# On the topology of black holes in higher dimensions

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Based on joint work with
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# Introduction

- In higher dimensional gravity one is naturally lead to consider questions about black holes in higher dimensions. In this talk we will focus on the issue of black hole topology.
- The natural starting point is Hawking's original theorem on black hole topology which states roughly:

  > *In a 4-dimensional spacetime obeying suitable energy conditions the surface of a steady state black hole is topologically a 2-sphere*

  This result also extends to *apparent horizons*.
- Aim is to present a generalization of Hawking's theorem to higher dimensions. The natural conclusion in higher dimensions is that the surface of a steady state black hole is of positive Yamabe type, i.e., admits a metric of positive scalar curvature.

# Hawking's theorem

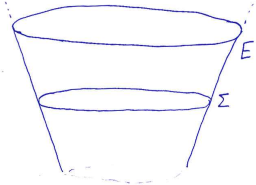
**Theorem**

*Suppose $(M, g)$ is a 4-dimensional AF stationary black hole spacetime obeying the dominant energy condition (DEC). Then cross sections of the event horizon are spherical.*

AF = admits a regular scri (conformal infinity) $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}-$

$E$ = event horizon $= \partial I^-(\mathscr{I}^+)$

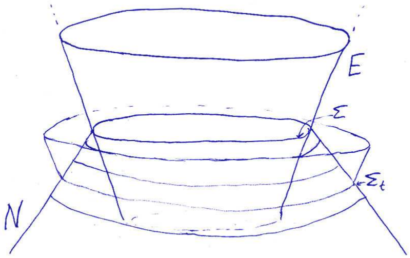
cross section = closed 2-surface $\Sigma$ obtained as intersection of $E$ with a spacelike hypersurface.



# Hawking's theorem

*Idea of proof:*

- By stationarity, $\Sigma$ is marginally outer trapped, $\theta = 0$.
- If $\Sigma \not\approx S^2$, i.e., if $g \geq 1$ then using Gauss-Bonnet and DEC, Hawking shows that $\Sigma$ can be deformed to an outer trapped surface, $\theta < 0$, outside the black hole region, which is forbidden by standard results.



# Hawking's theorem

*Remark:* Actually the torus $T^2$ is borderline for this argument.

$$g = 1 \quad \Rightarrow \quad \left.\frac{\partial \theta}{\partial t}\right|_{t=0} \leq 0$$

But can have $\Sigma \approx T^2$ only under special circumstances:

- $\Sigma$ must be flat
- null expansion and shear vanish on $\Sigma$
- A certain energy-momentum term vanishes along $\Sigma$.

So, by Hawkings' argument, generically, $\Sigma \approx S^2$.

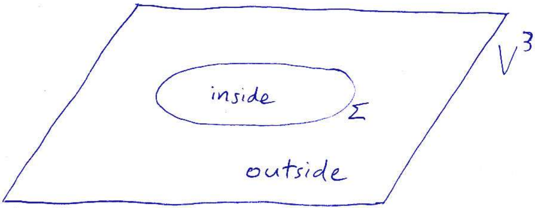
# Hawking's theorem for apparent horizons

The conclusion of Hawking's theorem holds for apparent horizons in spacetimes that needn't be stationary.

Consider,

$$\begin{aligned} M^4 &= \text{4-dim spacetime} \\ V^3 &= \text{spacelike hypersurface in } M^4 \\ \Sigma &= \text{closed 2-surface in } V^3 \end{aligned}$$

Suppose $\Sigma$ separates $V^3$ into an "inside" and an "outside":



# Hawking's theorem for apparent horizons

We say $\Sigma$ is an outer apparent horizon provided:

- $\Sigma$ is marginally outer trapped, i.e.,

$$\theta = 0 \quad \text{wrt outward null normal}$$

- there are no outer trapped surfaces outside of $\Sigma$.

Heuristically, $\Sigma$ is the "outer limit" of outer trapped surfaces.

**Theorem**

*Let $M^4$ be a spacetime satisfying the DEC. If $\Sigma$ is an outer apparent horizon in $V^3$ then $\Sigma \approx S^2$ (generically).*

*Remark:* This theorem includes the result on the topology of stationary black holes as a special case.

Our aim is to obtain a higher dimensional version of this theorem.

# Topological censorship

A completely different approach to studying black hole topology arose in the 90's based on the notion of topological censorship.

(Friedman, Schleich, Witt, Woolgar, G., ..)

From the point of view of topological censorship the domain of outer communications - the region of spacetime outside of all black holes and white holes - should have simple topology.

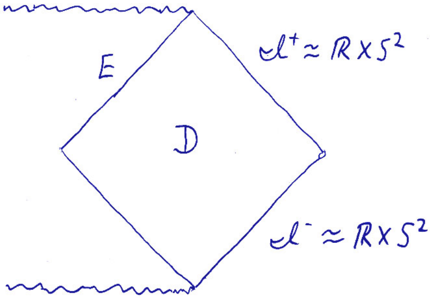
# Top cen and the topology of black holes

**Theorem (G.)**

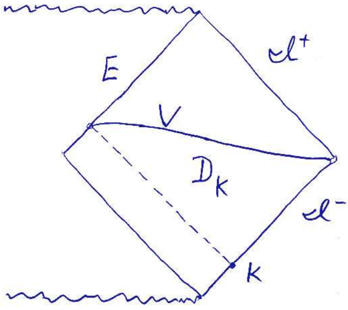
*Let $M$ be a 4-dim AF spacetime obeying the NEC. Suppose that the DOC*

$$D = I^{-}(\mathscr{I}^{+}) \cap I^{+}(\mathscr{I}-)$$

*is globally hyperbolic. Then $D$ is simply connected.*



# Top cen and the topology of black holes




<u>Fact:</u> $D_K$ is globally hyperbolic and simply connected.

$$
\begin{aligned}
\overline{V} &= \text{closure of } V \text{ in } M \\
&= V \cup \Sigma
\end{aligned}
$$

$\overline{V}$ simply connected $\Rightarrow \Sigma \approx S^2$

# Top cen and the topology of black holes

*Remark:* Topological censorship holds in arbitrary dimension. As long as scri is simply connected the DOC will be simply connected. However, this fact cannot be used to determine the topology of black holes in higher dimensions. Only in 3 spatial dimensions does the simple connectivity of space determine the topology of its boundary.

Thus, e.g., topological censorship does not appear to give any useful information in $D = 4 + 1$.

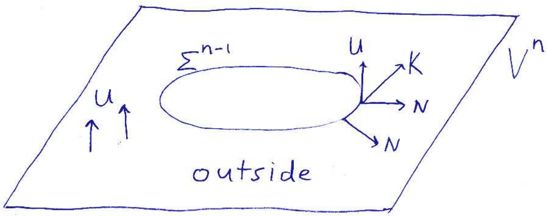
However, in a recent paper (hep-th/0509013) Helfgott, Oz and Yanay were able to get some additional mileage out of topological censorship in $D = 5 + 1$.

# Generalization of Hawking's Theorem

Let

$$\begin{aligned} M^{n+1} &= (n+1)\text{-dim spacetime, } n \geq 3 \\ V^n &= \text{spacelike hypersurface in } M^{n+1} \\ \Sigma^{n-1} &= \text{closed } (n-1)\text{-surface in } V^n \end{aligned}$$

Suppose $\Sigma$ separates $V^n$ into an "inside" and an "outside":



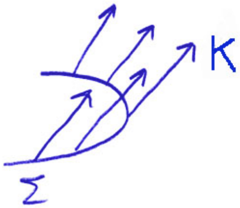

$$K = U + N = \text{ outward null normal to } \Sigma$$

# Generalization of Hawking's Theorem

*Null expansion of $\Sigma$ (wrt to $K$):*

$$\chi : T_p\Sigma \times T_p\Sigma \to \mathbb{R}\,,$$
$$\chi(X,Y) = \langle \nabla_X K, Y\rangle$$

$$\begin{aligned}\theta &= \operatorname{tr}\chi = h^{ab}\chi_{ab} \\ &= \operatorname{div}_\Sigma K\end{aligned}$$




Just as in $3+1$ case, we say $\Sigma^{n-1}$ is an outer apparent horizon provided:

- $\Sigma$ is marginally outer trapped, i.e.,

$$\theta = 0 \quad \text{wrt outward null normal}$$

- there are no outer trapped surfaces outside of $\Sigma$.

# Generalization of Hawking's Theorem

**Theorem (G. and Schoen)**

*Let $(M^{n+1}, g)$, $n \geq 3$, be a spacetime satisfying the DEC. If $\Sigma^{n-1}$ is an outer apparent horizon in $V^n$ then $\Sigma^{n-1}$ is of positive Yamabe type, i.e., admits a metric of positive scalar curvature, unless,*

- $\Sigma$ *is Ricci flat (flat if* $n = 3, 4$*) in the induced metric*
- $\chi \equiv 0 \quad$ *on* $\Sigma$
- $\mathcal{T}(U, K) = T_{ab}U^a V^b \equiv 0 \quad$ *on* $\Sigma$

I.e., *generically*, $\Sigma$ is of positive Yamabe type.

# Topological restrictions

There are many known topological obstructions to the existence of positive scalar curvature metrics in higher dimensions, beginning with the famous result of Lichnerowicz on the vanishing of the $\hat{A}$-genus (and generalizations by Hitchin).

A key advance was made in the late 70's/early 80's by Schoen-Yau and Gromov-Lawson.

Focus attention on $\dim \Sigma = 3$ ($\dim M = 4 + 1$) case.

**Fact**

*If $\Sigma$ is a closed orientable 3-manifold of positive Yamabe type then $\Sigma$ must be a connected sum of spherical manifolds and $S^2 \times S^1$'s.*

Thus, the basic horizon topologies in $\dim \Sigma = 3$ case are $S^3$ and $S^2 \times S^1$

# Topological restrictions

Here is a simple obstruction that holds in arbitrary dimensions:

**Fact (Gromov-Lawson)**

*A compact manifold that admits a metric of nonpositve sectional curvatures, $K \leq 0$, cannot carry a metric of positive scalar curvature.*

This rules out many obvious topologies.

# Proof of the theorem

*Proof:* We consider normal variations of $\Sigma$ in $V$, i.e., variations $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field

$$V = \frac{\partial}{\partial t}\Big|_{t=0} = \phi N, \quad \phi \in C^\infty(\Sigma)\,.$$

Let

$$\theta(t) = \text{ the null expansion of } \Sigma_t,$$

wrt $K_t = U + N_t$ and $N_t$ is the unit normal field to $\Sigma_t$ in $V$.

Since there are no outer trapped surfaces outside of $\Sigma$, one can show (Cai-G., Andersson-Mars-Simon) that there exists a normal variation of $\Sigma$ with

$$\phi > 0 \quad \text{and} \quad \left.\frac{\partial\theta}{\partial t}\right|_{t=0} \geq 0$$

# Proof, cont.

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = -\triangle\phi + 2\langle X, \nabla\phi\rangle + \left(Q + \operatorname{div} X - |X|^2\right)\phi\,,$$

where

$$Q = \frac{1}{2}S - \mathcal{T}(U,K) - \frac{1}{2}|\chi|^2 \quad \text{and} \quad X = \tan\left(\nabla_N U\right).$$

Since LHS $\geq 0$, completing the square on the RHS gives

$$-\triangle\phi + (Q + \operatorname{div} X)\,\phi + \phi|\nabla \ln\phi|^2 - \phi|X - \nabla\ln\phi|^2 \geq 0$$

Setting $u = \ln\phi$ we obtain,

$$\boxed{-\triangle u + Q + \operatorname{div} X - |X - \nabla u|^2 \geq 0}$$

# Proof, cont.

Absorbing laplacian term into divergence term,

$$Q + \mathrm{div}\,(X - \nabla u) - |X - \nabla u|^2 \geq 0$$

Setting $Y = X - \nabla u$, we have

$$-Q + |Y|^2 \leq \mathrm{div}\; Y$$

For any $\psi \in C^\infty(\Sigma)$, multiply through by $\psi^2$,

$$\begin{aligned}
-\psi^2 Q + \psi^2 |Y|^2 &\leq \psi^2 \mathrm{div}\; Y \\
&= \mathrm{div}\,(\psi^2 Y) - 2\psi \langle \nabla\psi, Y\rangle \qquad \text{(IBP)} \\
&\leq \mathrm{div}\,(\psi^2 Y) + 2|\psi||\nabla\psi||Y| \qquad \text{(Schwarz ineq)} \\
&\leq \mathrm{div}\,(\psi^2 Y) + |\nabla\psi|^2 + \psi^2 |Y|^2 \qquad (2ab \leq a^2 + b^2)
\end{aligned}$$

Canceling and integrating arrive at ...

# Proof, cont.

$$\int_\Sigma |\nabla\psi|^2 + Q\psi^2 \geq 0 \qquad \forall\psi \in C^\infty(\Sigma)$$

where $\qquad Q = \frac{1}{2}S - \mathcal{T}(U, K) - \frac{1}{2}|\chi|^2$

Consider equation

$$-\triangle\psi + Q\psi = 0$$

and corresponding eigenvalue problem,

$$-\triangle\psi + Q\psi = \lambda\psi$$

Have $\lambda_1 \geq 0$. Let $f$ be corresponding eigenfunction; can choose $f > 0$.

Then $\tilde{g} = f^{2/(n-2)}g$ has nonnegative scalar curvature.

# Final Comments

Consider case: dim $M = 5 + 1$ dim $\Sigma = 4$

- From Seiberg-Witten theory we know that there are many compact 4-manifolds - even simply connected ones - that do not admit metrics of positive scalar curvature.
- In the recent paper hep-th/0509013 of Helfgott, Oz and Yanay, the authors use topological censorship to argue that if the horizon is simply connected then either it is homeomorphic to $S^4$ or to a connected sum of $S^2 \times S^2$'s.

  They make use of results from 4-manifold topology and cobordism theory.