Associatives

(Y, ϕ) G_2 - manifold

ϕ ↑ 3-form

$P \subset Y$ associative if (a) $\mathrm{vol}_P = \phi|_P$

$\Updownarrow$

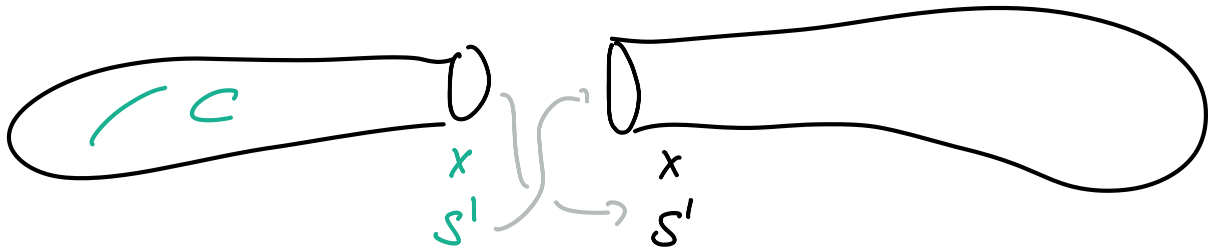
(b) $\forall v \in NP: \quad (i_v \psi)|_P = 0$

Why?

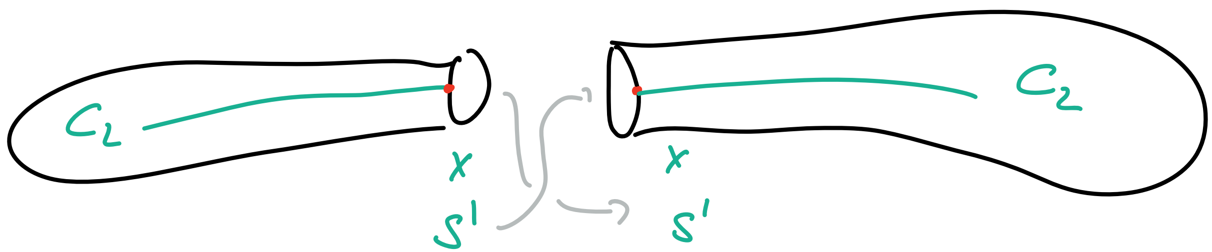
I. They are "natural objects" (~ holomorphic curves in CY3) and it is "self-evidently interesting" to construct them.

Joyce '96: associatives as fixed-point sets of ϕ-preserving involutions.

CHNP '15: associatives in twisted connected sums via rigid holomorphic curves in weak Fano 3-folds



Bera '22: associatives in twisted connected sums via matching holomorphic curves



(Nordström – Rodríguez Díaz – Sá Earp: ideas for concrete examples)

Open problem: Braun – del Zotto – Halverson – Larfors – Morrison – Schäfer-Nameki '18 propose a concrete example of a TCS with infinitely many associatives. Can they be constructed rigorously (using Bera's results)?

Aside: Acharya – Braun – Svanes – Valandro '13 have an orbifold version that needs no analysis.

It is also interesting to study associatives in special (almost) G_2-manifolds: Lotay, Kawai, Ball – Madnick, ...

II. G_2 Teichmüller space

$$\mathcal{T}(Y) := \{\phi \in \Omega^3(Y) \text{ torsion-free } G_2\text{-structure}\} / \mathrm{Diff}_0(Y)$$

$\pi \big\downarrow$ Joyce '96: Lagrangian immersion

$$H^3_{dR}(Y) \oplus H^4_{dR}(Y) \qquad \pi(\phi \cdot \mathrm{Diff}_0(Y)) := ([\phi], [*_\phi \phi])$$

Aside: Open problem 2: Is π an embedding?

FACTS: $\pi(\mathcal{J}(Y))$ is constrained by:

(1) $\int_Y \alpha\wedge\alpha\wedge\phi<0$ $\quad\forall[\alpha]\in H^2_{dR}(Y)\setminus\{0\}$ $\quad$ if $\pi_1(Y)$ finite

(2) $\int_Y p_1(V)\wedge\phi<0$ $\quad$ if $V\to Y$ admits a non-flat G_2-instanton

(3) $\int_P \phi = vol(P)>0$ $\quad$ if P is associative

(4) $\int_Q *_\phi\phi>0$ $\quad$ if Q is coassociative

Open problem 3: Do (2), (3) really impose more restrictions than (1)?

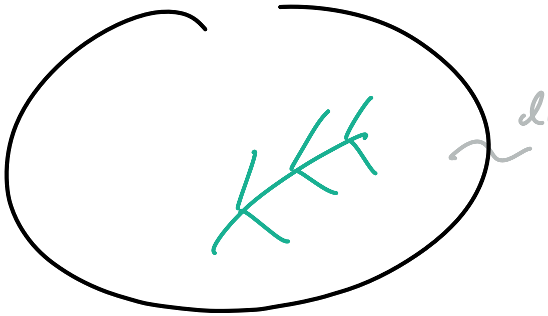
(2) w/ $V=TY$

Open problem 4/ Conjecture by Halverson–Morrison '16: Do (1)–(4) characterise the ideal boundary of $\mathcal{J}(Y)$?

Dwivedi–Platt–W. '22 construct examples of generalised

Kummer constructions $\{Y_t : t\in(0,T]\}$ in which the degeneration to the orbifold $Y_0 = T^7/\Gamma$ is witnessed by associating $P_t \subset Y_t$ with $\mathrm{vol}(P_t) \to 0$.

Sketch: Y_0 Y_t



* loc. $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$ ⇝ replace/resolve by Eguchi–Hanson $Bl_0(\mathbb{C}^2/\{\pm 1\})$ and perturb to torsion-free

* $\ell \times \mathbb{CP}^1$ is associative before perturbation and $\forall$ suitable $\ell \subset \mathbb{R}^3$ if ℓ is rational*, sometimes yields $P_t \subset Y_t$. //

Open problem 5: Extend DPW'22 to Joyce–Karigiannis' generalised generalised Kummer construction.

(related work in progress by Majewski)

III. Invariants?

OP4 is somewhat ill-posed b/c the notion of association itself depends on ϕ.

The situation would improve if there were invariants which enforce the persistence of associatives in certain homology classes.

Donaldson–Thomas IP (read between the lines of S&S)

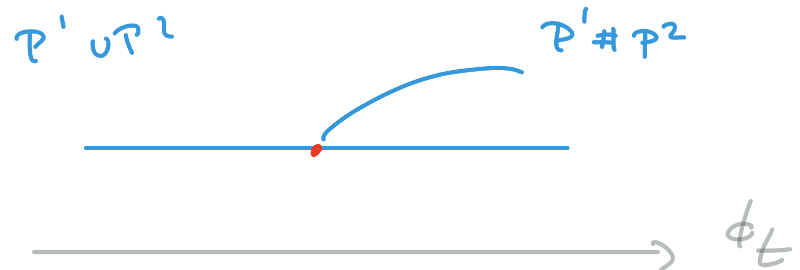
Associatives are crit. pts of a functional

$$\mathcal{A} : \{\text{submanifolds} + \text{decorations}\} \to \mathbb{R}.$$

Can one count $\mathrm{crit}(\mathcal{A})$? construct $HM(\mathcal{A})$?

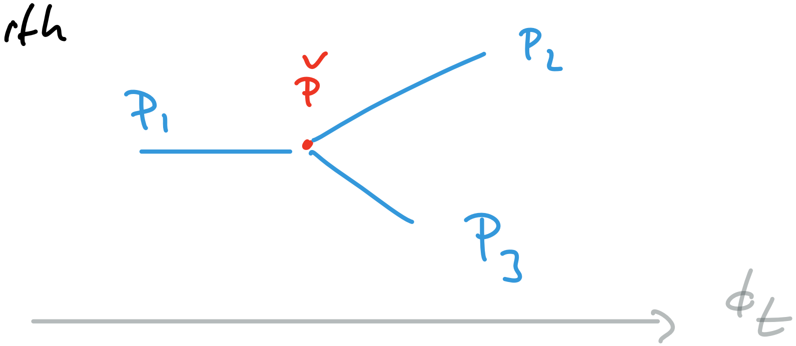
Joyce: there are a lot of problems:

(1) (self-) intersecting assoc. should produce connected sum associations



$\pm 1 \neq \pm 1 \pm 1$

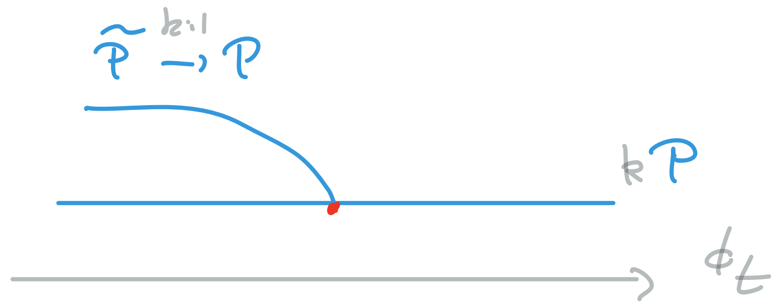
(2) $\exists$ resolutions of assoc. $\check{P}$ with sing. modelled on Clifford tori



(3) Multiple cover phenomena

Smith

$P \to Y$



and possibly much worse (cf. Zhenhua Liu '21)

Aside: Joyce has precise conj. about (1), (2) which will be proved in Bera's PhD thesis (and have part. been proved by Nordström).

Open problem 6: Which degenerations can occur in generic 1-parameter families?

Let us pretend only (1)–(3) occur.

Joyce '16 has a proposal for "superpotential counting QHS associatives" in tamed almost G_2-manifolds: (1), (2) ✓ (3) ?

Doan–W. '19 propose to count assoc. with (ADHM)-Seiberg–Witten monopoles:

* (1), (2) ✓ if there are no QHS associatives

  Morally:

* (3) might be okay but much more analytical work on generalised Seiberg–Witten equations, etc. needed

* If there are QHS associatives, then one can try

$$C_* = \bigoplus_{P} CM_*(P) + \cdots \quad \text{for ADHM SW monopoles}$$

Kronheimer–Mrowka's monopole cx.

w/ $\partial = \oplus\, \partial_p$ + bordism maps from Cayleys ($\rightsquigarrow$ $\nexists$ flow lines).

This deals with (1), (2) provided the relevant surgery traces are realised as Cayley bordisms.

Open problem 7: Are they?

Donaldson-Scaduto '20 started to study (1), (2) in the adiabatic limit.

Open problem D: What about multiple covers in the adiabatic limit?