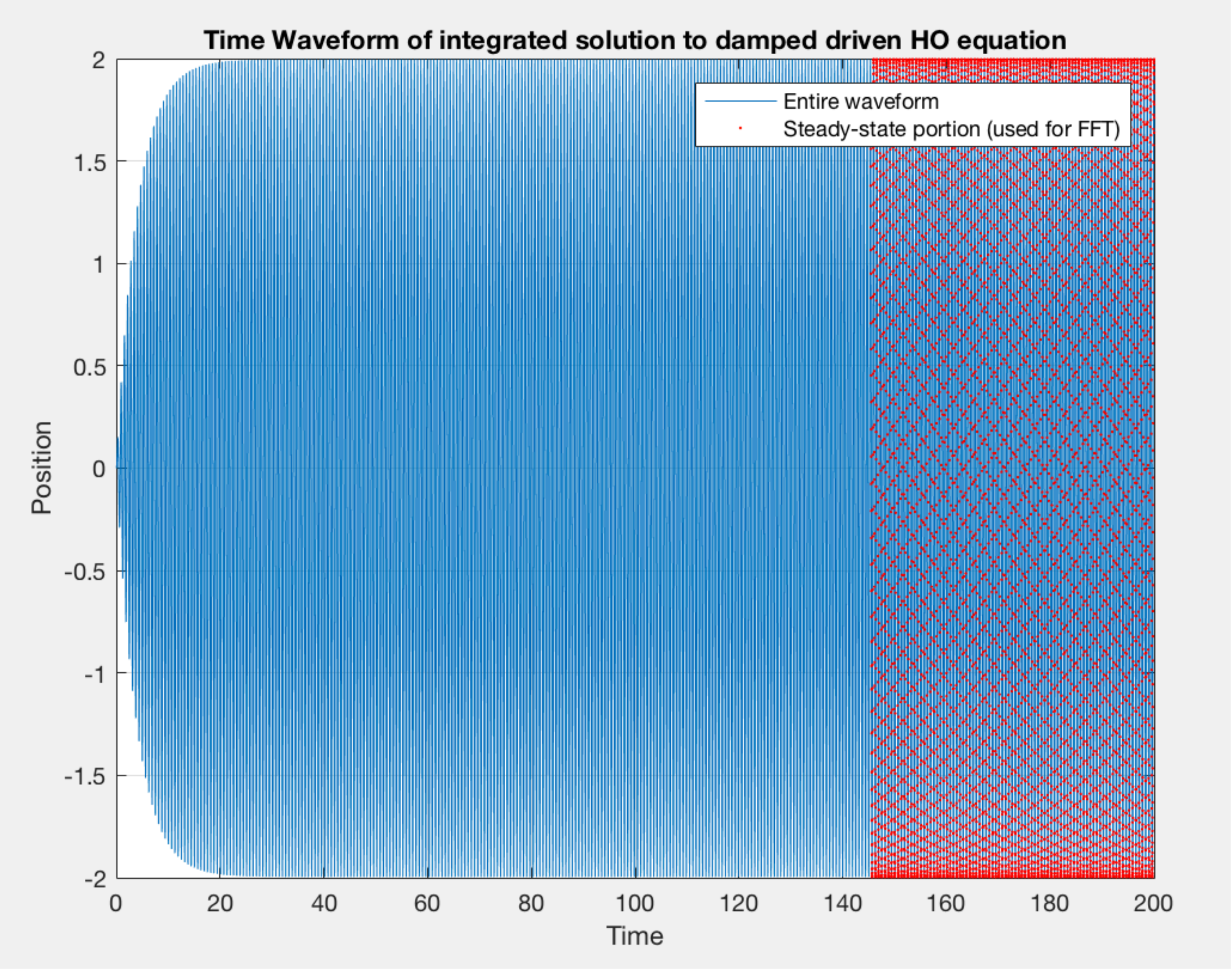
(c)

Fig. 4–11 (a) Response of an undamped harmonic oscillator to a periodic driving force, as described by Eq. (4–19). This beat pattern would continue indefinitely. (b) Transient behavior of a damped oscillator with a periodic driving force off resonance. (c) Transient behavior at exact resonance, showing smooth growth toward steady amplitude. (Photos by Jon Rosenfeld, Education Research Center, M.I.T.)

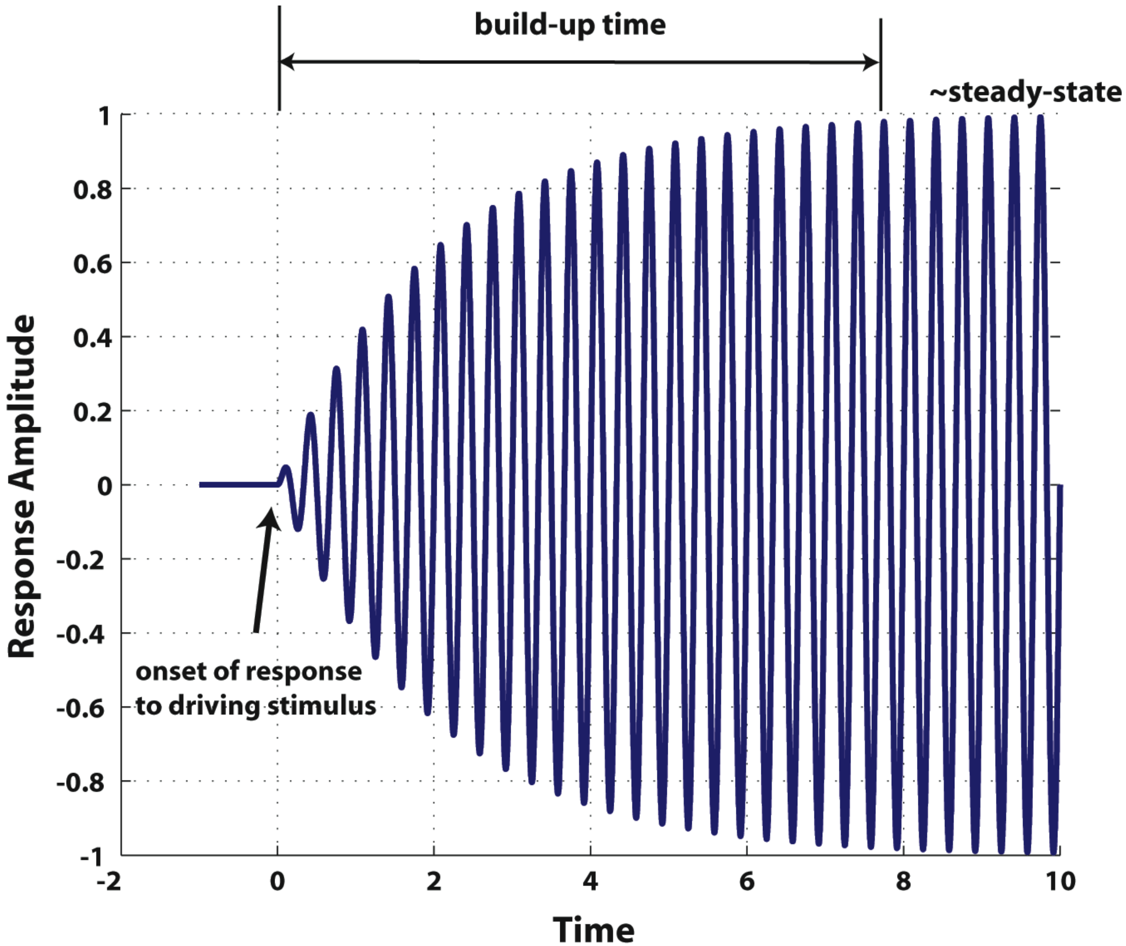
French (1971)

Approach 1 – Numeric + FFT

$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$



Approach 1 – Numeric + FFT



Note: Resonance takes time...

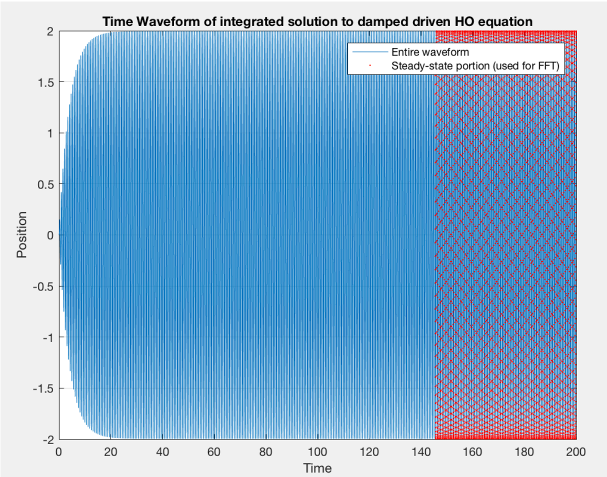
$$x(t) = A(\infty)\,[1 - e^{(-t/\tau)}]$$

$$\tau = 1/\gamma = Q\,/\,\omega_o$$

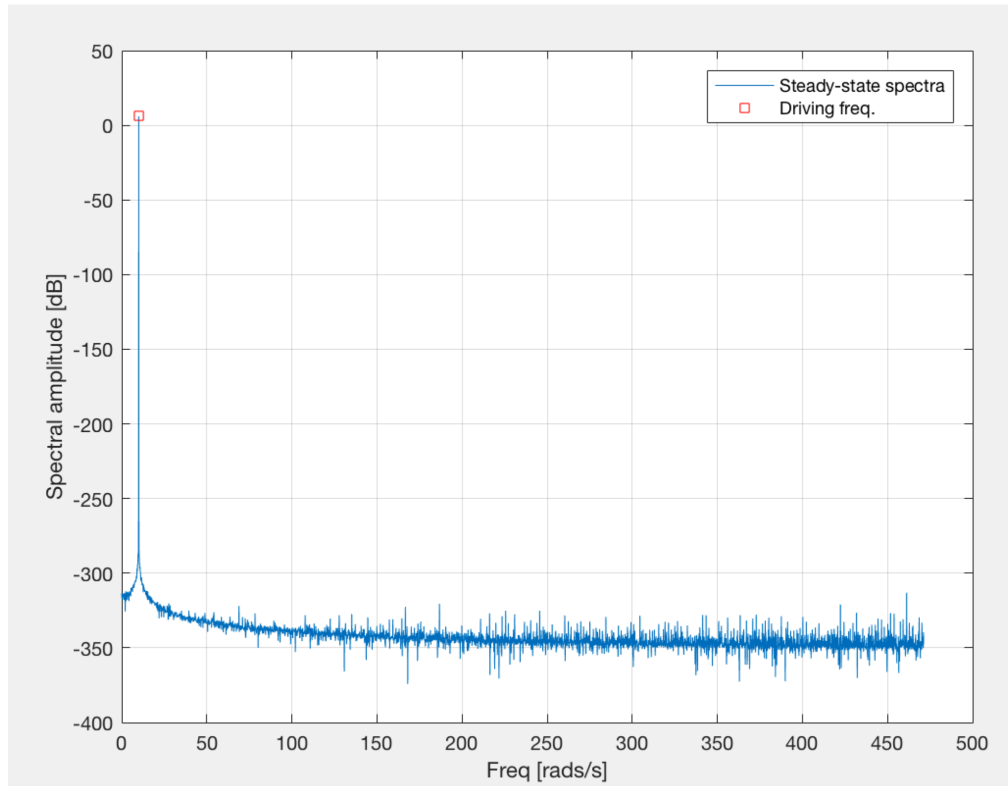
→ Hence the importance of "steady-state"

Approach 1 – Numeric + FFT

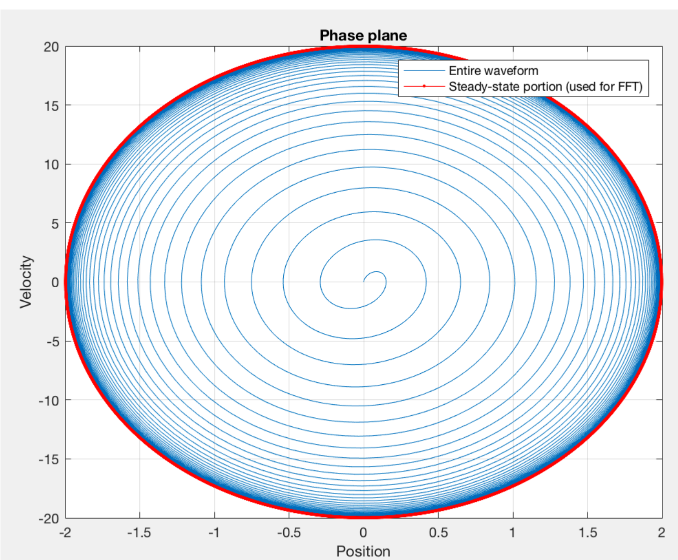
$$\ddot{x} = -\omega_o^2 x - \gamma\dot{x} + A\cos(\omega t)$$



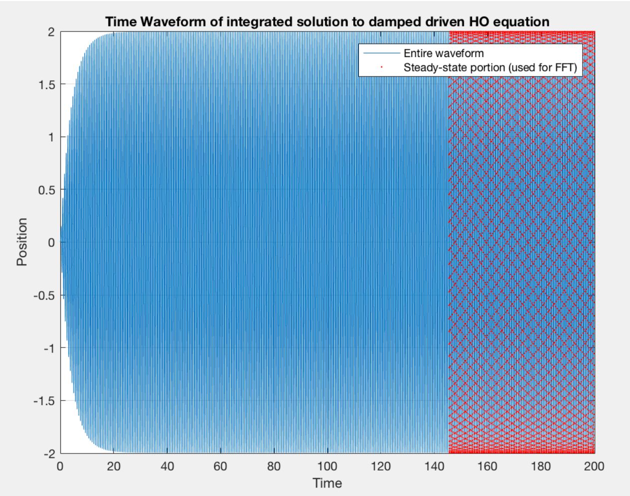
Compute the FFT to extract the magnitude (and phase; not shown)



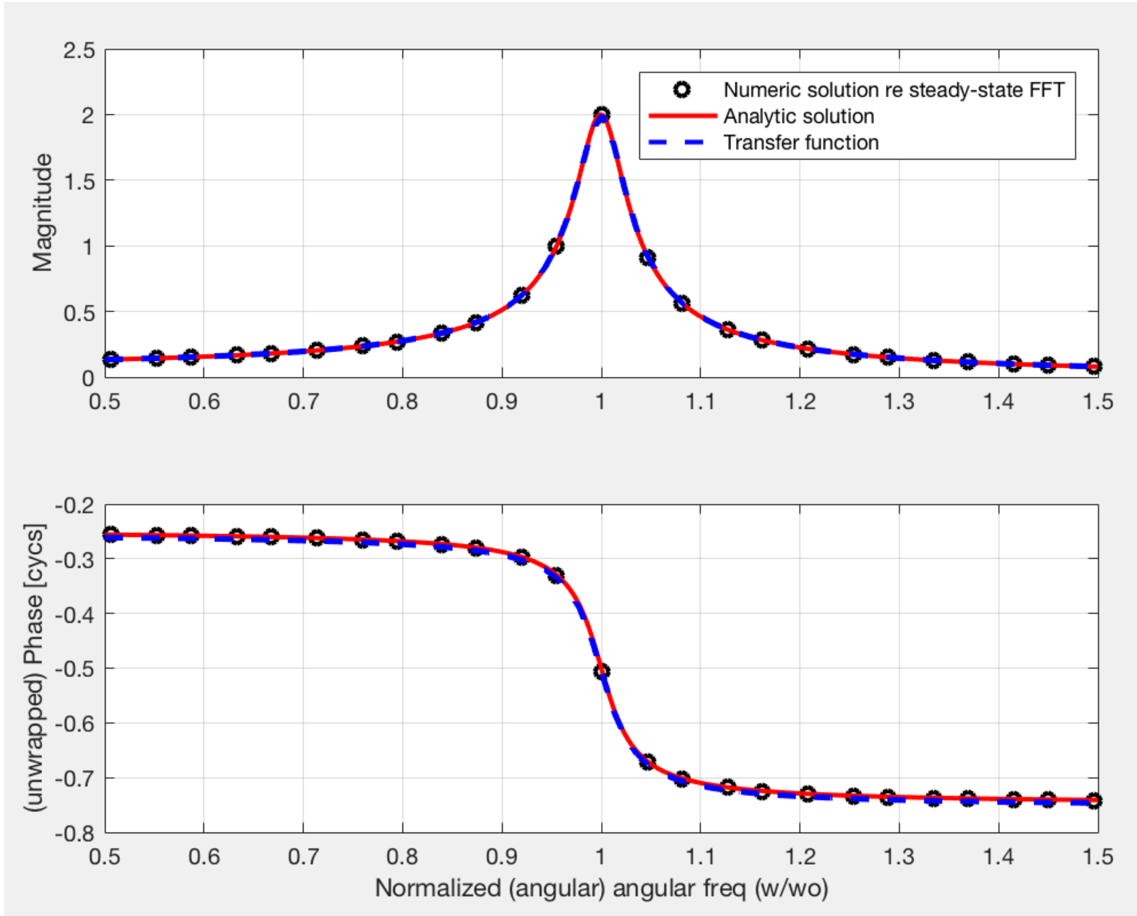
Can also plot in phase space...

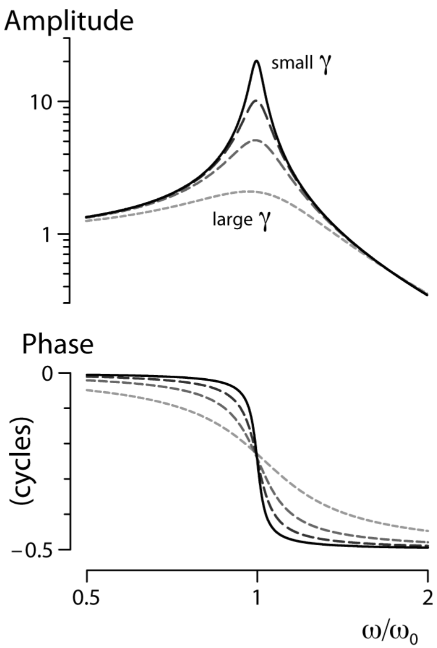


Approach 1 – Numeric + FFT



→ End up w/ the black circles....





Approach 2 – Analytic solution

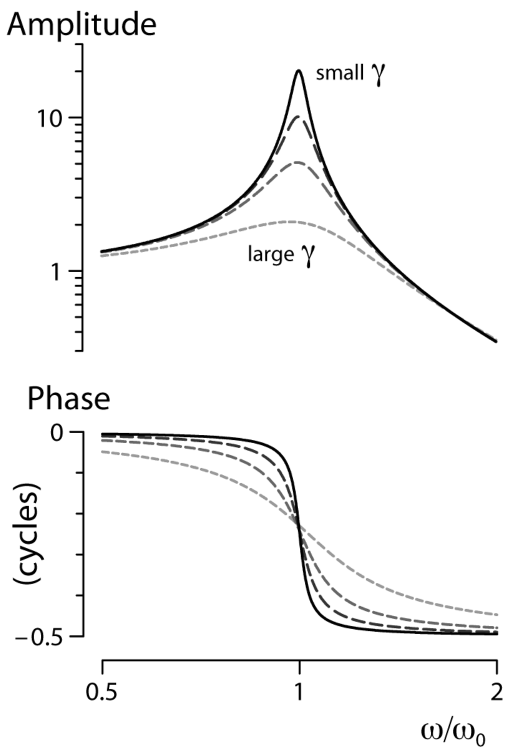
$$A(\omega) = \frac{F_o/m}{[(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}}$$

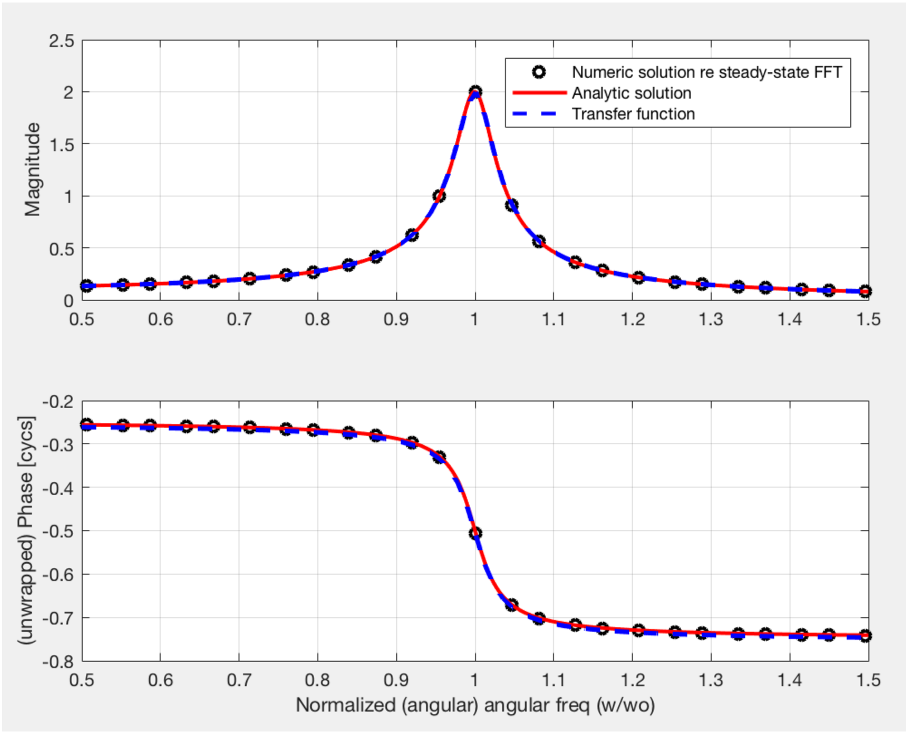
$$\delta(\omega) = \arctan\left(\frac{\gamma\omega}{\omega^2 - \omega_o^2}\right)$$

Can arrive here in a variety of ways

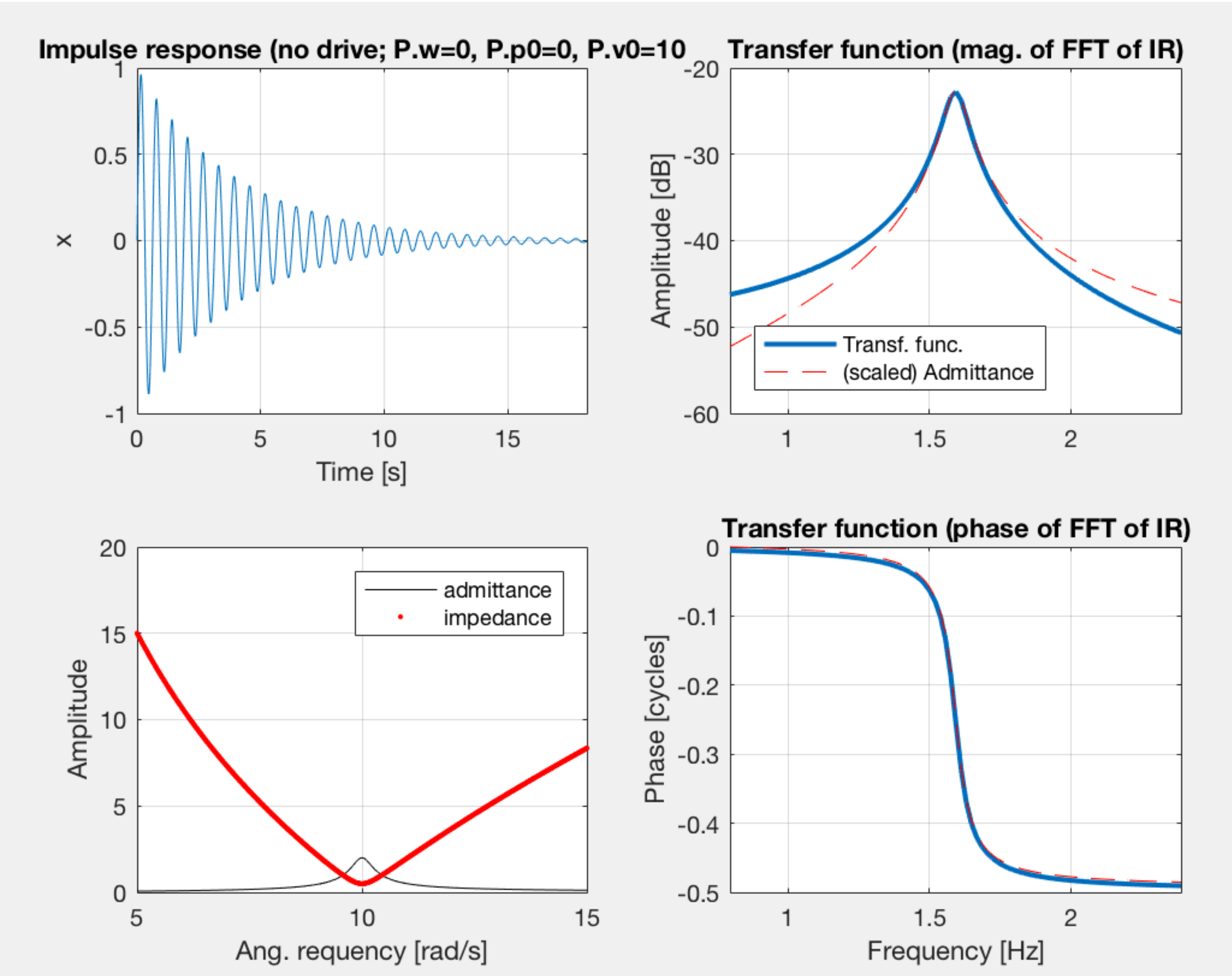
(including Fourier transforms; see notes at end)

→ End up w/ red line





Approach 3 – Transfer Function



Impulse Response

"Impulse response" (this fully characterizes the "system")

Input: Incoming "signal" (arbitrary)

Output: "filtered" signal (combination of the system and the input)

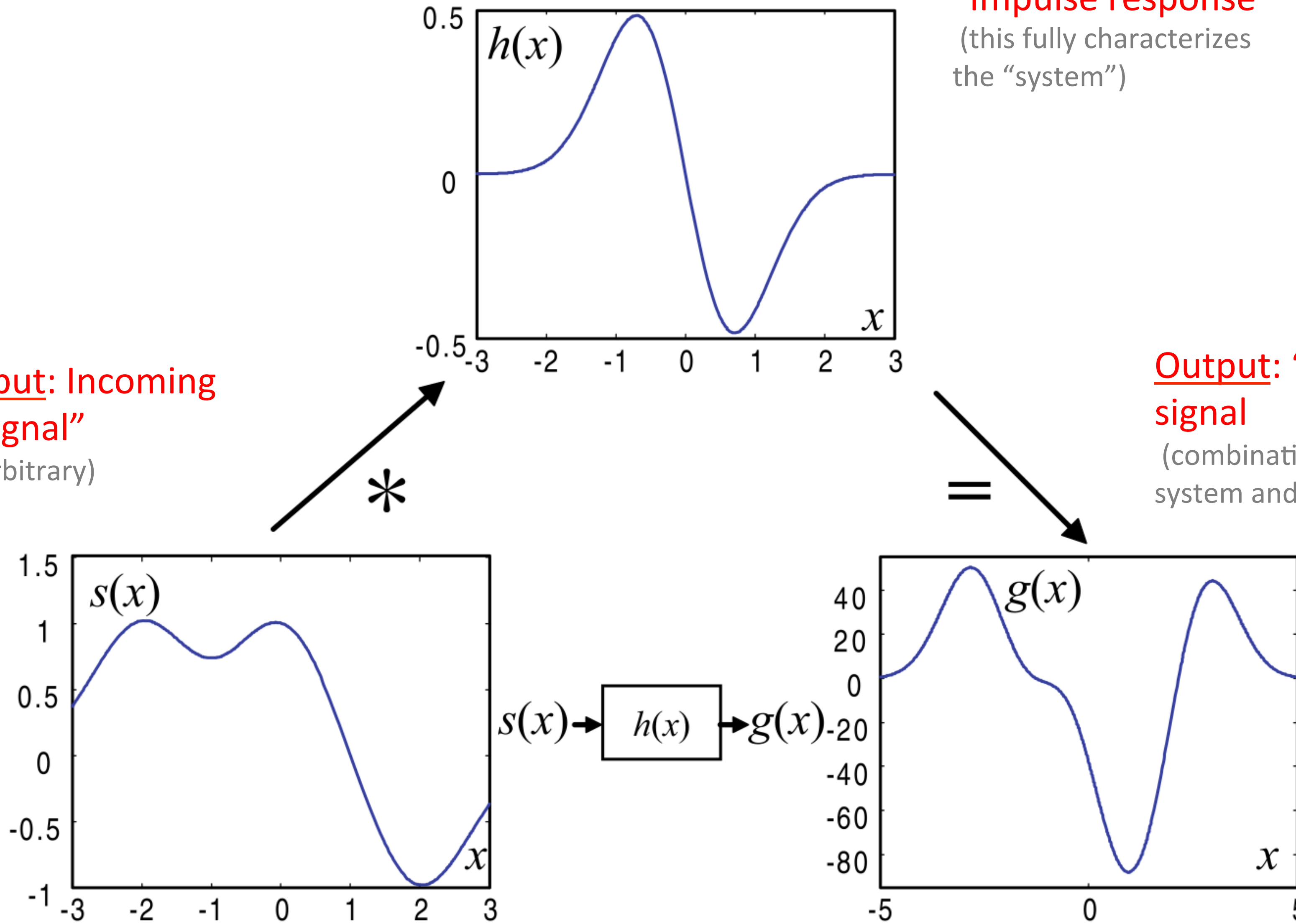


Fig. 4.11. Transmission of a signal. The transmitted signal is given by the convolution of the signal $s(x)$ with the system's impulse response $h(x)$

Buzug (2008)

Aside: Impulse

Resonance

CARL A. LUDEKE

JOURNAL OF APPLIED PHYSICS

VOLUME 13, JULY, 1942

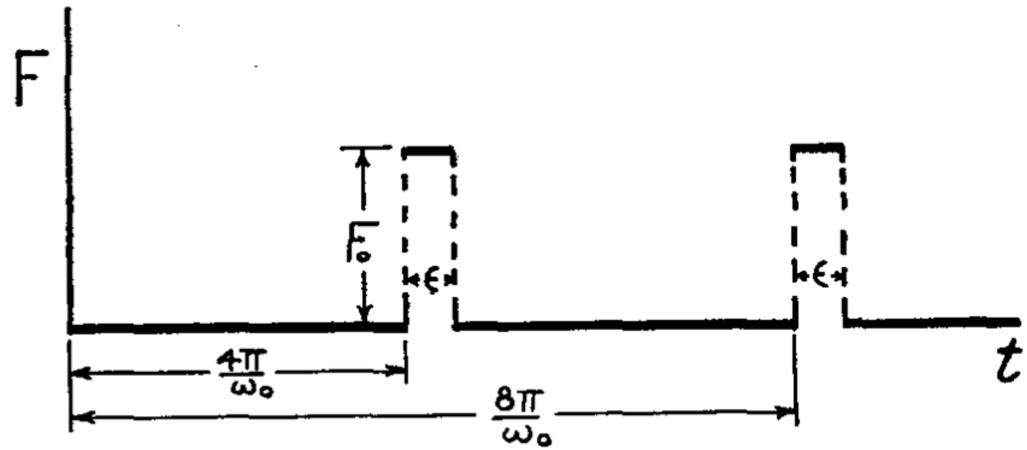


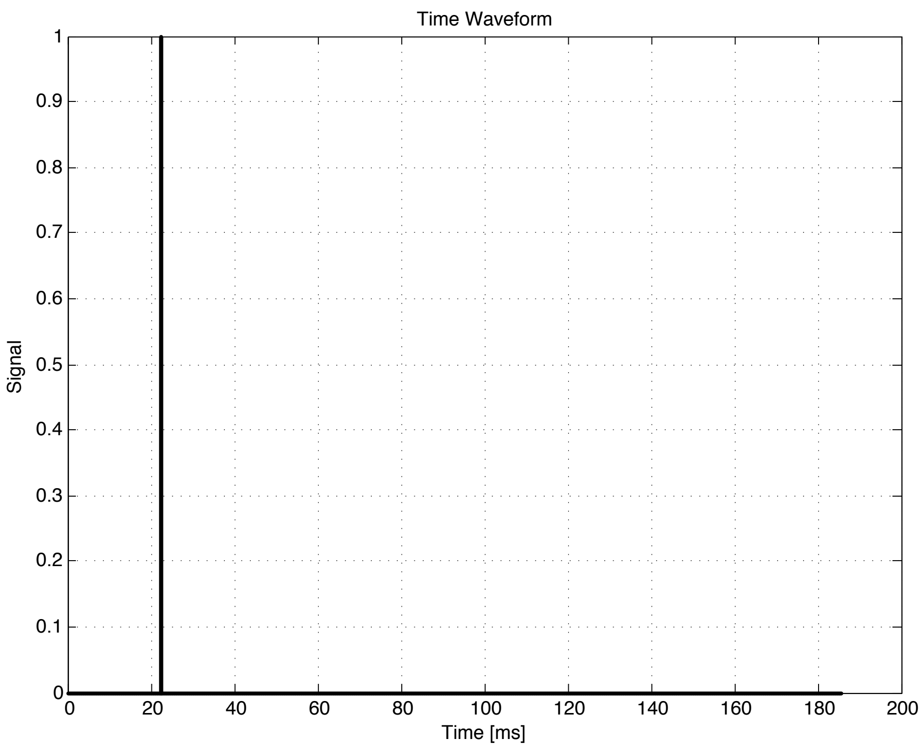
FIG. 4. The discontinuous force F supplied by the motor in Fig. 3, as a function of time t.

$$md^2x/dt^2+\beta dx/dt+kx = \sum A_n \sin n\omega t+\sum B_n \cos n\omega t.$$

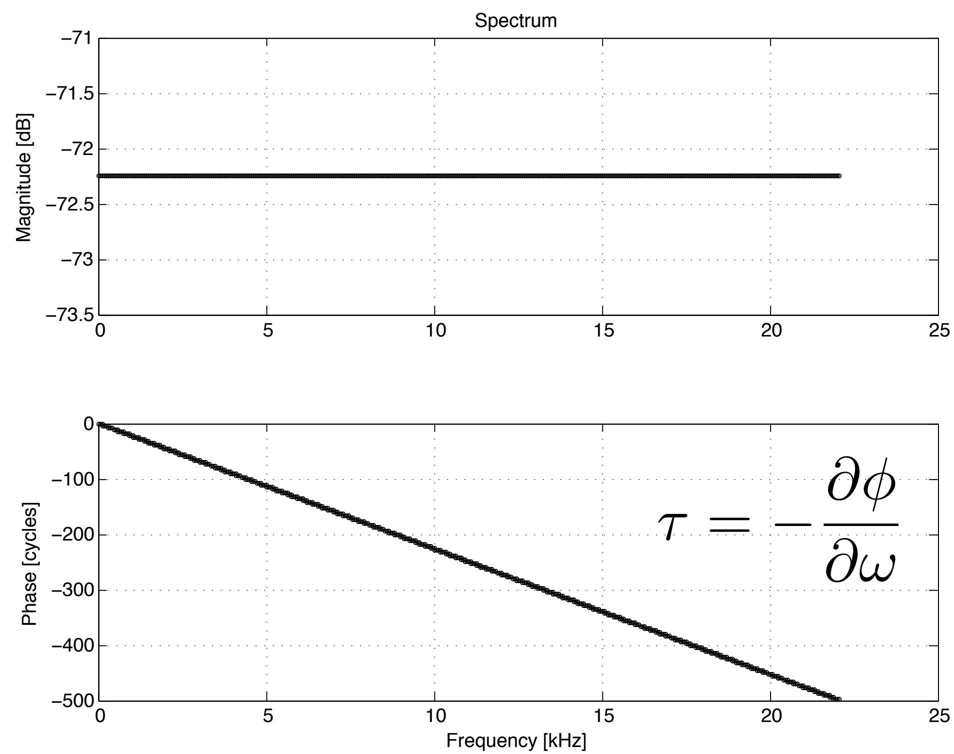
$$x=\sum\frac{A_n \sin(n\omega t-\varphi_n)}{[(k-mn^2\omega^2)^2+\beta^2n^2\omega^2]^{\frac{1}{2}}} +\sum\frac{B_n \cos(n\omega t-\alpha_n)}{[(k-mn^2\omega^2)^2+\beta^2n^2\omega^2]^{\frac{1}{2}}},$$

Aside: Impulse

Time domain



Spectral domain

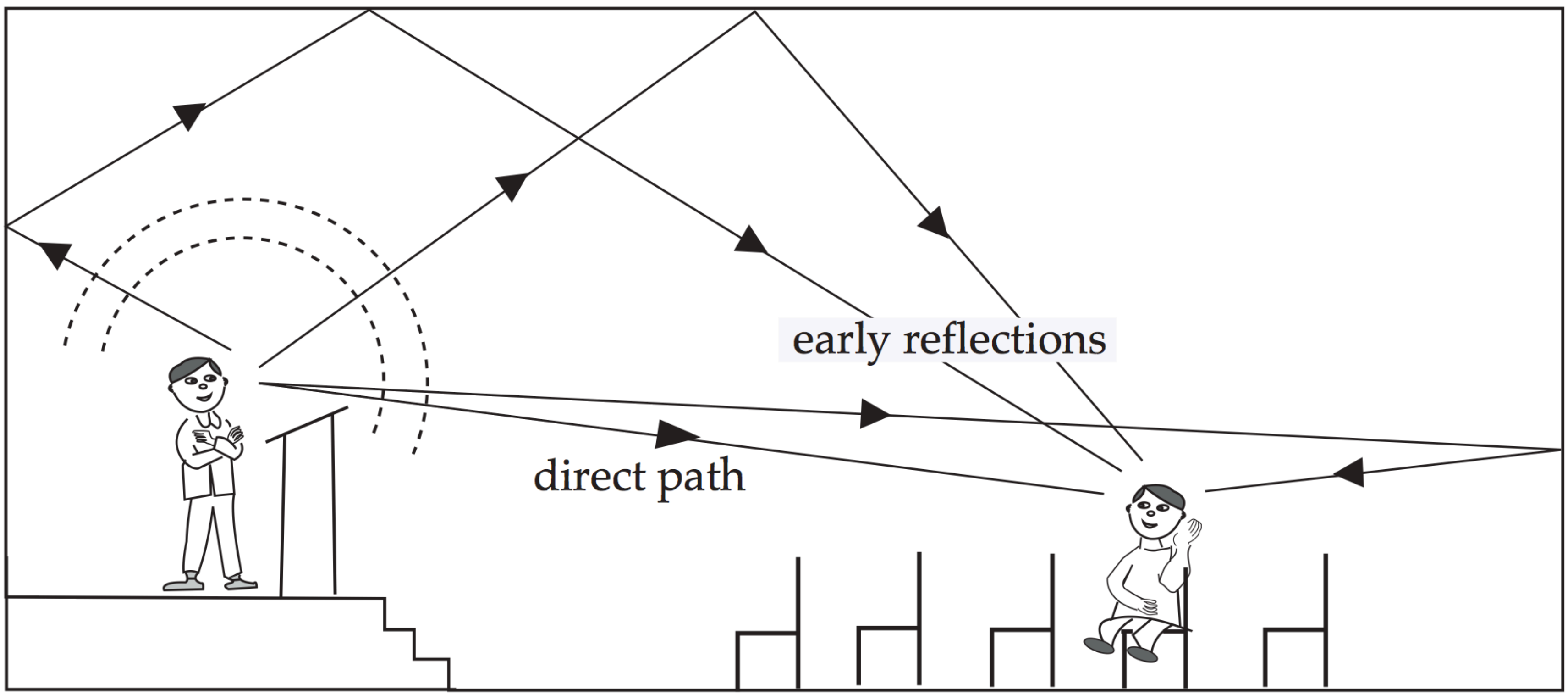


$$\tau = -\frac{\partial \phi}{\partial \omega}$$

Impulse (i.e., a "click") has a flat magnitude

(this is also a good place to mention the notion of a '*group delay*')

Ex. Acoustic Impulse Response



$$g(x) = \mathcal{L}\{s(x)\}$$

Room response (g) "filters" an input sound (s)

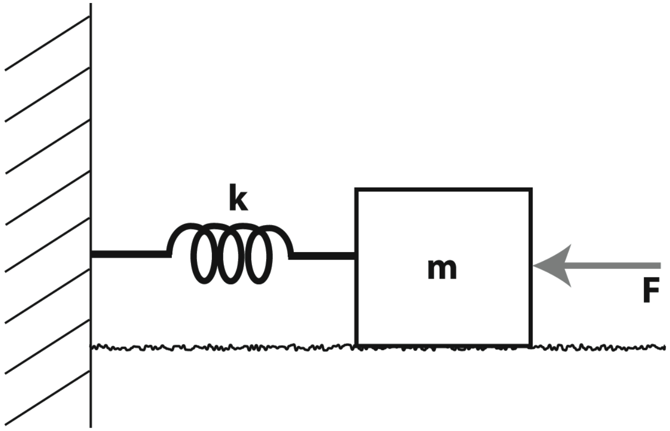
$$g(x) = s(x) * h(x) = \int_{-\infty}^{+\infty} s(\xi)h(x-\xi)\,\mathrm{d}\xi$$

Room response (g) is just "convolution" between s and room's impulse response (h)

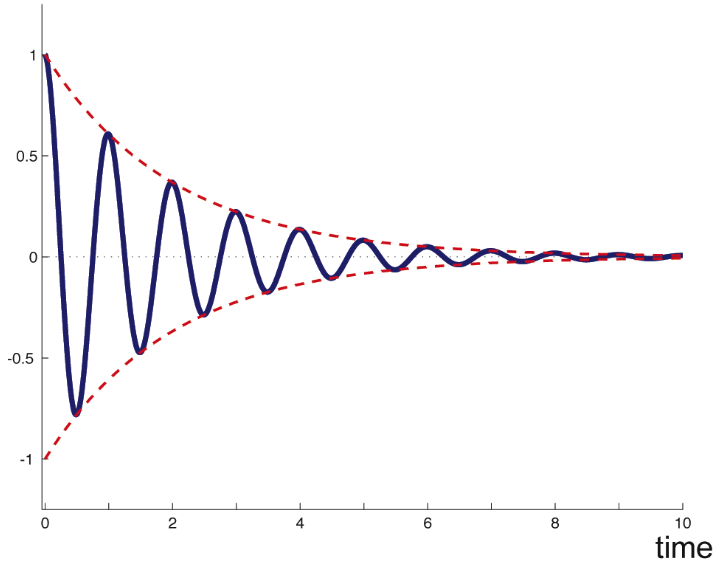
→ All the relevant bits of the room's acoustics are contained in h (which we can easily measure!)

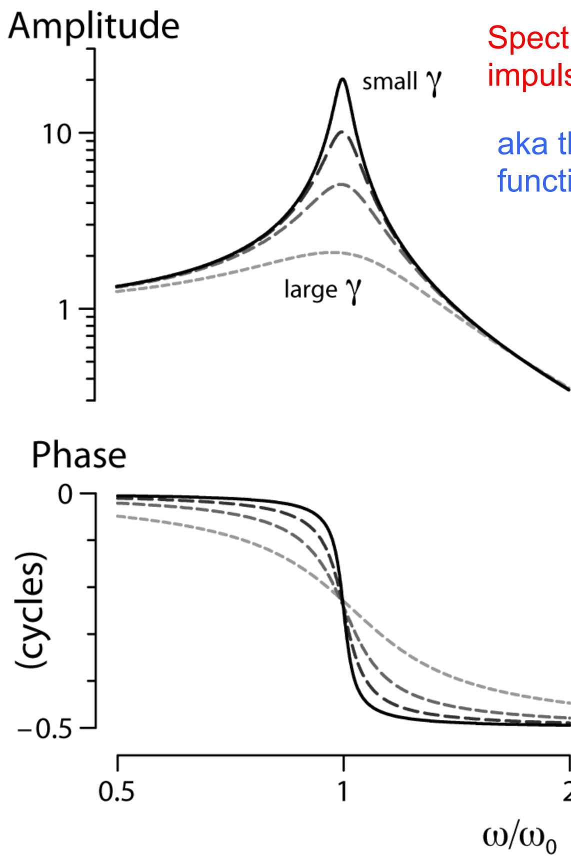
Pulkki & Karjalainen (2015)

Transfer functions



→ The "transfer function" is simply the Fourier transform of the impulse response





Buzug (2008)

Aside: Impedance & Pole/zero descriptions

$$\ddot{x} + \gamma\dot{x} + \omega_o^2 x = \frac{F_o}{m} e^{i\omega t}$$

$$Z \equiv \frac{F_{ext}}{\dot{x}} = \frac{x \cdot (-m\omega^2 + k + i\omega b)}{x \cdot i\omega} = b + i\left[m\omega - \frac{k}{\omega}\right]$$

Real part of Z (*resistance*) describes energy loss while imaginary part (*reactance*) describes energy storage

Poles and **Zeros** of a transfer function are the frequencies for which the value of the denominator and numerator of transfer function becomes zero respectively. The values of the poles and the zeros of a system determine whether the system is stable, and how well the system performs. Control systems, in the most simple sense, can be designed simply by assigning specific values to the poles and zeros of the system.

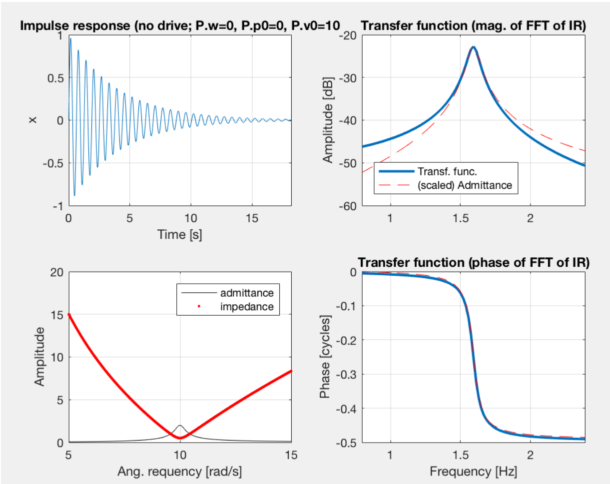
https://en.wikibooks.org/wiki/Control_Systems/Poles_and_Zeros

Useful reference:
http://web.mit.edu/2.14/www/Handouts/PoleZero.pdf

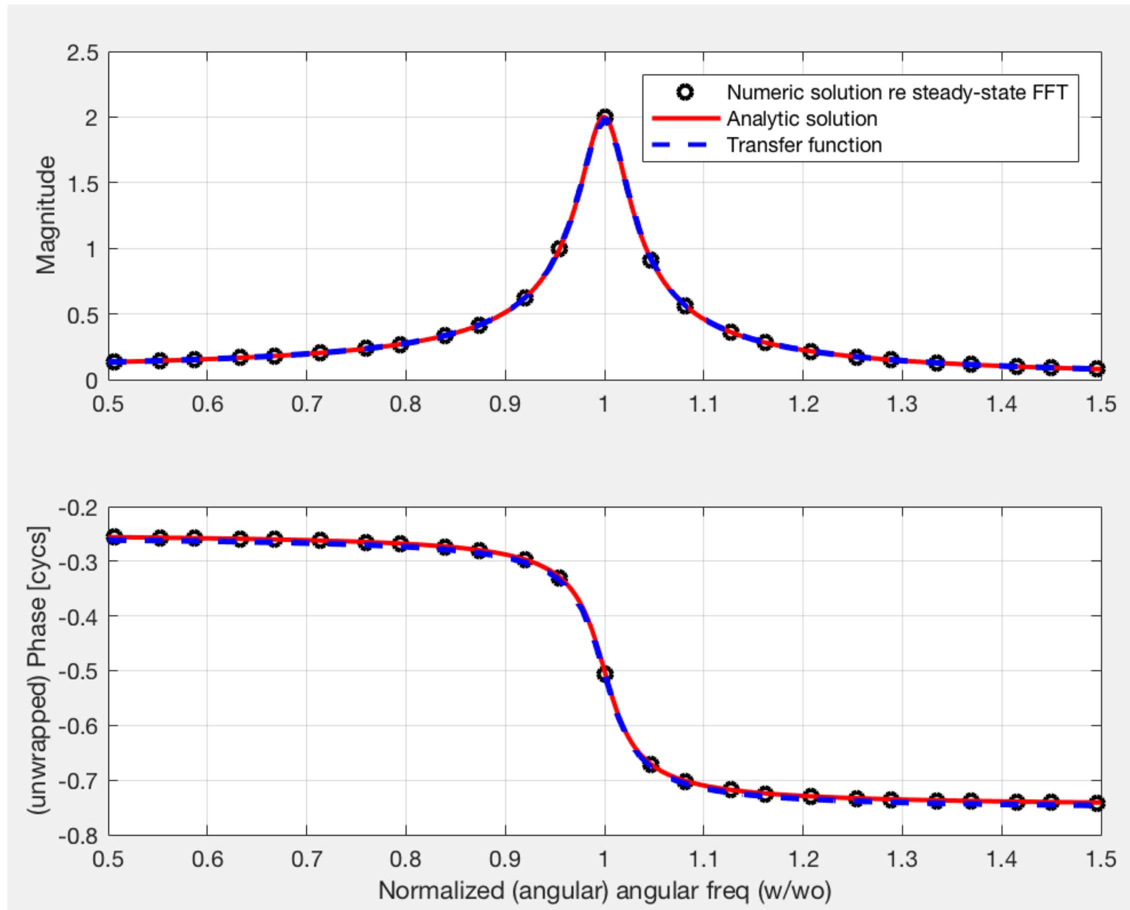
Note: Electrical engineers commonly use complex frequency (s) representation, tied back to Laplace transforms

EXhoResonance.m

Approach 3 – Transfer Function

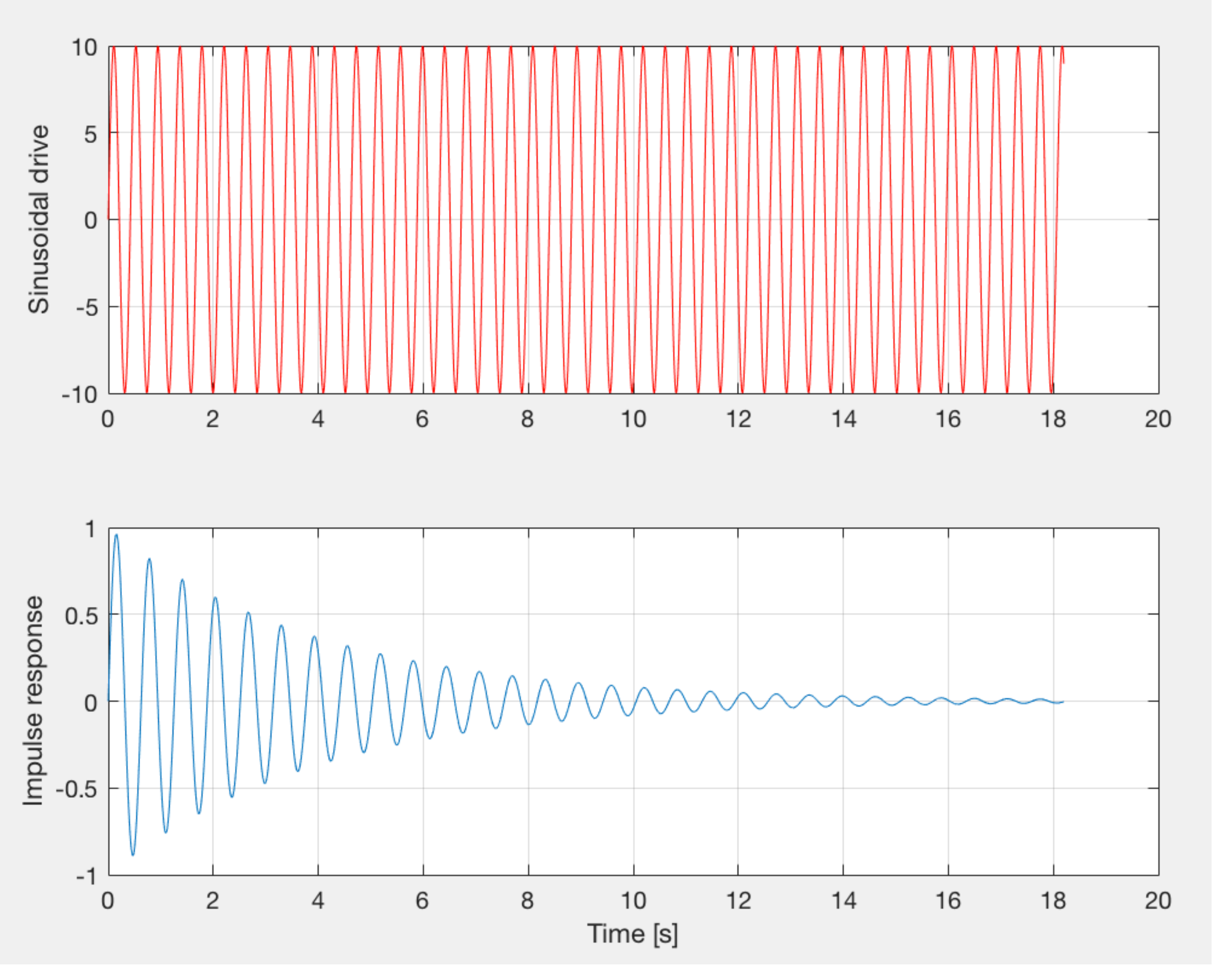


→ End up w/ dashed blue curve



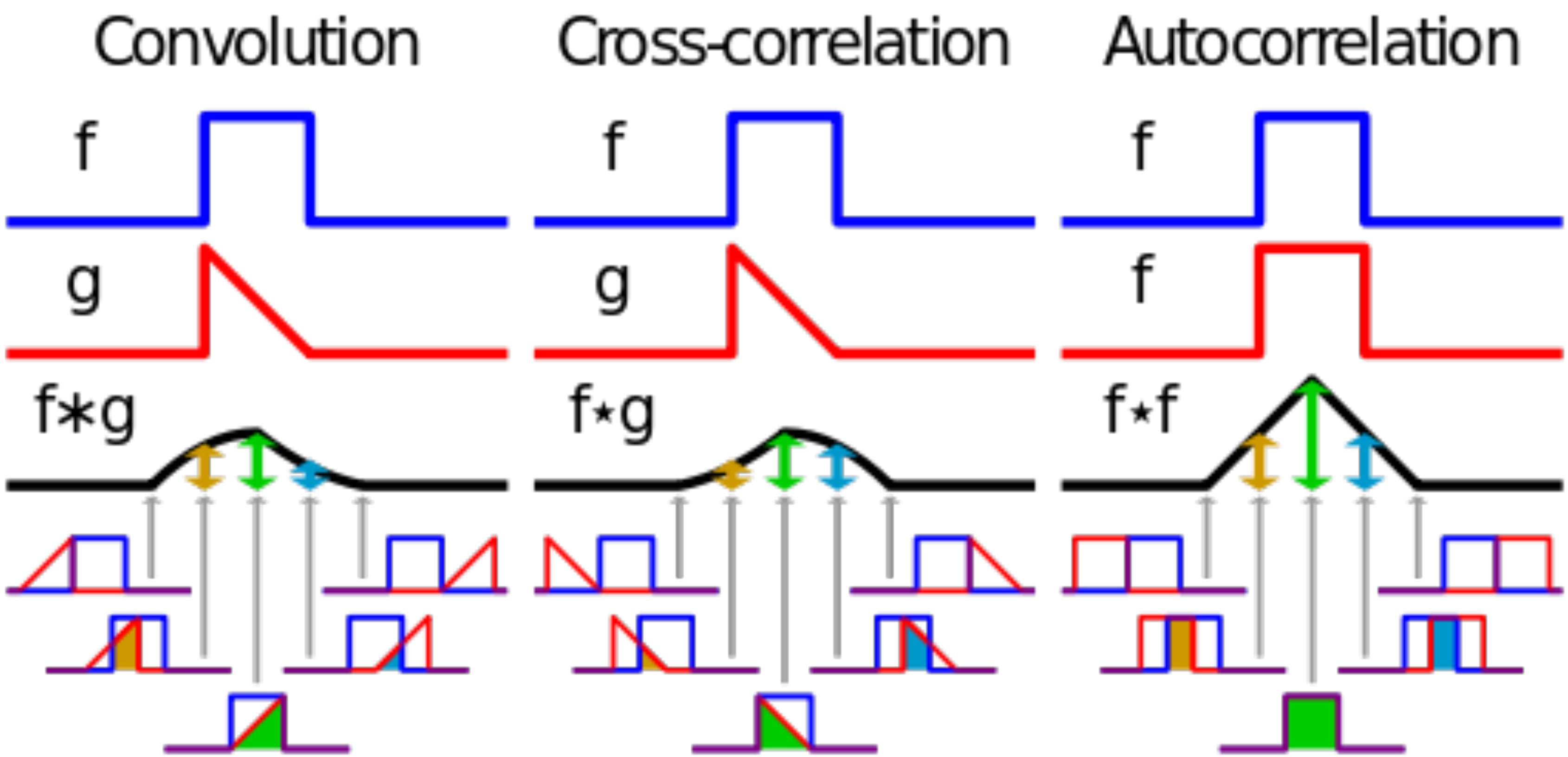
Approach 4 – Convolve the impulse response

Convolve (in time domain) input drive sinusoids w/ system's impulse response, then compute the FFT



Note: When convolving the impulse response and the drive, the initial transient is apparent, so we use the "long" time window and extract the "steady-state" portion of the convolved response

<u>Aside</u>: Convolutions et al....



Wikipedia

Aside: e.g., Microscope Imaging

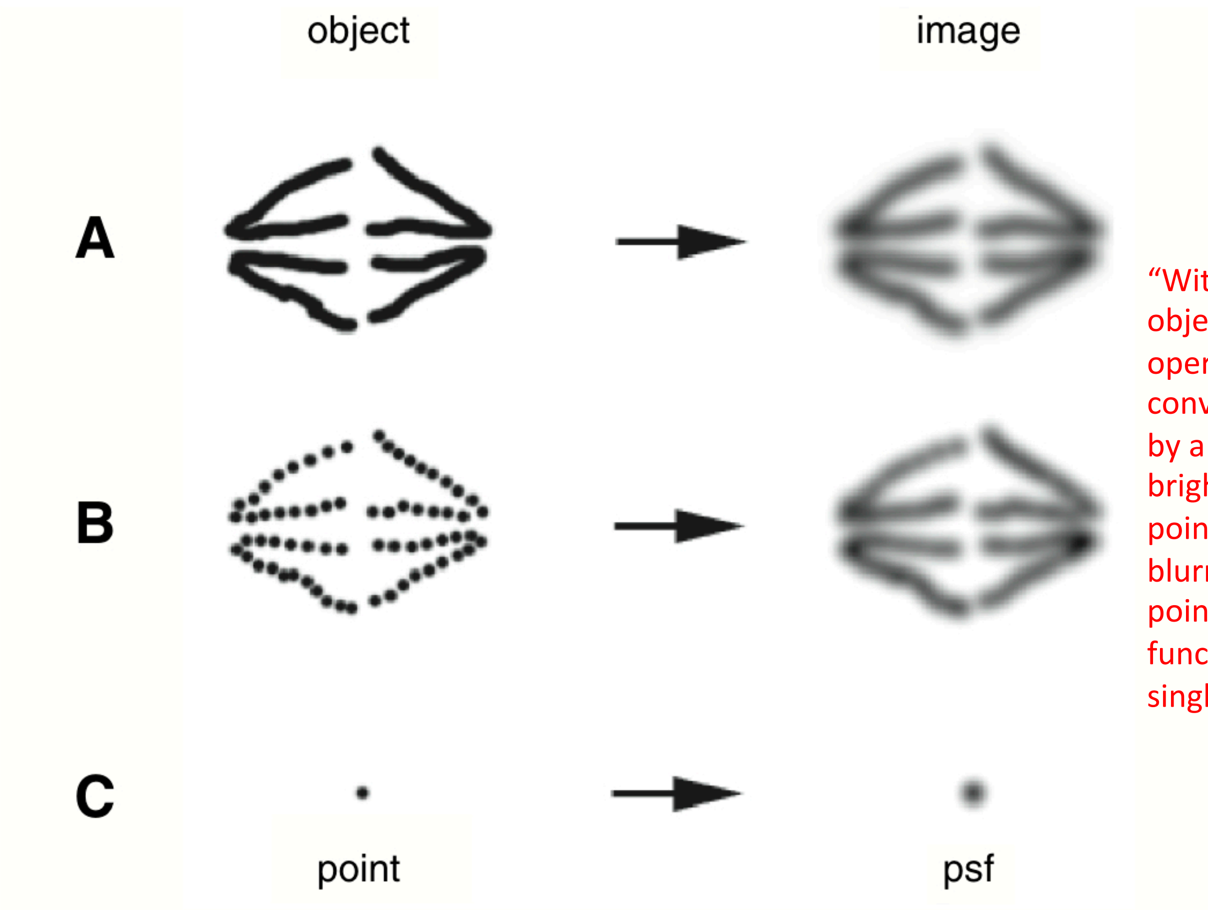


FIGURE 23.1. Diagram showing how a single point is imaged as the PSF by a microscope, and thus that the image of an extended object is the convolution of the object with the PSF.

"Within some quite general limitations, the object (specimen) and image are related by an operation known as convolution. In a convolution, each point of the object is replaced by a blurred image of the point having a relative brightness proportional to that of the object point. The final image is the sum of all these blurred point images. The way each individual point is blurred is described by the point spread function (PSF), which is simply the image of a single point."

Pawley (ch.23)

Aside: Connections to tomography (e.g., CT) & Radon transforms

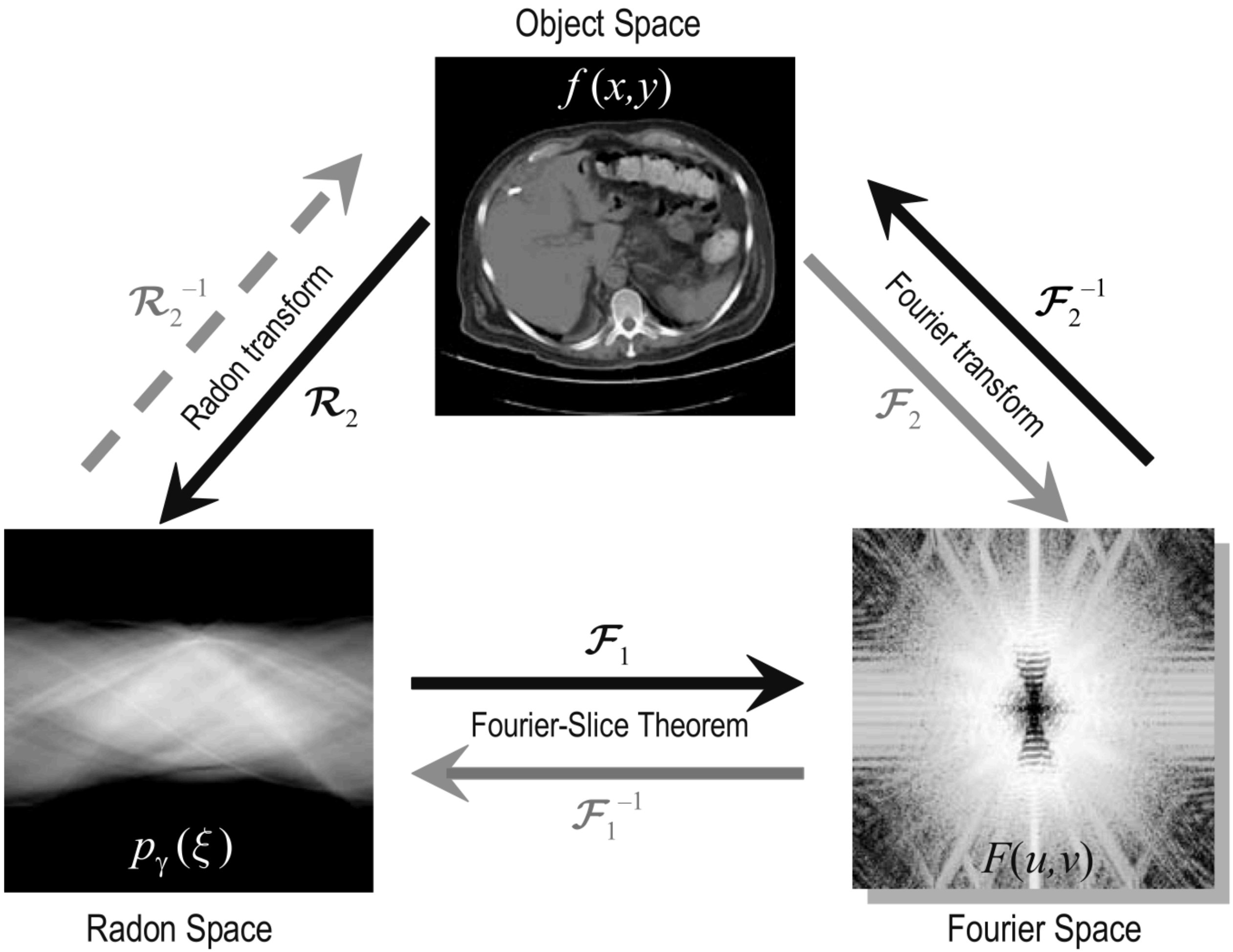
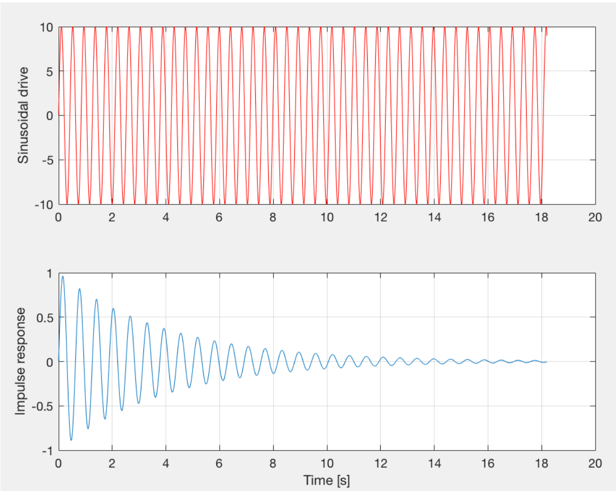
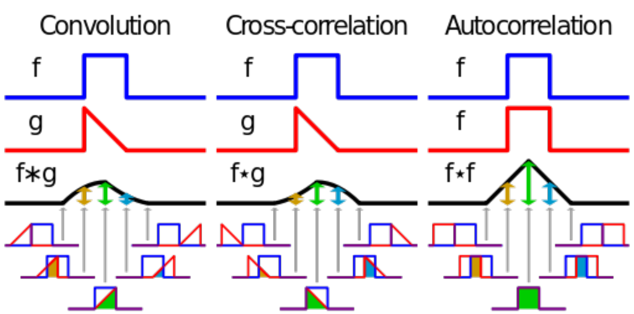


Fig. 5.15. Schematic summary of the relations among the spatial object domain (shown as an axial abdomen tomogram), the Radon space (given over an interval of 180° from the object), and the Fourier space (only absolute values are shown). The Fourier domain results directly from the spatial domain by a two-dimensional Fourier transform of the object, but can also be obtained by the Fourier slice theorem using a set of one-dimensional Fourier transforms of the projection profiles in the Radon space

Buzug (2008)

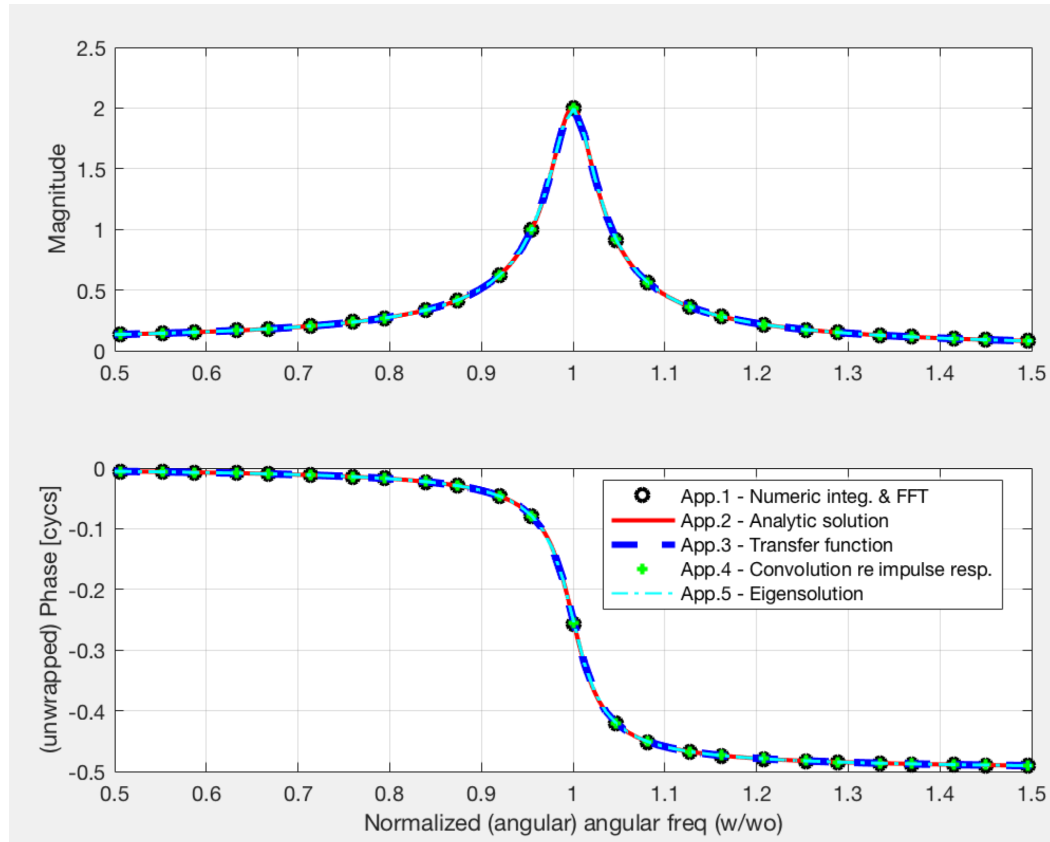
Approach 4 – Convolve the impulse response





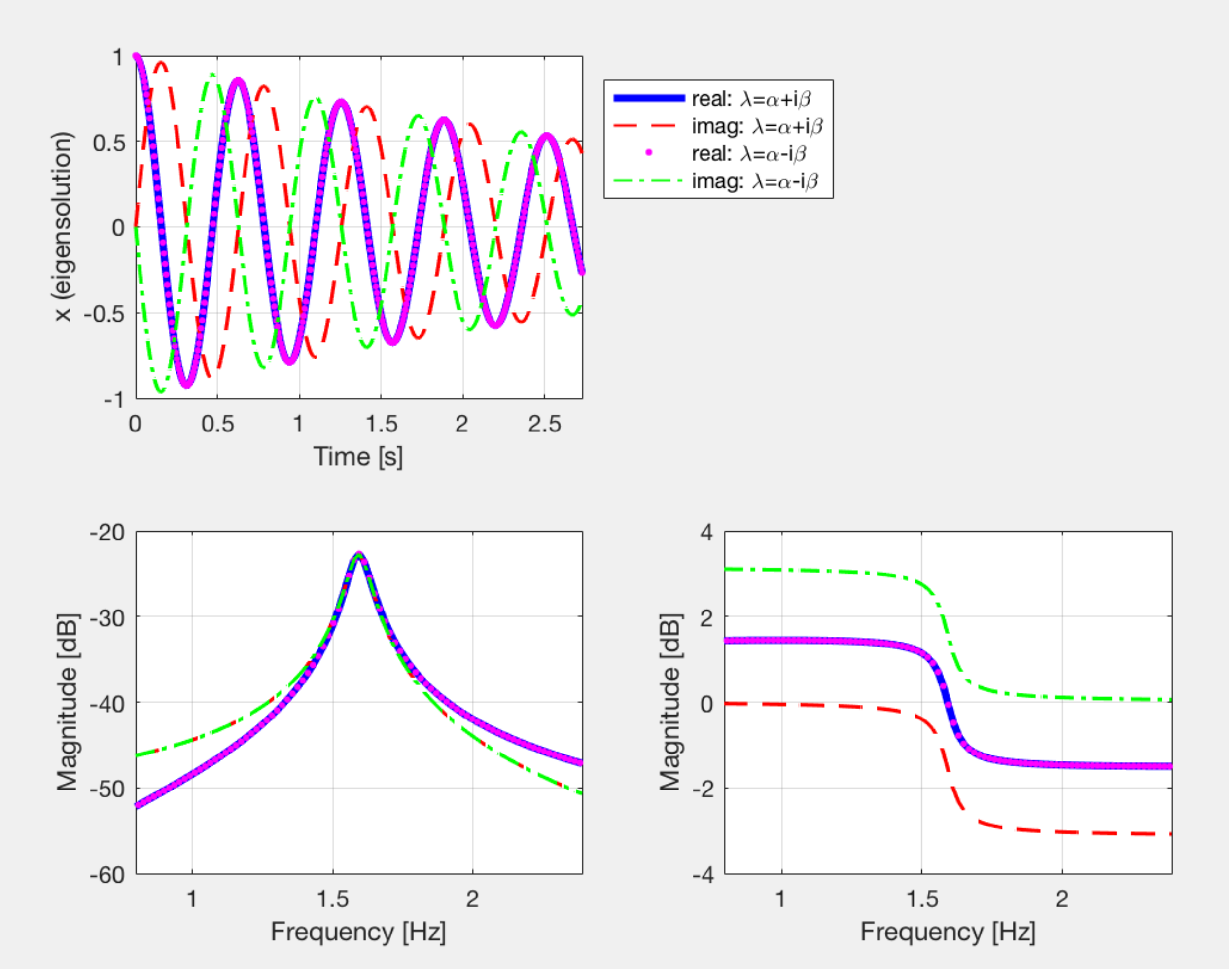
→ Now do this for a variety of different drive frequencies...

→ End up w/ the green +....

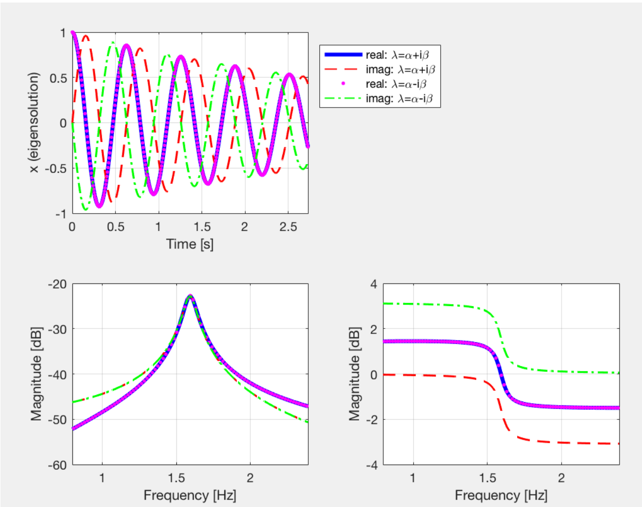


Approach 5 – Eigensolutions

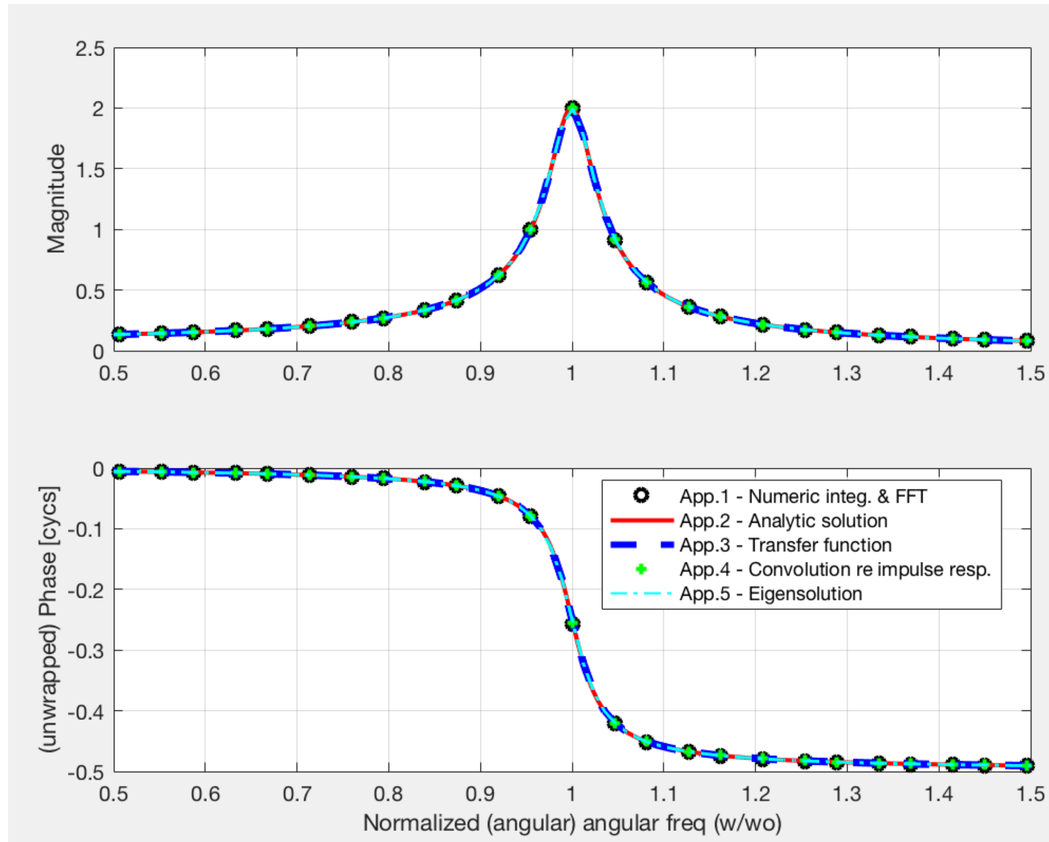
Determine eigenvalues for x=\dot{x}=0 (either numerically or analytically) as the resulting eigensolution, which is equivalent to the impulse response, then compute Fourier transform via FFT



Approach 5 – Eigensolutions

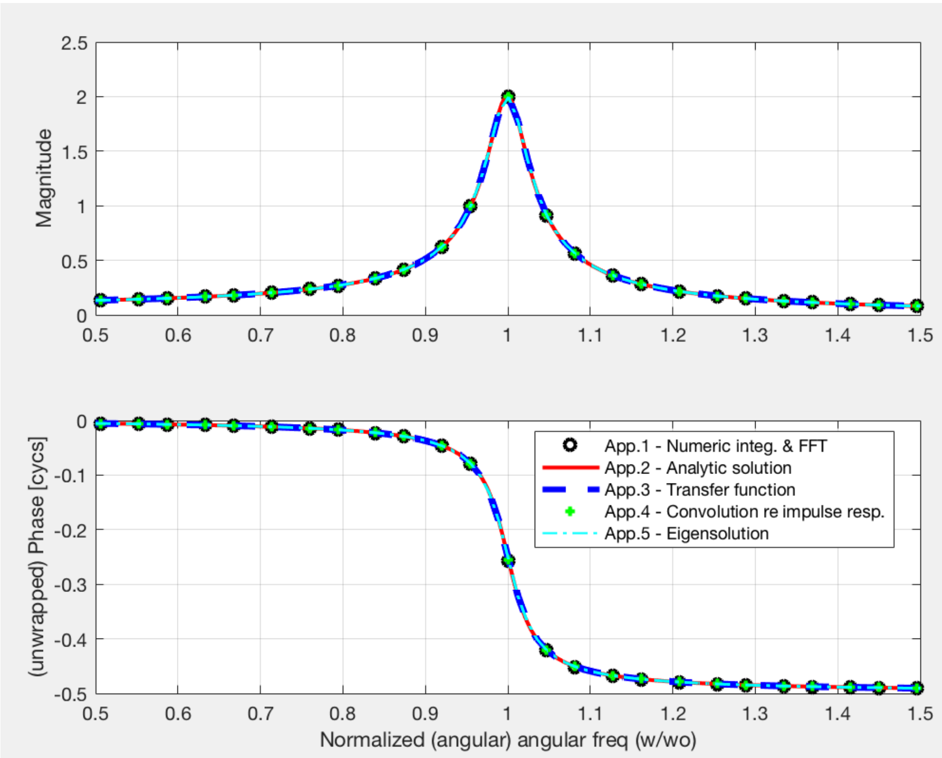


→ End up w/ cyan dashed line....



$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$

Several basic approaches:
(all arriving at the same answer)



1. Numerically solve the ODEs and extract the relevant magnitudes and phases (via an FFT)

2. Analytic solution I (via Fourier transforms)

3. Impulse response I (and associated *transfer function*)

4. Impulse response II (convolve in the time domain)

5. Analytic solution II (via eigensolutions)

Note (re still to do #6)
stochastic differential equation (SDE) for a purely noise-driven case

Aside (still to do)

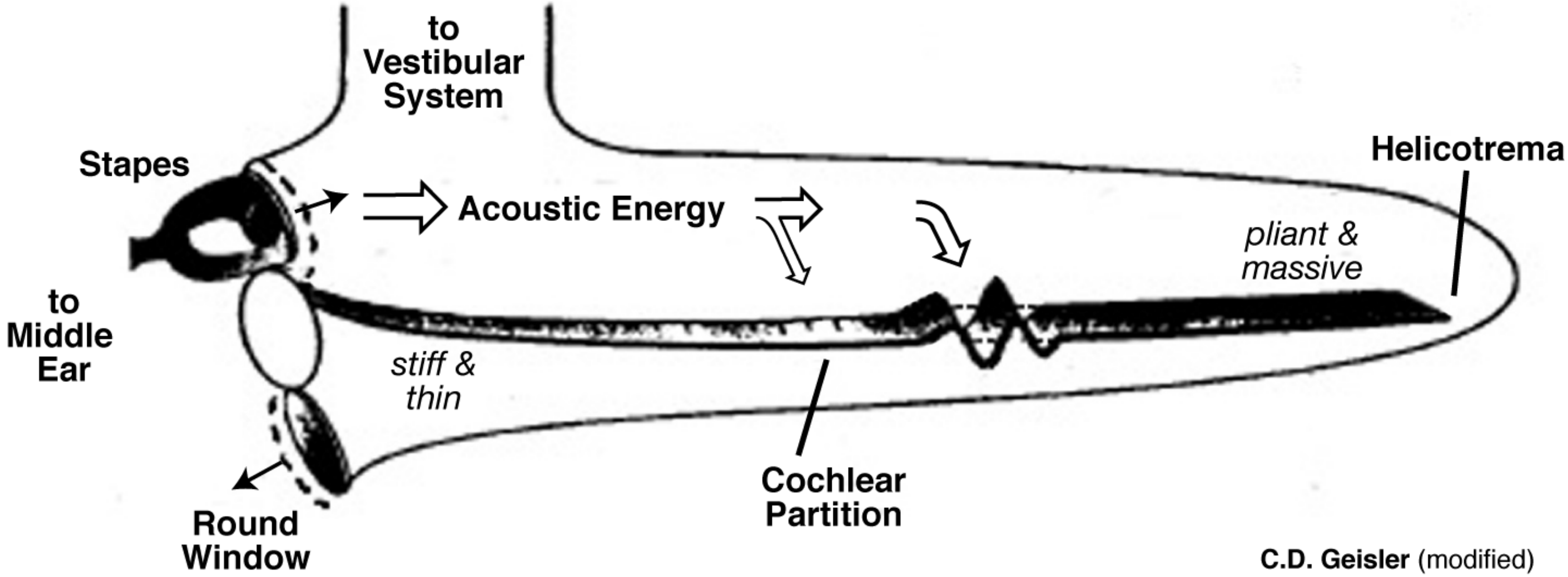
Noise-driven systems are very common physically....

Helmholtz resonator

in-situ earphone calibration

.... but harder to deal w/ analytically and computationally

Tonotopy REVISITED

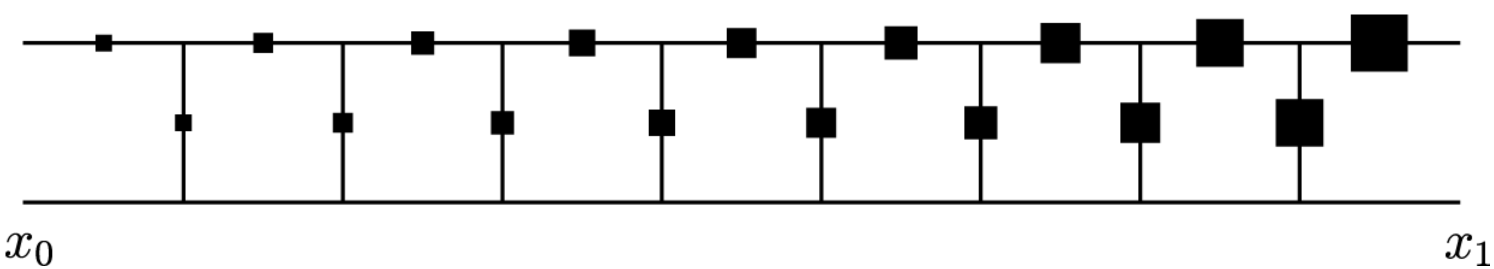


Model: Non-uniform transmission line

Ref: Zweig et al. 1976, Bergevin 2007

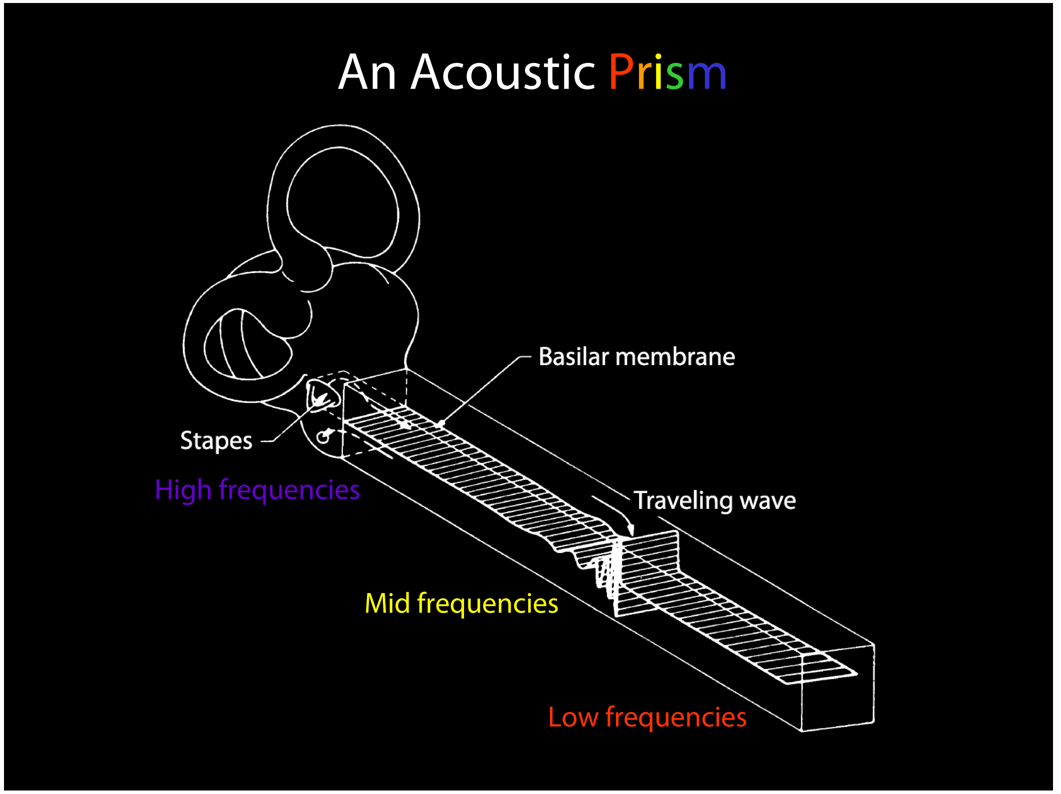
Several key ingredients:

- resonance
- longitudinal (e.g., fluid) coupling
- WKB approximation



→ Now in much better shape to understand this model!

Tonotopy & Traveling waves REVISITED



Theme/Question:
What is the (basic) physical basis for "tonotopy"?

Hint: It ain't a traveling wave per se.... (though such provides a useful framework)

→ This picture may be a bit more complicated....

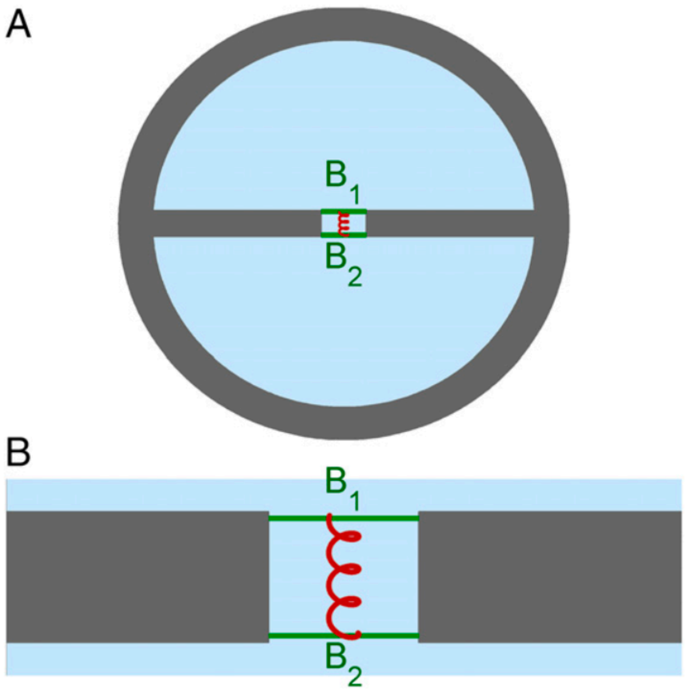
Tonotopy & Traveling waves REVISITED

Frequency selectivity without resonance in a fluid waveguide

14548–14552 | PNAS | October 7, 2014 | vol. 111 | no. 40

Marcel van der Heijden[1]

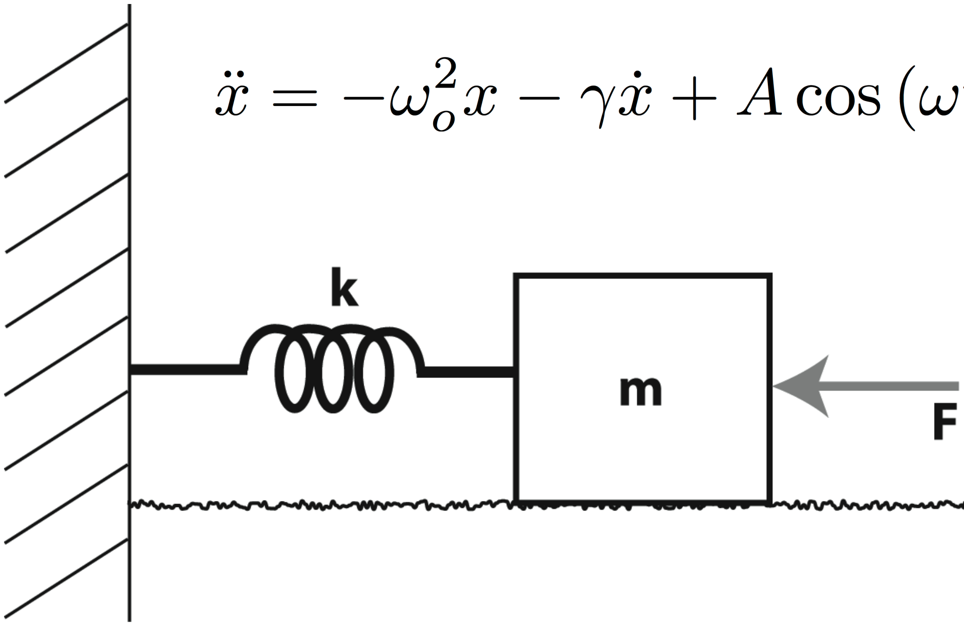
This work describes a simple waveguide that not only carries fluid waves, but also performs a spectral analysis. When driven by a complex input that contains several frequency components, it will spatially separate those components, in analogy to the separation of white light by a prism. The frequency tuning of the waveguide is not based on resonance, but on wave dispersion: Each wave has its own region in which it undergoes a steep deceleration, causing it to focus its energy and deliver it. This method of spectral analysis has not been described before. The waveguide bears a striking resemblance to the inner ear of mammals, both in terms of structure and behavior.



→ Call it what you will (or will not), but all the basic ingredients for “resonance” are there (e.g., elements that trade energy back and forth on a cycle-by-cycle basis, “stiffness gradient”, etc...)

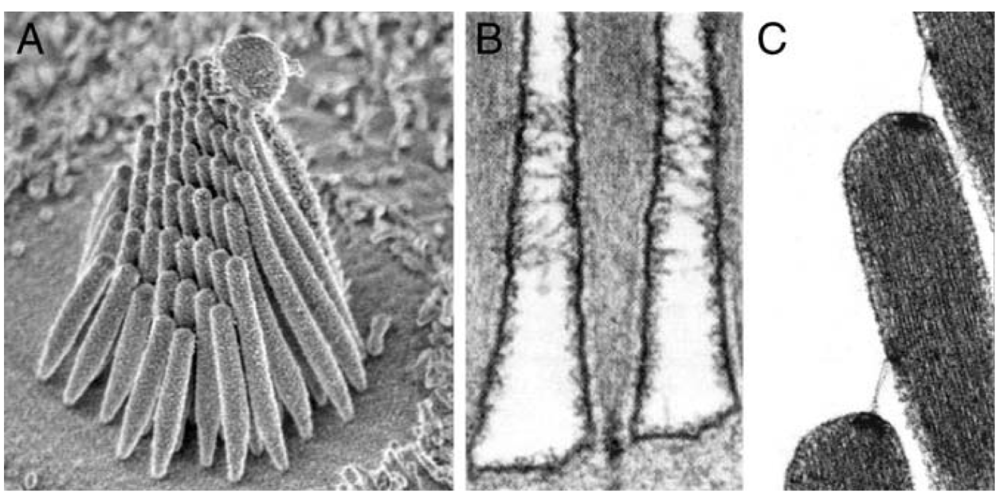
Tangent I: Resonance in the inner ear....

$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$

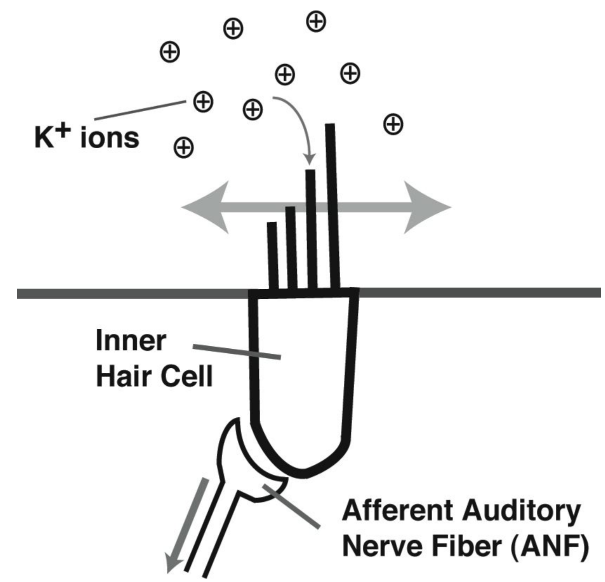


Resonance is chiefly a combination of two reactive forces:

- spring/stiffness
- inertial (i.e., moving mass)



Martin (2008)



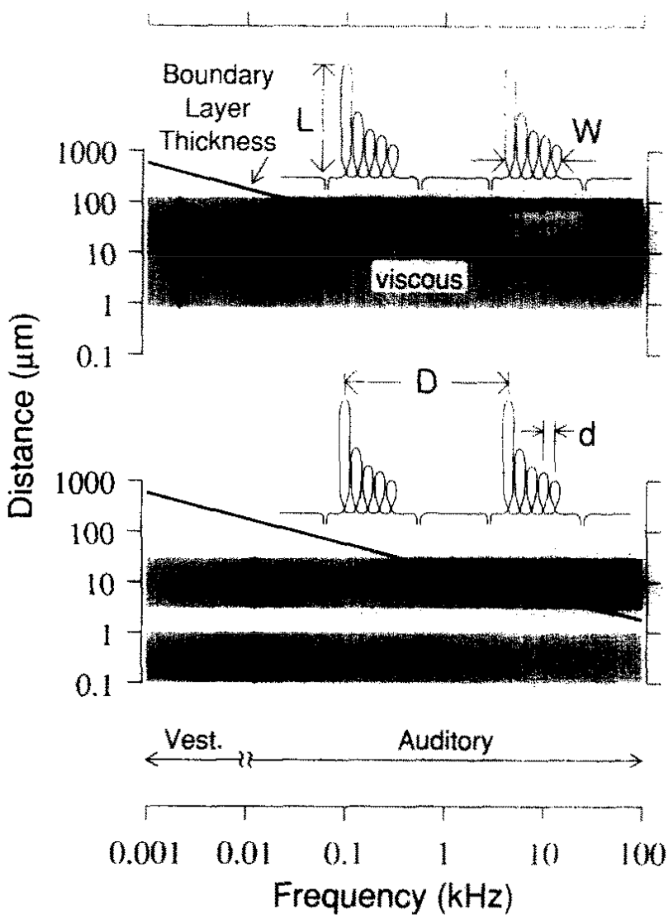
→ Can we model a hair cell bundle using this basic formulation?

Tangent I: Resonance in the inner ear....

The role of fluid inertia in mechanical stimulation of hair cells

Dennis M. Freeman and Thomas F. Weiss

Hearing Research, 35 (1988) 201–208



→ Hair cell bundles seem to operate in the region where viscous forces become relatively large....

Tangent I: Resonance in the inner ear....

Comparative Aspects of Hearing in Vertebrates and Insects with Antennal Ears

Joerg T. Albert[1] and Andrei S. Kozlov[2]

Current Biology *26*, R1050–R1061, October 24, 2016

Box 2. How liquid in the inner ear has shaped the hair bundle.

A hair bundle operates at small Reynolds numbers on the order of 10^{-4}. The Reynolds number (Re) is defined by: Re = $uL\rho/\mu$, where u is the velocity, L is a linear dimension (e.g., a hair bundle's size), ρ is the density and μ is the dynamic viscosity of the fluid. The Reynolds number indicates the relative importance of inertia over viscous forces for a particular type of flow. For the hair bundle, a Reynolds number of much lower than 1 indicates the relative importance of viscous forces.

The Scallop Theorem

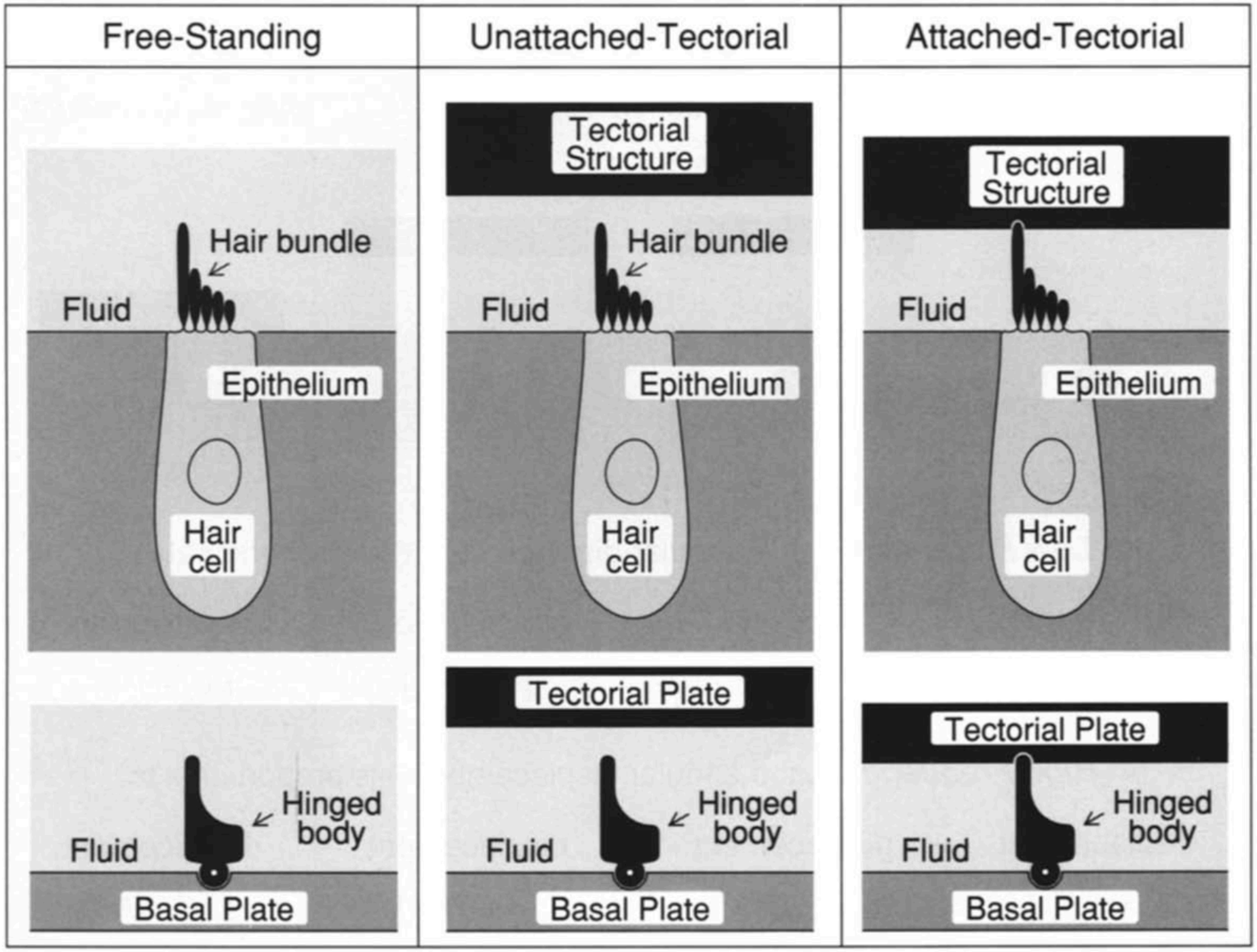
Life at low Reynolds number

E. M. Purcell
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 12 June 1976)

American Journal of Physics, Vol. 45, No. 1, January 1977

Tangent I: Resonance in the inner ear....

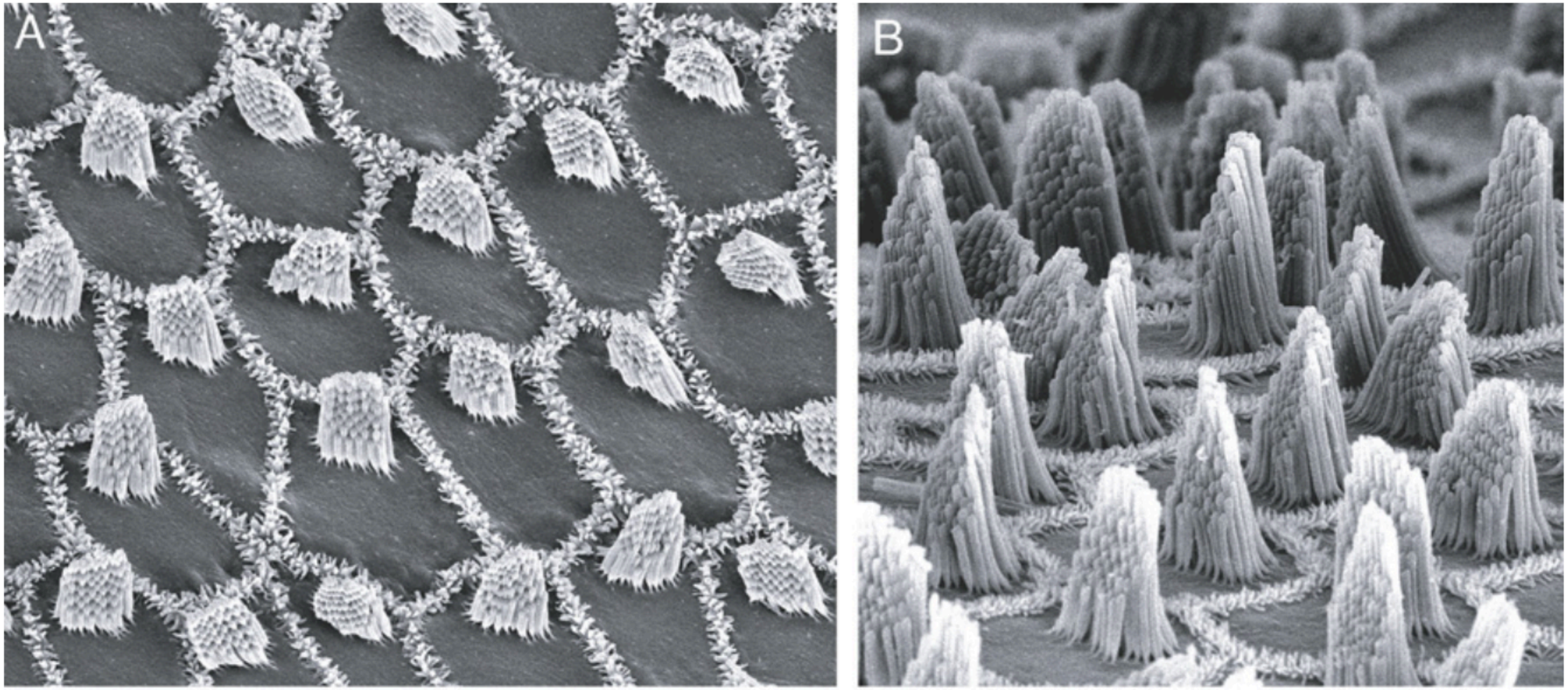
But things are a bit more complicated than just "what is the Reynold's # of a hair cell bundle?"....



Hearing Research, 35 (1988) 201–208

Tangent I: Resonance in the inner ear....

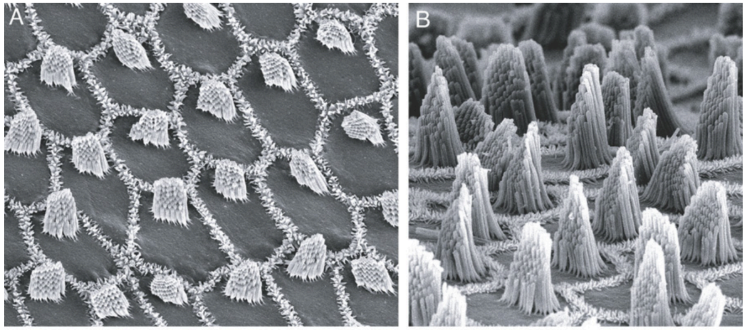
Chicken basilar papilla (i.e., auditory hair cells)

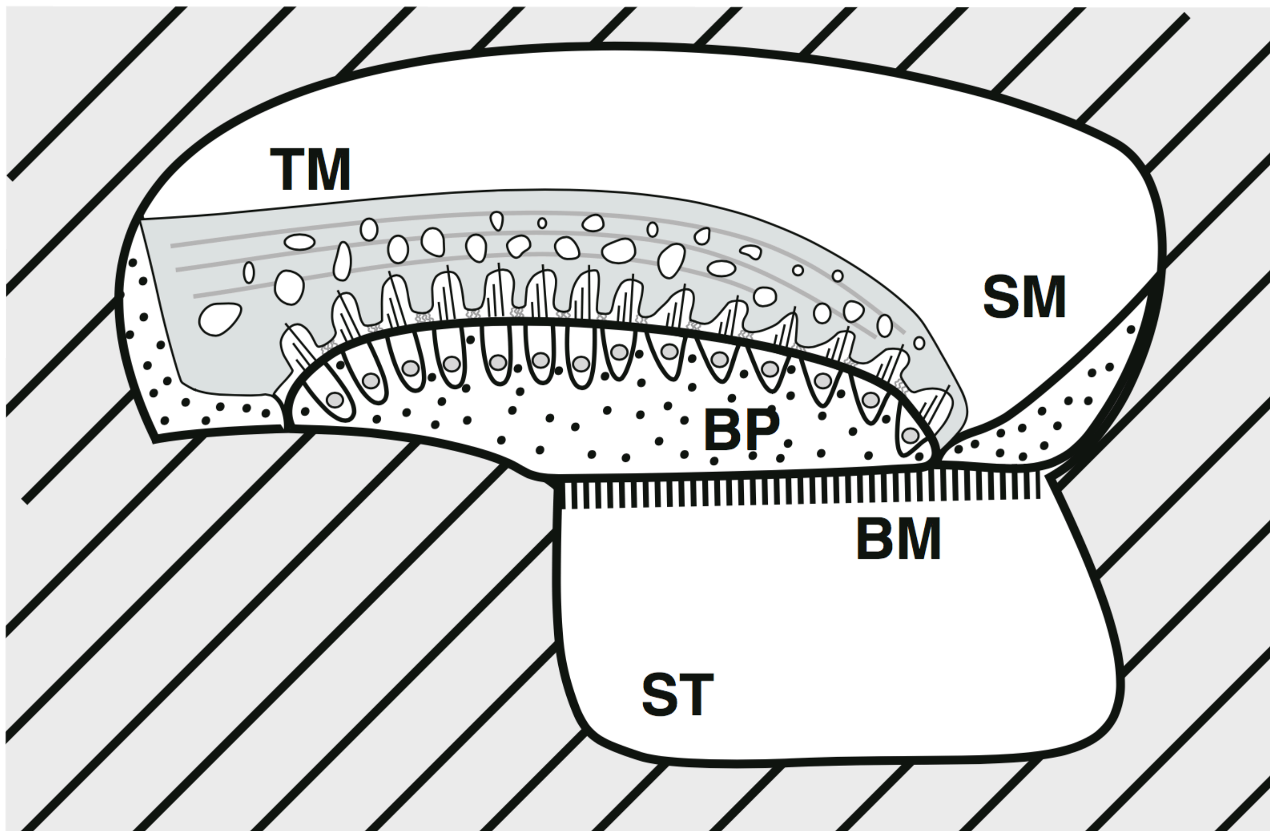


Hudspeth (2008)

Note: Be careful! This picture can be misleading. In-vivo, there is a massive tectorial membrane (**TM**) overlying these hair cells....

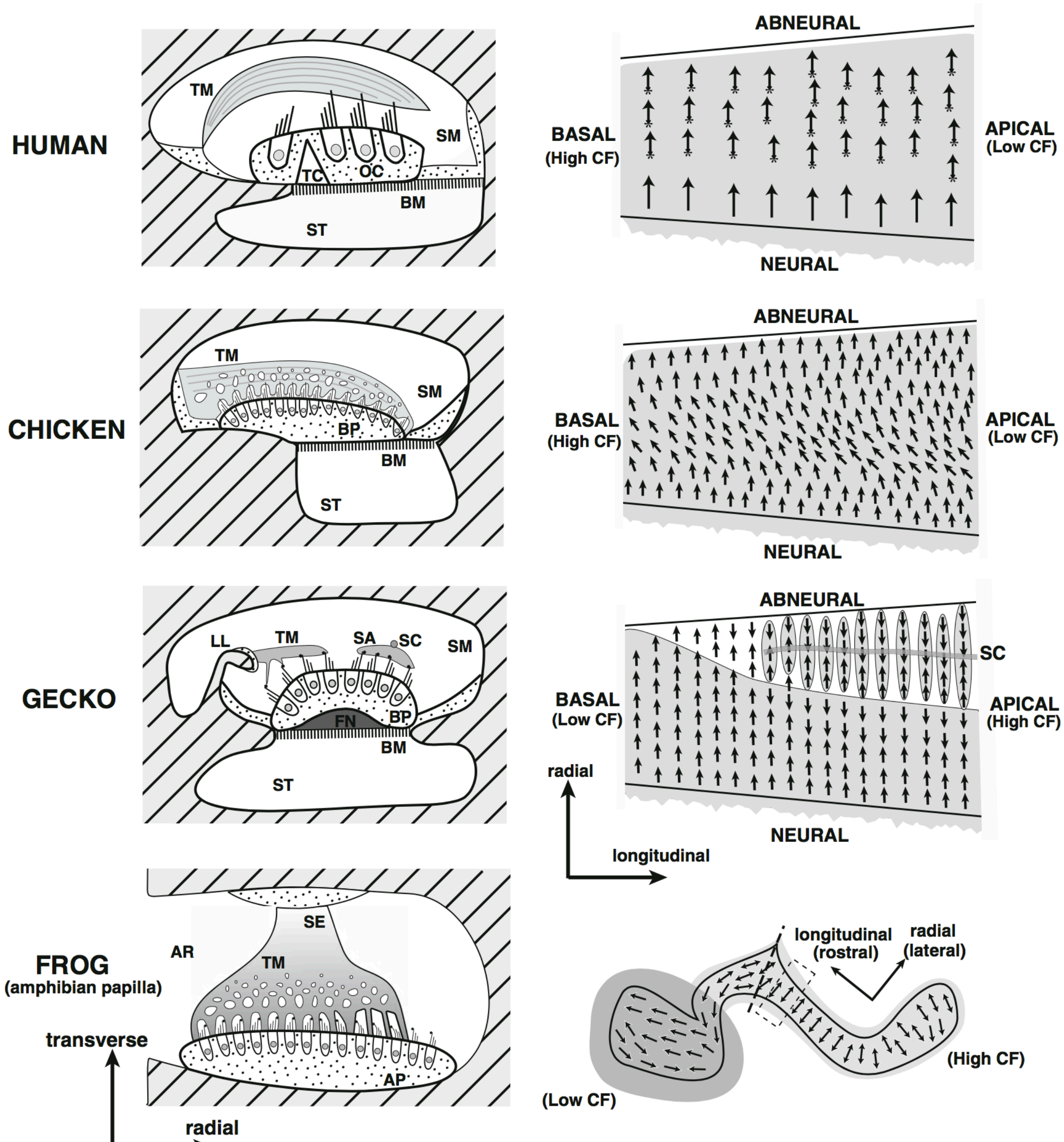
Tangent I: Resonance in the inner ear....





Bergevin et al (2008)

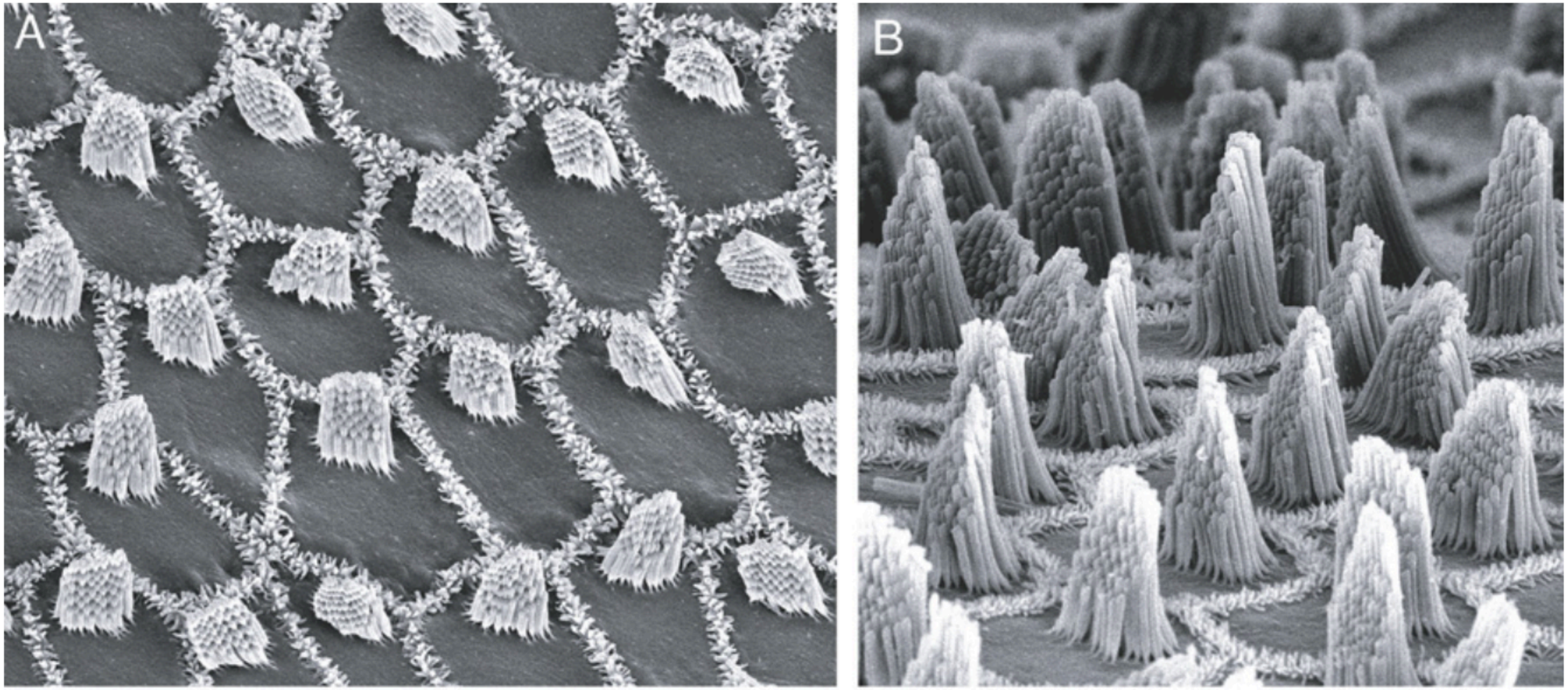
Tangent I: Resonance in the inner ear....



Bergevin et al (2008)

Tangent I: Resonance in the inner ear....

Chicken basilar papilla (i.e., auditory hair cells)



Hudspeth (2008)

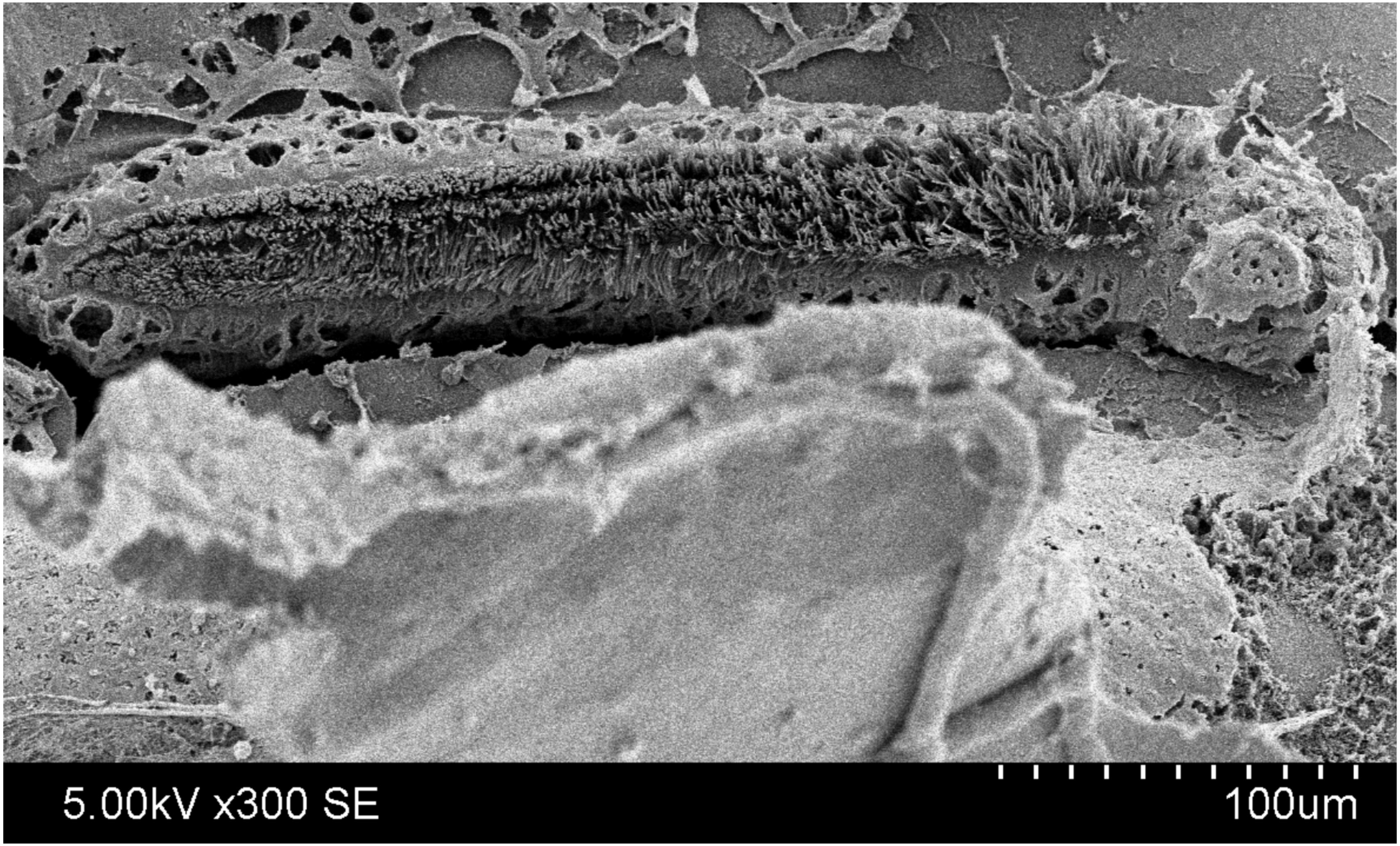
Note: Be careful! This picture can be misleading. In-vivo, there is a massive tectorial membrane (**TM**) overlying these hair cells....

→ Coupling between hair cells affects mechanical properties (e.g., tuning) how?

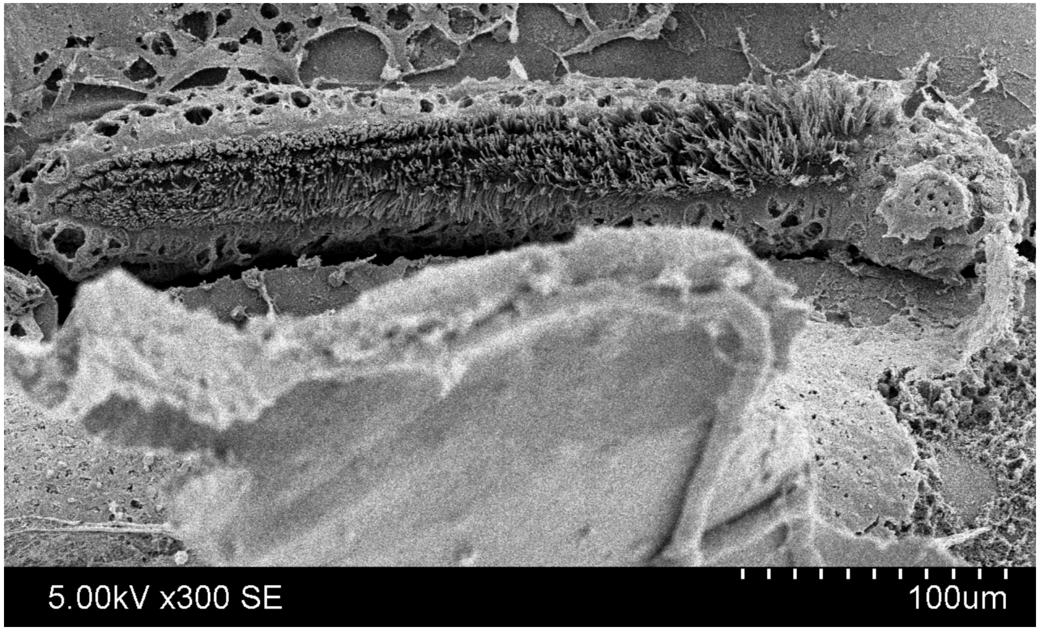
Tangent I: Resonance in the inner ear....

Tangent I: Resonance in the inner ear....

[in collaboration w/ Bob Harrison]

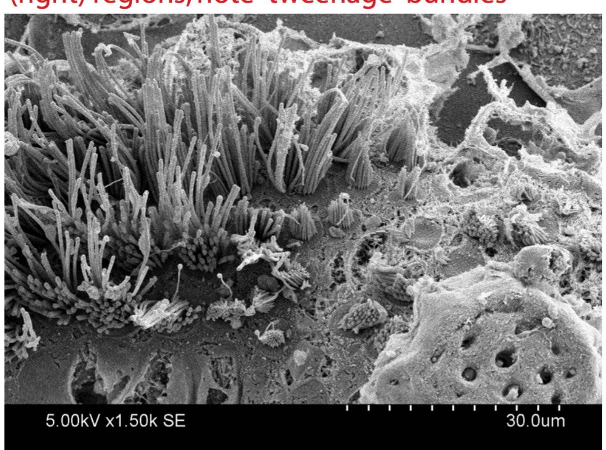


Tangent I: Resonance in the inner ear....

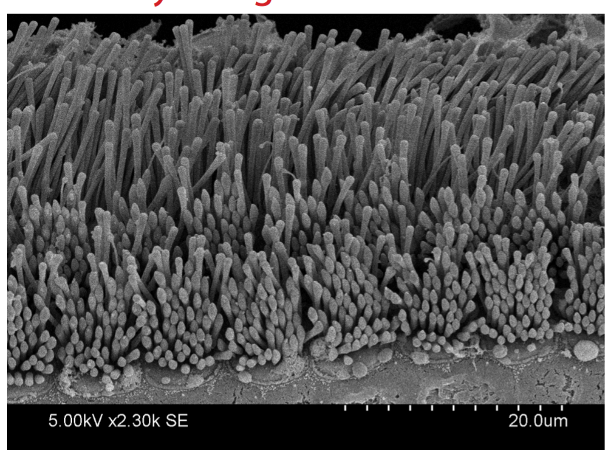


Note: These hair cells do not have an overlying TM (for the most part)

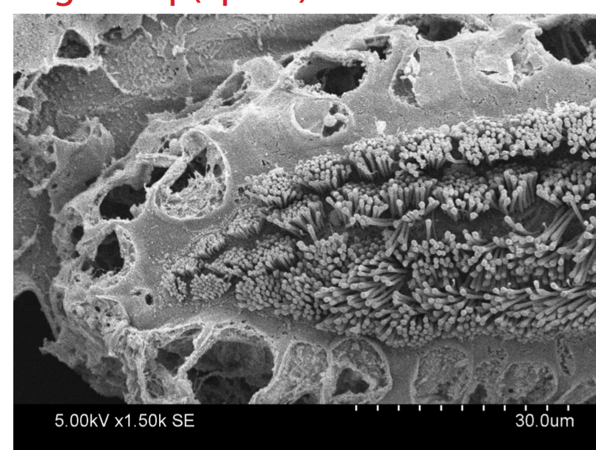
transition between free-standing (left) and TM (right) regions; note "tweenage" bundles



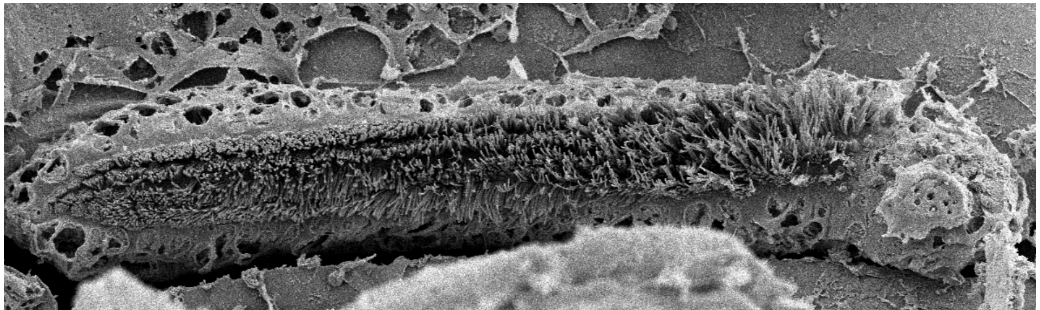
mid-way along

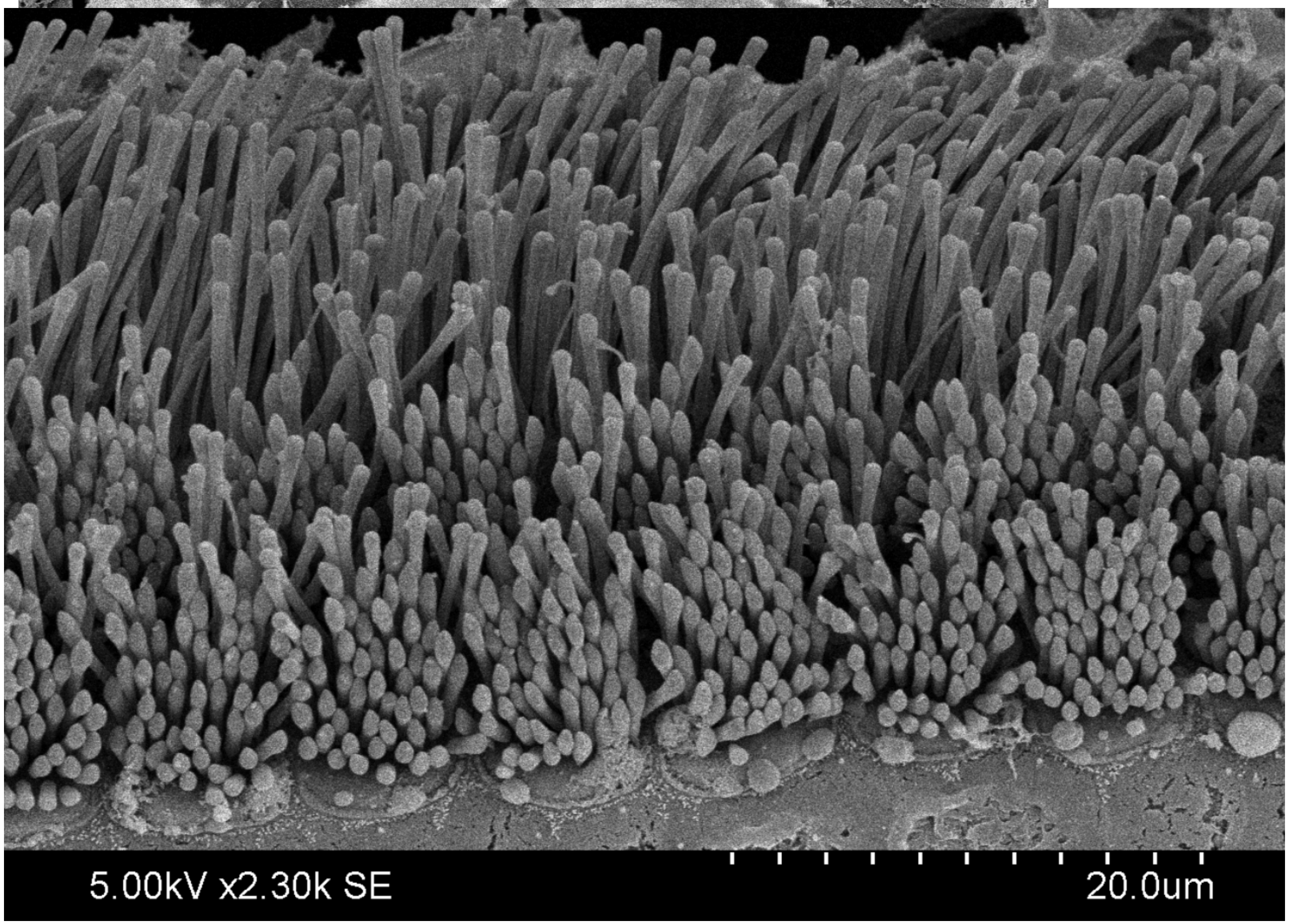


high-freq. (apical) end



Tangent I: Resonance in the inner ear....





- "phalanx" of hair cells
- Implications for inter-cell coupling?
- Longitudinal propagations? (e.g., traveling/standing waves)

Tangent I: Resonance in the inner ear....

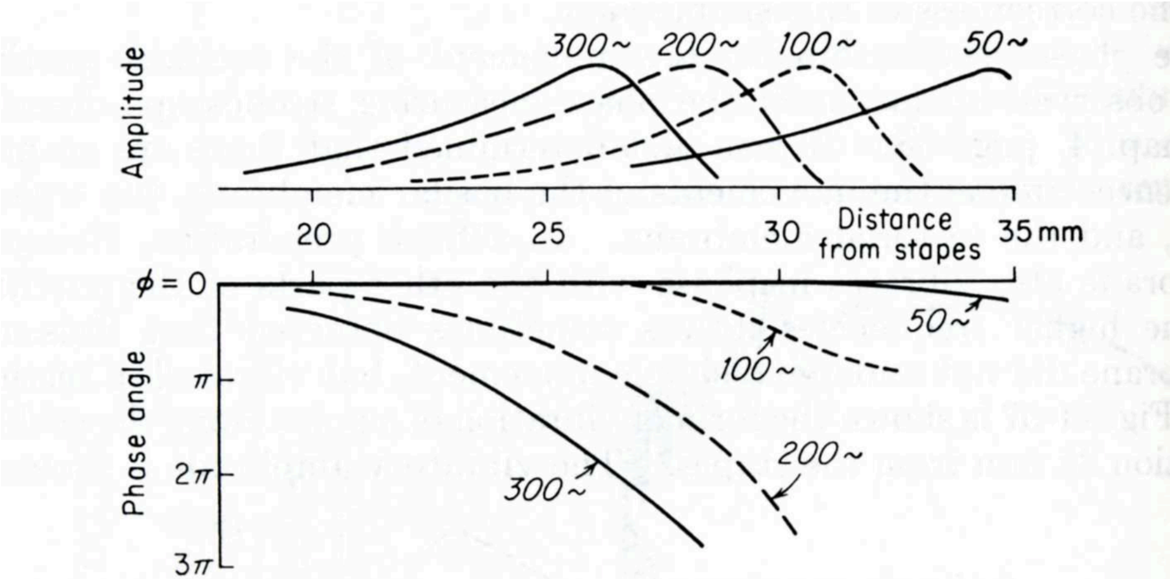
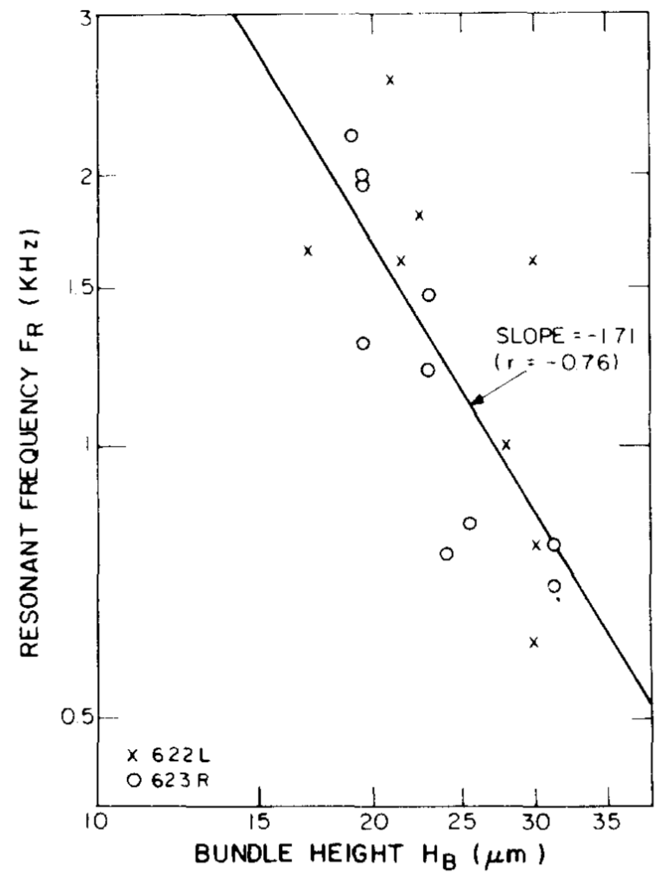


FIG. 11-58. Phase displacement and resonance curves for four low tones.

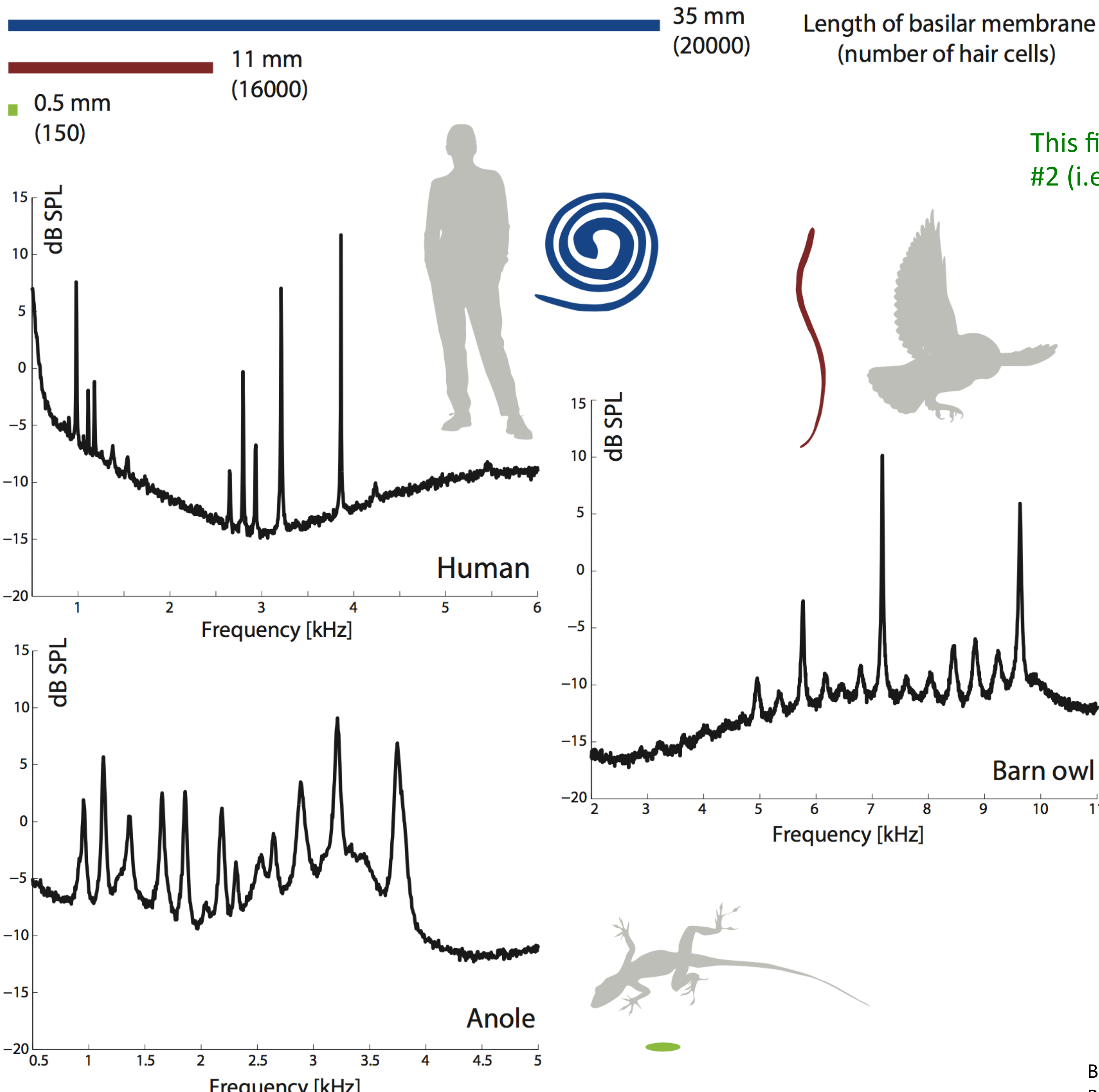
Bekesy (1960)

These are classic measurements from *dead ears*....



Frishkopf & DeRosier (1983)

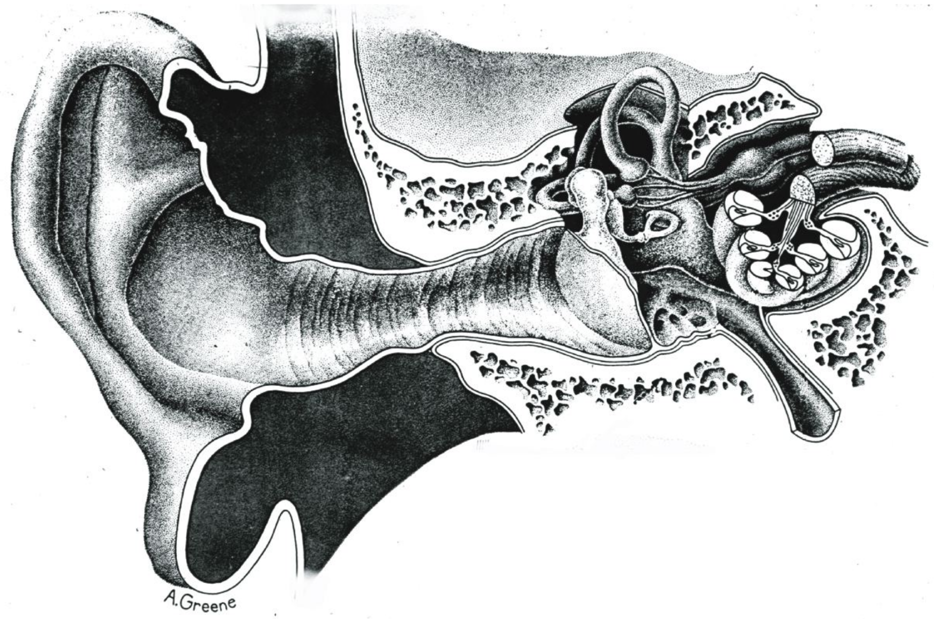
.... so clearly "tuned" responses can arise in the absence of an active process (see tangent #2)



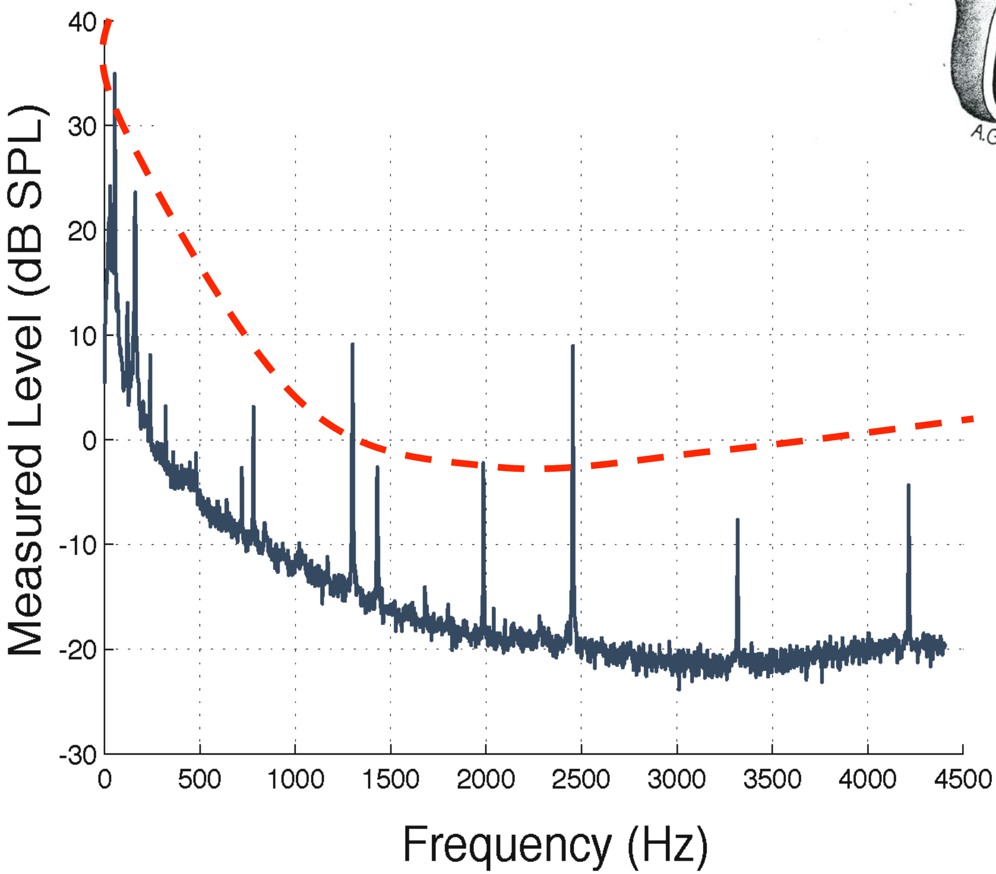
This figure leads us into tangent #2 (i.e., the ear ain't passive...)

Bergevin, Manley & Koppl (PNAS 2015)
Bergevin, Verhulst & van Dijk (SHAR 2017)

[Tangent II]: The ear isn't just passive.....



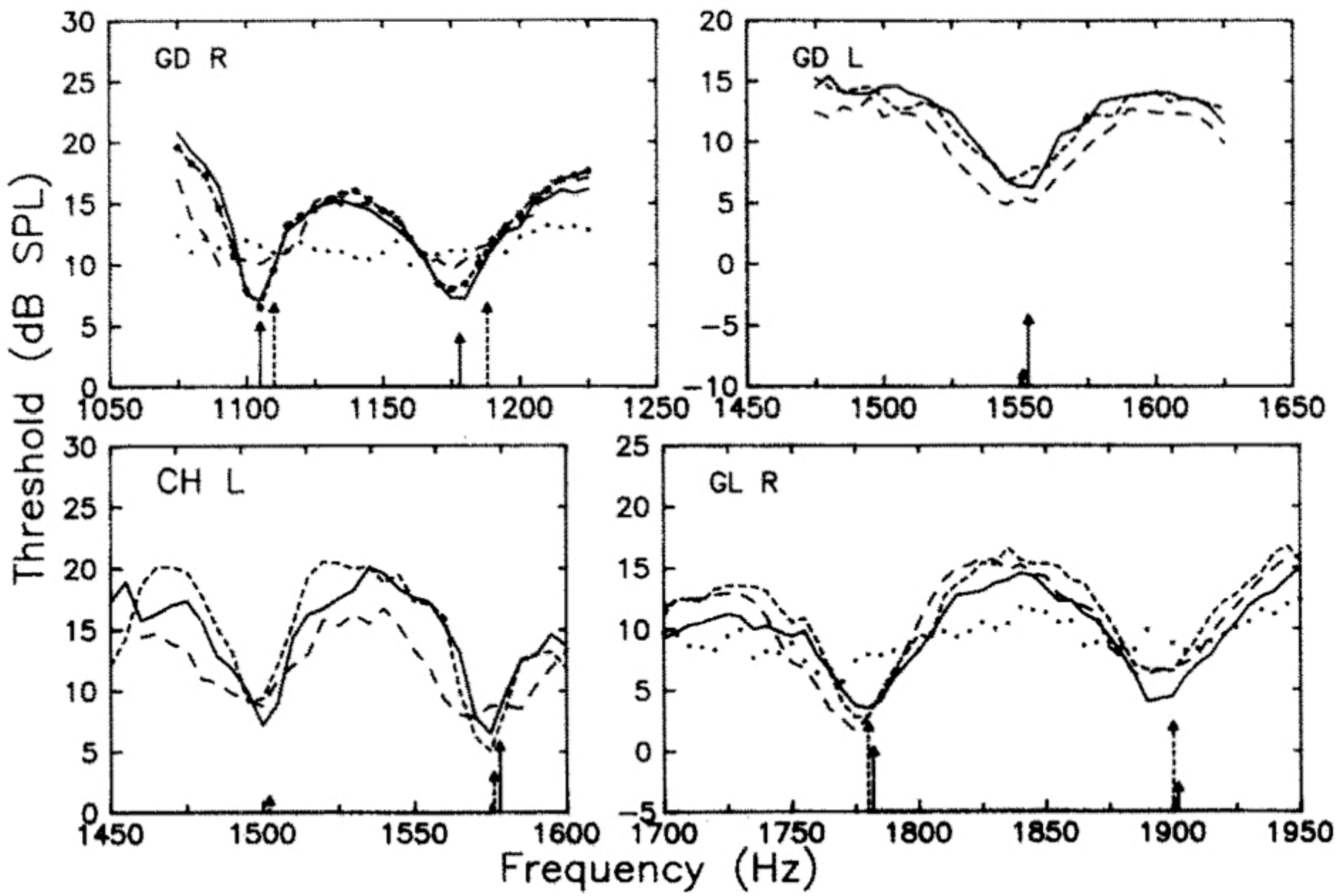
➤ Healthy ears actually *emit* sound



human threshold curve

Spontaneous otoacoustic emissions (SOAEs)

Tangent II: SOAEs & threshold



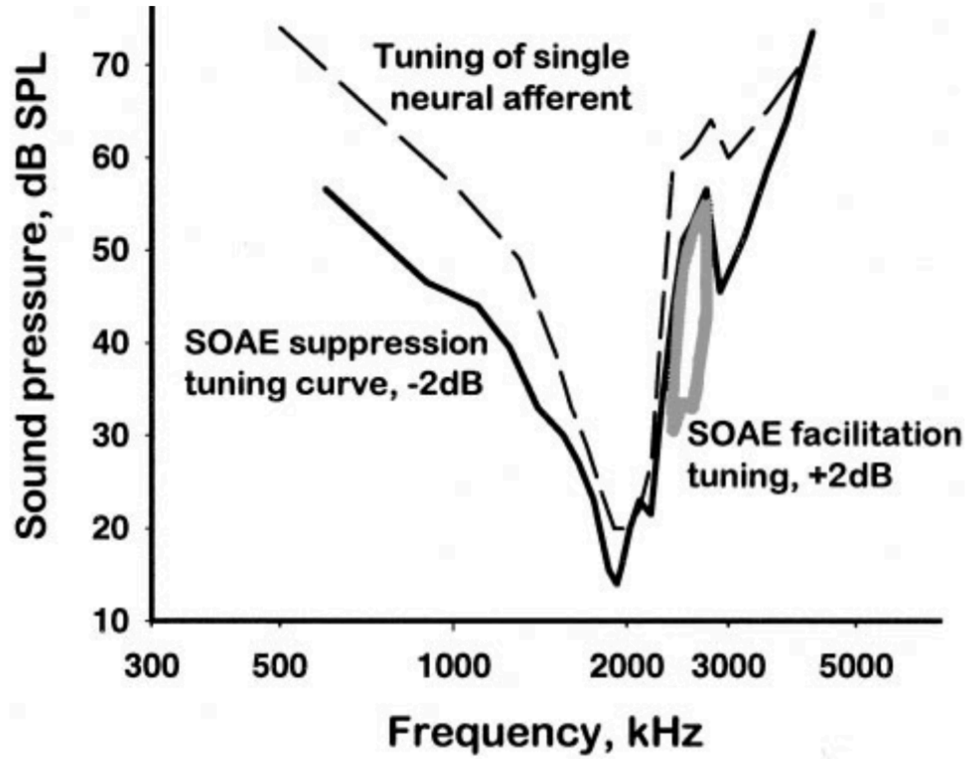
➢ SOAEs directly tied to forward auditory transduction (i.e., neural responses)

Long & Tubis (1988)

Tangent II: SOAEs & ANF responses

Evidence for an Active Process and a Cochlear Amplifier in Nonmammals

GEOFFREY A. MANLEY
Lehrstuhl für Zoologie, Technische Universität München, 85747 Garching, Germany



- ➢ SOAE “suppression” related to auditory nerve fiber tuning....
- ➢ ... probably in a complicated fashion

Tangent II: Simple heuristic for modeling SOAEs

$$\ddot{x} = -\omega_o^2 x - \gamma \dot{x} + A\cos(\omega t)$$

Passive, linear case doesn't do the trick....

Simple model to explain an SOAE peak:

$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$

van der Pol oscillator

Note: This equation comes in different flavors/ forms (e.g., "normal form", complex)

$$\dot{z} = -\mu z + i\omega_o z - |z|^2 z$$

where $z \in \mathbb{C}$, $\mu, \omega_o \in \Re$

[2] A (linear, undriven) harmonic oscillator can be described by a single, first–order ODE in terms of a complex variable z (e.g., [10]):

$$\dot{z} = -\mu z + i\omega_o z \ .$$

Via a change of variables, this can be re-expressed as a 2nd order (real–valued) ODE:

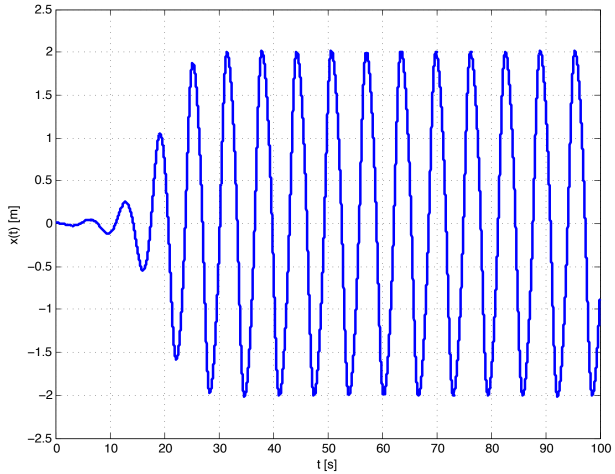
$$\ddot{x} + 2\mu\dot{x} + (\omega_o^2 + \mu^2)x = 0 \ .$$

Thus, the two notations are essentially equivalent. Note that in this case, no matter what the sign of ω_o is, the quantity $(\omega_o^2 + \mu^2)$ will always be positive. Thus the system will always have a positive stiffness, though the damping can be positive or negative (depending upon the sign of μ).

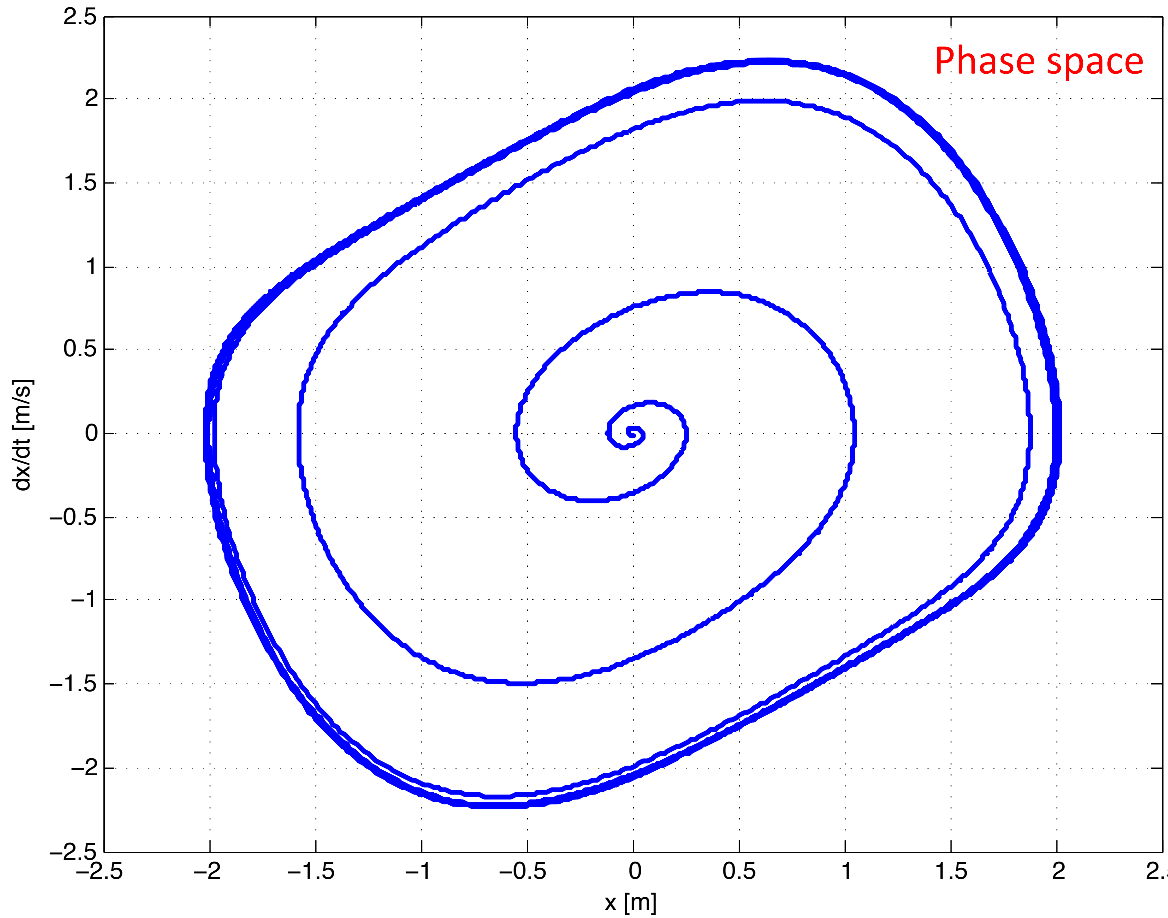
Binder et al (2011 MoH)

VDPode45EX.m

Tangent II: The ear isn't just passive.....



$$\ddot{x} = -x - \varepsilon(x^2 - 1)\dot{x}$$



Limit cycles:

- Negative damping for small displacements injects energy into system
- Nonlinearity stabilizes
- Self-sustained oscillation!

Tangent II: The ear isn't just passive.....

P.I.M. Johannesma
Workgroup Neurophysics, Lab. of Medical Physics and Biophysics, University of Nijmegen, Nijmegen, The Netherlands.

Narrow band filters and active resonators.

Psychophysical, Physiological and Behavioural Studies in Hearing

Proceedings of the 5th International Symposium on Hearing

Noordwijkerhout, The Netherlands
April, 8-12, 1980

Are spontaneous otoacoustic emissions generated by self-sustained cochlear oscillators?

Carrick L. Talmadge and Arnold Tubis
Department of Physics, Purdue University, West Lafayette, Indiana 47907

Hero P. Wit
Institute of Audiology, University Hospital, P. O. Box 30.001, 9700 RB Groningen, The Netherlands

Glenis R. Long
Department of Audiology and Speech Science, Purdue University, West Lafayette, Indiana 47907

J. Acoust. Soc. Am. **89** (5), May 1991

Tangent II: The ear isn't just passive.....

4622 Biophysical Journal Volume 95 November 2008 4622–4630

Frequency Clustering in Spontaneous Otoacoustic Emissions from a Lizard's Ear

Andrej Vilfan* and Thomas Duke†
*J. Stefan Institute, Ljubljana, Slovenia; and †London Center for Nanotechnology and Department of Physics & Astronomy, London, United Kingdom

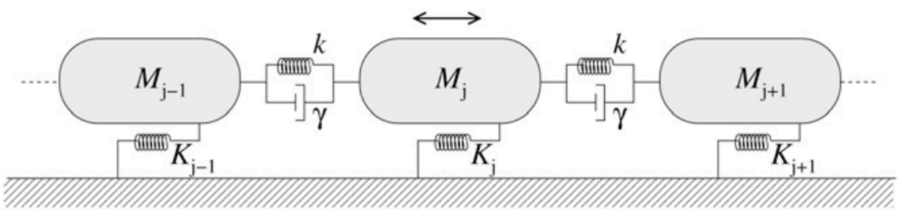
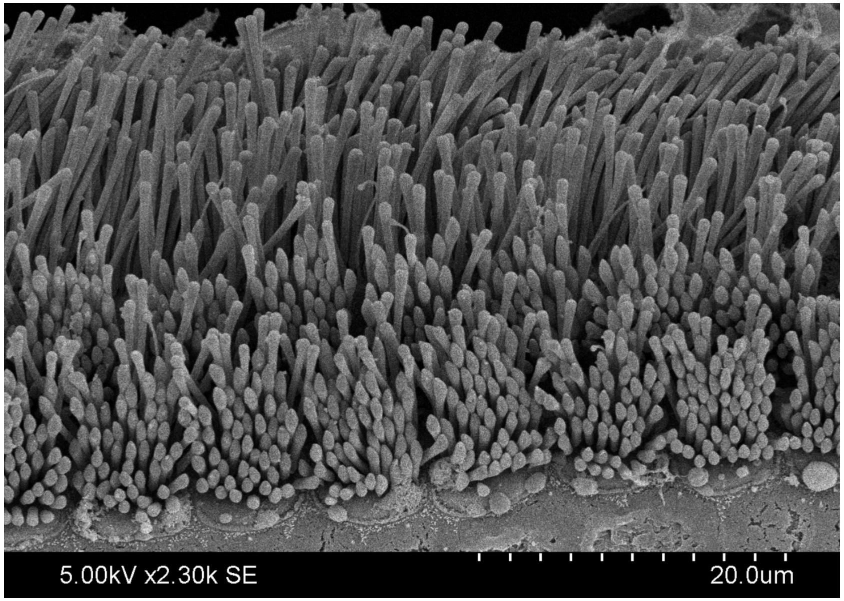
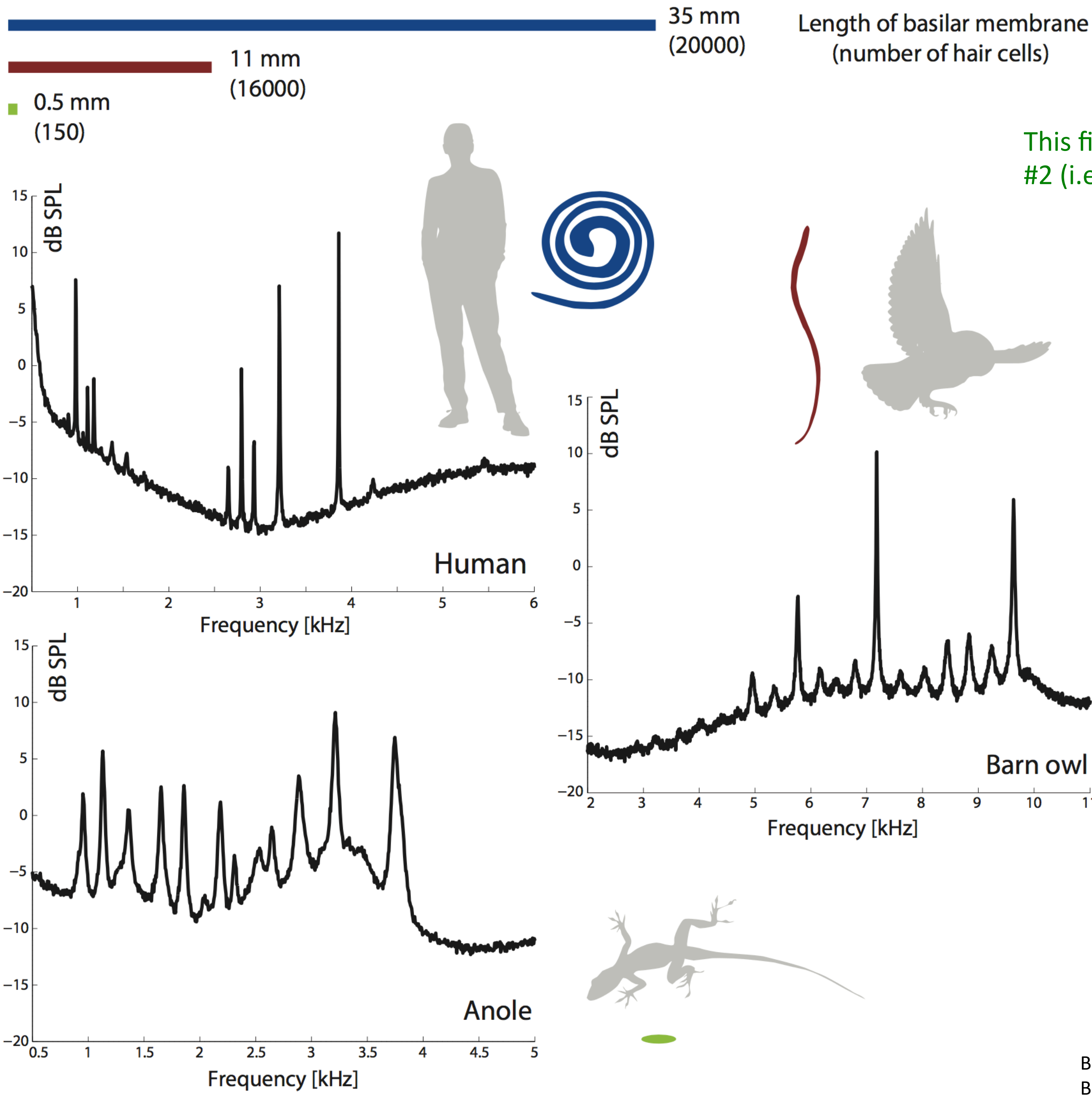


FIGURE 1 Mechanical equivalent of the model. Sallets are represented as inertial oscillators (mass M_j, spring K_j), coupled to their neighbors by elastic (constant k) and damping (constant γ) elements. In addition, there exists an active driving mechanism within each oscillator (not shown).

Unclear what the "correct" coupling should be...





This figure leads us into tangent #2 (i.e., the ear ain't passive...)

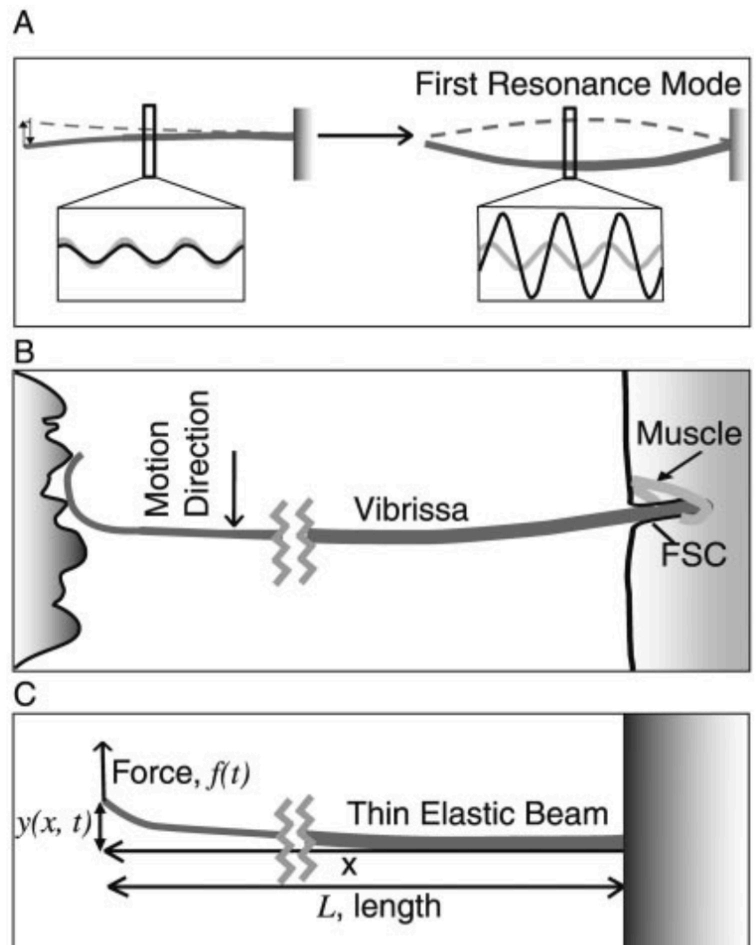
Bergevin, Manley & Koppl (PNAS 2015)
Bergevin, Verhulst & van Dijk (SHAR 2017)

Tangent III: Other examples of "oscillators" in biology...

Vibrissa Resonance as a Transduction Mechanism for Tactile Encoding

Maria A. Neimark,[1*] Mark L. Andermann,[1,2*] John J. Hopfield,[3] and Christopher I. Moore[4]

The Journal of Neuroscience, July 23, 2003 • 23(16):6499–6509



vi·bris·sa
/vīˈbrisə/

noun ZOOLOGY

any of the long stiff hairs growing around the mouth or elsewhere on the face of many mammals, used as organs of touch; whiskers.

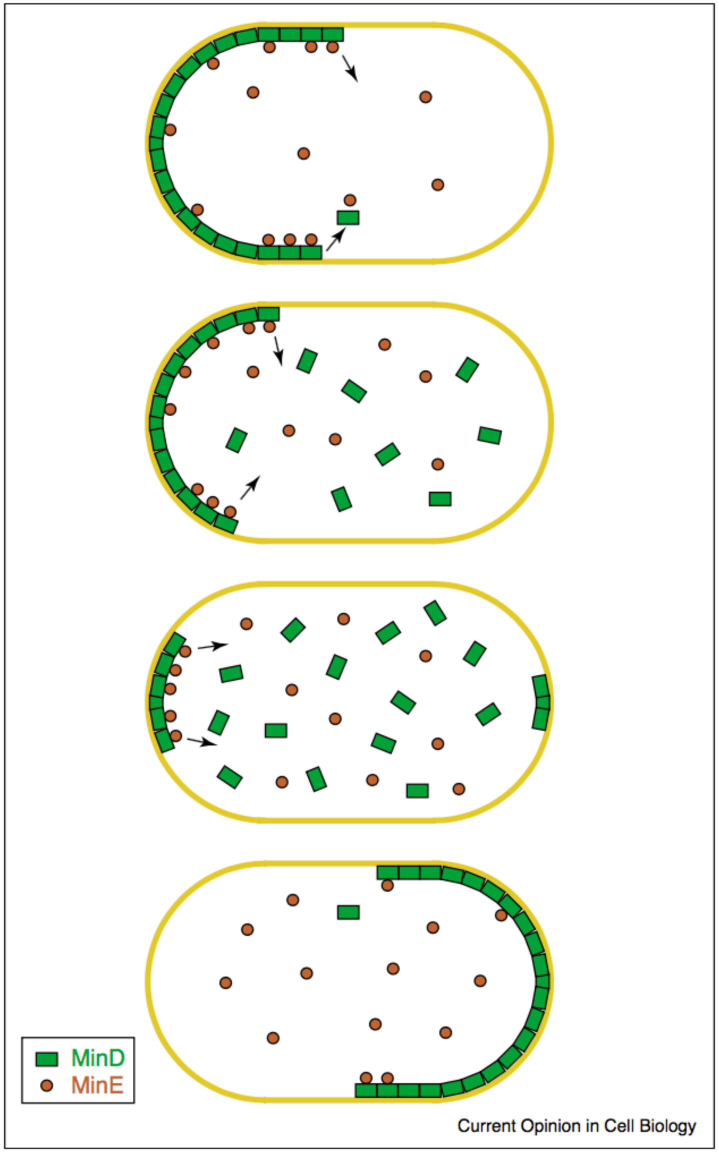
- ORNITHOLOGY
  each of the coarse bristlelike feathers growing around the gape of certain insectivorous birds that catch insects in flight.

Tangent III: Other examples of "oscillators" in biology...

Oscillations in cell biology

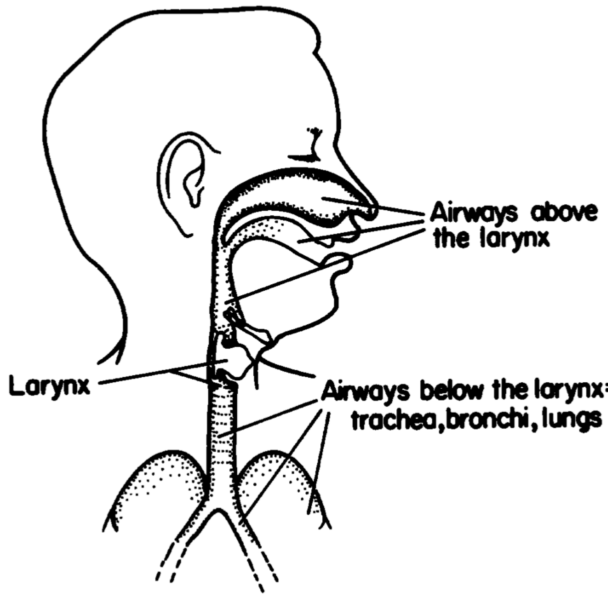
Karsten Kruse and Frank Jülicher

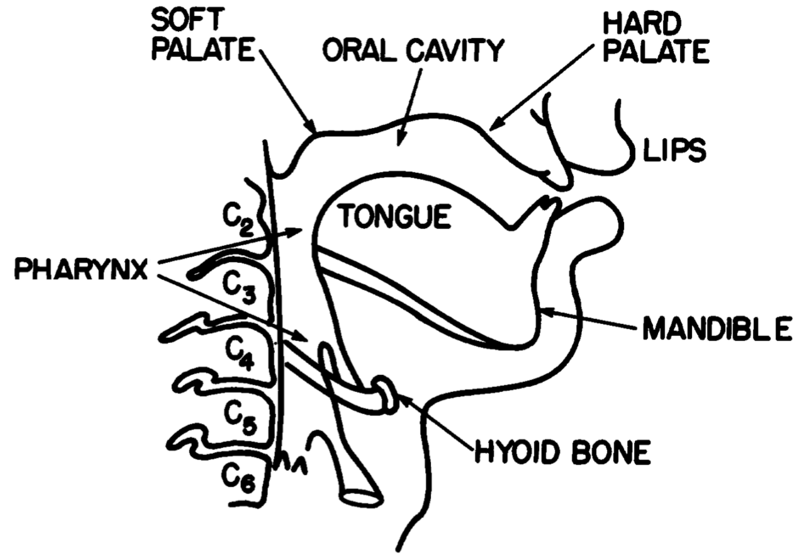
Current Opinion in Cell Biology 2005, **17**:20–26



Schematic representation of Min oscillations in *E. coli*. MinD (green) is localized on the inner bacterial membrane (yellow) on one side of the cell, where it aggregates. MinE (red) induces disassembly of the MinD aggregates and detachment of MinD molecules into the cytoplasm. MinD then assembles on the membrane of the opposite side of the cell and the process is repeated.

Tangent III: Other examples of “oscillators” in biology...

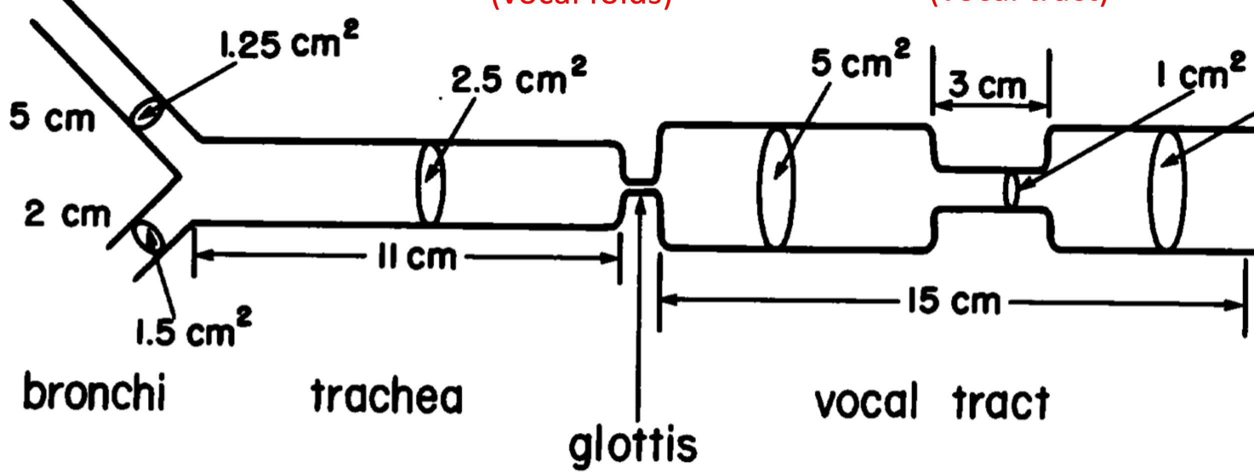




Pressure source
(lungs)

Vibration source
(vocal folds)

Filter
(vocal tract)



Output
(mouth)

Stevens (2000)

Tangent III

- **Source**: Vibrating vocal folds make ‘broadband’ sound

→ Harmonics commonly referred to as “***overtones***”

- **Filter**: Vocal tract shapes source sound

→ Resulting ‘shape’ emphasizes features (e.g., formants)

- **Formants**: Filtered harmonics; basis for vowels

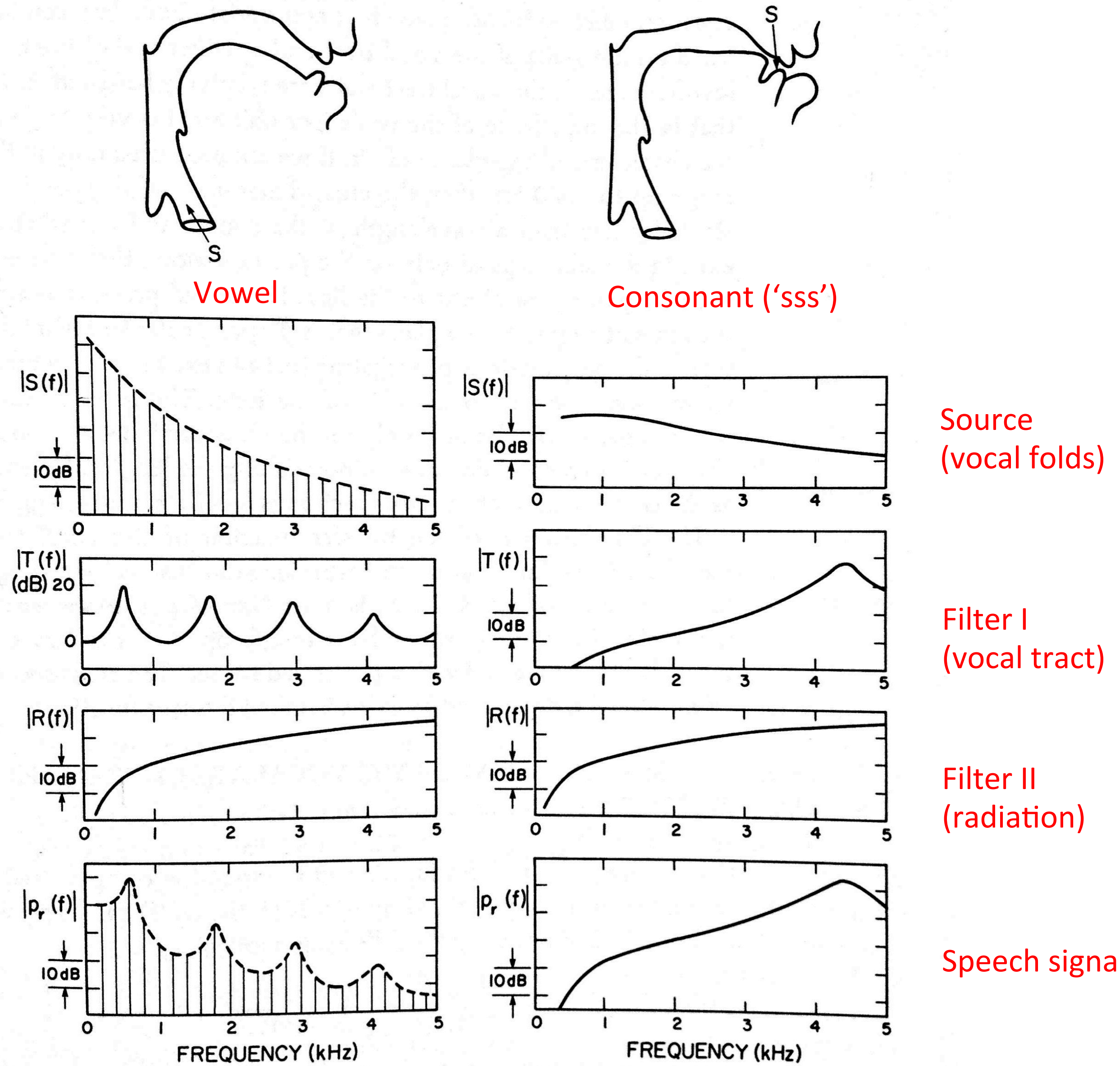
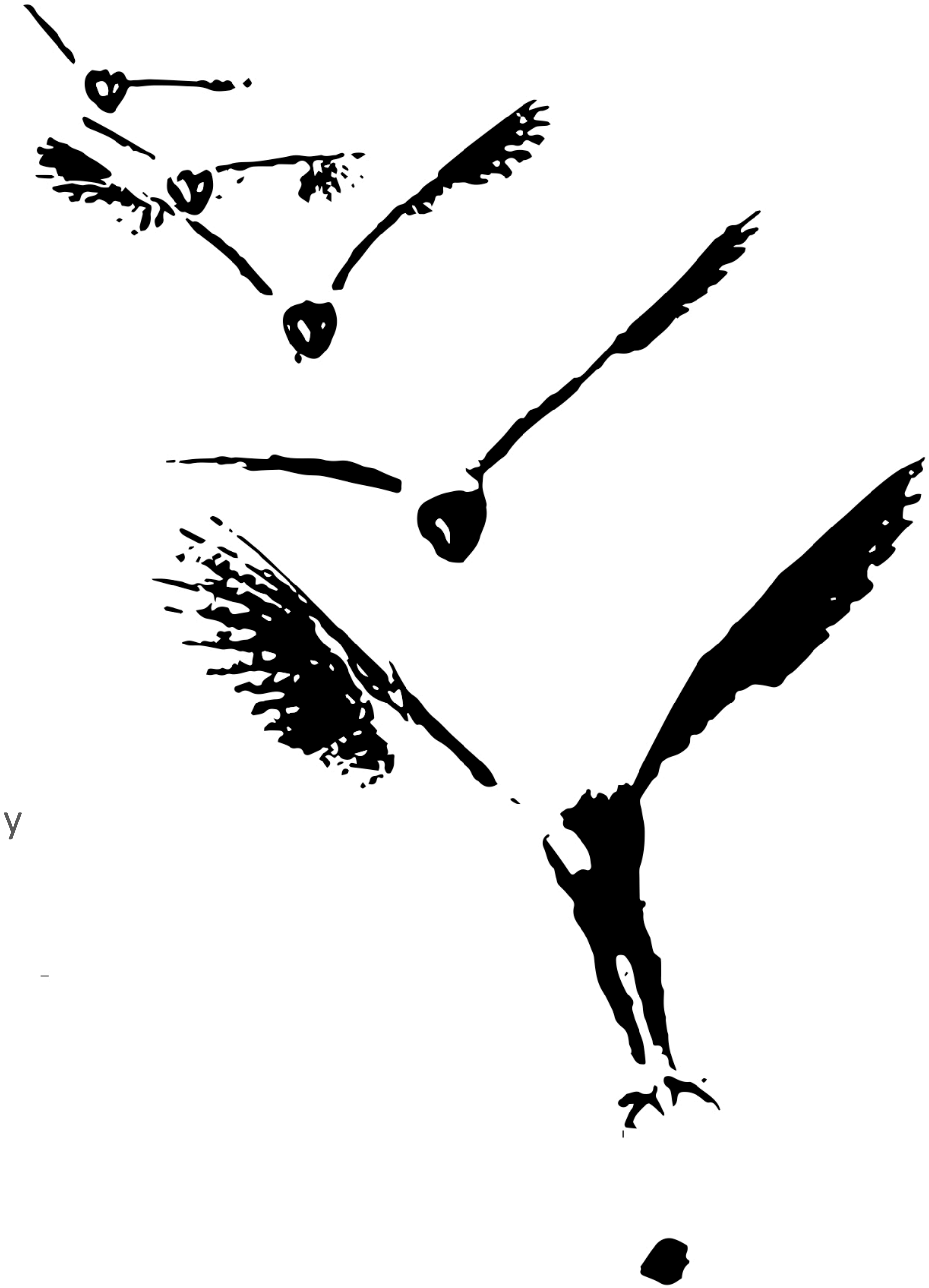


Figure 3.1 Sketches indicating components of the output spectrum $|p_r(f)|$ for a vowel and a fricative consonant. The output spectrum is the product of a source spectrum $S(f)$, a transfer function $T(f)$, and a radiation characteristic $R(f)$. The source spectra are similar to those derived in figures 2.10 and 2.33 in chapter 2. For the periodic source, $S(f)$ represents the amplitudes of spectral components; for the noise source, $S(f)$ is amplitude in a specified bandwidth. See text.

Stevens (2000)

Fini



Christopher Bergevin
York University, Dept. of Physics & Astronomy

Email: cberge@yorku.ca
URL: http://www.yorku.ca/cberge/

Ex. RLC circuit = Damped, Driven Harmonic Oscillator

Mechanical		Electrical
F (force)	$\leftrightarrow$	V (potential)
v (velocity)	$\leftrightarrow$	I (current)
x (position)	$\leftrightarrow$	q (charge)
m (mass)	$\leftrightarrow$	L (inductance)
b (damping)	$\leftrightarrow$	R (resistance)
k (spring)	$\leftrightarrow$	$1/C$ (capacitance)

state variables (F, v, x ↔ V, I, q)

system properties (m, b, k ↔ L, R, $1/C$)