Deformations of G_2-structures, String Dualities and Flat Higgs Bundles

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Flat Riemannian Geometry

A subgroup $\pi \leq \text{Iso}(\mathbb{R}^n) := O(n) \ltimes \mathbb{R}^n$ is *Bieberbach* if π acts freely and properly discontinuously on $\mathbb{R}^n$, and $Q := \mathbb{R}^n/\pi$ is compact. Q is then a compact flat Riemannian manifold with $\pi_1(Q) = \pi$. There is an exact sequence:

$$1 \to \Lambda \to \pi \to H \to 1$$

where $\Lambda := \pi \cap \mathbb{R}^n$ and H is the *holonomy* of π. Note that the n-torus $\mathbb{T}$ is a Bieberbach manifold with trivial holonomy.

Bieberbach's Theorems

1. Λ is a lattice and H is finite. Equivalently: there is a finite normal covering $\mathbb{T} \to Q$ which is a local isometry.
2. Isomorphisms between Bieberbach subgroups of $\text{Iso}(\mathbb{R}^n)$ are conjugations of $\text{Aff}(\mathbb{R}^n)$. Equivalently: two Bieberbach manifolds of the same dimension and with isomorphic π_1's are affinely isomorphic.
3. There are only finitely many isomorphism classes of Bieberbach subgroups of $\text{Iso}(\mathbb{R}^n)$. Equivalently: there are only finitely many affine classes of Bieberbach manifolds of dimension n.

Platycosms

Bieberbach manifolds of dimension 3 are called *platycosms*.

Classification of Platycosms

There are only 10 affine equivalence classes of platycosms. Out of those, 4 are non-orientable. The orientable ones are:

1. The torocosm $\mathcal{G}_1 = \mathbb{T}$ with $H_{\mathcal{G}_1} = \{0\}$
2. The dicosm $\mathcal{G}_2$ with $H_{\mathcal{G}_2} = \mathbb{Z}_2$
3. The tricosm $\mathcal{G}_3$ with $H_{\mathcal{G}_3} = \mathbb{Z}_3$
4. The tetracosm $\mathcal{G}_4$ with $H_{\mathcal{G}_4} = \mathbb{Z}_4$
5. The hexacosm $\mathcal{G}_5$ with $H_{\mathcal{G}_5} = \mathbb{Z}_6$
6. The didicosm, a.k.a. the *Hantzsche-Wendt manifold* $\mathcal{G}_6$ with $H_{\mathcal{G}_6} = \mathbb{Z}_2 \times \mathbb{Z}_2 =: \mathbb{K}$

$$H_{\mathcal{G}_6} = \left\langle A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle \subset SO(3)$$

G_2-manifolds

Let (M^7, g) be an oriented Riemannian manifold. A *G_2-structure on M* is an element $\varphi \in \Omega^3(M, \mathbb{R})$ such that $\forall x \in M$, φ_x is stabilized by $G_2 \subset SO(7)$ acting on $\Lambda^3 T^*M$ (we say that φ is *positive*). Equivalently, a G_2-structure is a reduction of the structure group of the frame bundle $FM \to M$ down to G_2.

Properties of G_2

- G_2 is the compact real Lie group with Lie algebra $\mathfrak{g}_2$.
- $\dim GL(7) - \dim G_2 = 49 - 14 = 35 = \dim \Lambda^3 T^*M$. The set of positive 3-forms is open in $\Lambda^3 T^*M$.
- Connected Lie subgroups: $U(1) \subset SU(2) \subset SU(3) \subset G_2$

A G_2-structure is closed if $d\varphi = 0$, and torsion-free if $d \star \varphi = 0$.

Theorem

Hol$(M, g) \subseteq G_2 \iff d\varphi = d \star \varphi = 0 \iff \nabla_g \varphi = 0$

Basic model

Let $N = \mathbb{C}^2 \times \mathbb{T}^3$, g the flat product metric, $(\omega_1, \omega_2, \omega_3)$ the flat hyperkähler structure of $\mathbb{C}^2$, and dx_1, dx_2, dx_3 a basis of flat 1-forms on $\mathbb{T}^3$. Then:

$$\varphi = \sum_{i=1}^{3} dx_i \wedge \omega_i + dx_1 \wedge dx_2 \wedge dx_3$$

is a closed, torsion-free G_2-structure, so g is a flat G_2-metric.

- Here is a slightly better model: let $N = \mathbb{C}^2 \times \mathcal{G}_6$ and choose local flat sections dx_i of $T^*\mathcal{G}_6$. Then $\mu = dx_1 \wedge dx_2 \wedge dx_3$ is a global flat 3-form, and if one chooses $(\omega_1, \omega_2, \omega_3)$ to transform by the inverse action of $\mathbb{K}$ on a flat trivialization of $T^*\mathcal{G}_6$, then $\eta = \sum dx_i \wedge \omega_i$ is also globally defined. Thus $\varphi = \eta + \mu$ is a closed G_2-structure. In fact, it is also torsion-free, and the holonomy of the G_2-metric is $\mathbb{K}$.
- Now let $\Gamma \leq SU(2)$ and $\mathbb{K}$ act (compatibly) on $\mathbb{C}^2$ and consider the flat bundle $\widetilde{M} = \widetilde{\mathbb{C}^2/\Gamma} \times_{\mathbb{K}} \mathcal{G}_6 \to \mathcal{G}_6$. There is an induced closed, torsion-free G_2-structure $\widetilde{\varphi}$ on $\widetilde{M}$ whose holonomy is $SU(2) \rtimes \mathbb{K} \subset G_2$ [Acharya 99].

In the last example we have allowed the ω_i's to have non-trivial monodromy by replacing $\mathbb{C}^2$ by a flat rank 2 complex vector bundle over the platycosm Q whose monodromy is the ADE group $\mathbb{K}$.

ADE G_2-orbifolds

Let (Q,δ) be an oriented platycosm with $\pi_1(Q)=\pi$. Fix the following data:

ADE/G_2 data for (Q,δ)

- $p:\mathcal{V}\to Q$ a rank 1 quaternionic vector bundle
- $\Gamma \leq Sp(1)$ a finite subgroup (and hence a fiberwise action on $\mathcal{V}$)
- $\mathbf{H}\subset T\mathcal{V}$ a flat quaternionic connection on $\mathcal{V}$ compatible with the Γ-action
- $\mu\in\Omega^3(Q)$ a flat volume form
- $\eta\in\Omega^2(\mathcal{V}/Q)\otimes\Gamma(Q,\mathbf{H}^*)$ a Γ-invariant “vertical hyperkähler element"

This can be chosen in most cases. We then call $M=\mathcal{V}/\Gamma$ an *ADE G_2-platyfold of type* Γ. The G_2-structure on M can be written as $\varphi=\eta+\mu$.

- Let $V=\mathrm{Ker}(dp)$ be the vertical space. There is a decomposition $d=d_V+d_{\mathbf{H}}$ and, moreover:

$$d\varphi=0 \iff d_V\eta=d_V\mu=d_{\mathbf{H}}\eta=d_{\mathbf{H}}\mu=0$$

Donaldson's theorem

A closed G_2-structure on a coassociative fibration $M \to Q$ with orientation compatible with those of M and Q is equivalent to the following data:

1. A connection $\mathbf{H} \subset TM$ on $M \to Q$
2. A hypersymplectic element $\eta \in \mathbf{H}^* \oplus \Lambda^2 V^*$
3. A "horizontal volume form" $\mu \in \Lambda^3 \mathbf{H}^*$

satisfying the following equations:

$$\begin{aligned} d_{\mathbf{H}}\eta &= 0 \qquad & d_{\mathbf{H}}\mu &= 0 \\ d_V\eta &= 0 \qquad & d_V\mu &= -F_{\mathbf{H}}(\eta) \end{aligned}$$

where $F_{\mathbf{H}}$ is the curvature operator of $\mathbf{H}$.

We call $(\mathbf{H}, \eta, \mu)$ *Donaldson data* for $M \to Q$.

Slodowy slices and the Kronheimer family

Fix $(M_0 \to Q, \varphi_0)$ an ADE G_2-platyfold of type Γ, with Donaldson data $(\mathbf{H}_0, \eta_0, \mu_0)$. We have $\varphi_0 = \eta_0 + \mu_0$.
We would like to define a deformation family $f : \mathcal{F} \to \mathcal{B}$ with central fiber M_0 and such that φ_0 extends to a section of $\Omega^{3,+}_{\mathrm{cl}}(\mathcal{F}/\mathcal{B})$. That is, $\forall s \in \mathcal{B}$, $M_s := f^{-1}(s)$ has a closed G_2-structure.

- The deformation space of $\mathbb{C}^2/\Gamma$ can be embedded in $\mathfrak{g}_c$: choose $x \in \mathfrak{g}_c$ nilpotent and subregular, complete it to a $\mathfrak{sl}_2(\mathbb{C})$-triple (x, h, y) and consider the *Slodowy slice*: $\mathcal{S} := x + \mathfrak{z}_c(y)$, where $\mathfrak{z}_c(y)$ is the centralizer of y. Then the adjoint quotient $ad : \mathfrak{g}_c \to \mathfrak{h}_c/W$ restricts to:

$$\Psi : \mathcal{S} \to \mathfrak{h}_c/W$$

a flat map with $\Psi^{-1}(0) = \mathbb{C}^2/\Gamma$. This is the $\mathbb{C}^*$-miniversal deformation of $\mathbb{C}^2/\Gamma$.

Slodowy slices and the Kronheimer family

- Fix $\omega \in \mathfrak{h}$. The *Kronheimer family* $\mathcal{K}_\omega \to \mathfrak{h}_c$ is a simultaneous resolution of all fibers of Ψ over the projection $\mathfrak{h}_c \to \mathfrak{h}_c/W$. All hyperkähler ALE-spaces are fibers of $\mathcal{K}_\omega$ for some ω.
  We enlarge the family slightly in order to include all ω's: let Z be the adjoint representation of $SU(2)$. Consider $\bigsqcup_\omega \left(\mathcal{K}_\omega \to \{\omega\} \times \mathfrak{h}_c\right)$. This gives us a family of hyperkähler ALE-spaces $\mathcal{K} \to \mathfrak{h}_Z := \mathfrak{h} \otimes Z$.
- The idea to construct $f : \mathcal{F} \to \mathcal{B}$ is to define a "fibration of Kronheimer families" over Q. Then a section of the fibration will pick a hyperkähler deformation of $\mathbb{C}^2/\Gamma$ changing with $x \in Q$. The condition for Donaldson data will be a condition on the section, and $\mathcal{B}$ will be the space of allowed sections.

The existence of f will be a consequence of the following result:

Hyperkähler deformations over a platycosm

There is a rank $3\dim(\mathfrak{h})$ flat vector bundle $t:\mathcal{E}\to Q$ and a family $u:\mathcal{U}\to\mathcal{E}$ of complex surfaces, equipped with Donaldson data:

- $\mathbf{H}: u^*T\mathcal{E}\to T\mathcal{U}$ a connection
- $\eta\in\Omega^2(\mathcal{U}/\mathcal{E})\otimes u^*\Omega^1(\mathcal{E})$
- $\mu\in u^*\Omega^3(\mathcal{E})$

The family has the following properties:

1. $\mathcal{U}|_{\underline{\mathbf{0}}(Q)}\cong M_0$
2. $(\eta+\mu)|_{M_0}=\varphi_0$
3. $\forall x\in Q$ we have $\mathcal{U}|_{t^{-1}(x)}\cong\mathcal{K}$

where $\underline{\mathbf{0}}:Q\to\mathcal{E}$ denotes the zero-section.
Moreover, given a flat section $s:Q\to\mathcal{E}$, let $M_s:=u^{-1}(s(Q))$. Then the restrictions $(\eta|_{M_s},\mu|_{M_s},\mathbf{H}|_{M_s})$ satisfy Donaldson's criteria, and hence define a closed G_2-structure $\varphi_s:=(\eta+\mu)|_{M_s}$ on M_s.

Sketch of proof:

$$q : \mathcal{U} \xrightarrow{u} \mathcal{E} \xrightarrow{t} Q$$

1. *Construct the flat bundle $t : \mathcal{E} \to Q$:* Choose a flat trivialization of Q common to V and T^*Q. Glue $\mathfrak{h} \otimes T^*U_i \cong \mathfrak{h}_Z$ using cocycle of T^*Q.
2. *Construct the family of complex surfaces $u : \mathcal{U} \to \mathcal{E}$:* Pullback $\mathcal{K} \to \mathfrak{h}_Z$ by local maps $\psi_i : U_i \times \mathfrak{h}_Z \to \mathfrak{h}_Z$. Glue using cocycle of $\mathcal{V}$.
3. *Construct Donaldson data (η, μ, H) on $\mathcal{U}$:*
   - $\mu \in \Omega^{0,3}(\mathcal{U})$ is just a pullback from Q.
   - $\eta \in \Omega^{2,1}(\mathcal{U})$ is constructed locally by wedging $\psi_i^* \omega^a_{\mathsf{unf}}$ with local sections $\mu_a \in t^*\Omega^1(U_i)$, $a = 1, 2, 3$. Gluing construction from the previous step guarantees this is well-defined globally.
   - $\mathbf{H}$ is the most delicate step. It is essentially determined from a connection $\mathbf{H}_q$ on $q : \mathcal{U} \to Q$, which is in turn constructed from $\mathbf{H}_0$ through the dilation action of $\mathbb{R}^3$ on $\mathfrak{h}_Z$.
4. Induce Donaldson data on $M_s := u^{-1}(s(Q))$, where s is a flat section of t ("flat spectral cover")
5. $\mathcal{B} = \Gamma_{\mathsf{flat}}(Q, \mathcal{E})$ and the family $f : \mathcal{F} \to \mathcal{B}$ is the pullback of $\mathcal{U}$ by the tautological map $\tau : Q \times \Gamma_{\mathsf{flat}}(Q, \mathcal{E}) \to \mathcal{E}$. $\square$

The Hantzsche-Wendt G_2-platyfold

Our main example will be the *Hantzsche-Wendt G_2-platyfold*

$$M := \mathbb{C}^2/\Gamma \times_{\mathbb{K}} \mathcal{G}_6$$

especially when $\Gamma = \mathbb{Z}_2$. Among the ADE G_2-platyfolds, this is the only possible $\mathcal{N} = 1$ background. This is because Hol$(\widetilde{M}) = SU(2) \rtimes \mathbb{K}$ cannot be conjugated to a subgroup of $SU(3)$, while all others fix a direction in $\mathbb{R}^7$.

- From the theorem, when $\Gamma = \mathbb{Z}_2$, the deformation space is $\mathcal{B} = \Gamma(\mathcal{G}_6, T^*\mathcal{G}_6 \otimes \mathfrak{u}(1))$. The moduli space $\mathcal{M}_{G_2}$ is determined from the symmetries of the cover by $(\mathcal{B}/\mathbb{Z}_2)^{\mathbb{K}}$. Topologically, it is given by

  $$\mathcal{M}_{G_2} = \mathbf{Y} := \text{the three positive axes in } \mathbb{R}^3$$

  This agrees with a computation by D. Joyce [Joyce 00].
- The M-theory moduli space $\mathcal{M}_{G_2}^{\mathbb{C}}$ is obtained by adding the holonomies of C-fields, which are elements of $\exp(i\mathfrak{u}(1)) = \mathbb{R}/2\pi\mathbb{Z}$. Thus $\mathcal{M}_{G_2}^{\mathbb{C}}$ is the complexification of $\mathcal{M}_{G_2}$, given by a trident consisting of three copies of $\mathbb{C}$ touching at a point. We write this as:

  $$\mathcal{M}_{G_2}^{\mathbb{C}} \cong \mathbf{Y}_{\mathbb{C}}$$

M-theory/IIA duality

The string duality we will use relates geometric structures on a G_2-space (M, φ) and a "dual" Calabi-Yau threefold X.
Suppose $Q \subset M$ is an associative submanifold (i.e., $\varphi|_Q = d\mathrm{vol}_Q$) and $U(1)$ acts by isometries on M fixing Q. Then its *IIA dual* is

$$X := M/U(1)$$

The Calabi-Yau structure on X is required to have a real structure such that, under the projection map $d : M \to X$, $d(Q) \cong Q$ is a totally real special Lagrangian submanifold. There is also a condition on the behavior of the metric near $Q \subset X$.

- The case when $M \to Q$ is an ADE G_2-orbifold of type Γ corresponds to the "large volume limit" on X: essentially, $X = T^*Q$ with a semi-flat Calabi-Yau metric that blows-up along Q in a specified way.
- When $\Gamma = \mathbb{Z}_n$ this signals there is a stack of n $D6$-branes "wrapping" $Q \subset T^*Q$.
  Thus we expect the moduli space $\mathcal{M}_{IIA}$ to parametrize special Lagrangian deformations of Q and the (still undefined) data of n $D6$-branes on Q.

Hermitian-Yang-Mills Equations

Here is a more precise description. The supersymmetry condition for type IIA strings on T^*Q with n $D6$-branes is given by the *Hermitian-Yang-Mills (HYM) equations:*

$$\begin{cases} \mathcal{F}^{2,0} = 0 \\ \Lambda \mathcal{F} = 0 \end{cases} \tag{1}$$

Here $\mathcal{F}$ is the curvature of a $SU(n)$-connection $\mathcal{A}$ on a holomorphic vector bundle $\mathcal{E}$ over T^*Q endowed with a hermitian metric, and Λ is the Lefschetz operator of contraction by the Kähler form. Note that because $\mathcal{A}$ is hermitian, the first equation implies $\mathcal{F}^{0,2} = \overline{\mathcal{F}^{2,0}} = 0$.
The further condition that the $D6$-branes "wrap" Q is obtained by dimensional reduction of (1) down to Q. This yields:

$$\begin{cases} F_A = \theta \wedge \theta \\ D_A \theta = 0 \\ D_A \star \theta = 0 \end{cases} \tag{2}$$

for a $SU(n)$-connection A on a complex vector bundle $E \to Q$ and a "Higgs field" $\theta \in \Omega^1(Q, Ad(E))$. We call this the *Pantev-Wijnholt (PW) system* - see also [Donaldson 87] and [Corlette 88].

The IIA moduli space parametrizes solutions of PW. Recall the following result:

Donaldson-Corlette Theorem

Let G be a semisimple algebraic group and K a maximal compact subgroup. Let (Q, g) be a compact Riemannian manifold with fundamental group π, and let $(\widetilde{Q}, \widetilde{g})$ be its universal cover. Fix a homomorphism $\rho : \pi \to G$ and let $h : \widetilde{Q} \to G/K$ be a ρ-equivariant map. Then the following are equivalent:

1. $h : \widetilde{Q} \to G/K$ is a harmonic map of Riemannian manifolds
2. The Zariski closure of $\rho(\pi)$ is a reductive subgroup of G (i.e., ρ is semisimple)

Moreover, if ρ is irreducible, the harmonic map is unique.

This result allows us to prove:

Proposition

Solutions to PW are the same as flat reductive bundles on Q. It follows that $\mathcal{M}_{IIA}$ is the *character variety:*

$$\mathrm{Char}(Q, G) := \mathrm{Hom}(\pi, G)//G$$

The Hantzsche-Wendt Calabi-Yau

The IIA dual of M is the Hantzsche-Wendt Calabi-Yau $X = T^*\mathcal{G}_6$. We now describe some of its properties:

- *Character Variety:* For $\Gamma = \mathbb{Z}_2$:

$$\mathcal{M}_{IIA}(X) = \text{Char}(\mathcal{G}_6, SL(2,\mathbb{C})) \cong \mathbf{Y}_{\mathbb{C}}$$

  which matches $\mathcal{M}^{\mathbb{C}}_{G_2}(M)$. There are similar descriptions for higher n.
- *SYZ fibration:* Recall there is a finite Galois cover $\mathbb{T} \to \mathcal{G}_6$. Since $T\mathbb{T}$ is a trivial flat bundle, we can identify all fibers with $\mathbb{R}^3$. The map $T\mathbb{T} \to \mathbb{R}^3$ induces $(T\mathbb{T})/\mathbb{K} \to \mathbb{R}^3/\mathbb{K}$, where $\mathbb{K}$ acts via the differential action. We then prove that $(T\mathbb{T})/\mathbb{K} \cong T^*\mathcal{G}_6$. Thus, we have:

$$g : X \to \mathbb{R}^3_{\mathbf{Y}}$$

  where $\mathbb{R}^3_{\mathbf{Y}} := \mathbb{R}^3/\mathbb{K}$ is called the $\mathbf{Y}$*-vertex.* Geometrically, it is a cone over a thrice-"punctured" two-sphere. The isotropy is $\mathbb{Z}_2$ at the punctures and $\mathbb{K}$ at the vertex.
- It is known [Loftin, Yau, Zaslow 05] that $\mathbb{R}^3_{\mathbf{Y}}$ admits affine Hessian metrics solving the Monge-Ampère equation, and so $T^*\mathbb{R}^3_{\mathbf{Y}}$ admits semi-flat CY metrics.

A G_2-conifold transition?

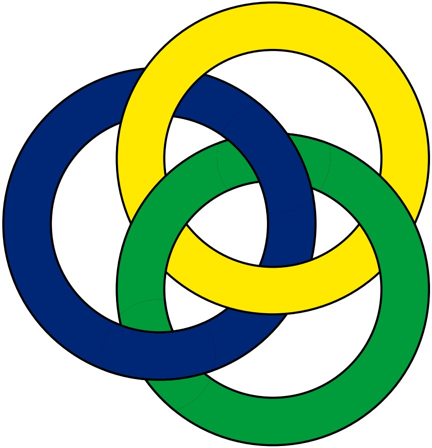

Figure: Left: Borromean rings. Right: Three fully linked unknots.
LaTeX code by Dan Drake, available at http://math.kaist.ac.kr/∼drake

Flat Higgs bundles

We will now discuss special solutions of the PW system.

Definition

A *flat* $GL(r,\mathbb{C})$*-Higgs bundle* on a compact Riemannian manifold Q is a tuple (E,h,A,θ) consisting of a complex rank r vector bundle $E \to Q$ with a hermitian metric h, a unitary flat connection $A \in \Omega^1(End(E))$, and a $\mathcal{C}_Q^\infty$-linear bundle map $\theta : \Gamma(E) \to \Gamma(E \otimes T^*Q)$ satisfying $\theta \wedge \theta = 0$, and such that the following *flatness conditions* are satisfied:

- $D_A\theta = 0$
- $D_A \star \theta = 0$

If furthermore (Q,δ) is flat, we require θ to be compatible with δ.
The condition $\theta \wedge \theta = 0$ means that the three matrices $\theta_1, \theta_2, \theta_3$ are simultaneously diagonalizable. So we can describe a flat Higgs bundle in terms of *flat spectral data*.

Flat spectral data

Definition

Let (E, h, A, θ) be a flat Higgs bundle over Q. The *Spectral Cover* associated to θ is the subvariety $S_\theta \subset T^*Q$ defined via its characteristic polynomial:

$$S_\theta = \{(q, \lambda); \det(\lambda \otimes \mathbf{1}_E - \theta) = 0\} \tag{3}$$

Definition: *Flat Spectral Data*

Let (E, h, A, θ) be a rank n flat Higgs bundle over a platycosm (Q, δ). Assume θ is regular. We define *flat spectral data* to be:

1. A n-sheeted covering map $\pi : S_\theta \to Q$.
2. A line bundle $\mathcal{L} \to S_\theta$ determined by the eigenlines of θ
3. A hermitian metric $\widetilde{h}$ on $\mathcal{L}$ determined by h
4. A hermitian flat connection $\widetilde{A}$ on $\mathcal{L}$ determined by A
5. A Lagrangian embedding $\ell : S_\theta \to T^*Q$ satisfying $Im(d\ell) \subset \mathbf{H}^*_\delta$.

Flat Spectral Correspondence

There is an equivalence:

$$\textbf{FlatHiggs} \longleftrightarrow \textbf{FlatSpec}$$

between flat Higgs bundles and flat spectral data.

Flat Higgs bundles admit a Hitchin map similar (although not as nice) to their holomorphic cousins:

Definition

Let $p_i(\theta)$ be the coefficient of λ^{n-i} in the expansion of $\det(\lambda\mathbf{1} - \theta) \in \mathbb{C}[\lambda]$. The *Hitchin map* is defined by:

$$\mathfrak{H} : \textbf{FlatHiggs} \to \bigoplus_{i=1}^{n} H^0(Q, (T^*Q)^{\otimes i})$$
$$(E, h, A, \theta) \mapsto (p_1(\theta), \ldots, p_n(\theta))$$

An important property is that the spectral cover S_θ depends only on $\mathfrak{H}(\theta)$.

SYZ Mirror Symmetry

Recall the Hantzsche-Wendt Calabi-Yau has a SYZ fibration $g : X \to \mathbb{R}^3_{\mathbf{Y}}$, obtained as a $\mathbb{K}$-quotient of the smooth sLag torus fibration $d : T^*\mathbb{T} \to \mathbb{R}^3$. We would like to describe the mirror SYZ fibration $g^\vee : X^\vee \to \mathbb{R}^3_{\mathbf{Y}}$.

- SYZ Mirror Symmetry tells us that, away from singular fibers, $g^\vee$ is given by the dual torus fibration: if $\mathbb{T}_b = g^{-1}(b)$ is a smooth fiber, then $(g^\vee)^{-1}(b) = \mathrm{Hom}(\pi_1(\mathbb{T}_b), U(1)) =: \mathbb{T}_b^\vee$ parametrizes $U(1)$-local systems on $\mathbb{T}_b$.
  Our proposal: $g^\vee$ should be obtained as an appropriate $\mathbb{K}$-quotient of the SYZ mirror of d, $d^\vee : (\mathbb{C}^*)^3 \to \mathbb{R}^3$.
- There is a natural induced action of $\mathbb{K}$ on $\mathbb{T}^\vee$ by pullback of local systems. This is *not* a free action. Our proposed mirror is then an orbifold:

$$g^\vee : [(\mathbb{C}^*)^3/\mathbb{K}] \to \mathbb{R}^3_{\mathbf{Y}}$$

Mirror symmetry maps IIA string theory on X to IIB string theory on $X^\vee$. At the level of branes, the mirror map sends a $D6$-brane on X to a $D3$-brane on $X^\vee$. Mathematically, it is given by a Fourier-Mukai functor

$$\mathcal{FM} : \mathsf{Fuk}(X) \to \mathcal{D}(X^\vee)$$

and sends a $U(1)$-local system on a sLag fiber $Q_b \subset X$ to a skyscraper sheaf over the associated point on $Q_b^\vee$. More generally, it send a $SU(n)$-local system on Q_b to a direct sum of skyscraper sheaves supported on $Q_b^\vee$. Thus, Mirror Symmetry for branes predicts an identification:

$$\underbrace{\mathsf{Char}(\pi, SL(n,\mathbb{C}))}_{\mathcal{M}_{IIA}} \cong \underbrace{\mathsf{Hilb}^n_{\mathbf{Y}}(X^\vee)}_{\mathcal{M}_{IIB}}$$

where $\mathsf{Hilb}^n_{\mathbf{Y}}(X^\vee)$ is the *punctual Hilbert scheme supported at the vertex fiber over* $\mathbb{R}^3_{\mathbf{Y}}$.

The Spectral Mirror

To avoid working with Hilbert schemes of orbifolds, we will construct a crepant resolution $X^\wedge$ of $X^\vee$ that gives the correct moduli space. As a bonus, we will get a new interpretation of SYZ.
The idea is to use mirror symmetry and the flat spectral construction to build $X^\wedge$ as the solution to a moduli problem on X. For this reason, we call $X^\wedge$ the *spectral mirror*.

- Recall that a smooth fiber $\mathbb{T}_b^\vee$ of $X^\vee$ parametrizes local systems $(\mathcal{L}_b, A_b)$ on the sLag $\mathbb{T}_b \hookrightarrow X = T^*\mathcal{G}_6$. Recall also that $\forall b$ there is an unramified $\mathbb{K}$-cover $p_b : \mathbb{T}_b \to \mathcal{G}_6$. Thus $(p_b, \mathcal{L}_b, A_b)$ can be thought as spectral data over $\mathcal{G}_6$.
- Assume $\forall b$, $(\mathcal{L}_b, A_b)$ is a deformation of a local system $(\mathcal{L}_0, A_0)$ on $\mathcal{G}_6$. Then we are looking at the moduli space of $\mathbb{K}$-spectral data over $\mathcal{G}_6$. By the flat spectral correspondence, this is equivalent to a certain moduli space of flat Higgs bundles. If we unpack the Higgs data, we are led to define the spectral mirror as:

$$X^\wedge := \mathcal{M}_{\text{Higgs}}^{\mathbb{K}} \subset \mathcal{M}_{\text{Higgs}}^{SO(4,\mathbb{C})}$$

the moduli space of flat $SO(4,\mathbb{C})$-Higgs bundles over $\mathcal{G}_6$ whose spectral covers have Galois group $\mathbb{K}$.

Proposition

The Hitchin map $\mathfrak{H} : X^\wedge \to \mathcal{B}$ defines a smooth model for $g^\vee : X^\vee \to \mathbb{R}^3_{\mathbf{Y}}$.

The proof consists of showing that $Im(\mathfrak{H}) \cong \mathbb{R}^3_{\mathbf{Y}}$ and the smooth Hitchin fibers are $\mathfrak{H}^{-1}(b) \cong \mathbb{T}^\vee_b$.
A topological model for $X^\wedge$ is given by $\mathrm{Char}^0(\pi, SO(4,\mathbb{C}))$. The same methods used previously can be used here to describe it explicitly. Roughly speaking, $X^\wedge$ is obtained from $X^\vee$ by replacing $(g^\vee)^{-1}(\mathbf{Y})$ by $\mathrm{Sym}^2(\mathbf{Y}_\mathbb{C})$. For $n = 2$, the Hilbert scheme is a length 2 thickening of the diagonal $\mathbf{Y}_\mathbb{C}$, so topologically:

Proposition

$$\mathrm{Hilb}^2_{\mathbf{Y}}(X^\wedge) \cong \mathbf{Y}_\mathbb{C} = \mathrm{Char}(\pi, SL(2,\mathbb{C}))$$

This confirms the mirror symmetry prediction. We expect that the same result holds for higher n.

- This result suggests an approach to SYZ mirror symmetry, useful when a singular fiber is covered by a smooth fiber: construct smooth models for the mirrors as moduli spaces of flat Higgs bundles on the singular fiber.

Thank you!