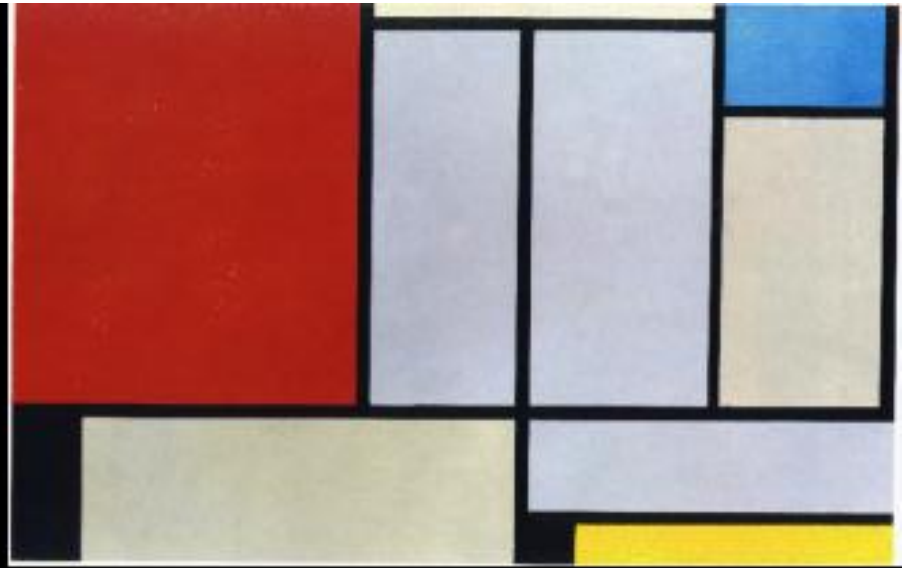


*Robustness
and
Complexity*



*John Doyle
Control and Dynamical Systems
Caltech*



Research interests



- Complex networks applications
 - Biological regulatory networks
 - Ubiquitous, pervasive, embedded control, computing, and communication networks
- New mathematics and algorithms
 - robustness analysis
 - systematic design
 - multiscale physics



Collaborators and contributors (partial list)



Biology: Csete, Yi, Borisuk, Bolouri, Kitano, Kurata, Khammash, El-Samad, ...

Alliance for Cellular Signaling: Gilman, Simon, Sternberg, Arkin, ...

HOT: Carlson, Zhou, ...

Theory: Lall, Parrilo, Paganini, Barahona, D'Andrea, ...

Web/Internet: Low, Effros, Zhu, Yu, Chandy, Willinger, ...

Turbulence: Bamieh, Dahleh, Gharib, Marsden, Bobba, ...

Physics: Mabuchi, Doherty, Marsden, Asimakapoulos, ...

Engineering CAD: Ortiz, Murray, Schroder, Burdick, Barr, ...

Disturbance ecology: Moritz, Carlson, Robert, ...

Power systems: Verghese, Lesieutre, ...

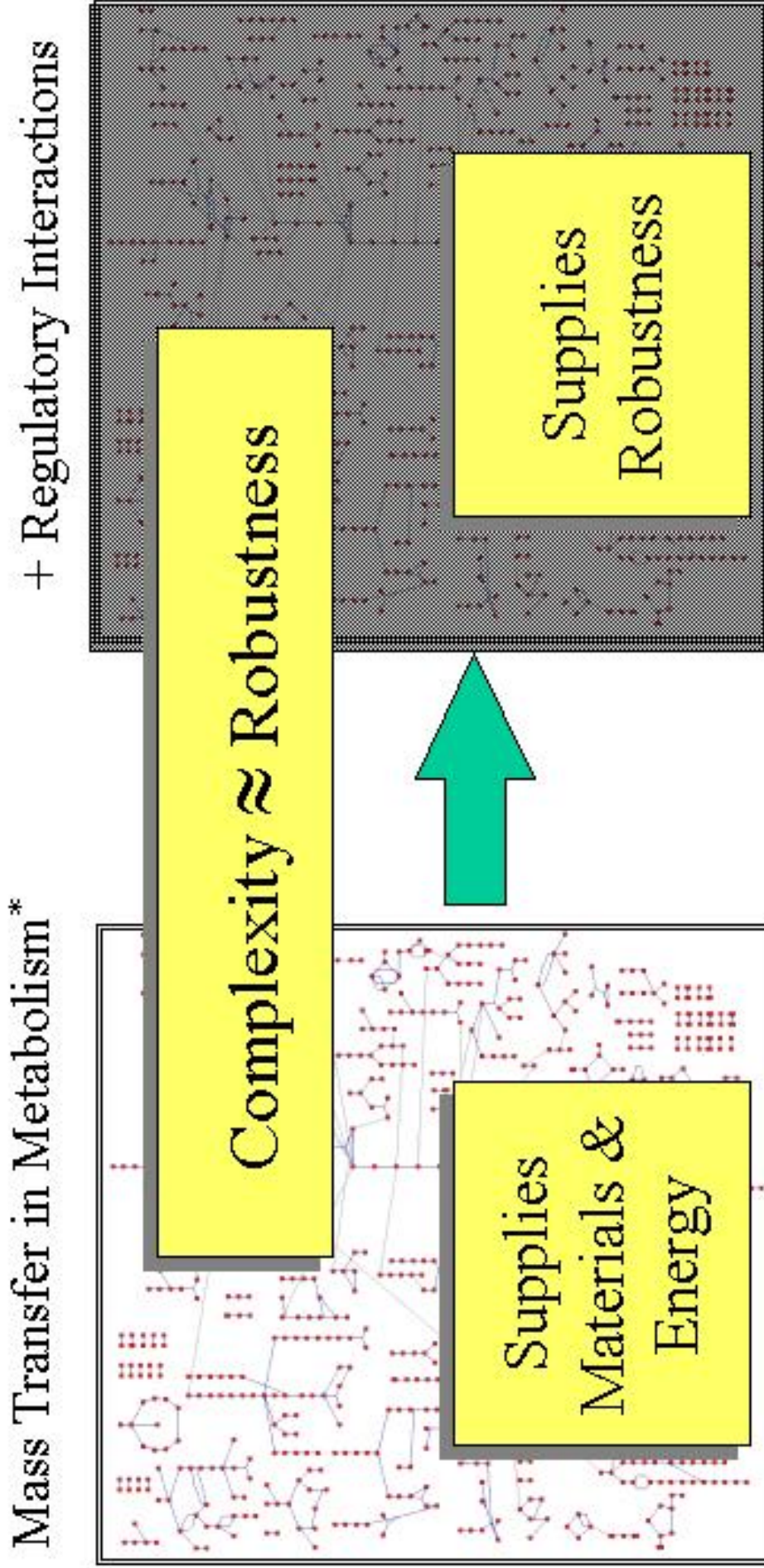
Finance: Primbs, Yamada, Giannelli, ...

... and casts of thousands ...

Background reading online

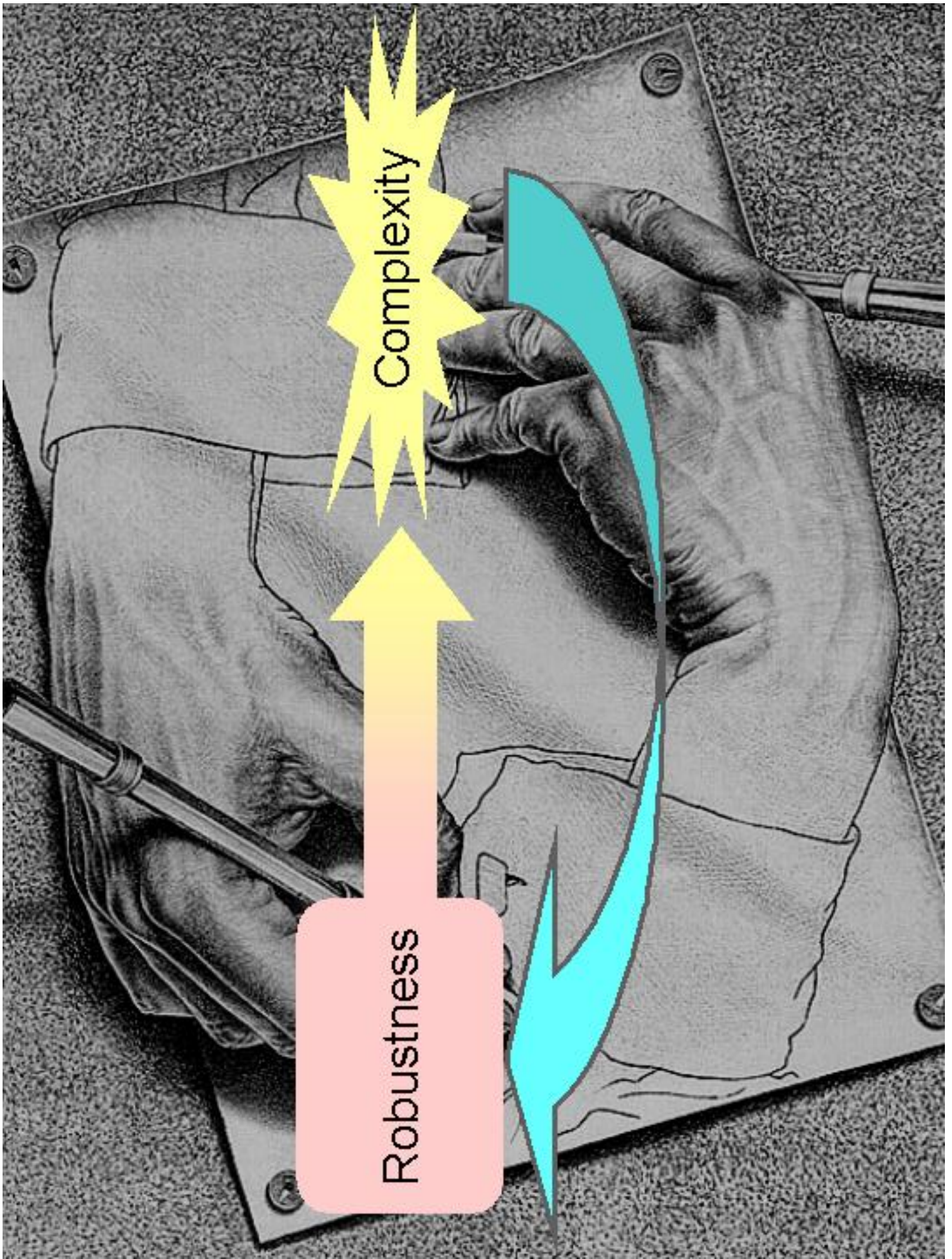
- " On website accessible from my homepage
- " <http://www.cds.caltech.edu/~doyle/Networks/>
- " Papers with minimal math
 - Chemotaxis, Heat shock in E. Coli
 - HOT and power laws (w/ Jean Carlson, UCSB)
 - Web & Internet traffic, protocols, future issues
- " Recommended books
 - **A course in Robust Control Theory**, Dullerud and Paganini, Springer
 - **Essentials of Robust Control**, Zhou, Prentice-Hall
 - **Cells, Embryos, and Evolution**, Gerhart and Kirschner
- " Thesis: Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization (Parrilo)

Biochemical Network: *E. Coli* Metabolism



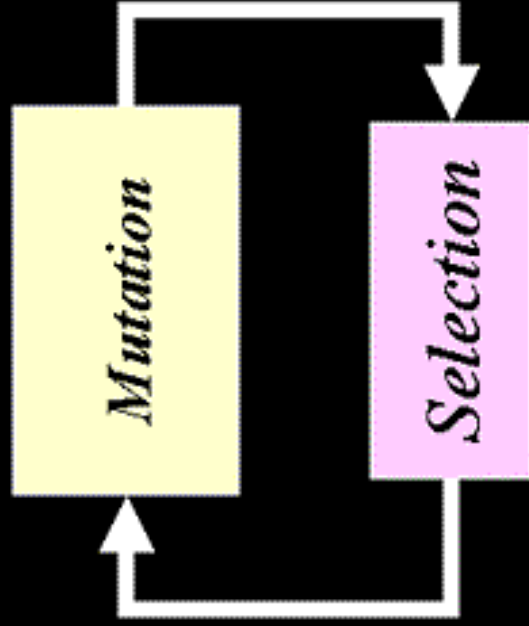
From Adam Arkin

* from: EcoCYC by Peter Karp

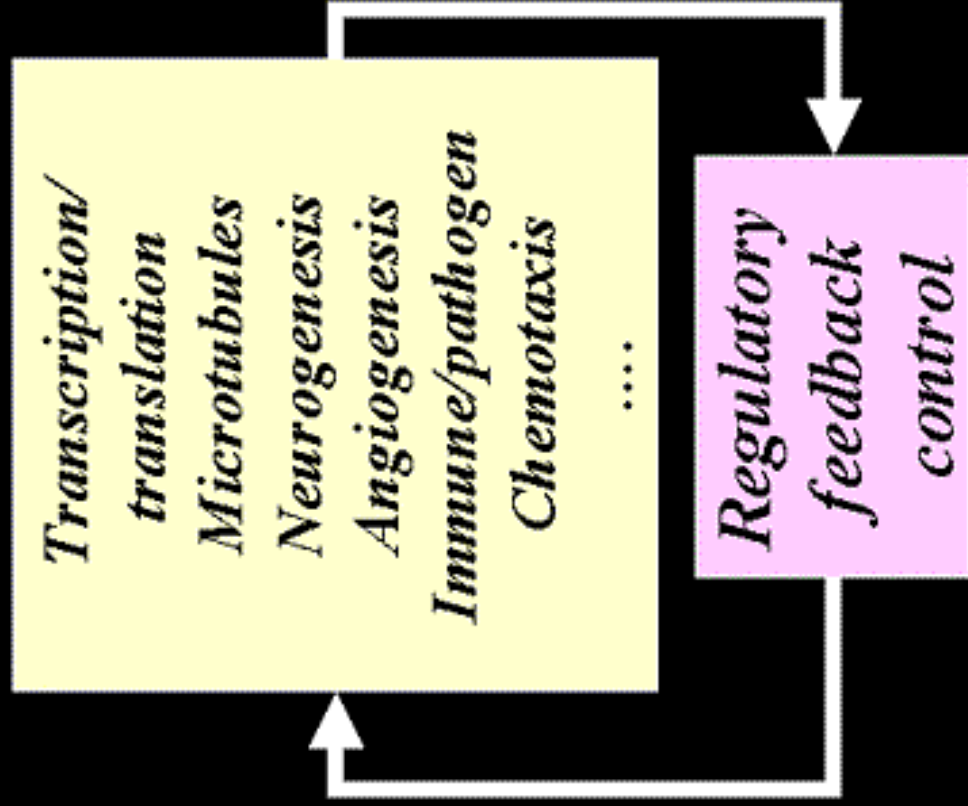


An apparent paradox

Component behavior seems to be *gratuitously* uncertain, the networks *gratuitously* complex, yet the systems have robust performance.

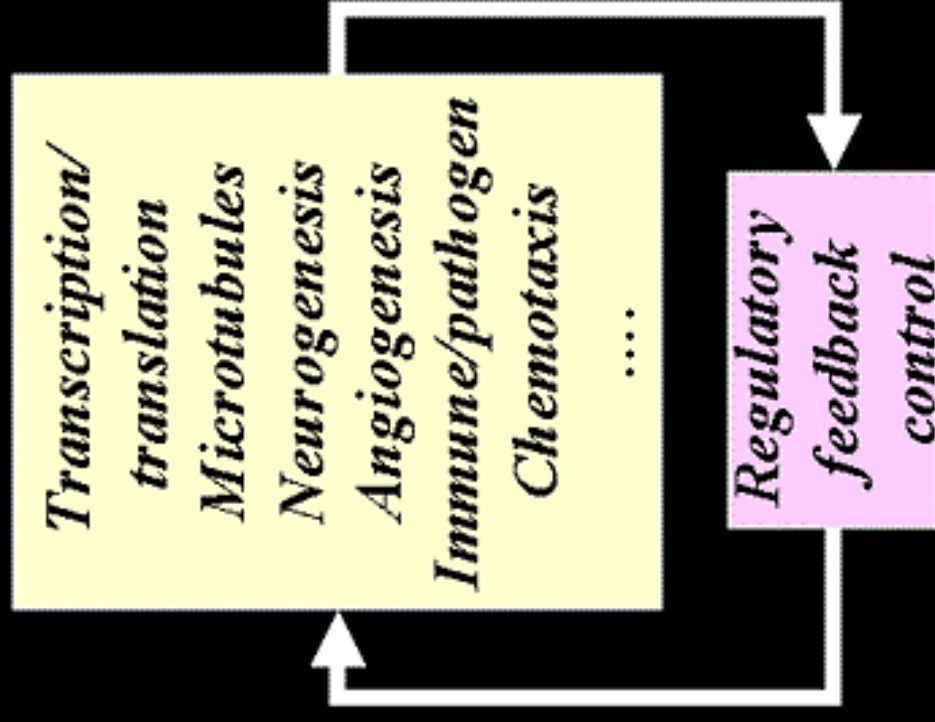


Darwinian evolution uses selection on random mutations to create complexity.



- Such feedback strategies appear throughout biology (and advanced technology).
- Gerhart and Kirschner (correctly) emphasize that this “exploratory” behavior is ubiquitous in biology...
- ... but claim it is rare in our machines.
- This is true of primitive, but not advanced, technologies.
- Robust control theory provides a clear explanation.

Component behavior seems to be *gratuitously* uncertain, yet the systems have robust performance.



Motivation

- Continuing themes already discussed in this program...
- Without extensive engineering theory and math, even “reverse engineering” complex engineering systems would be hopeless.
- Modeling and simulation alone is inadequate
- Why should biology be much easier?
- We would not expect to have much success reverse engineering this laptop with:
 - Reductionism: try to find the single transistor or group of transistors responsible for *this slide*
 - Emergence: “emerging” a random collection of silicon and metal

Engineering theory

- It turns out, that with respect to robustness and complexity, there is *too much* theory, not too little.
- Two great abstractions of the 20th Century:
 - Separate systems engineering into control, communications, and computing
 - Theory
 - Applications
 - Separate systems from physical substrate
- Facilitated massive, wildly successful, and explosive growth in both mathematical theory and technology...
- ... but creating a new Tower of Babel where even the experts do not read papers or understand systems outside their subspecialty.

Tower of Babel

- Issues for theory
 - Rigor
 - Relevance
 - Accessibility
- Spectacular success on the first two
- Little success on the last one
- Perhaps all three is impossible?
- (In contrast, there are whole research programs in “complex systems” devoted exclusively to accessibility. They have been relatively “popular,” but can be safely ignored in biology.)

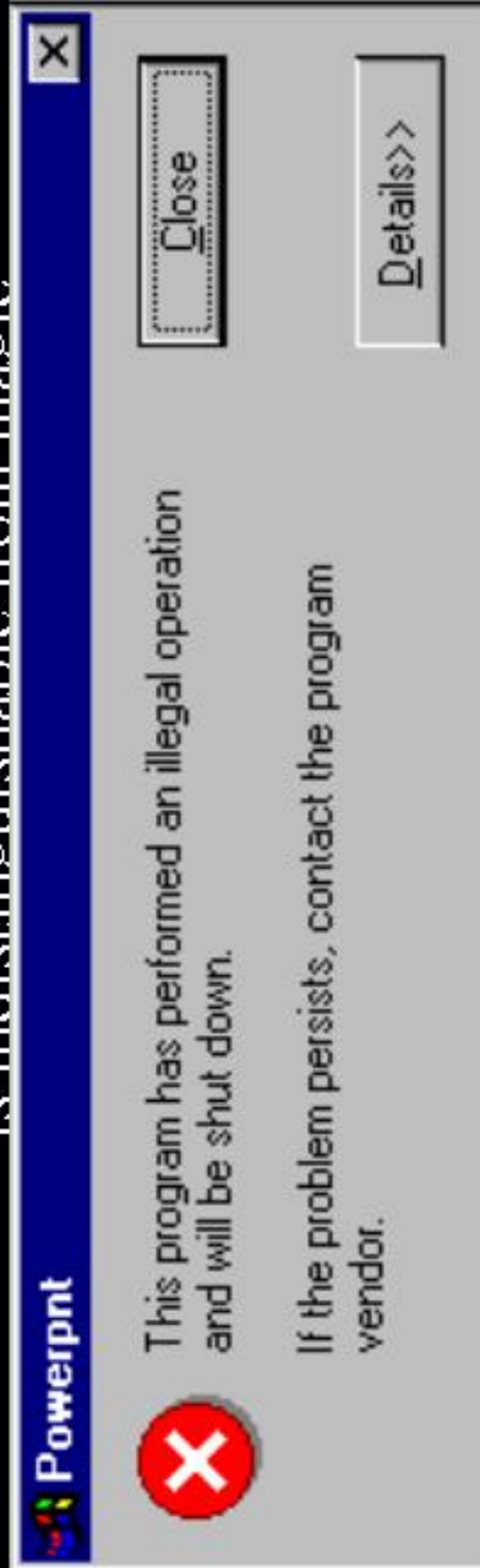
Today's goal

- Aim to tell you something not easily obtained elsewhere (papers online, other talks, texts, etc)
- Blending biology with math and engineering
- Introduce basic ideas about robustness and complexity
- Minimal math details, but still suggestive of what is possible
- Hopefully familiar (but unconventional) example systems, not requiring specialized expertise
 - Biology: Heat shock, chemotaxis
 - Engineering: Cars and planes

Caveats

- The “real thing” is much more complicated
- I’m not a biologist (and I’m not *really* an engineer)
- Perhaps any accessible “simple story” is necessarily very misleading

“Any sufficiently advanced technology
is indistinguishable from magic.”



“Any sufficiently advanced technology
is indistinguishable from magic.”

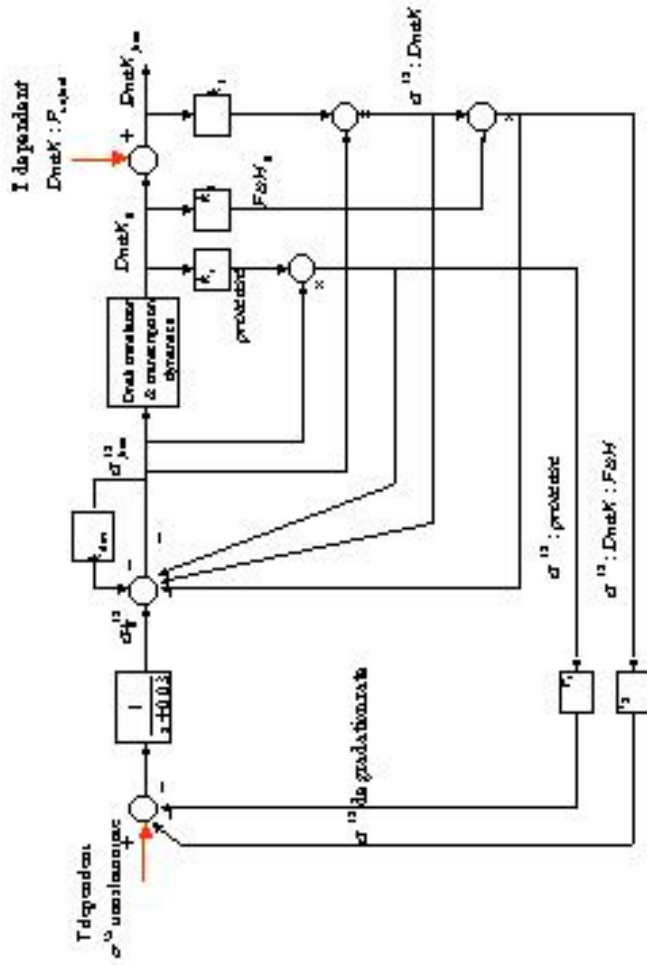
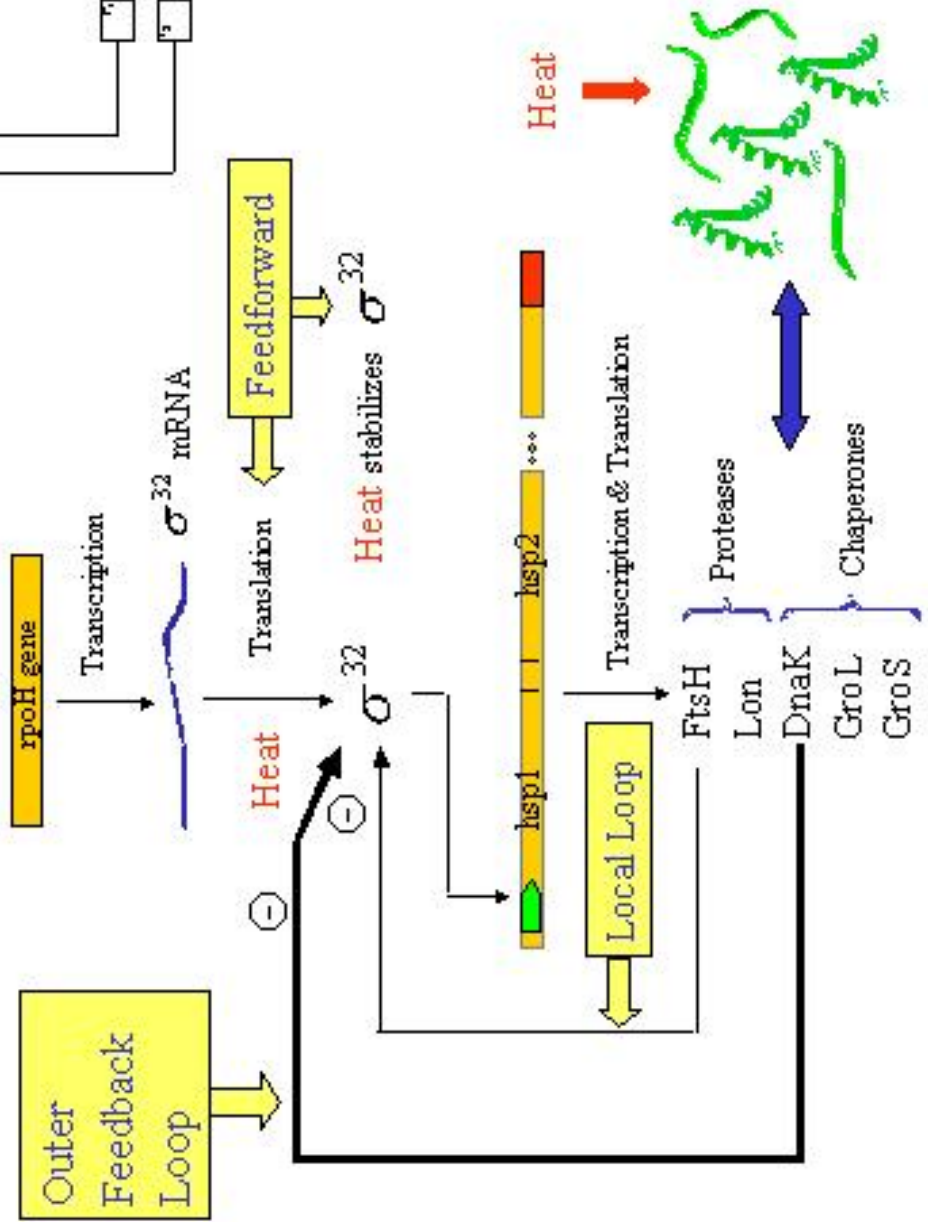
Arthur C. Clarke

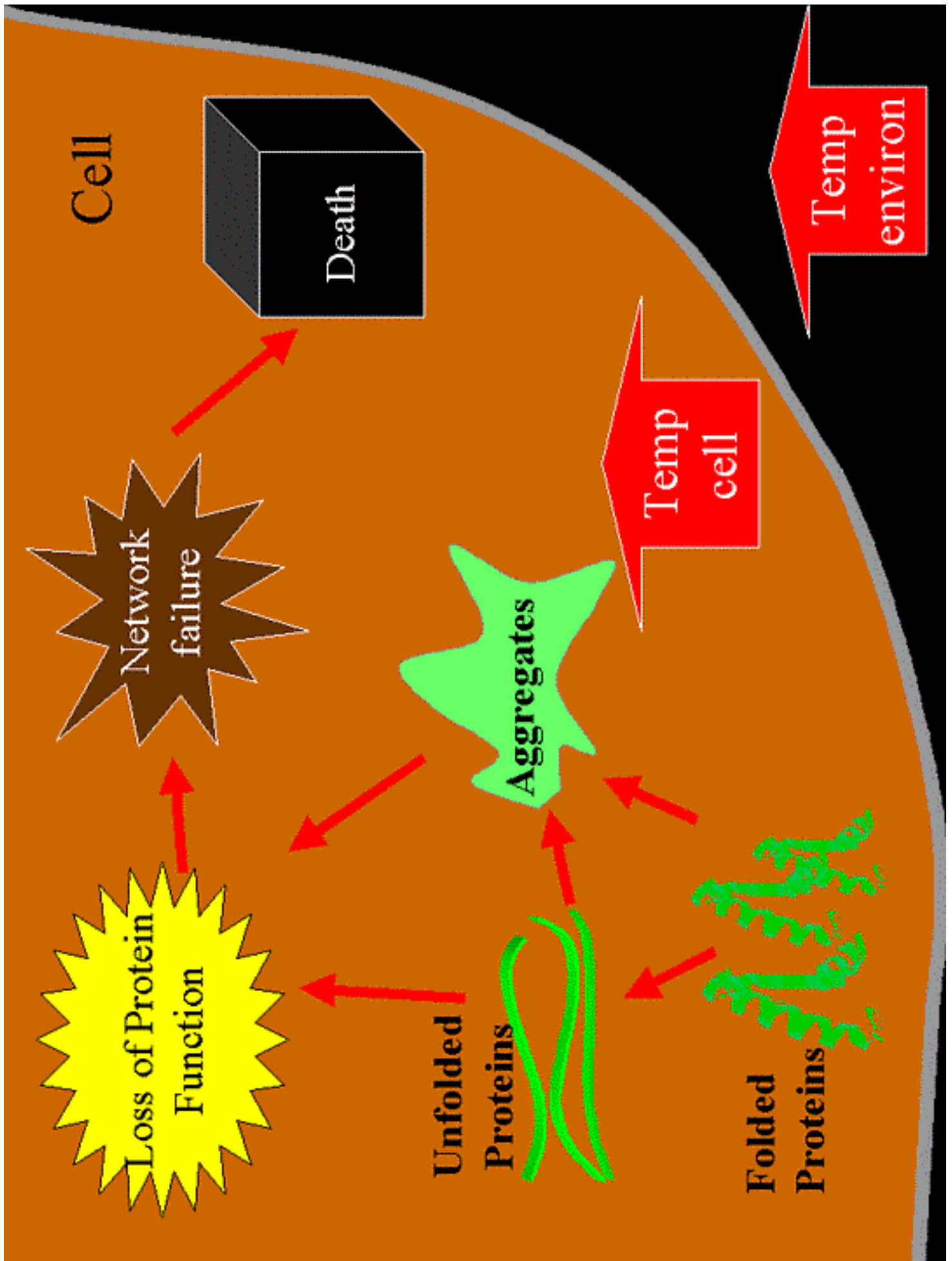
“Those who say do not know,
those who know do not say.”

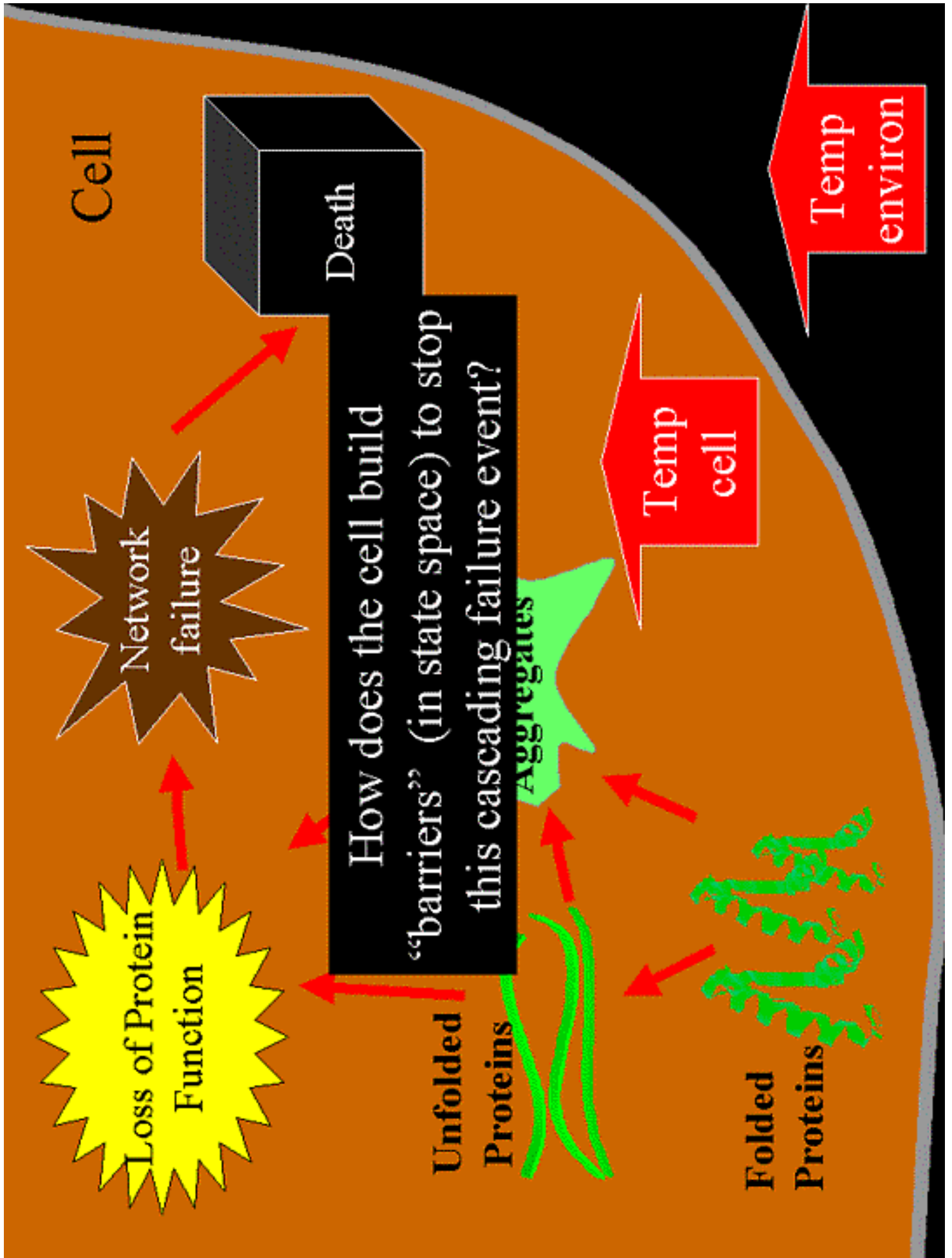
Zen saying

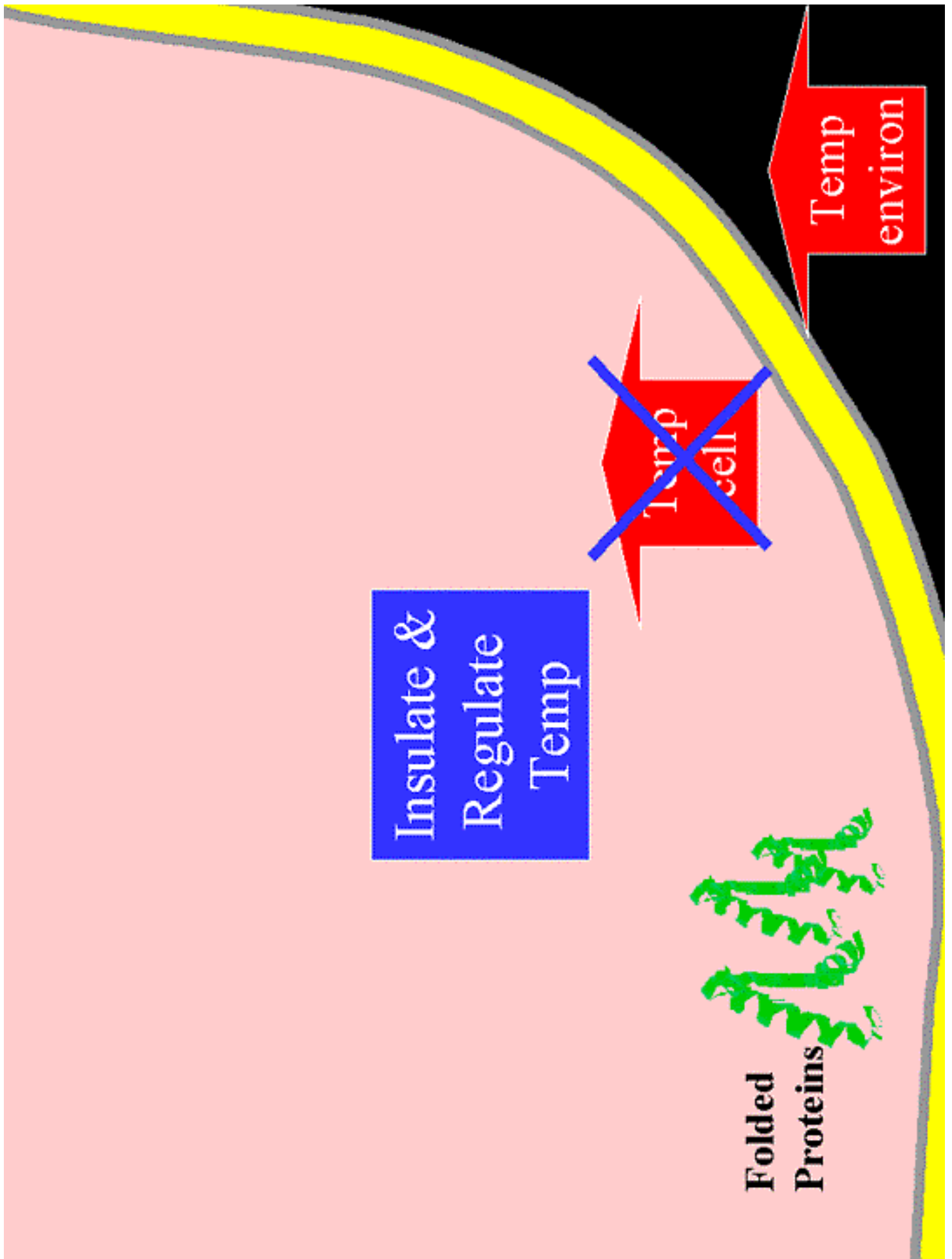
E. Coli Heat Shock

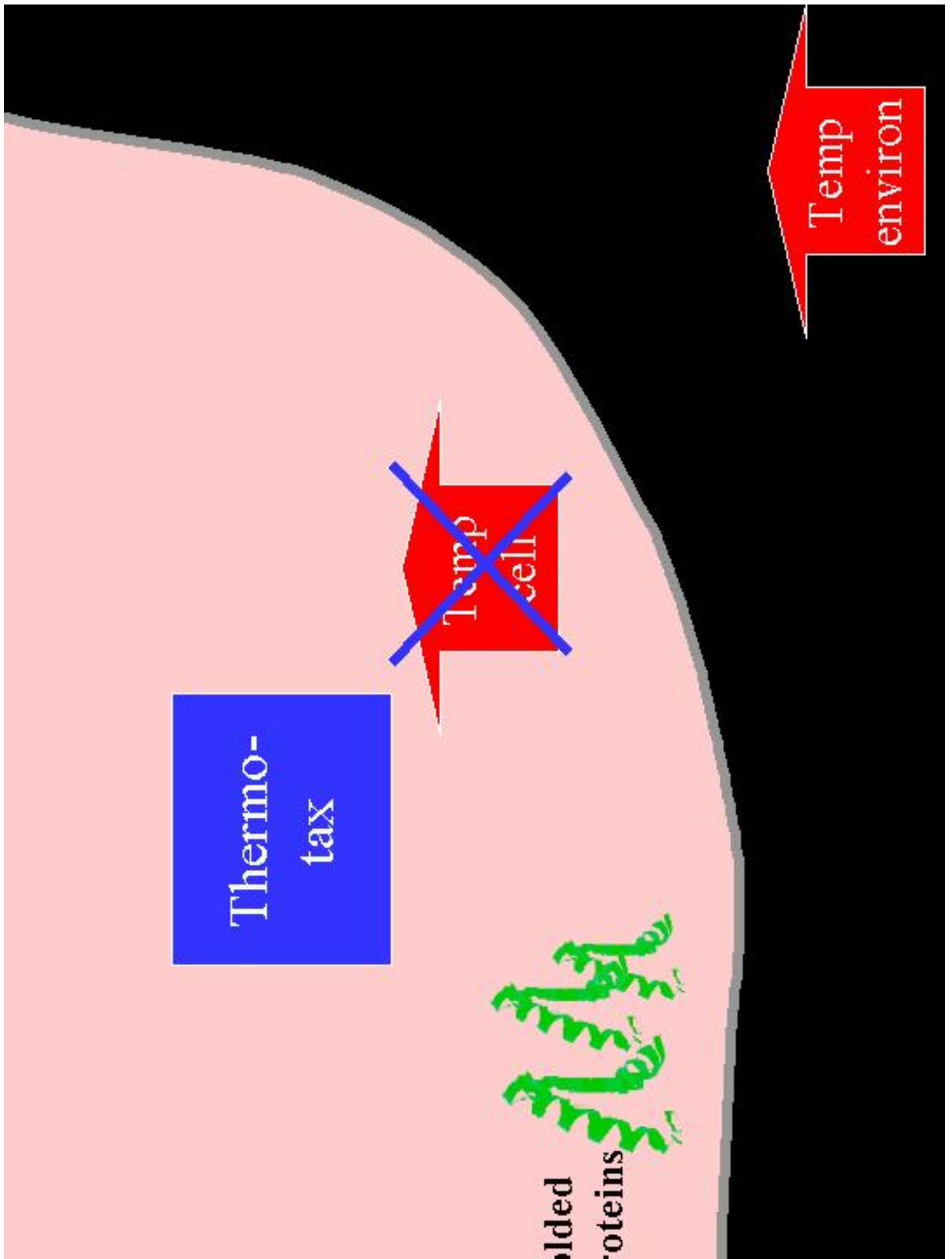
(with Kurata, El-Samad, Khammash, Yi)

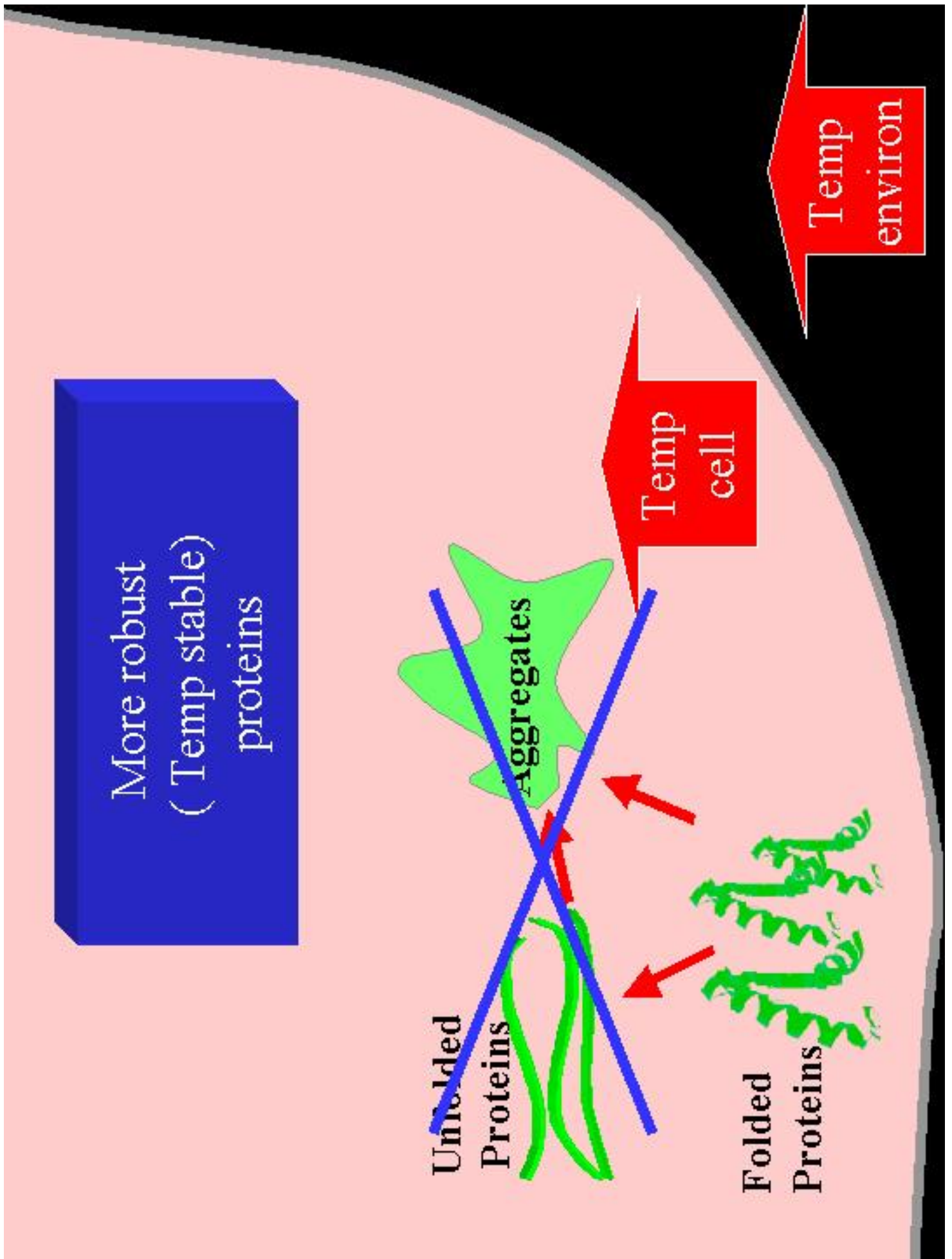




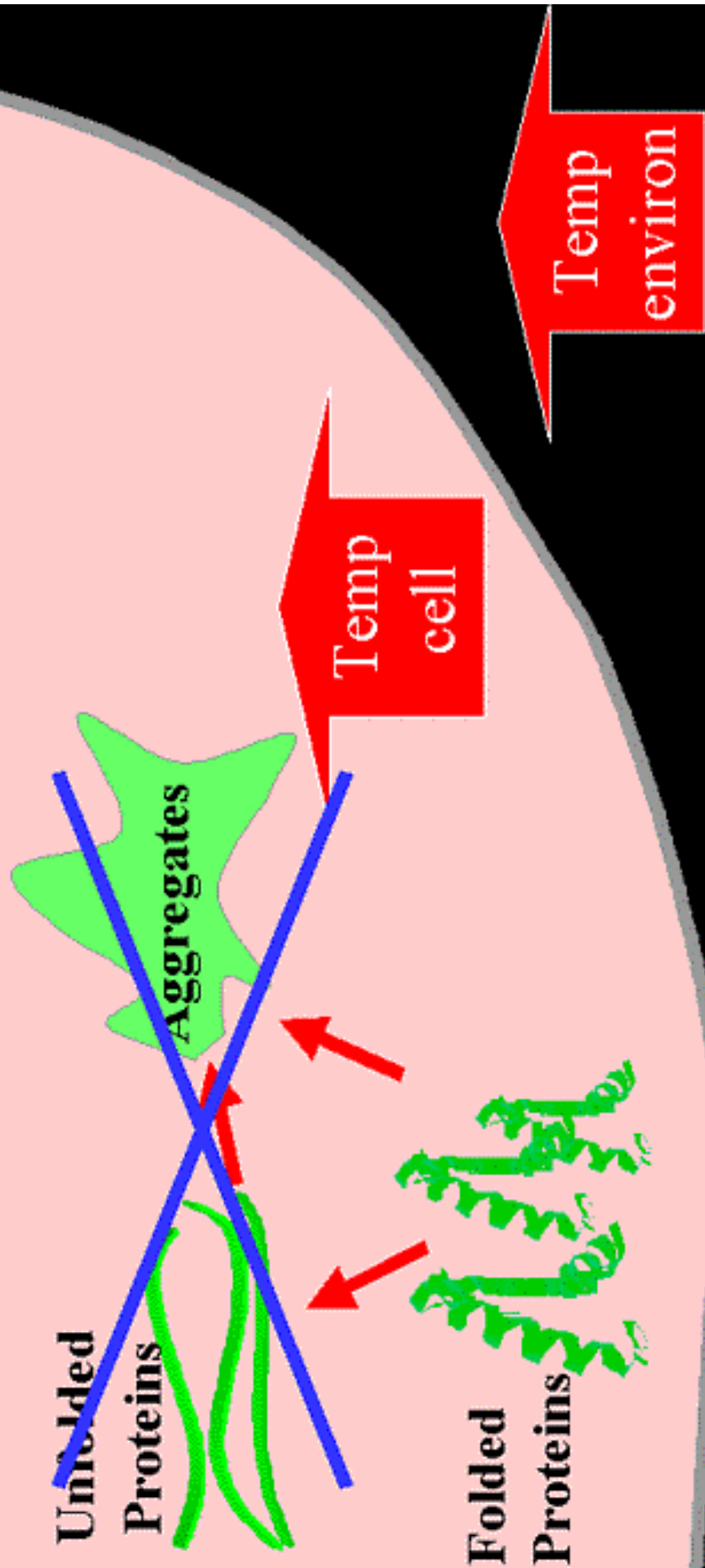


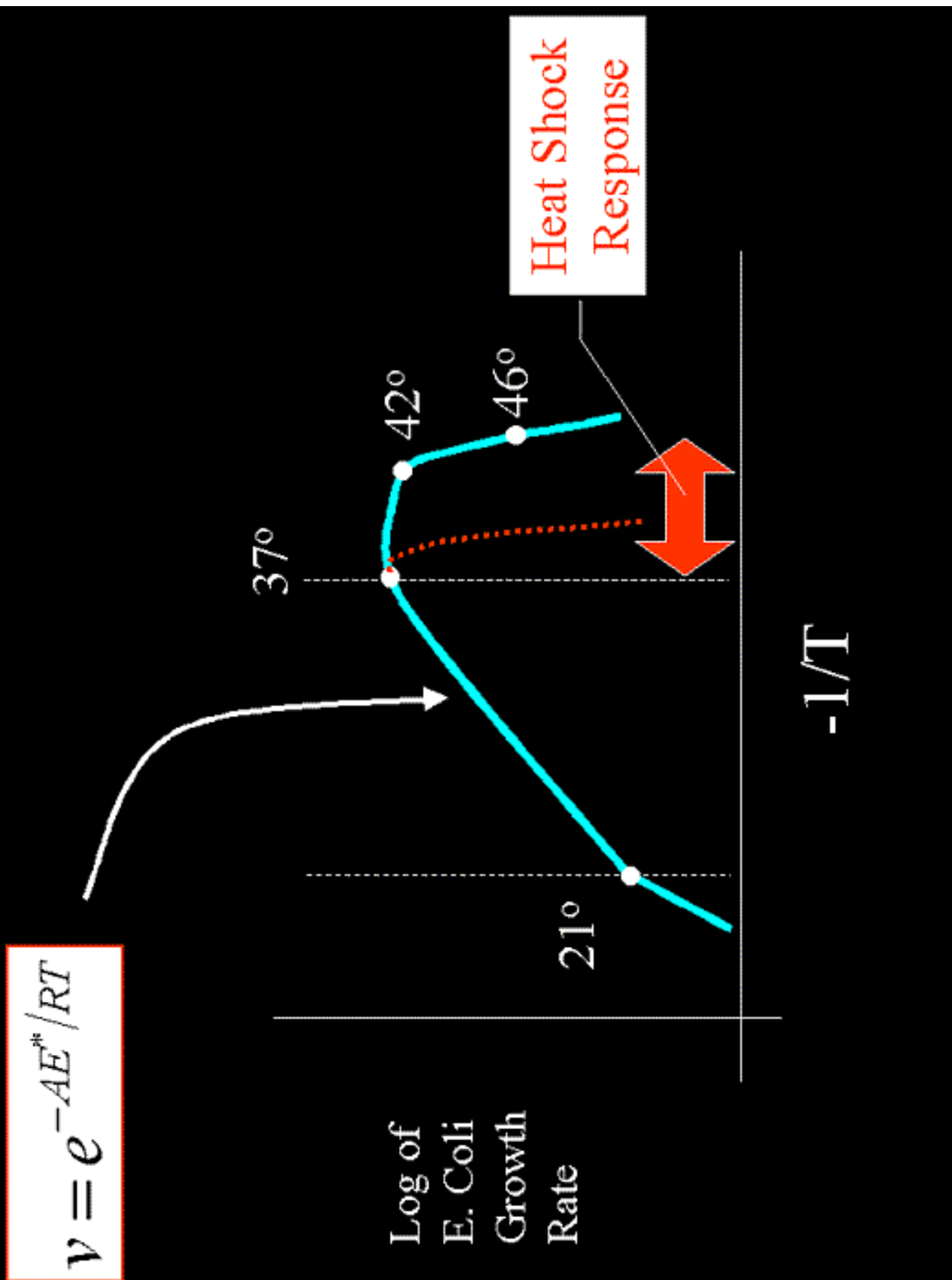




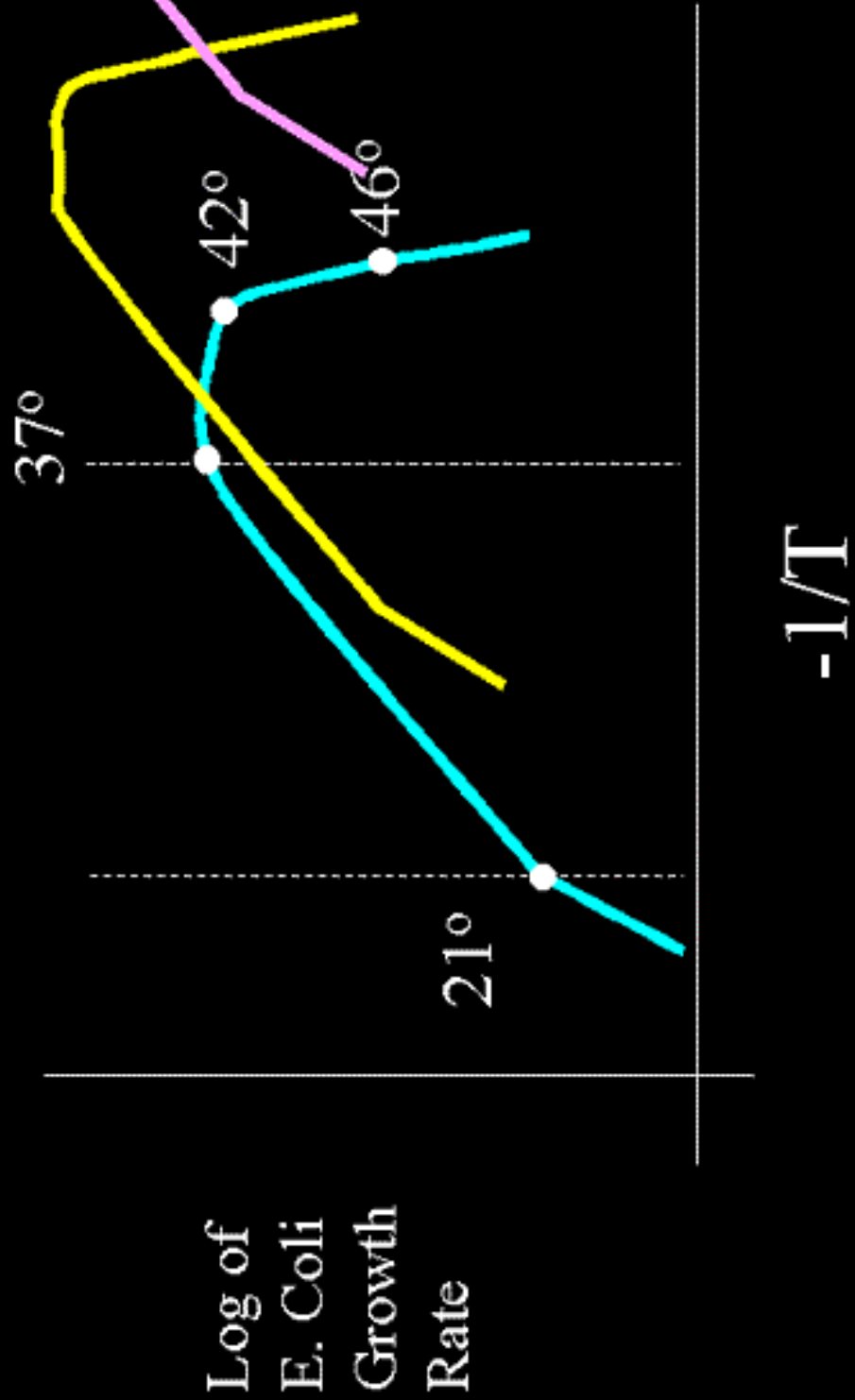


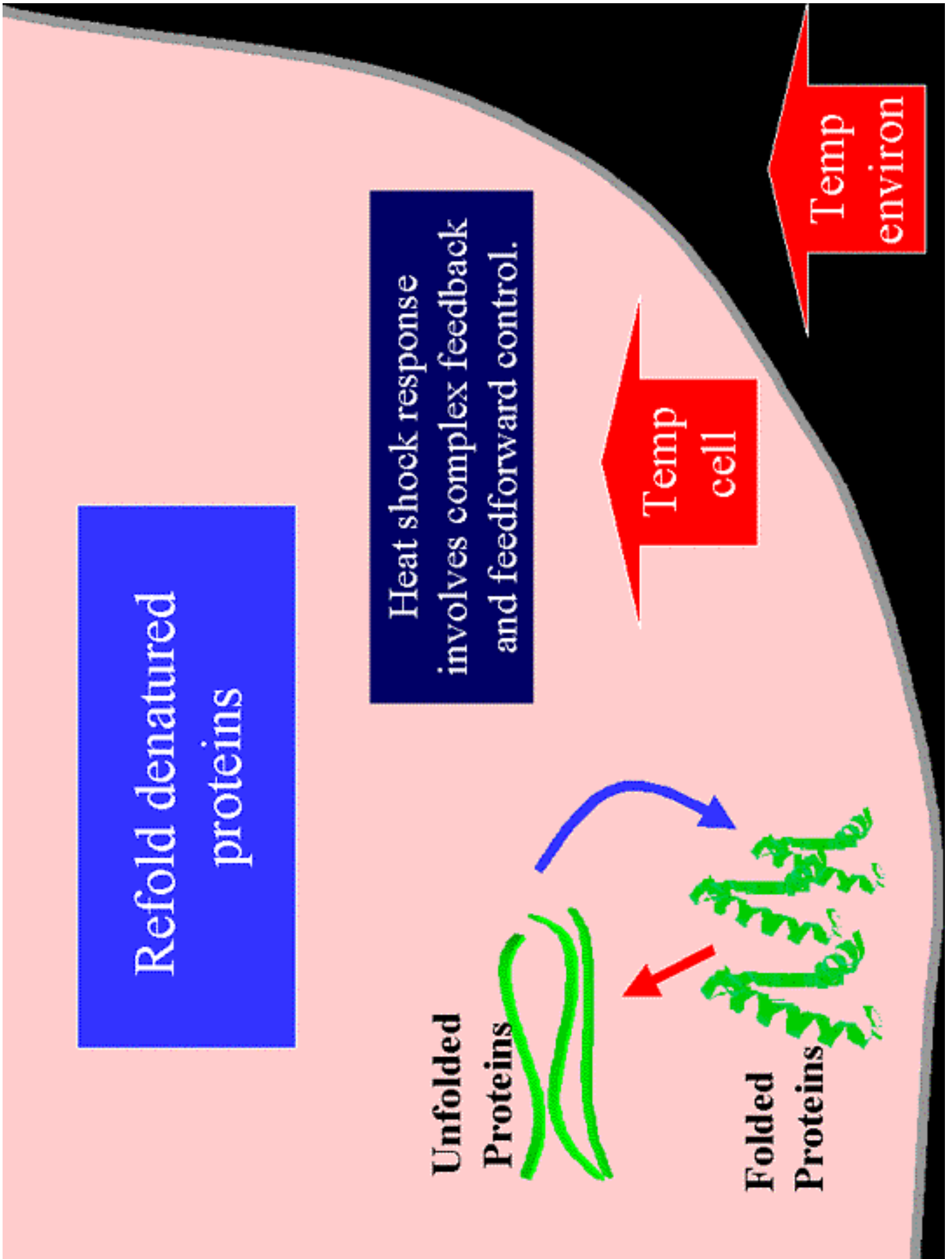
- Key proteins can have multiple (allelic or paralogous) variants
- Allelic variants allow populations to adapt
- Regulated multiple gene loci allow individuals to adapt





Robustness/performance tradeoff?





Refold denatured proteins

Heat shock response involves complex feedback and feedforward control.

Temp cell

Temp environ

Unfolded Proteins

Proteins

Folded Proteins

Proteins

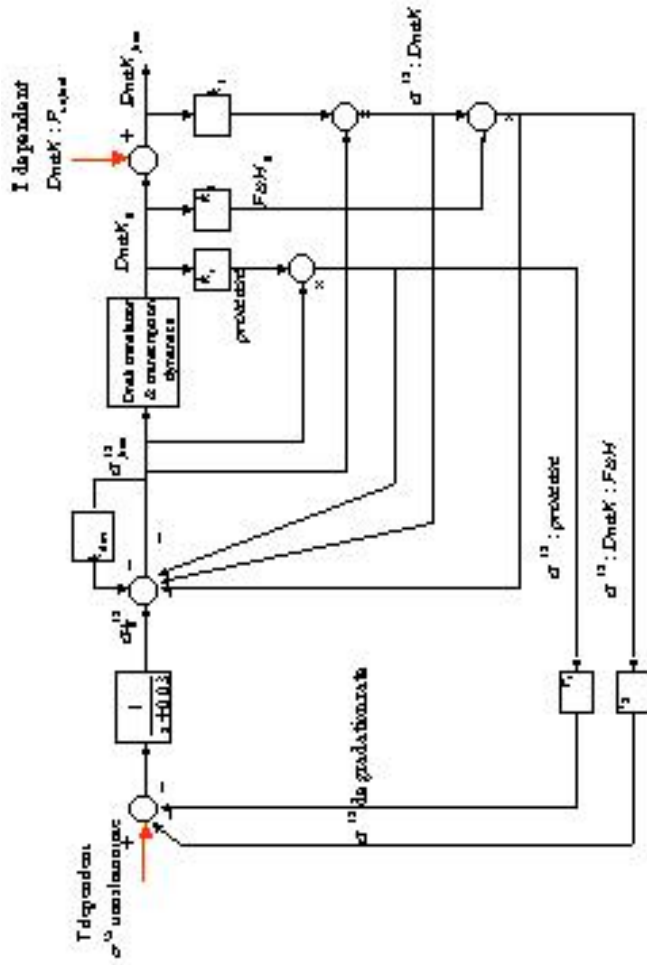
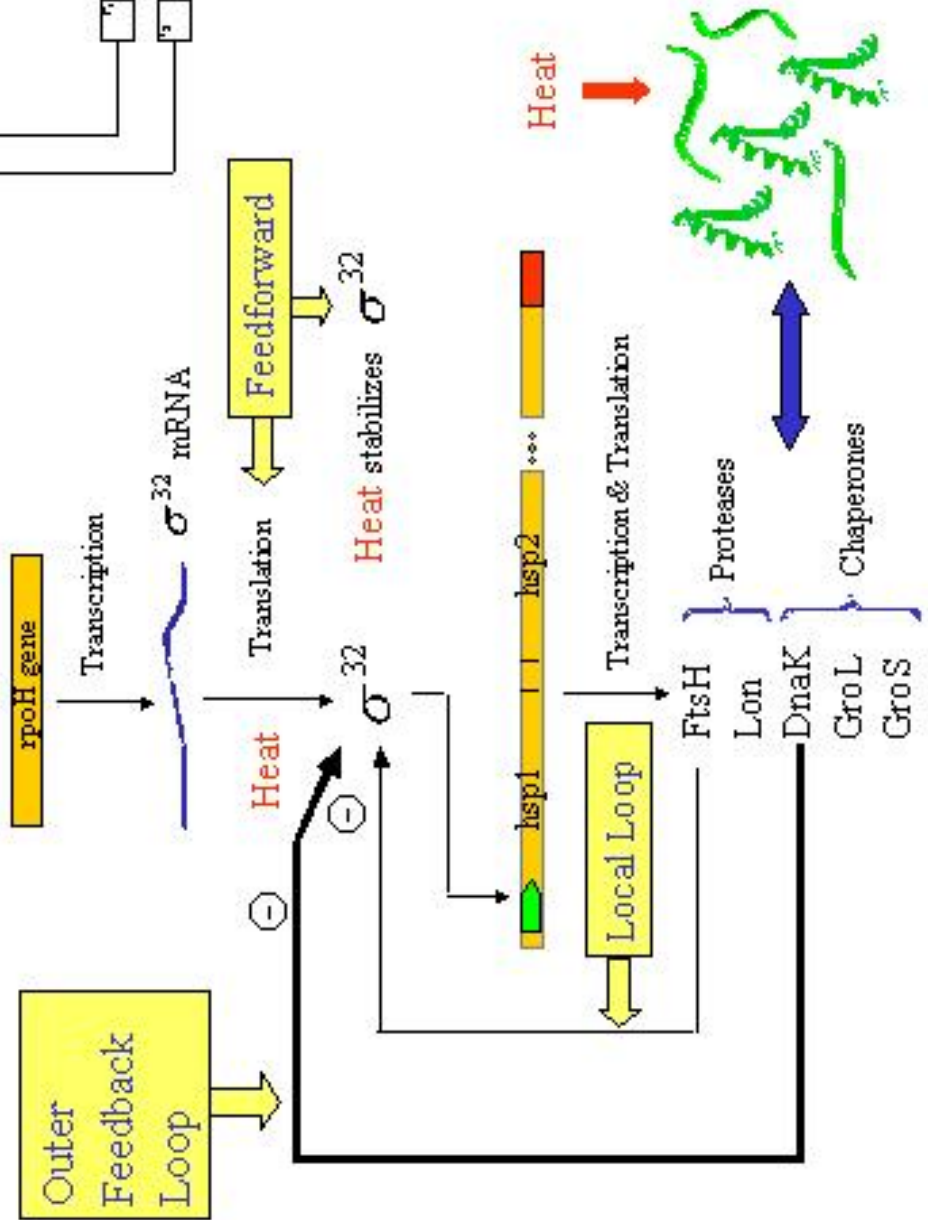
Alternative strategies

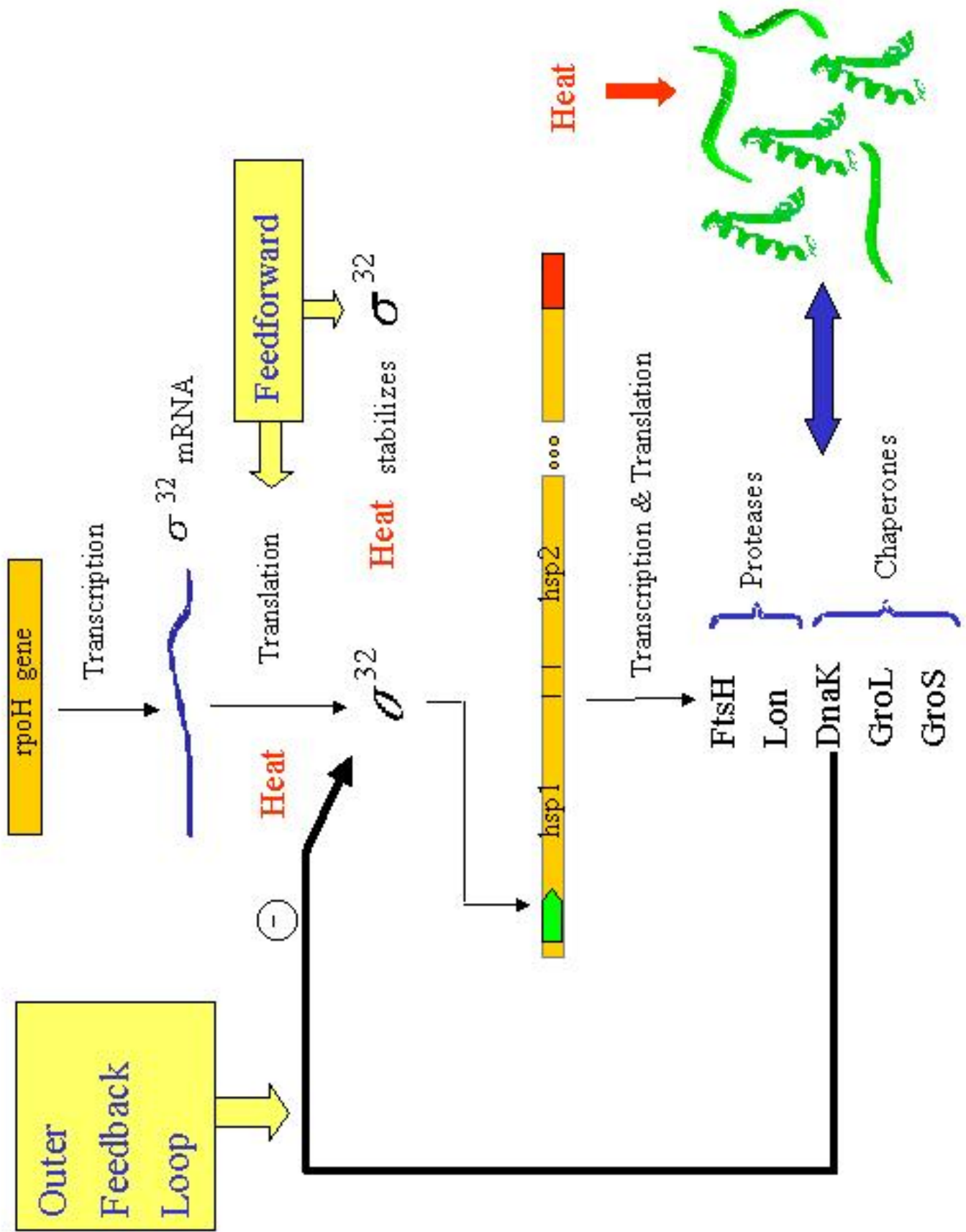
Why does biology (and advanced technology) overwhelmingly opt for the complex control systems instead of just robust components?

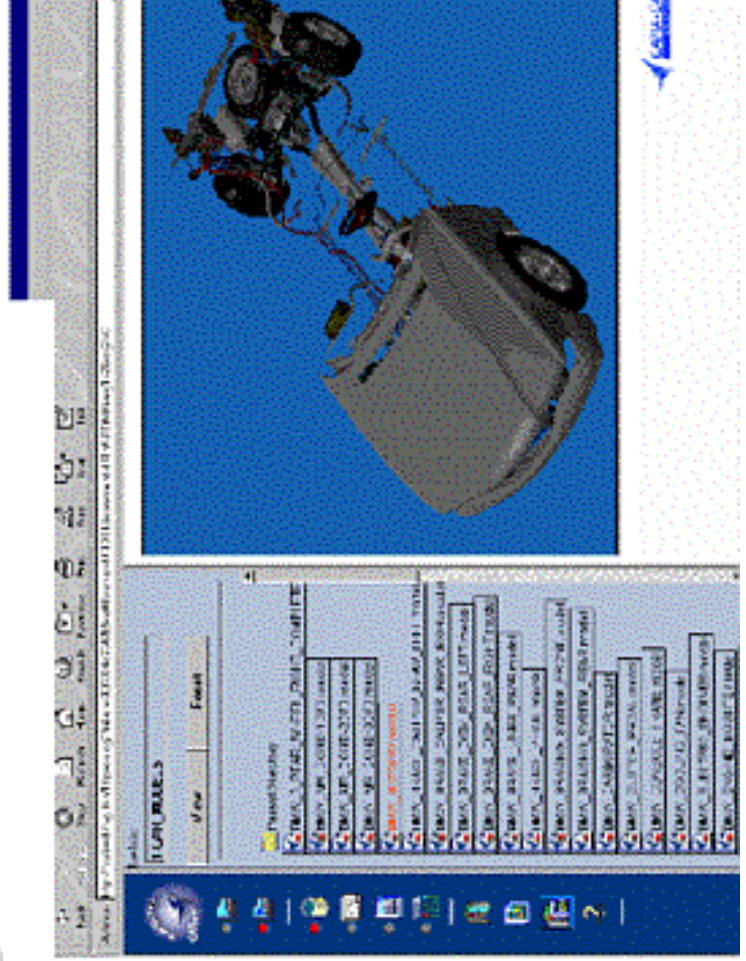
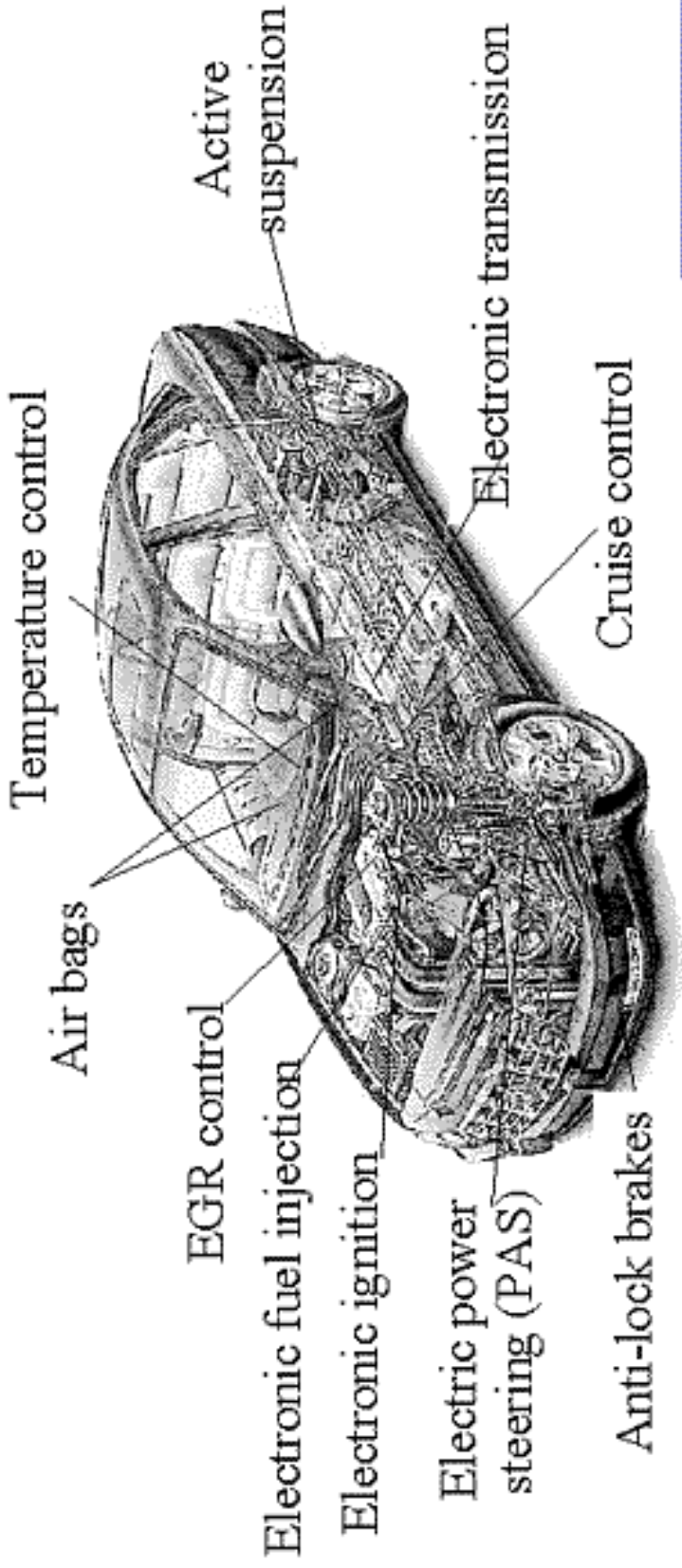
- Robust proteins
 - Temperature stability
 - Allelic variants
 - Paralogous isozymes
- Regulate temperature
- Thermotax
- Heat shock response
 - Up regulate chaperones and proteases
 - Refold or degraded denatured proteins

E. Coli Heat Shock

(with Kurata, El-Samad, Khammash, Yi)



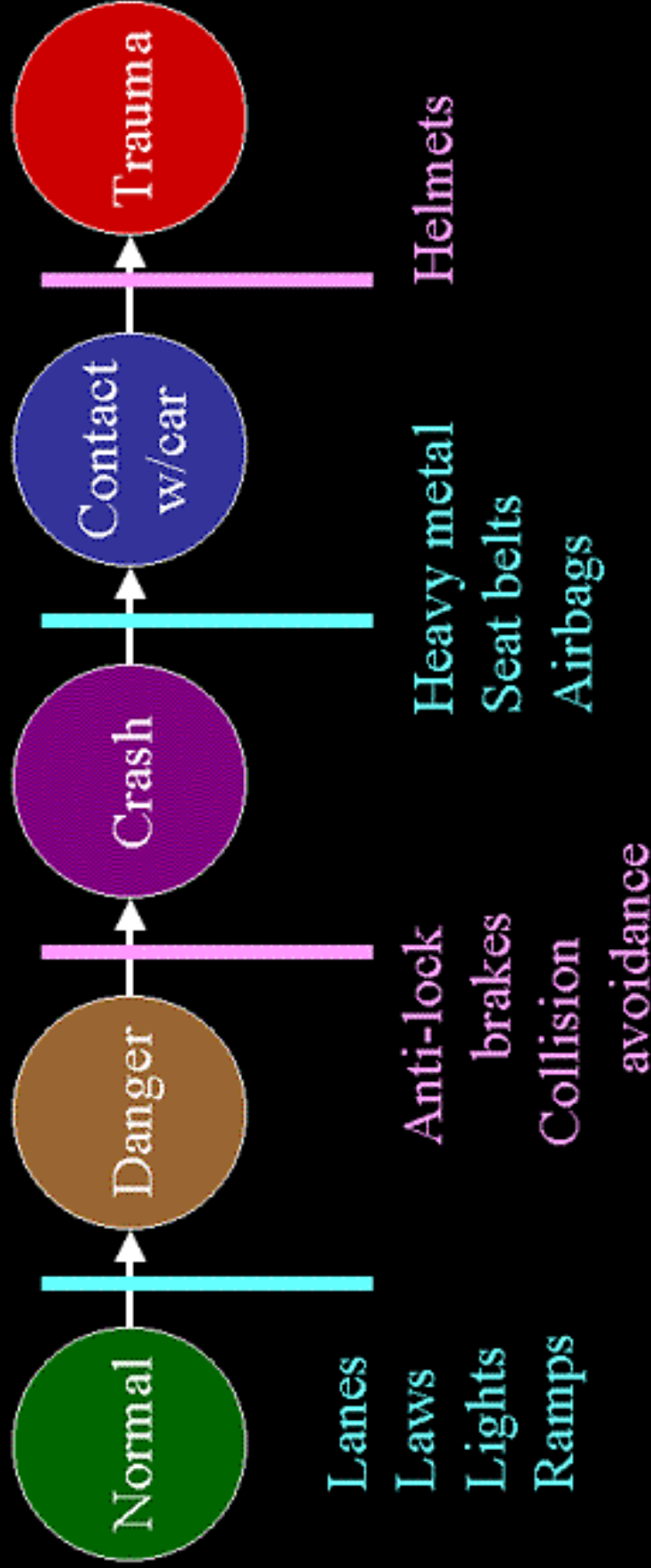




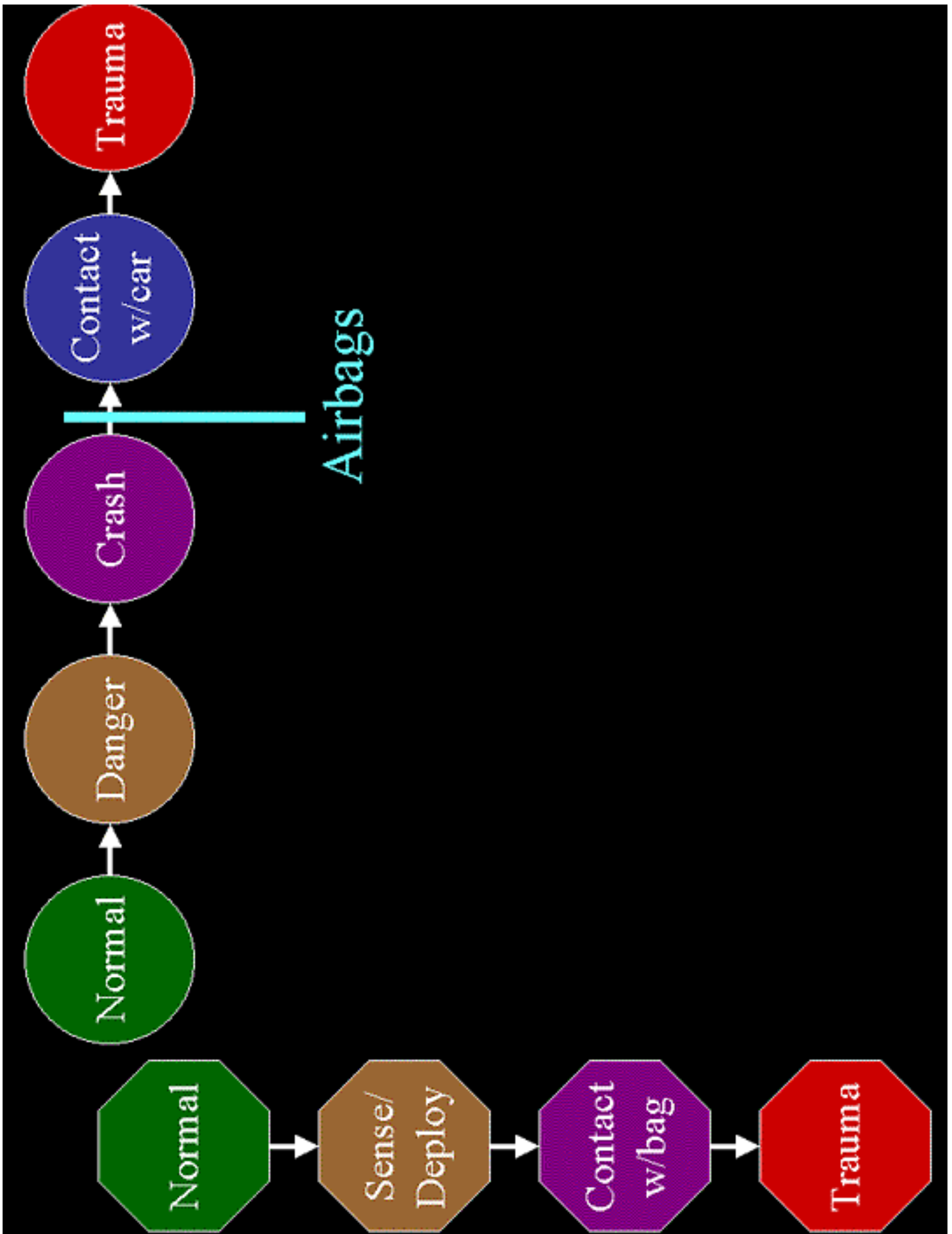
In development:

- drive-by-wire
- steering/traction control
- collision avoidance

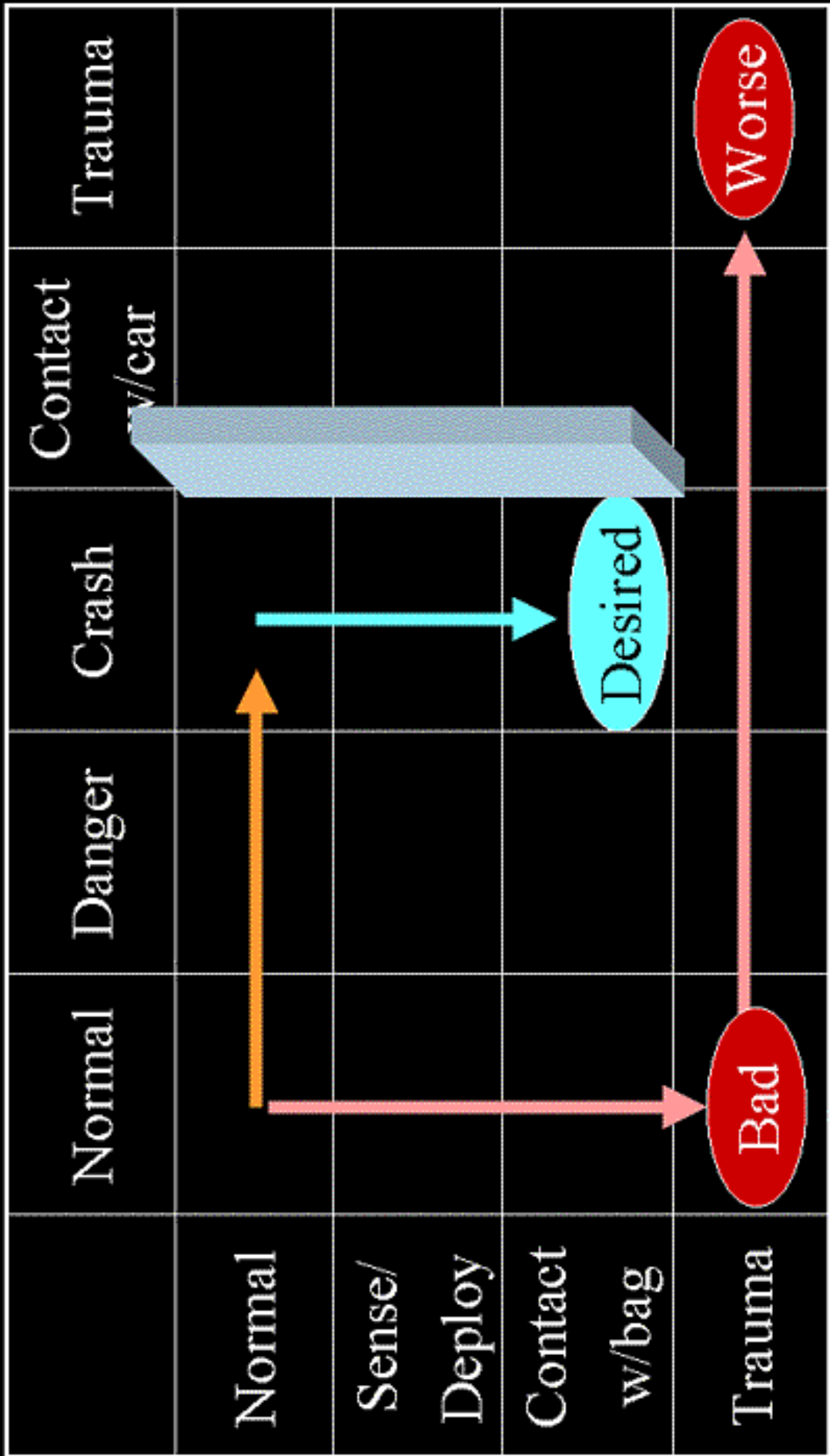
Cascading events in car crashes



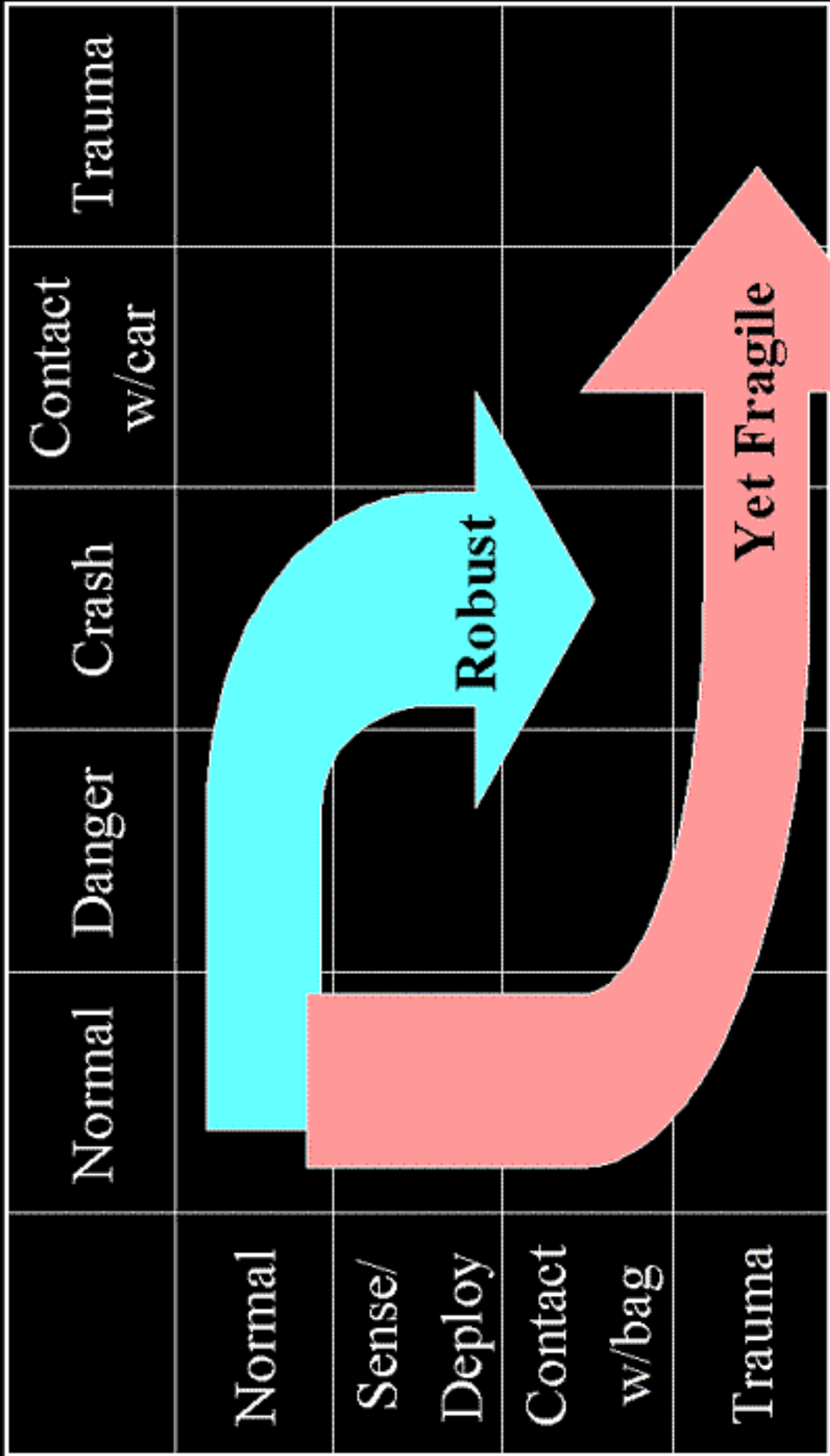
Barriers in state space



Full state space



Full state space



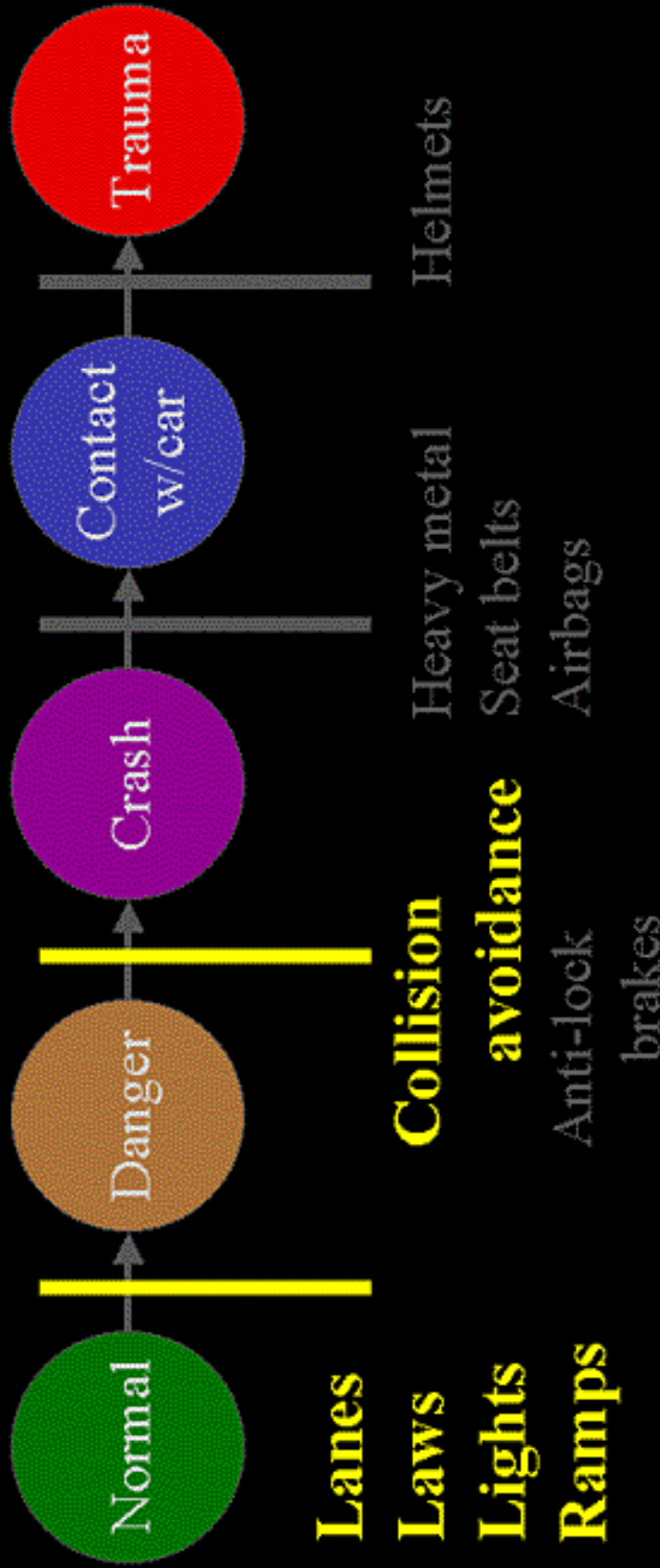


“Robust, yet fragile”

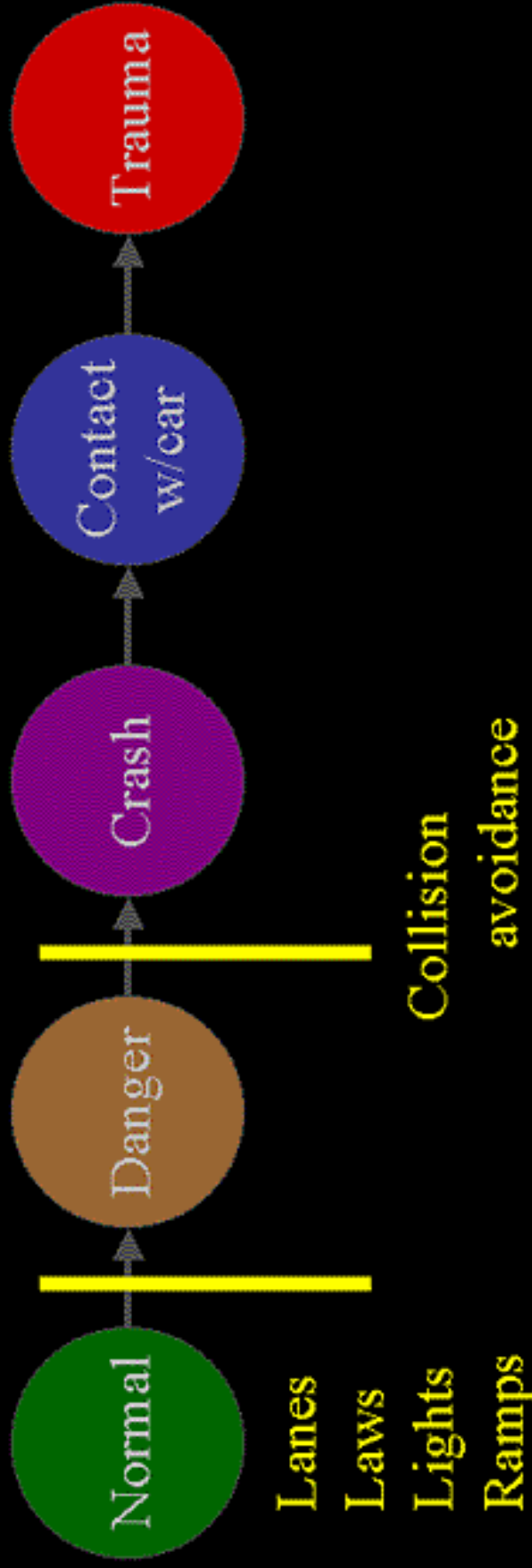
- **Robust** to uncertainties
 - that are common,
 - the system was designed for, or
 - has evolved to handle,
- **...yet fragile** otherwise
- This is *the* most important feature of complex systems (the essence of HOT).



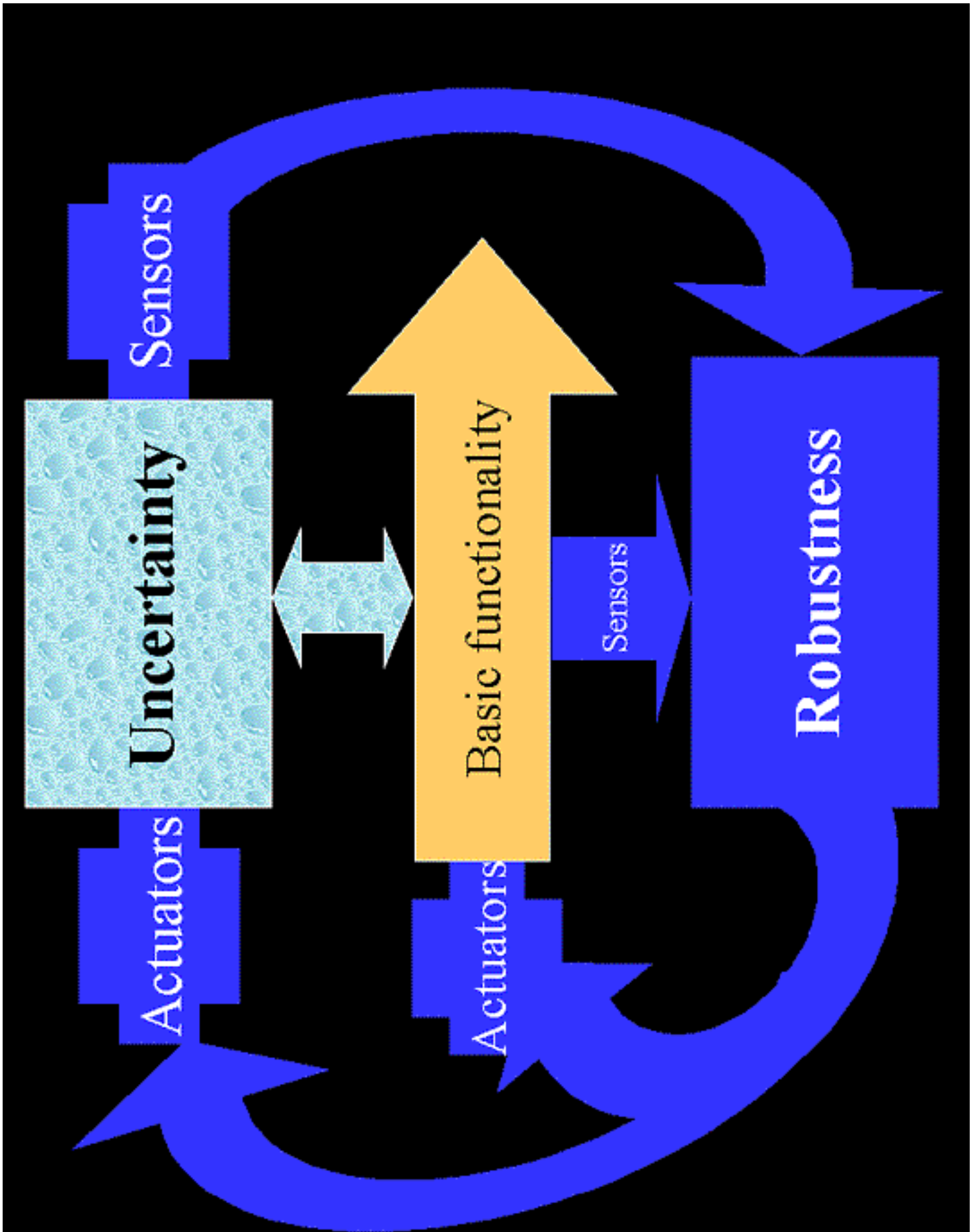
Humans supply most feedback control

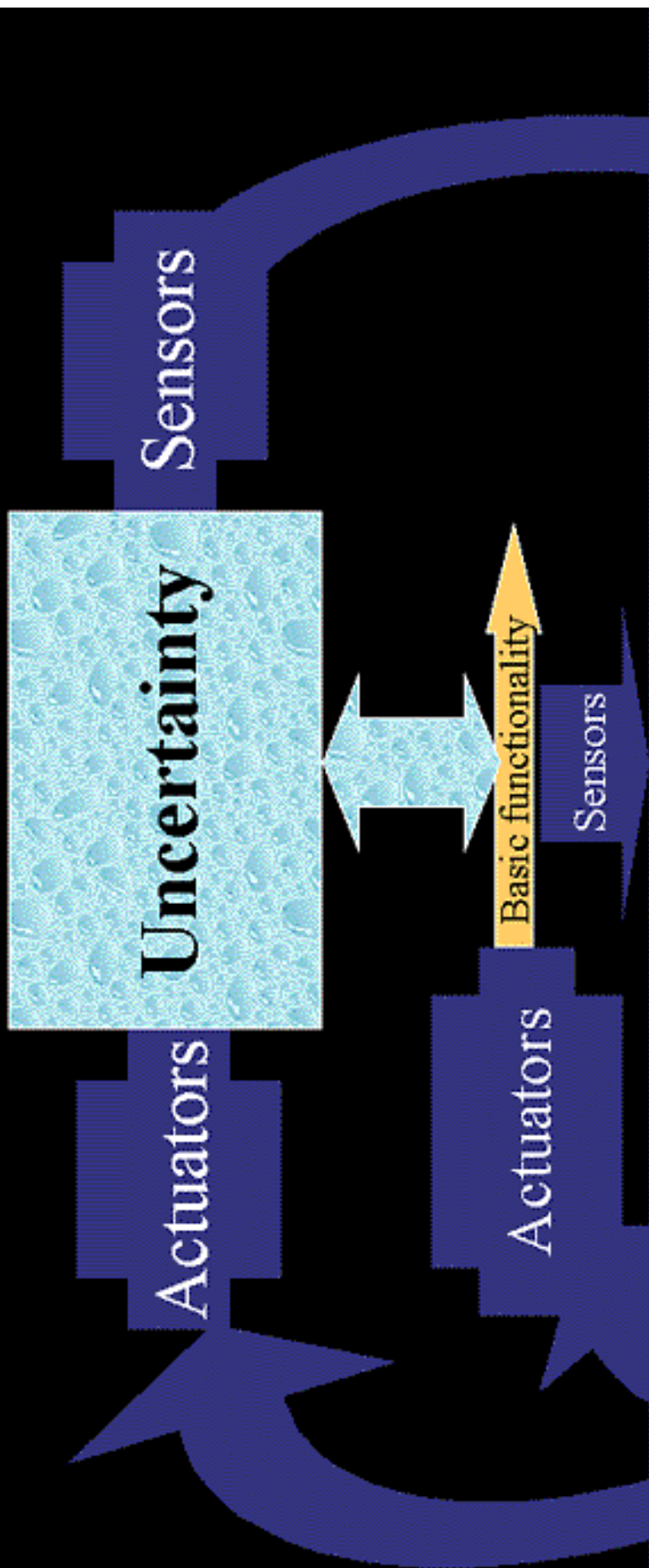


Fully automated systems?



- Internally: unimaginably more complex.
- Superficially: much simpler?





Complexity is dominated by

Robustness

(through regulatory feedback networks)

Sensors

Uncertainty

Actuators

Basic functionality

Sensors

Actuators

But scientific research has ignored almost all real complexity.

Sensors

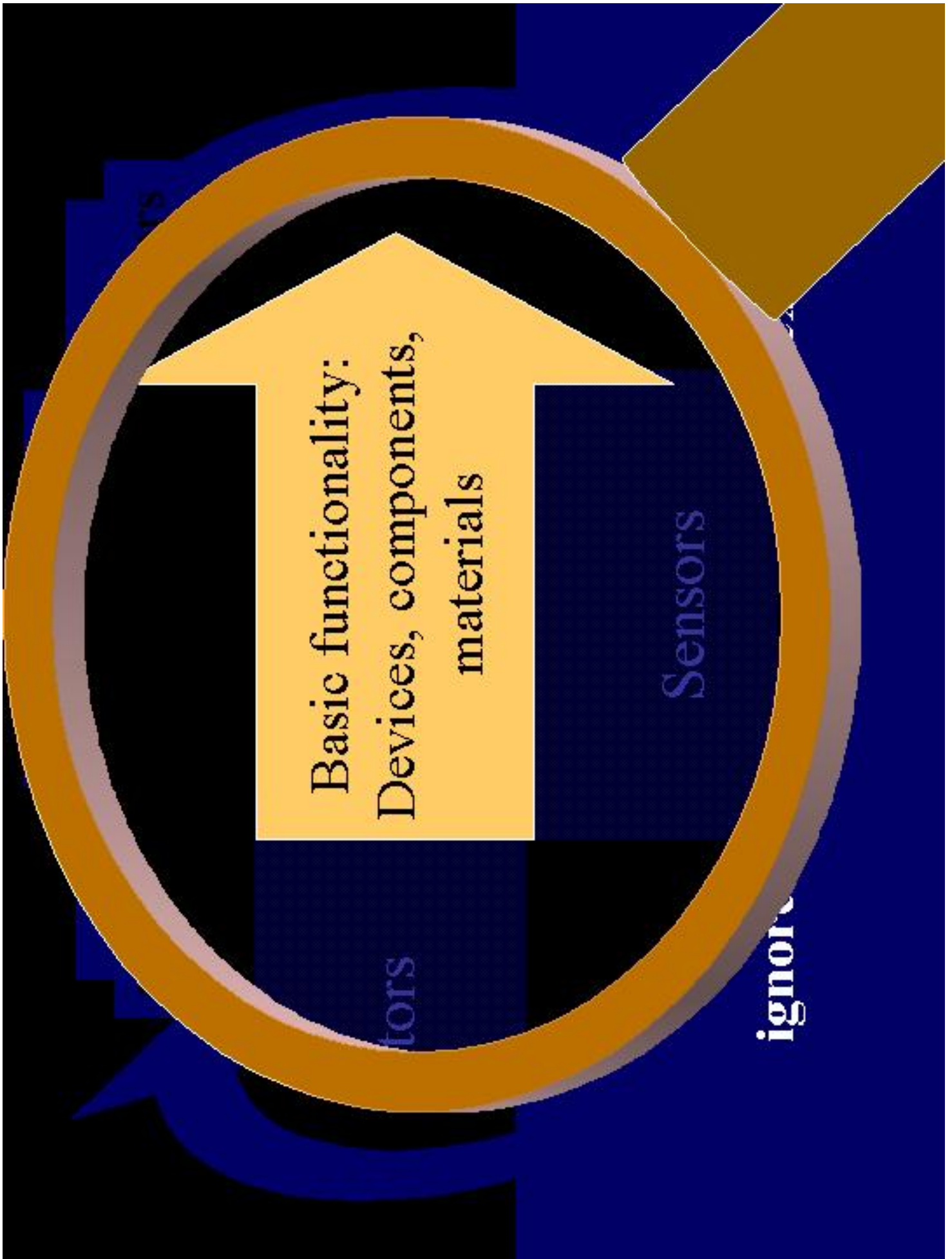
Uncertainty

Actuators

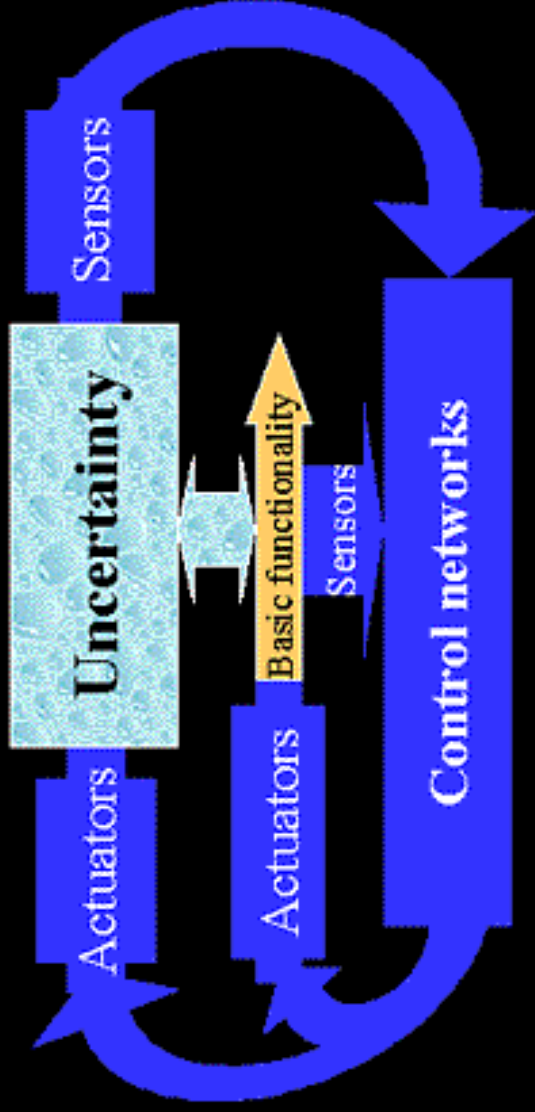
Actuators



But scientific research has ignored almost all real complexity.



Control, communications, computing



- Sense data
- Communications:
 - Information: Focus on what is surprising in data
 - Reliably store or transmit information
- Control
 - Extract what is *useful* (not merely surprising)
 - Compute decisions from *useful* information
 - Take appropriate action

Theoretical foundations

- Control theory: feedback, optimization, games
- Information theory: source and channel coding
- Computational complexity: decidability, P-NP-coNP-...
- Dynamical systems: dynamics, bifurcation, chaos
- Statistical physics: phase transitions, critical phenomena, multiscale physics
- These are largely fragmented within isolated technical disciplines.
- Unified theory would be both intellectually satisfying and of enormous practical value.



Complexity and robustness



- Complexity *phenotype* : robust, yet fragile
- Complexity *genotype*: internally complicated
- New theoretical framework: HOT (Highly optimized tolerance, with Jean Carlson, Physics, UCSB)
- Applies to biological and technological systems
 - Pre-technology: simple tools
 - Primitive technologies use simple strategies to build fragile machines from precision parts.
 - Advanced technologies use complicated architectures to create robust systems from sloppy components ...
 - ... but are also vulnerable to cascading failures ...

Robust, yet fragile phenotype

- *Robust* to large variations in environment and component parts (reliable, insensitive, resilient, evolvable, simple, scaleable, verifiable, ...)
- *Fragile*, often catastrophically so, to cascading failures events (sensitive, brittle,...)
- Cascading failures can be initiated by small perturbations (Cryptic mutations, viruses and other infectious agents, exotic species, ...)
- There is a tradeoff between
 - ideal or nominal performance (no uncertainty)
 - robust performance (with uncertainty)
- Greater “pheno-complexity”= more extreme robust, yet fragile

Robust, yet fragile phenotype

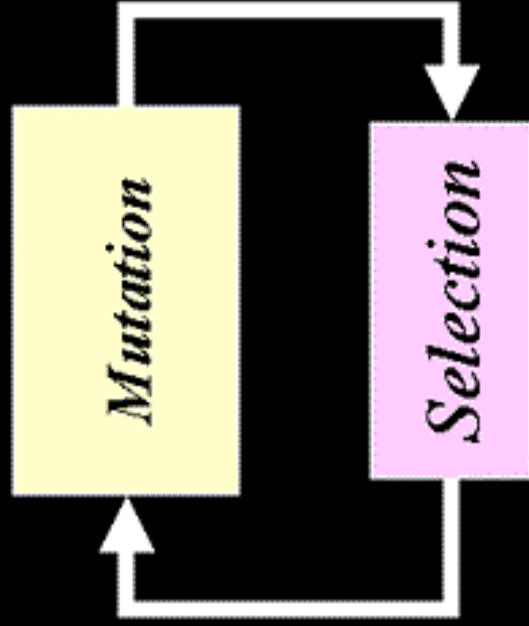
- Robustness is not just to genetic variation, but includes perturbations to both components and environment
- Cascading failures can be initiated by small perturbations (Cryptic mutations, viruses and other infectious agents, exotic species, ...) or large
- In many complex systems, the size of cascading failure events are often unrelated to the size of the initiating perturbations
- Fragility is most interesting when it does not arise because of large perturbations, but catastrophic responses to small variations

Complicated genotype

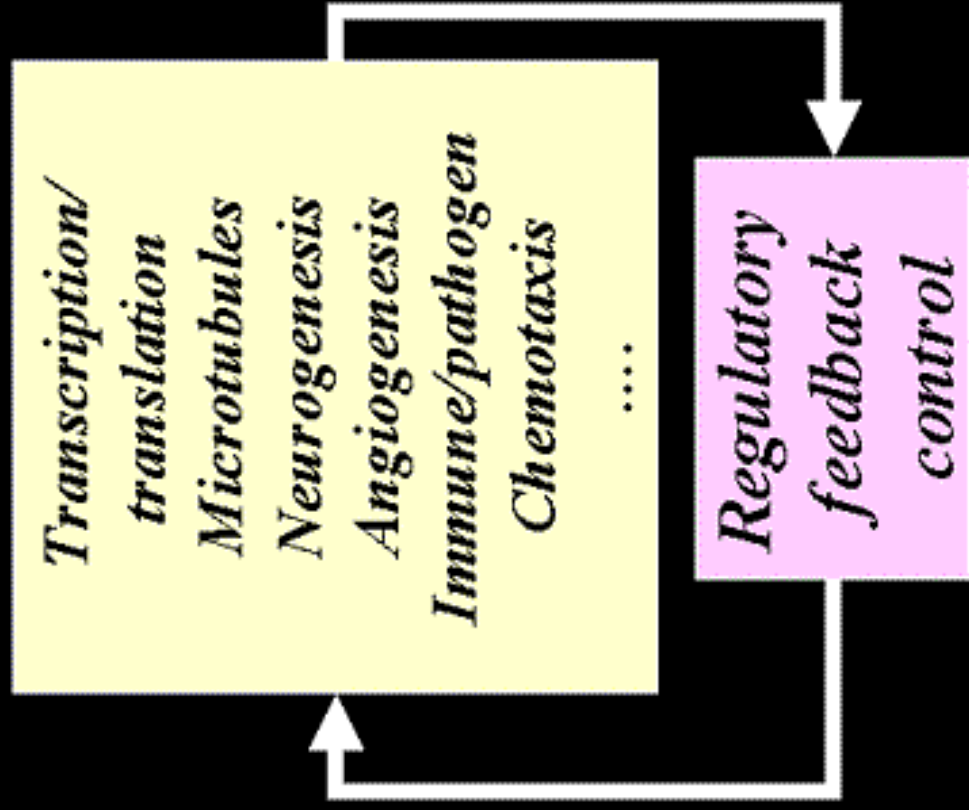
- Robustness is achieved by building barriers to cascading failures
- This often requires complicated internal structure, hierarchies, self-dissimilarity, layers of feedback, signaling, regulation, computation, protocols, ...
- Greater “geno-complexity” = more parts, more structure
- Molecular biology is about biological simplicity, what are the parts and how do they interact.
- If the complexity phenotypes and genotypes are linked, then robustness is the key to biological complexity.
- “Nominal function” may tell little.

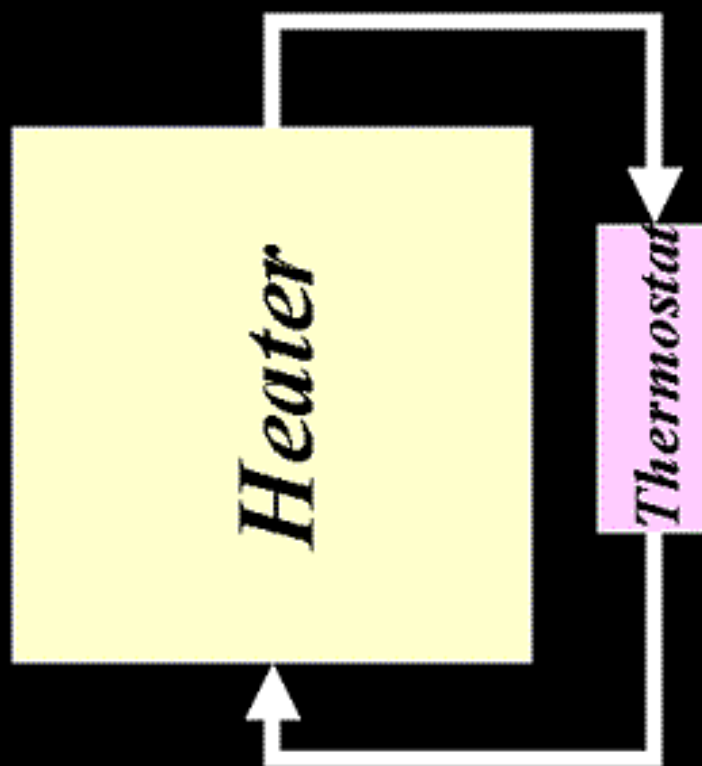
An apparent paradox

Gratuitously uncertain components and complex networks, but robust system performance.

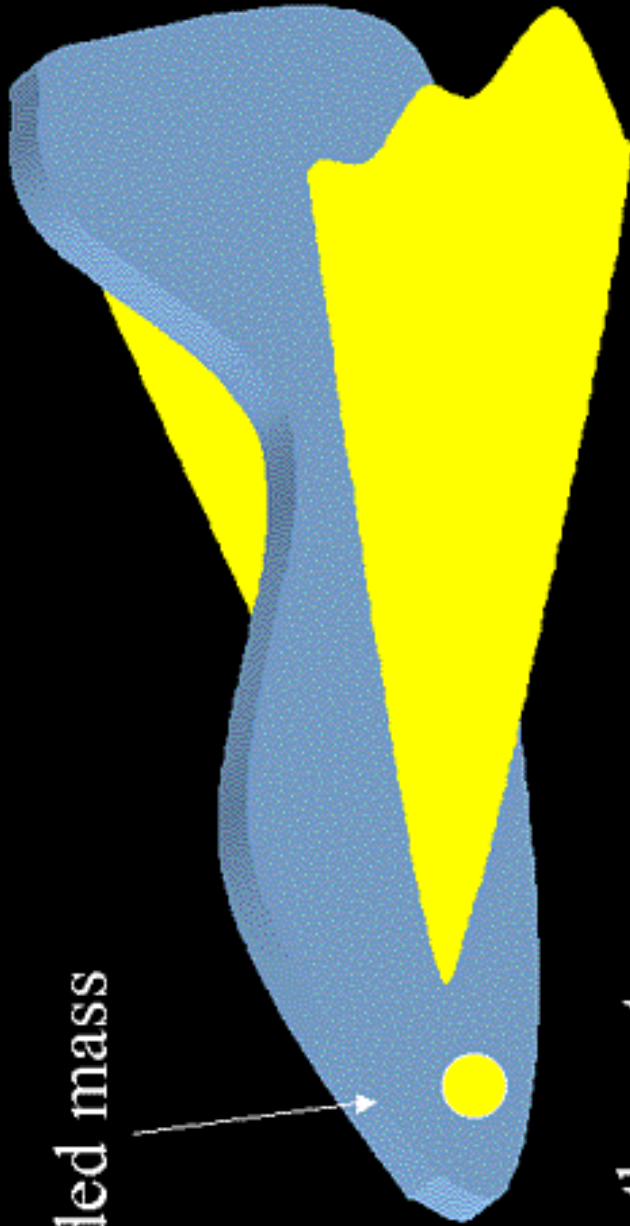


Darwinian evolution uses selection on random mutations to create complexity.





Tail



Added mass

Moves the center of pressure aft.

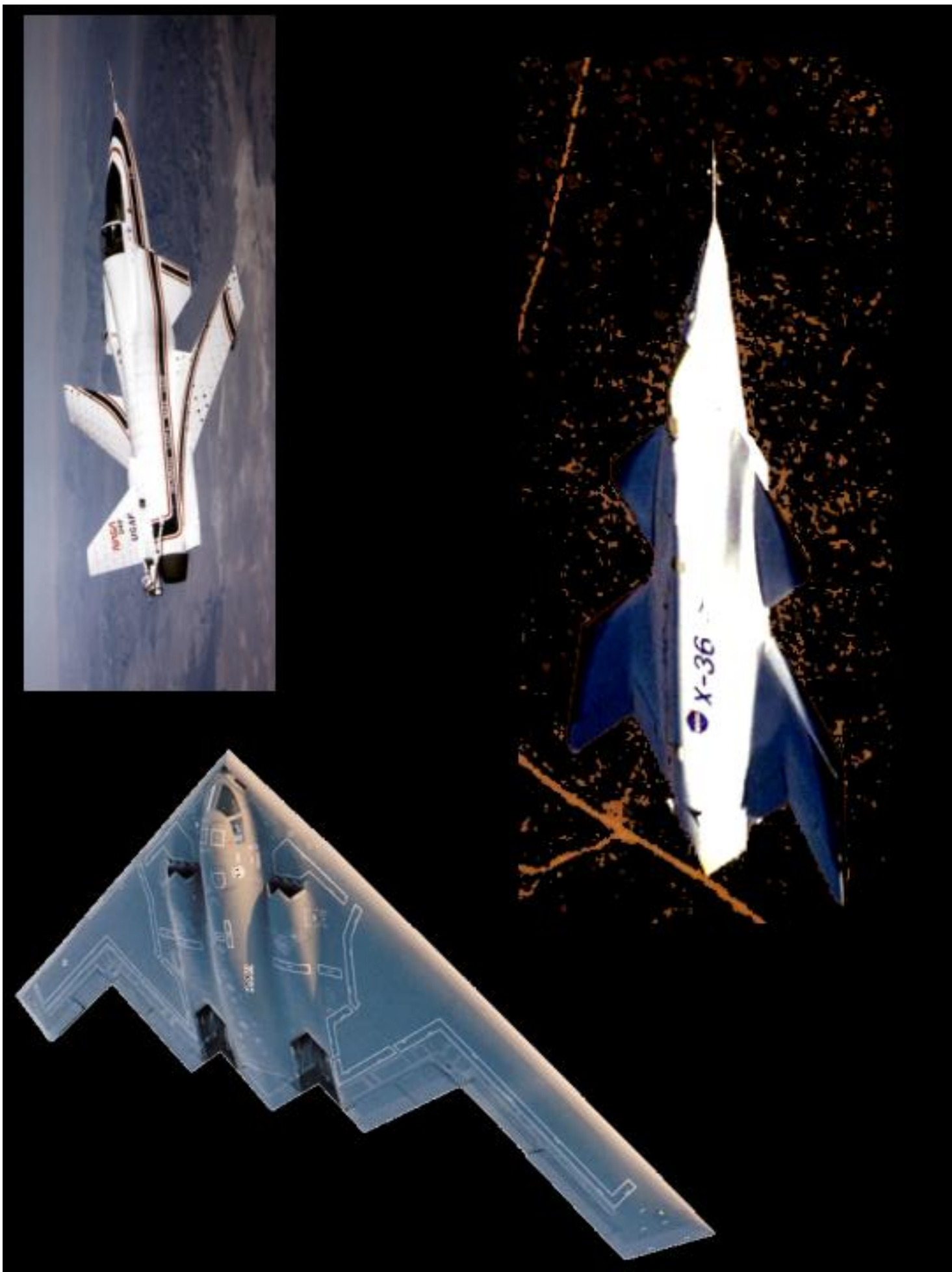
Moves the center of mass forward.

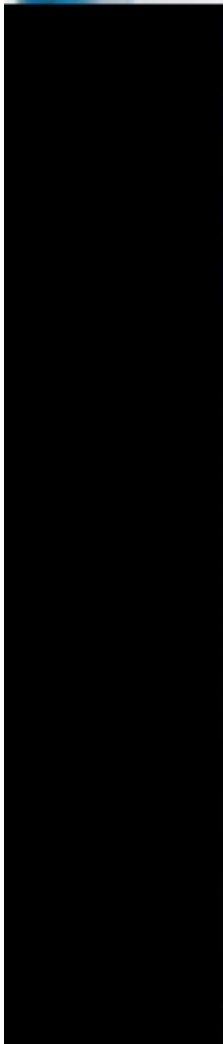
Thus stabilizing forward flight.

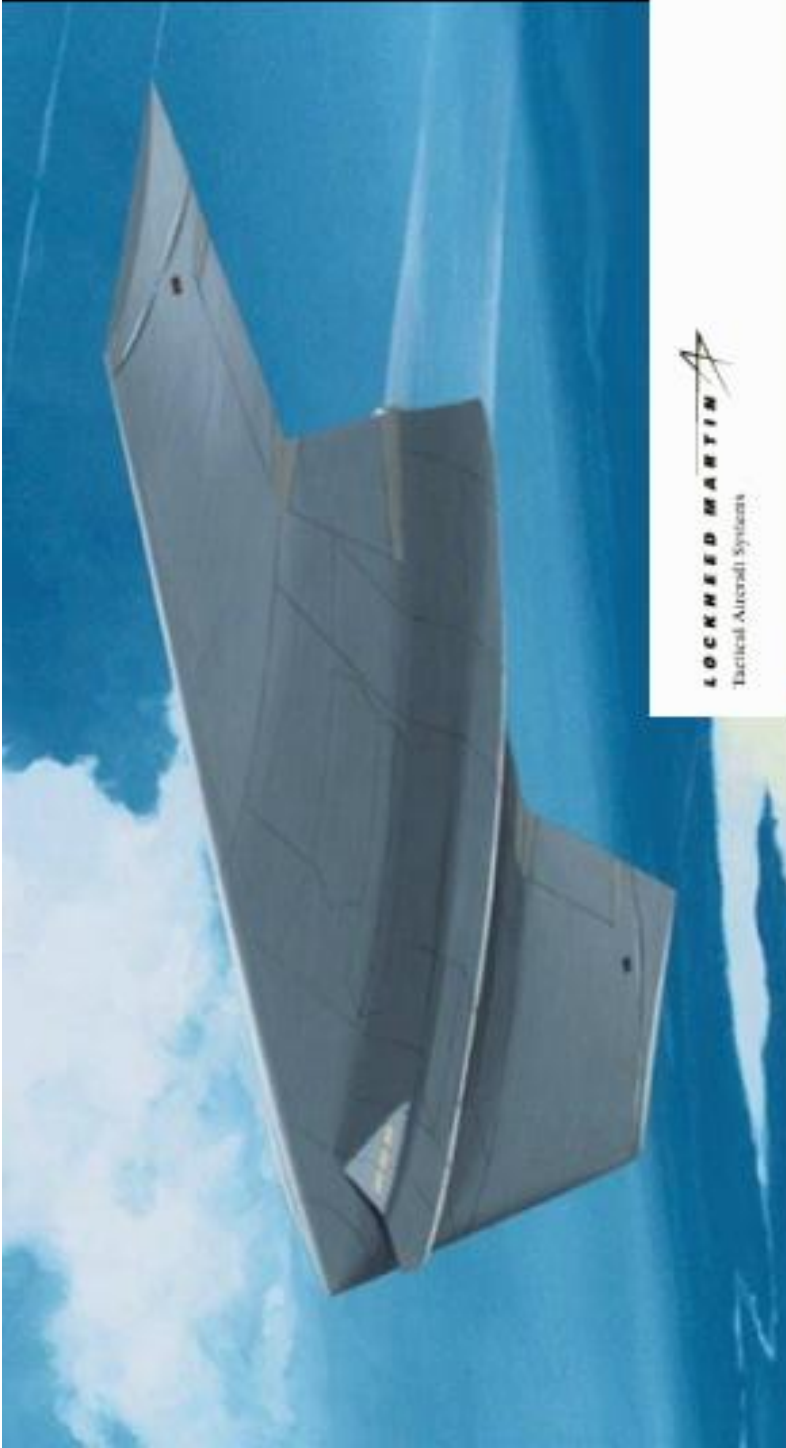
At the expense of extra weight and drag.



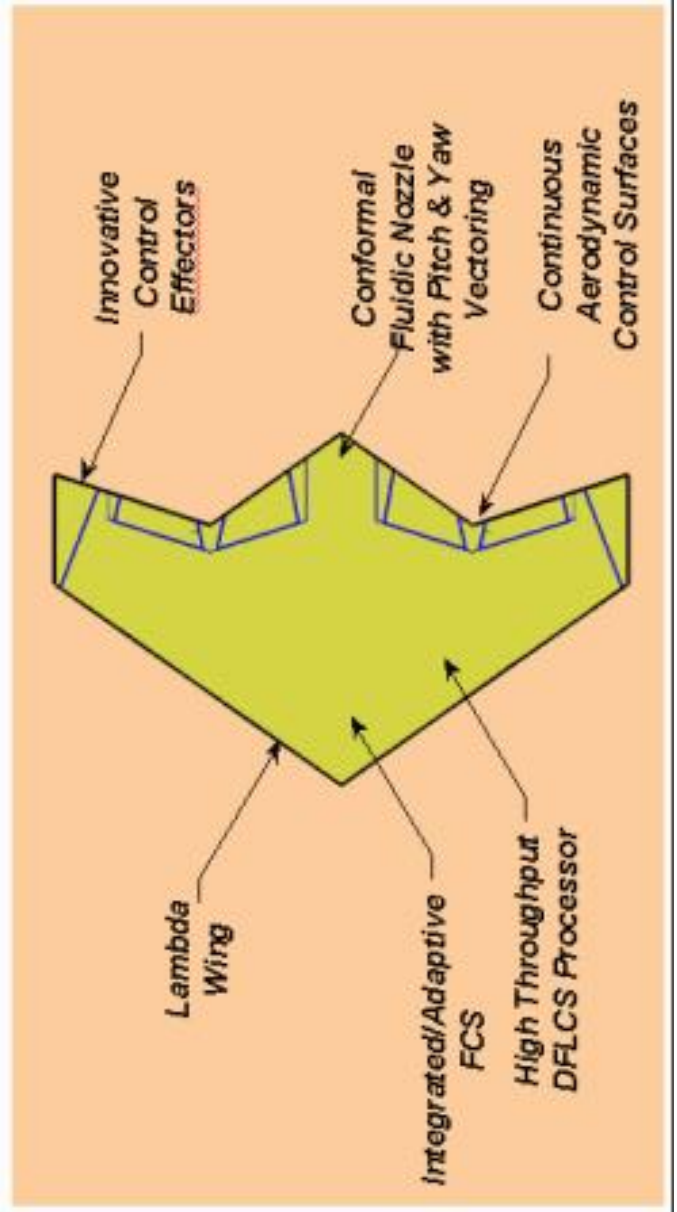
For minimum weight & drag,
(and other performance issues)
eliminate fuselage and tail.

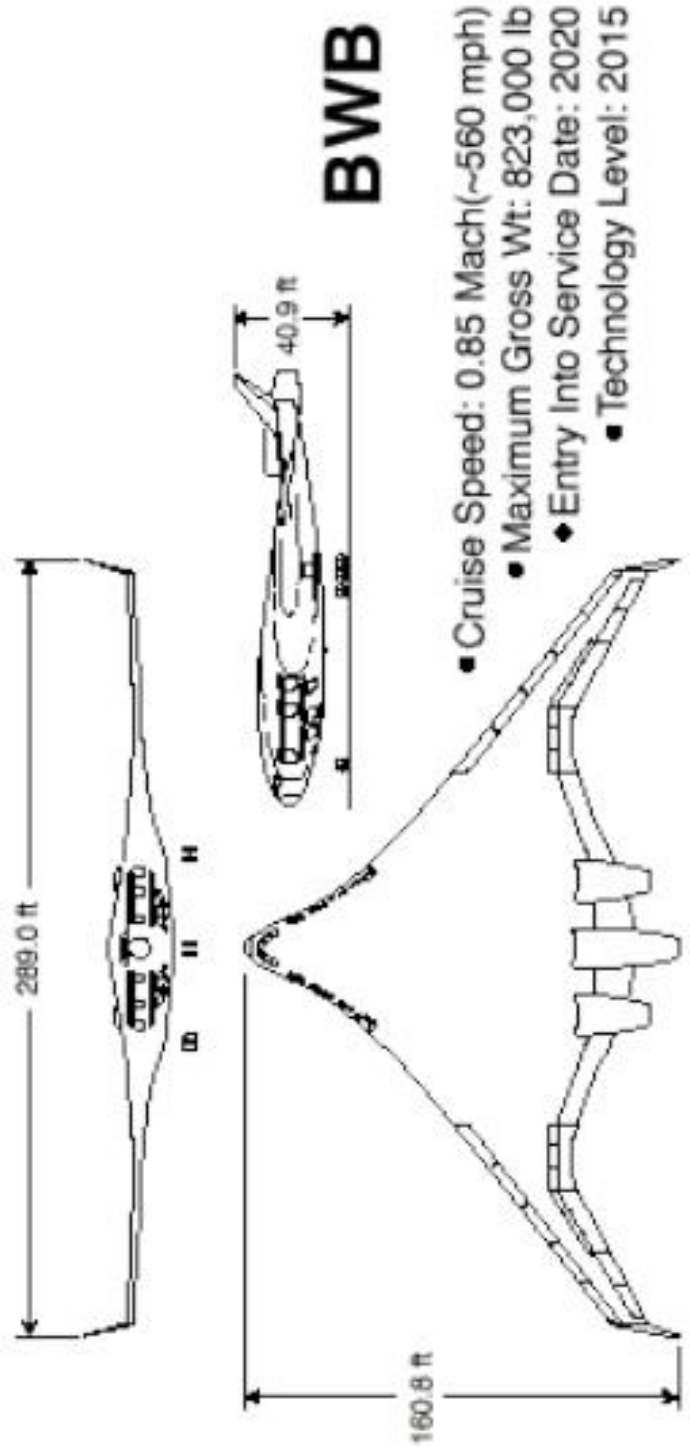




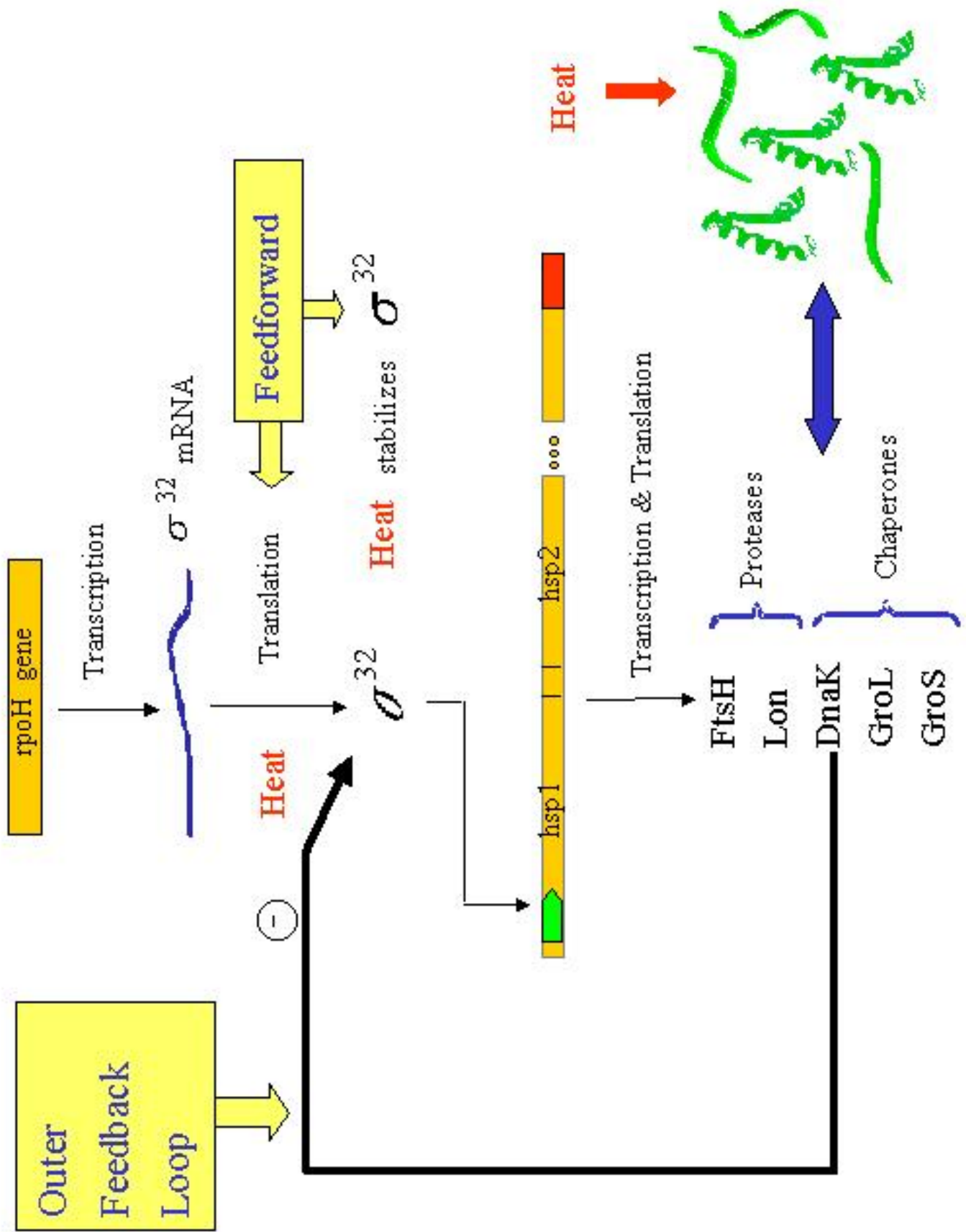


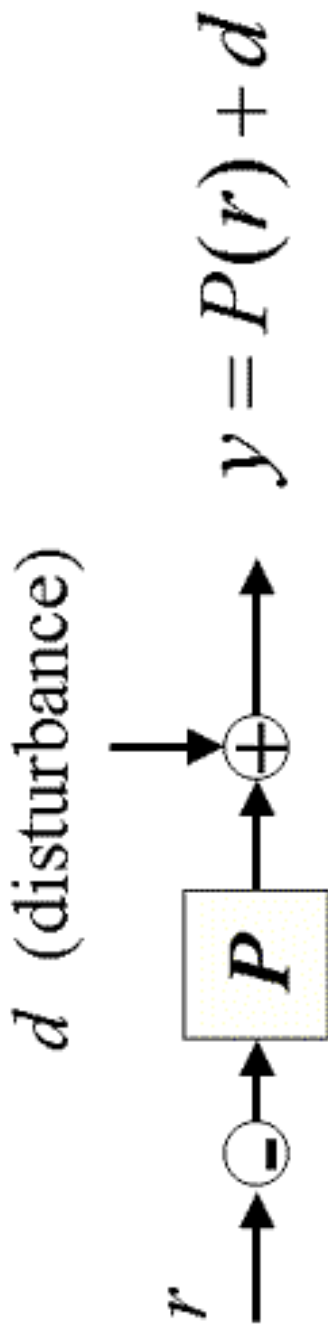
Uninhabited Vehicle 2001 Technologies





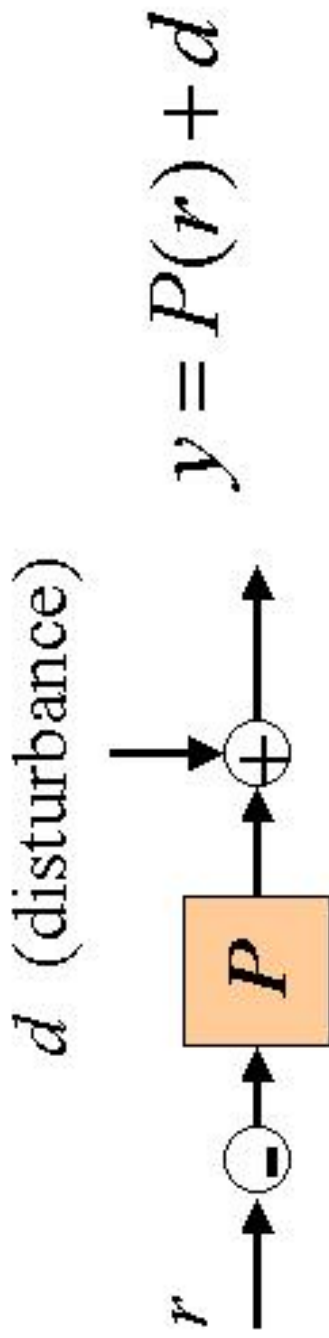
Why do we love building robust
systems from highly uncertain
and unstable components?



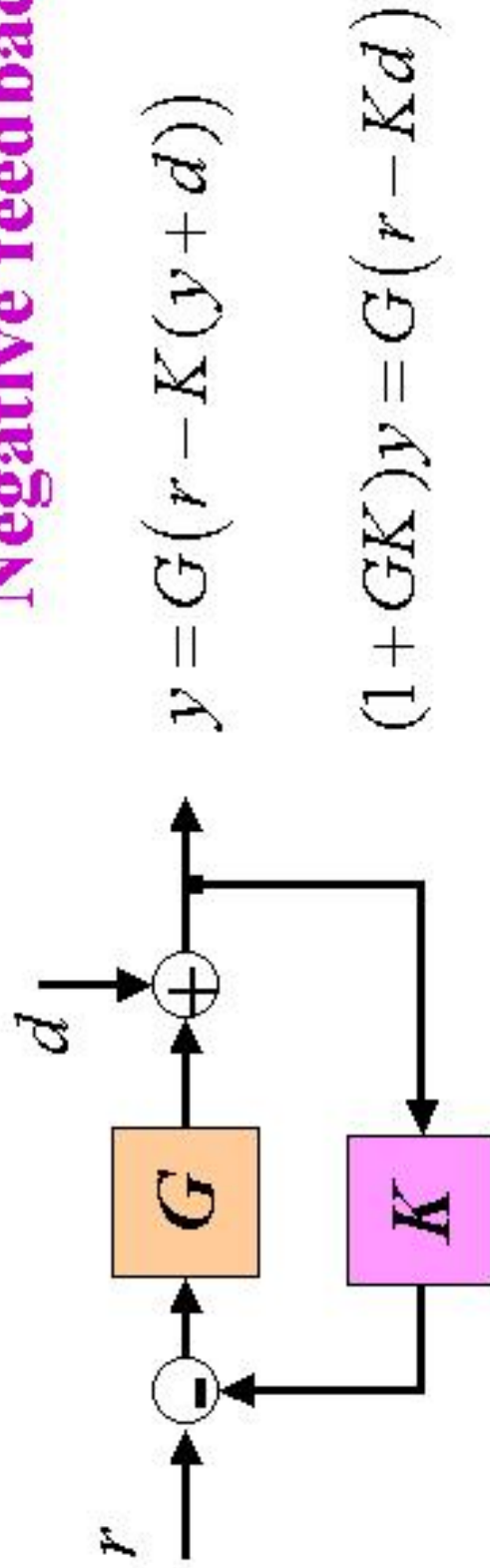


Assumptions on components:

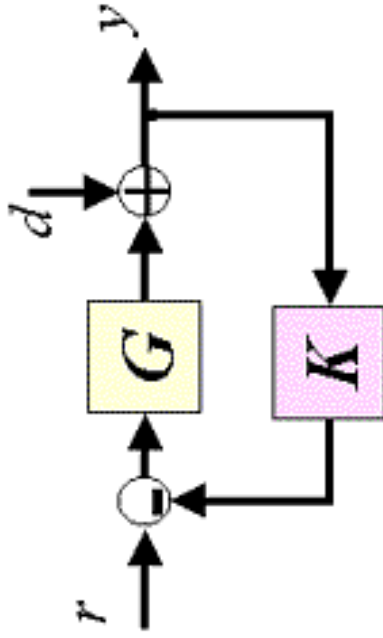
- Everything just numbers $y = (P + \Delta P)r + d$
 - Uncertainty in P
 - Higher gain = more uncertain
- $$P_1 > P_2 \Rightarrow \frac{\Delta P_1}{P_1} > \frac{\Delta P_2}{P_2}$$



Negative feedback



$$y = G S r + S d = \frac{1}{K} (1 - S) r + S d \quad S \square \left(\frac{1}{1 + GK} \right)$$



Design recipe:

- $1 \gg \gg K \gg \gg 1/G$
- $G \gg \gg 1/K \gg \gg 1$
- G maximally uncertain!
- K small, low uncertainty

$$G \gg \gg \frac{1}{K} > 1 \Rightarrow GK \gg \gg 1$$

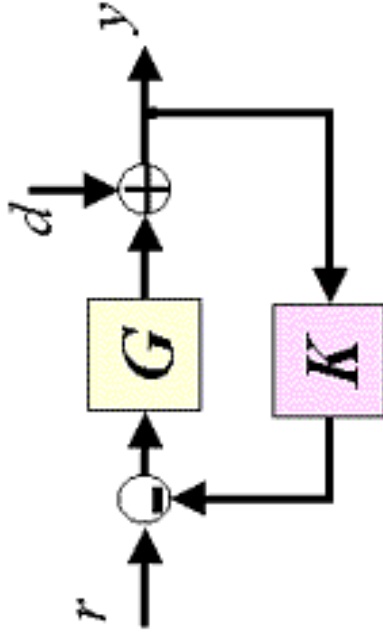
$$\Rightarrow S \ll \ll 1 \Rightarrow y \approx \frac{1}{K}r$$

Results for $y \approx (1/K)r$:

- high gain
- low uncertainty
- d attenuated

$$y = G_S r + S d = \frac{1}{K} (1 - S) r + S d \quad S \ll \left(\frac{1}{1 + GK} \right)$$

S = sensitivity function



Design recipe:

- $I \gg K \gg 1/G$
- $G \gg 1/K \gg I$
- G maximally uncertain!
- K small, low uncertainty

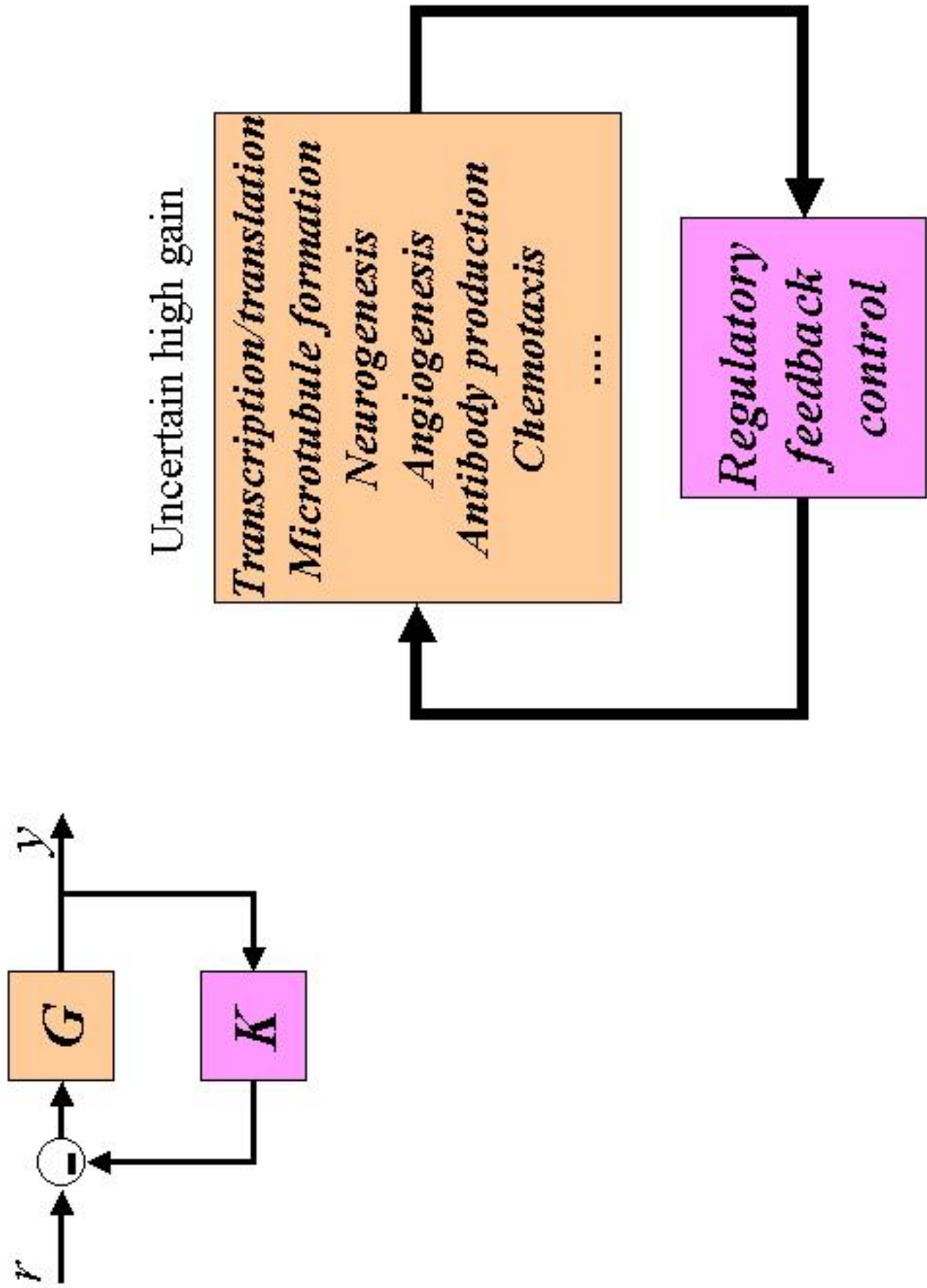
Extensions to:

- Dynamics
- Multivariable
- Nonlinear
- Structured uncertainty

All cost more computationally.

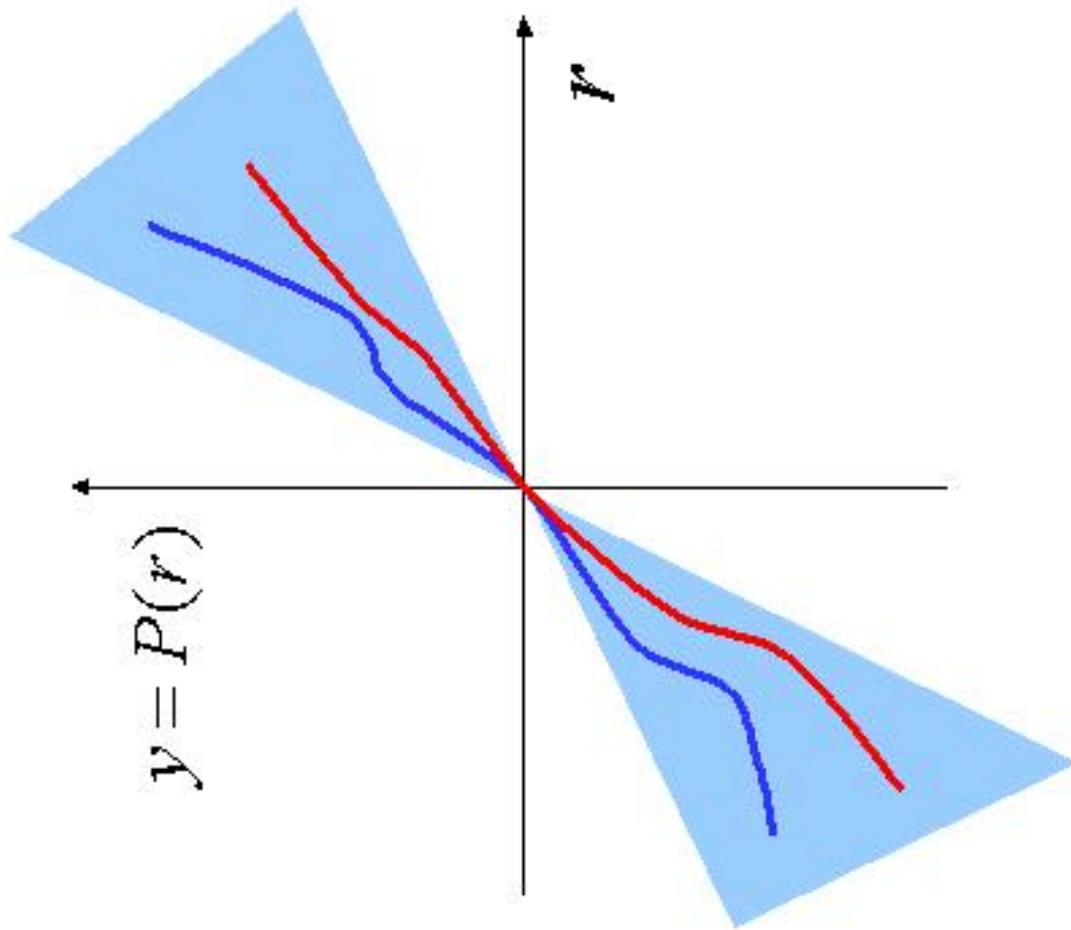
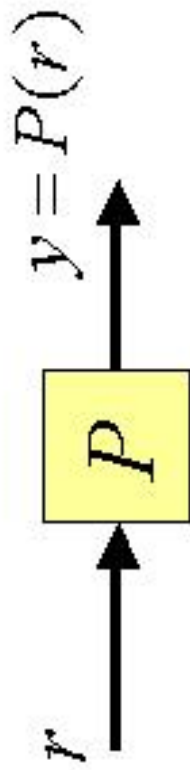
Results for $y \approx (1/K)r$:

- high gain
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- d attenuated

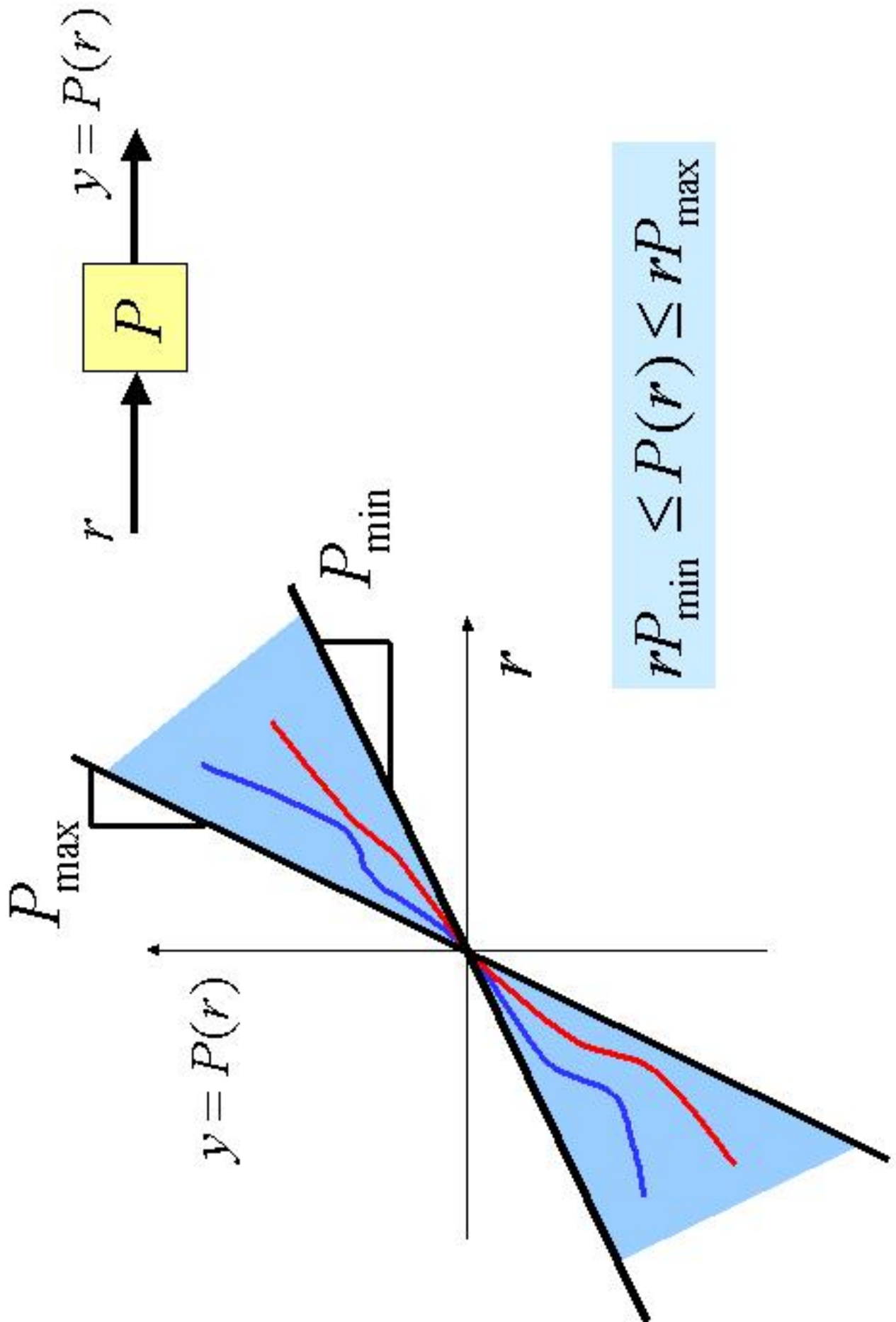


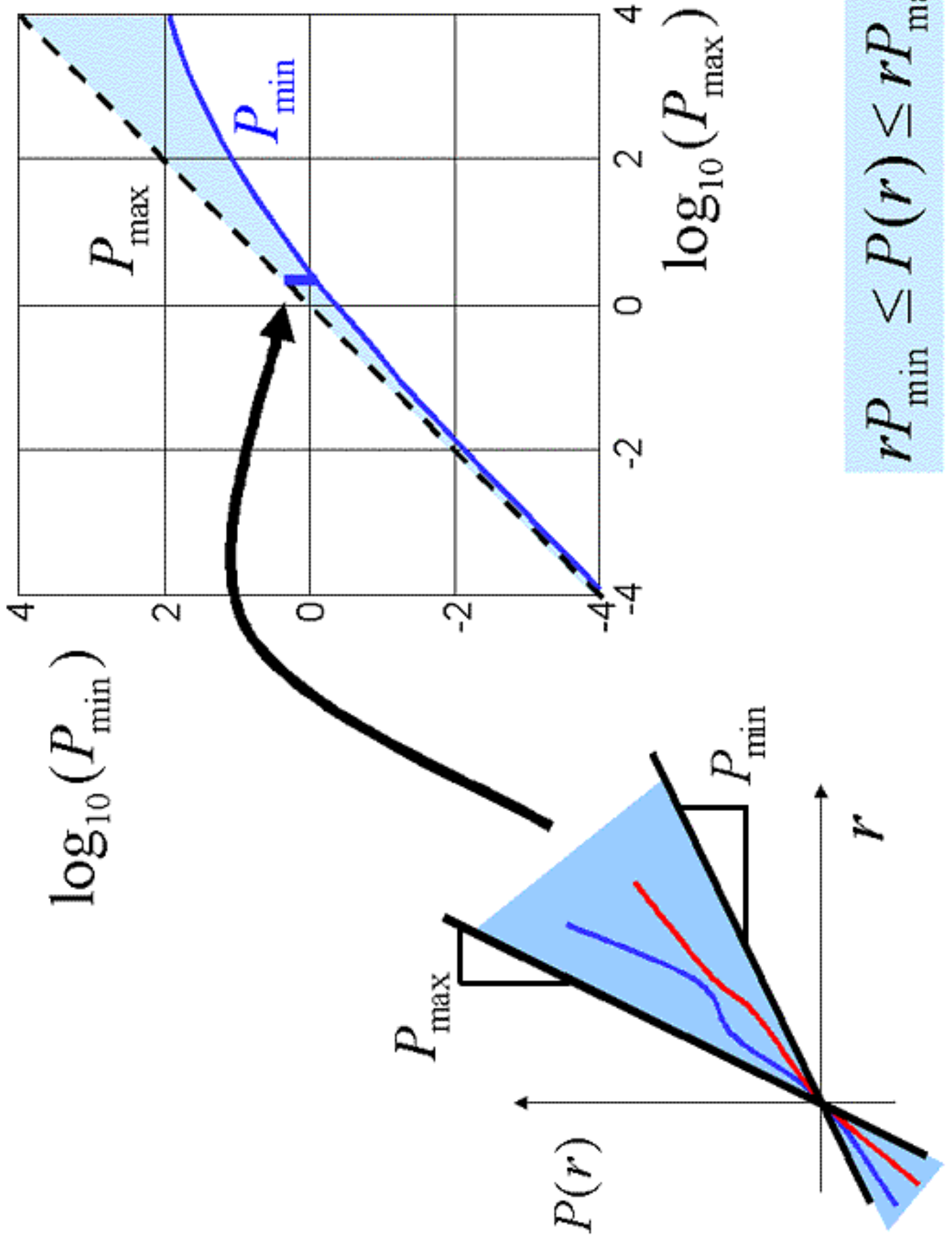
A slightly more detailed description....

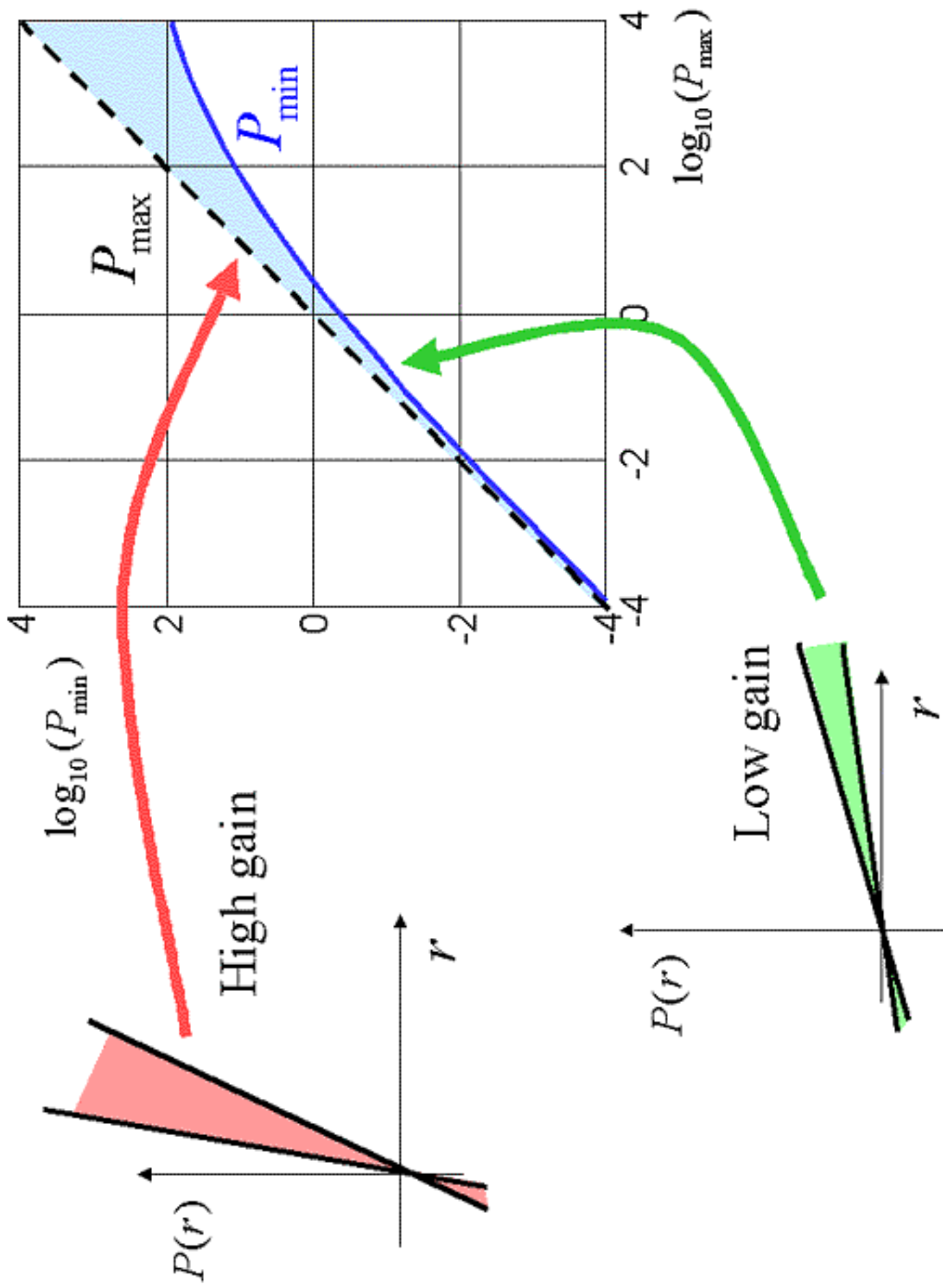
Components

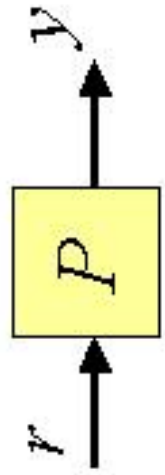
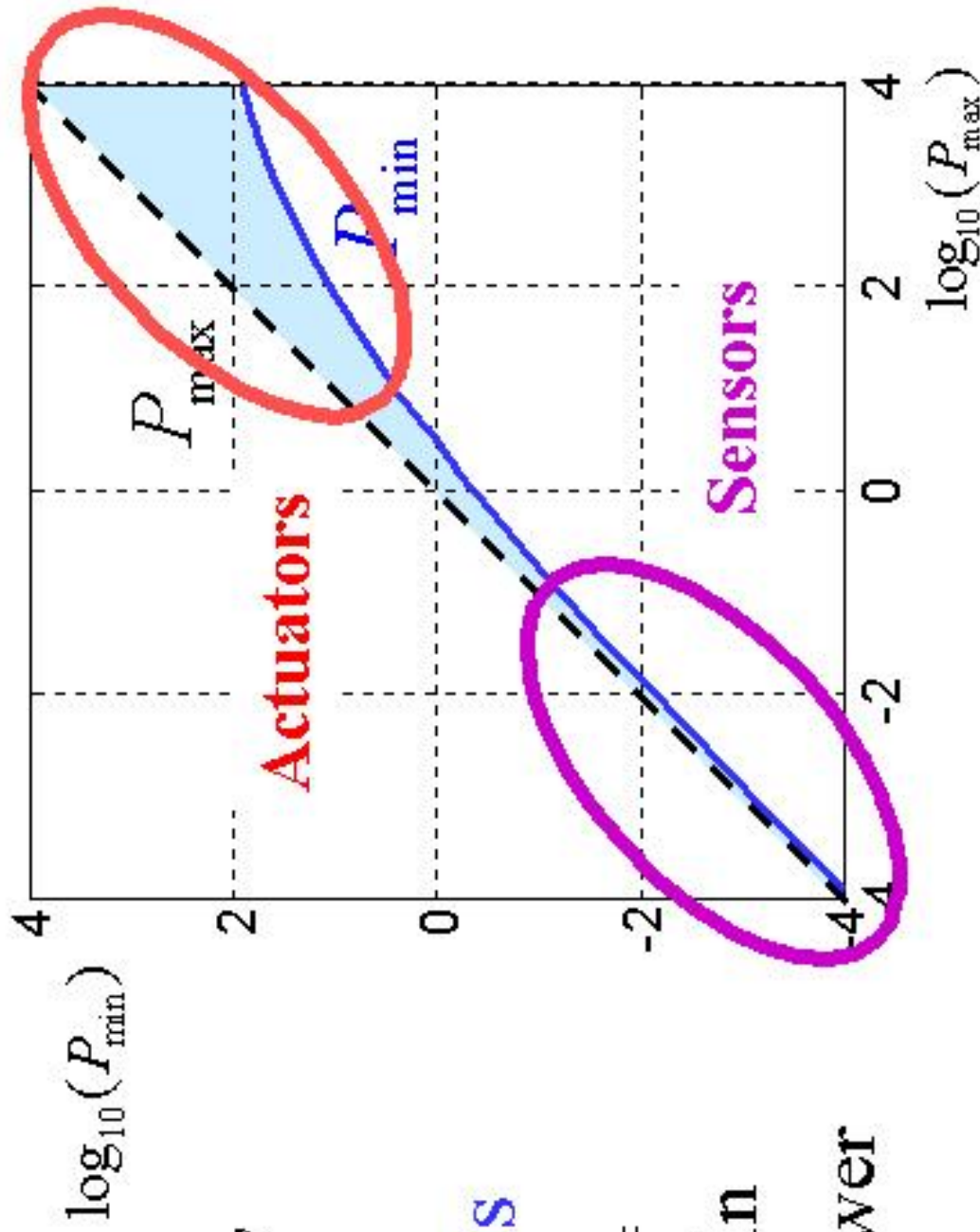


Uncertainty





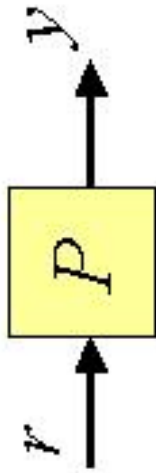




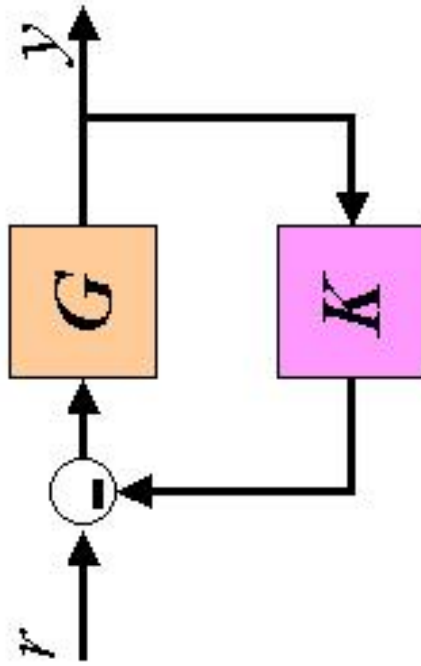
Assumptions

- Higher gain = more uncertain
- Upper and lower limits on achievable P_{\max}

Negative feedback

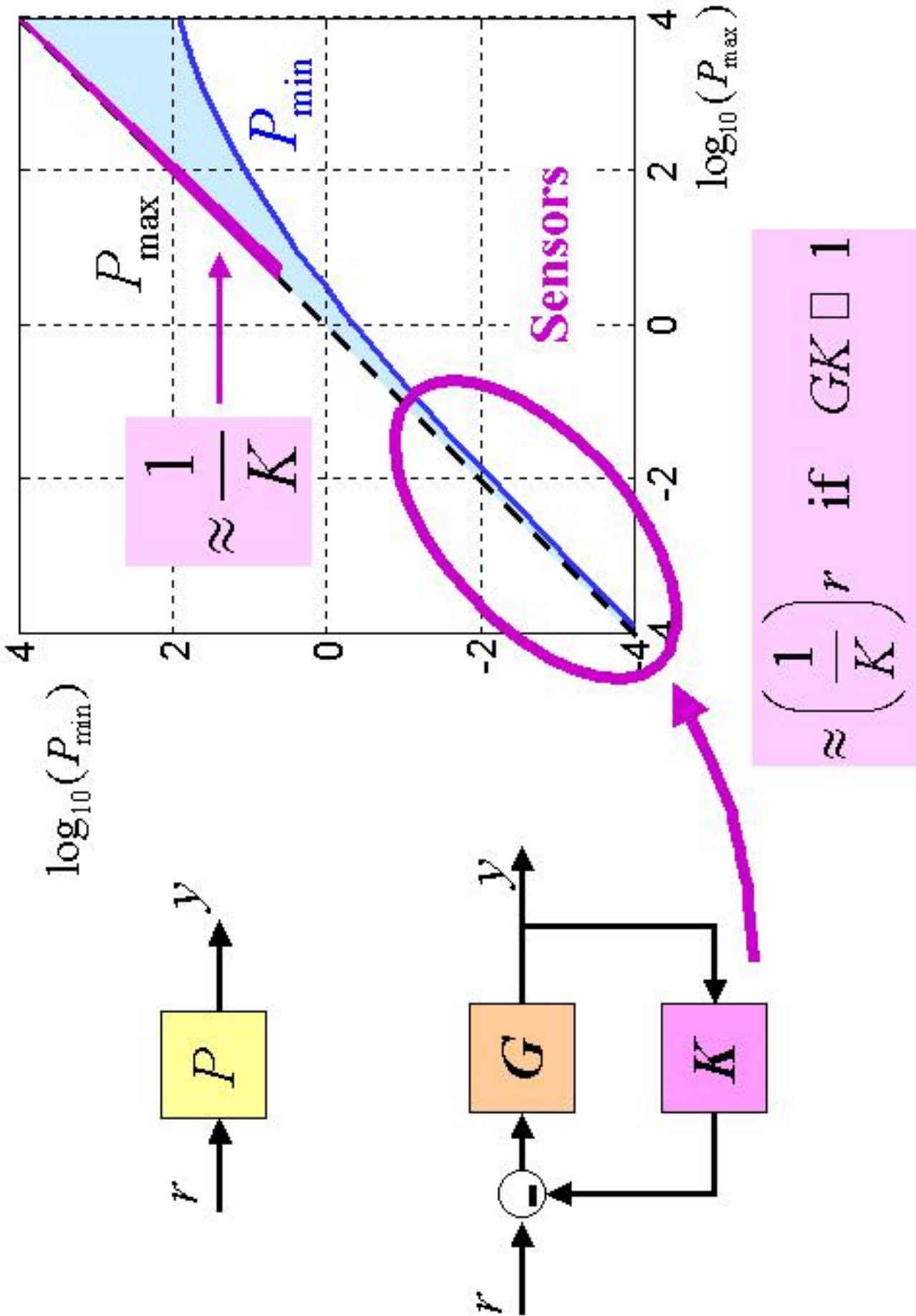


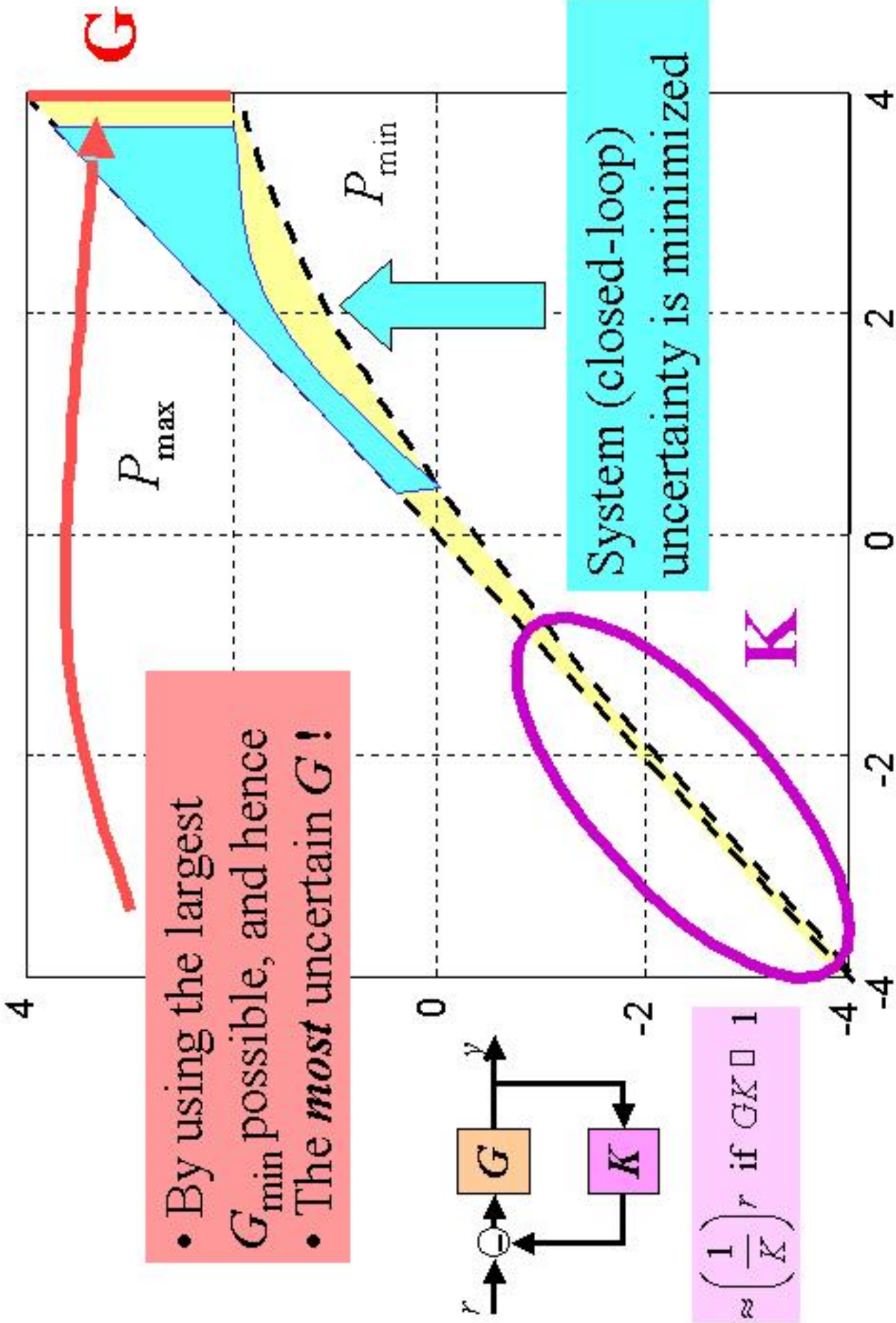
$$y = G(r - GK(y))$$

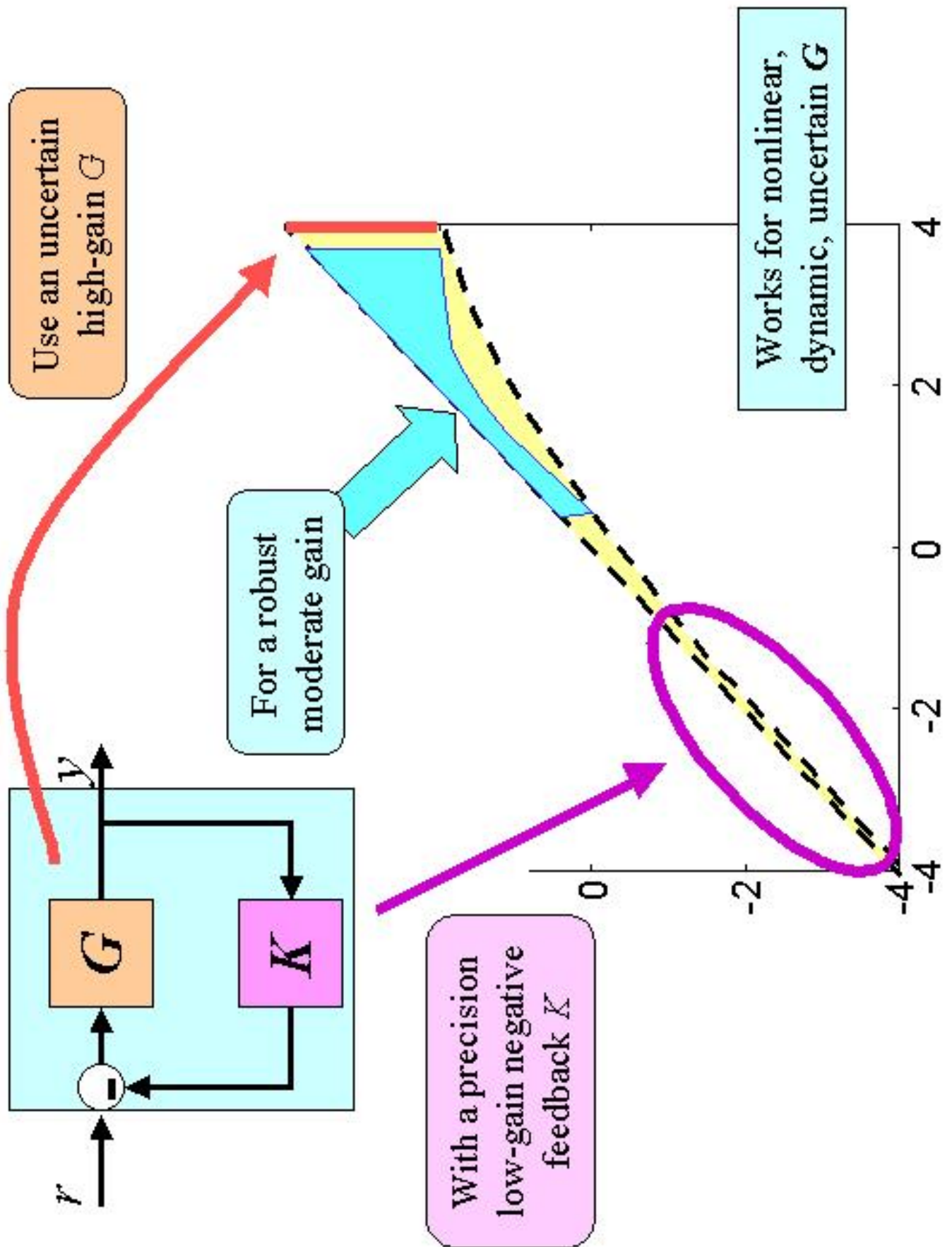


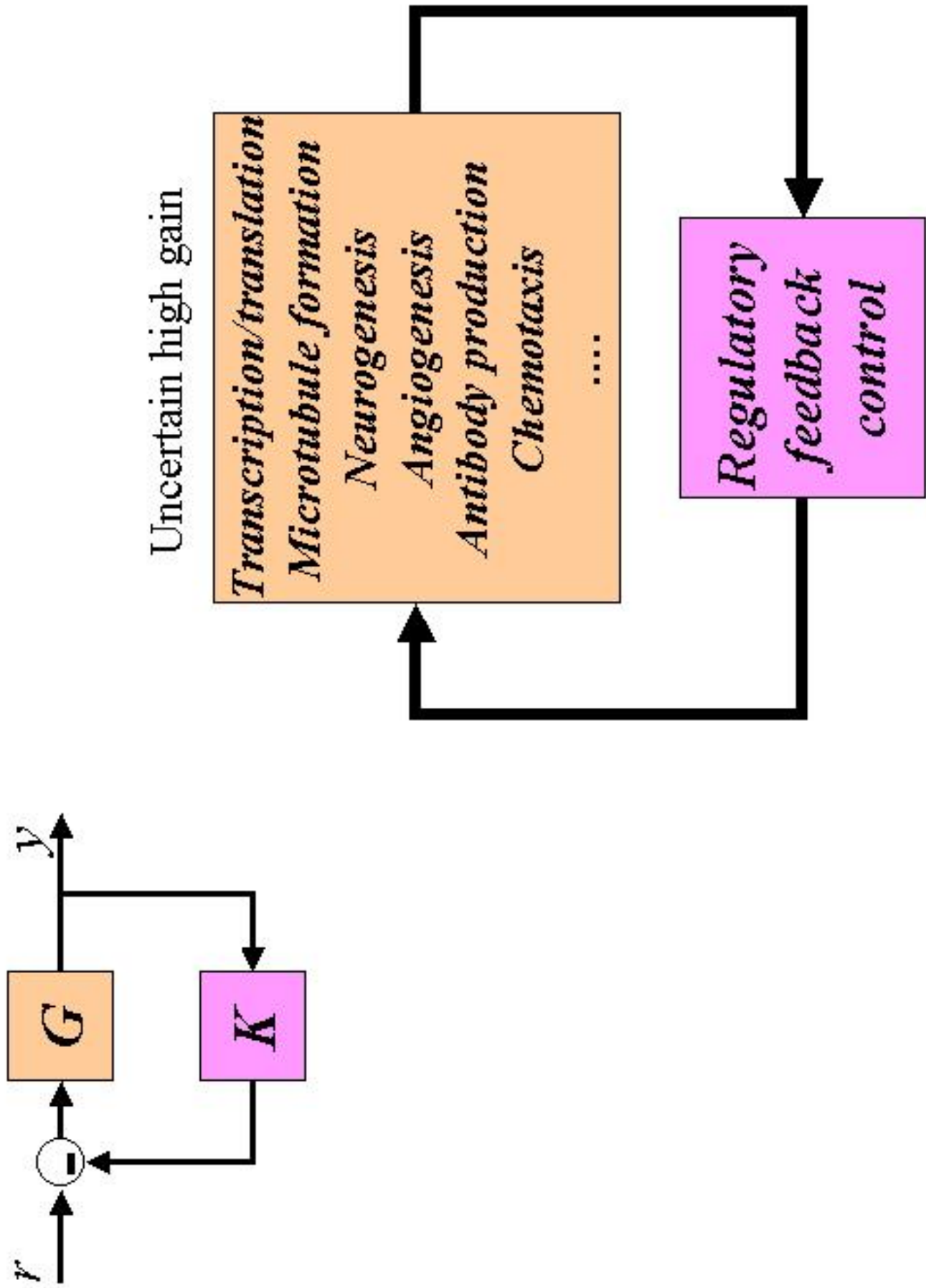
$$y \approx \left(\frac{G}{1+GK} \right) r = \left(\frac{1}{1/G+K} \right) r$$

$$\approx \left(\frac{1}{K} \right) r \quad \text{if } GK \gg 1$$



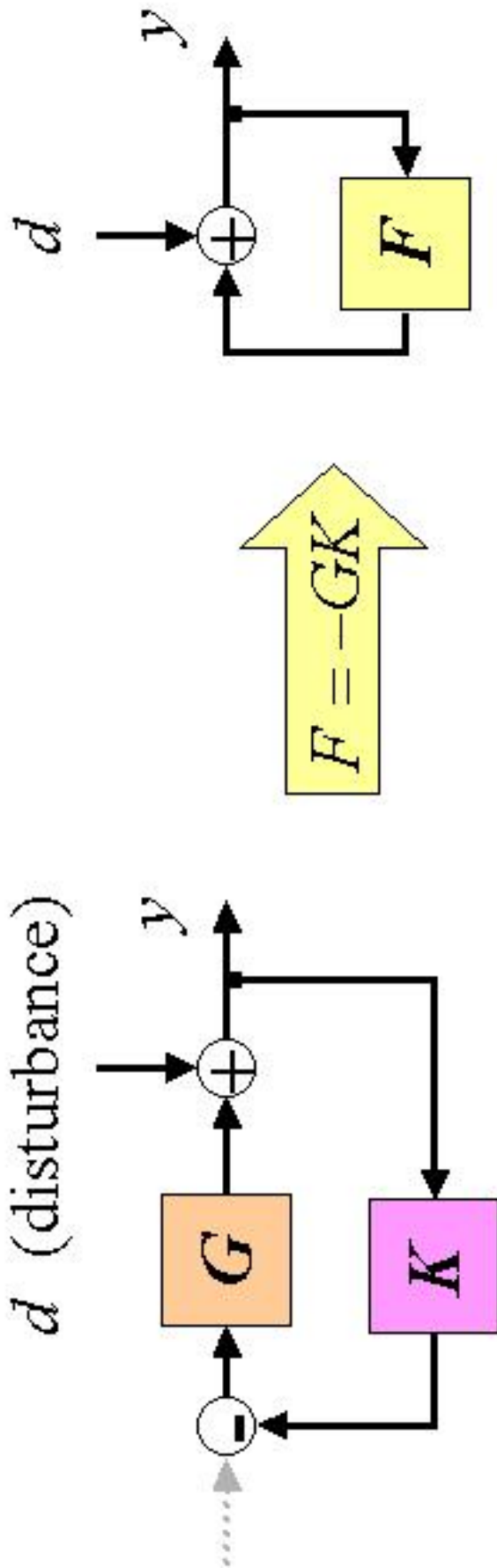






Summary

- Primitive technologies build fragile systems from precision components.
- Advanced technologies build robust systems from sloppy components.
- There are many other examples of regulator strategies deliberately employing uncertain and stochastic components...
- ... to create robust *systems*.
- High gain negative feedback is the most powerful mechanism, and also the most dangerous.
- In addition to the added complexity, what can go wrong?



$$y = F(y) + d$$

$$y \approx \left(\frac{1}{1-F} \right) d \approx \left(-\frac{1}{F} \right) d \quad \text{if} \quad |F| \ll 1$$

$$y \approx \left(\frac{1}{1-F} \right) d$$

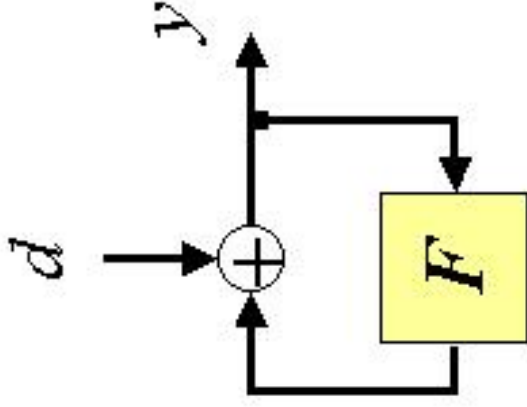
If y , d and F are just numbers:

$$S \equiv \frac{y}{d} = \frac{1}{1-F}$$

$S = \text{sensitivity function}$

S measures disturbance rejection.

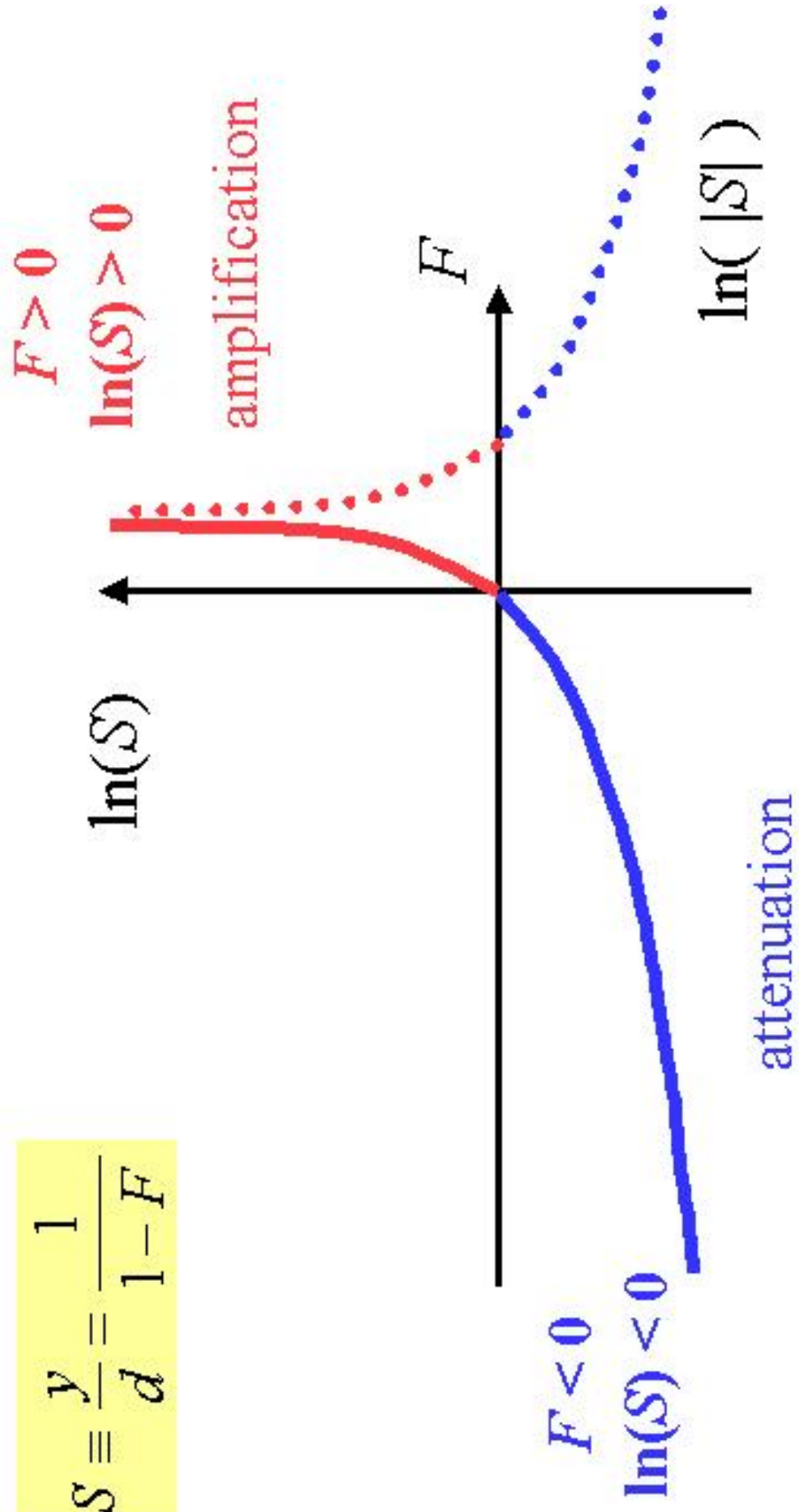
It's convenient to study $\ln(S)$.



Negative F ($F < 0$) $\Rightarrow \ln(S) < 0 \Rightarrow$ Disturbance attenuated

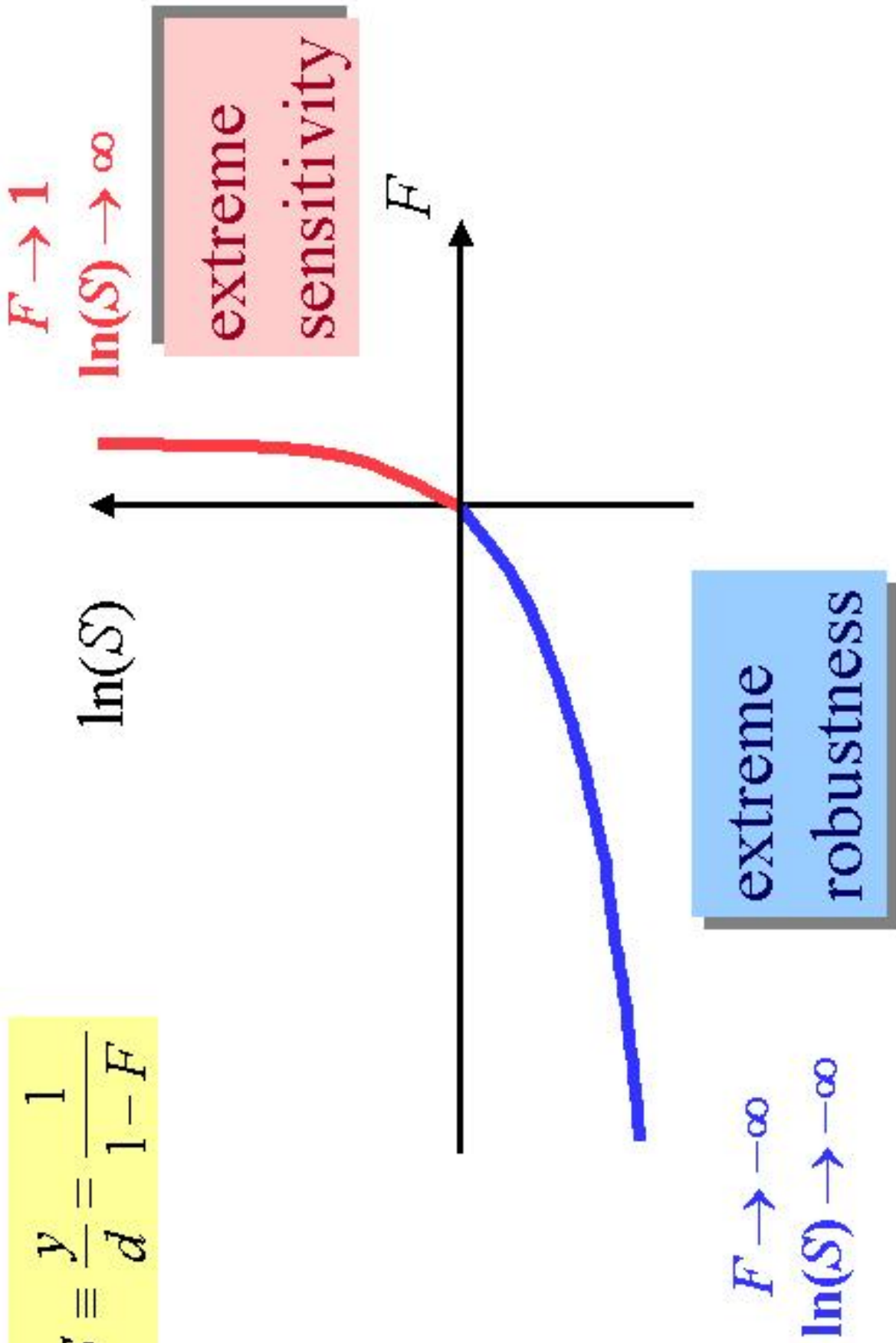
Positive F ($F > 0$) $\Rightarrow \ln(S) > 0 \Rightarrow$ Disturbance amplified

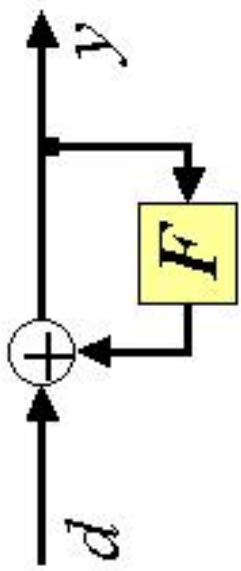
$$S \equiv \frac{y}{d} = \frac{1}{1-F}$$



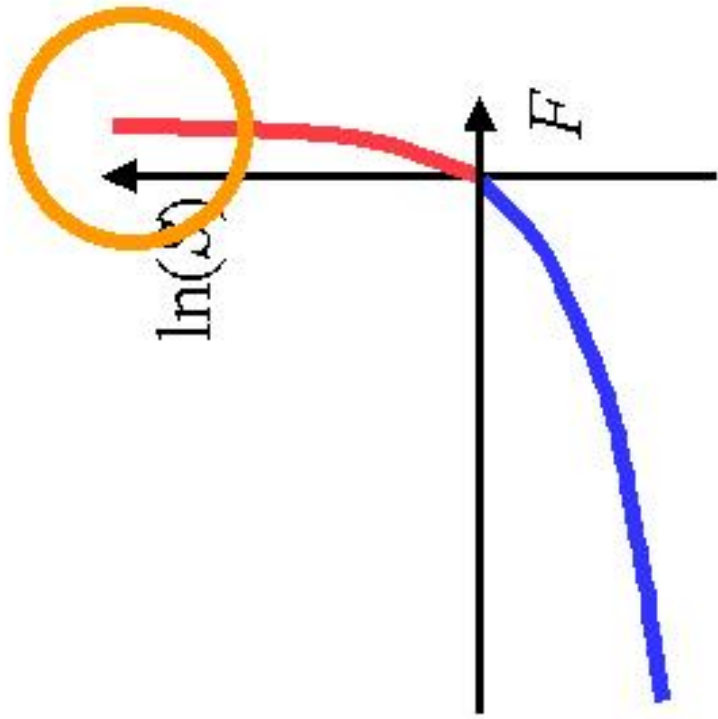
Negative F ($F < 0$) $\Rightarrow \ln(S) < 0 \Rightarrow$ Disturbance attenuated
 Positive F ($F > 0$) $\Rightarrow \ln(S) > 0 \Rightarrow$ Disturbance amplified

$$S \equiv \frac{y}{d} = \frac{1}{1-F}$$





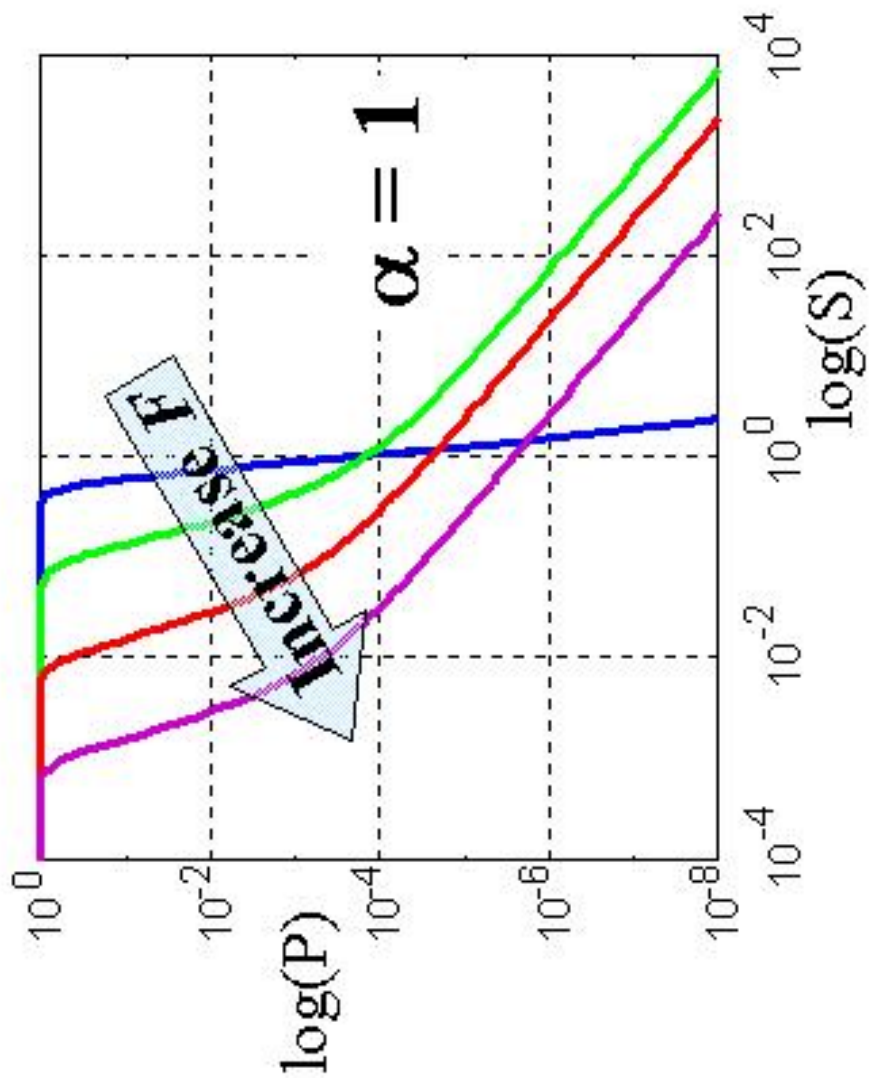
$$S = \frac{1}{1-F}$$

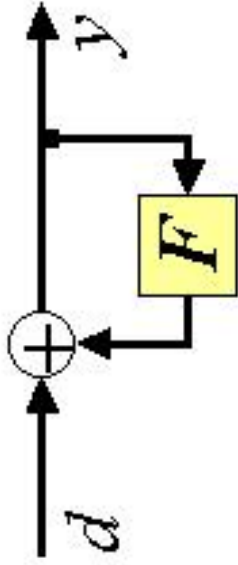


Assume:

1. F (and S) random variables
2. $\text{Prob}(F = -1) > 0$

$\Rightarrow |S|$ has power law tail with $\alpha=1$





$$S = \frac{1}{1 - F}$$

If these model physical processes, then d and y are signals and F is an operator. We can still define

$$S(\omega) = |Y(\omega) / D(\omega)|$$

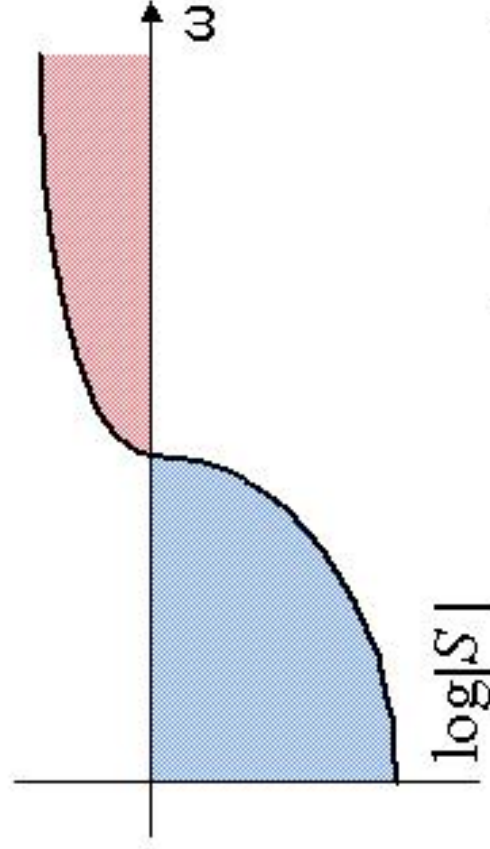
where E and D are the Fourier transforms of y and d . (If F is linear, then S is independent of D .)

Under assumptions that are consistent with F and d modeling physical systems (in particular, causality), it is possible to prove that:

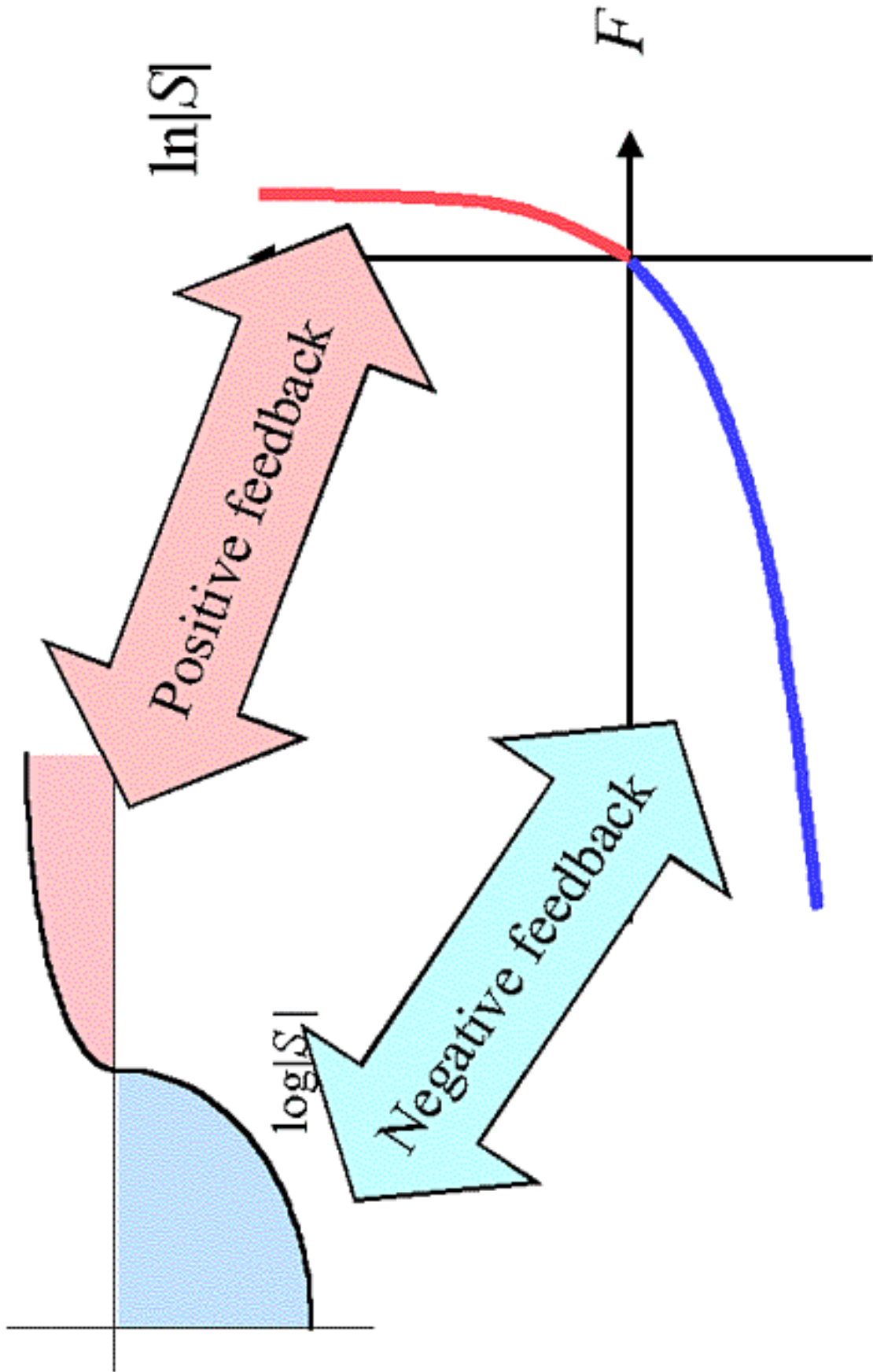
$$\begin{aligned} (F < 0) &\Rightarrow \ln(S) < 0 \Rightarrow \text{attenuate} \\ (F > 0) &\Rightarrow \ln(S) > 0 \Rightarrow \text{amplify} \end{aligned}$$

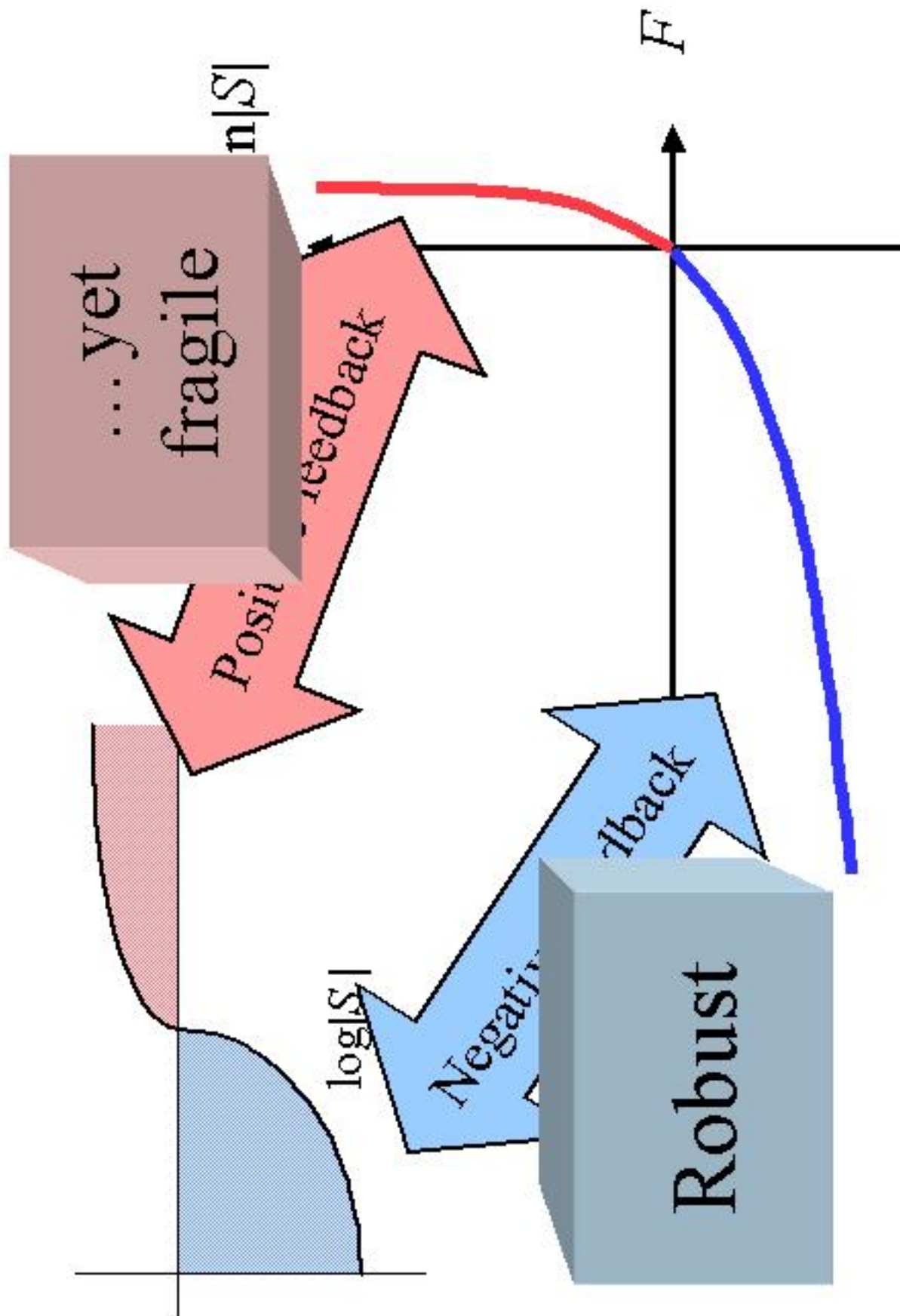
The **amplification** ($F > 0$) must at least balance the **attenuation** ($F < 0$).

$$\int \log |S(\omega)| d\omega \geq 0$$

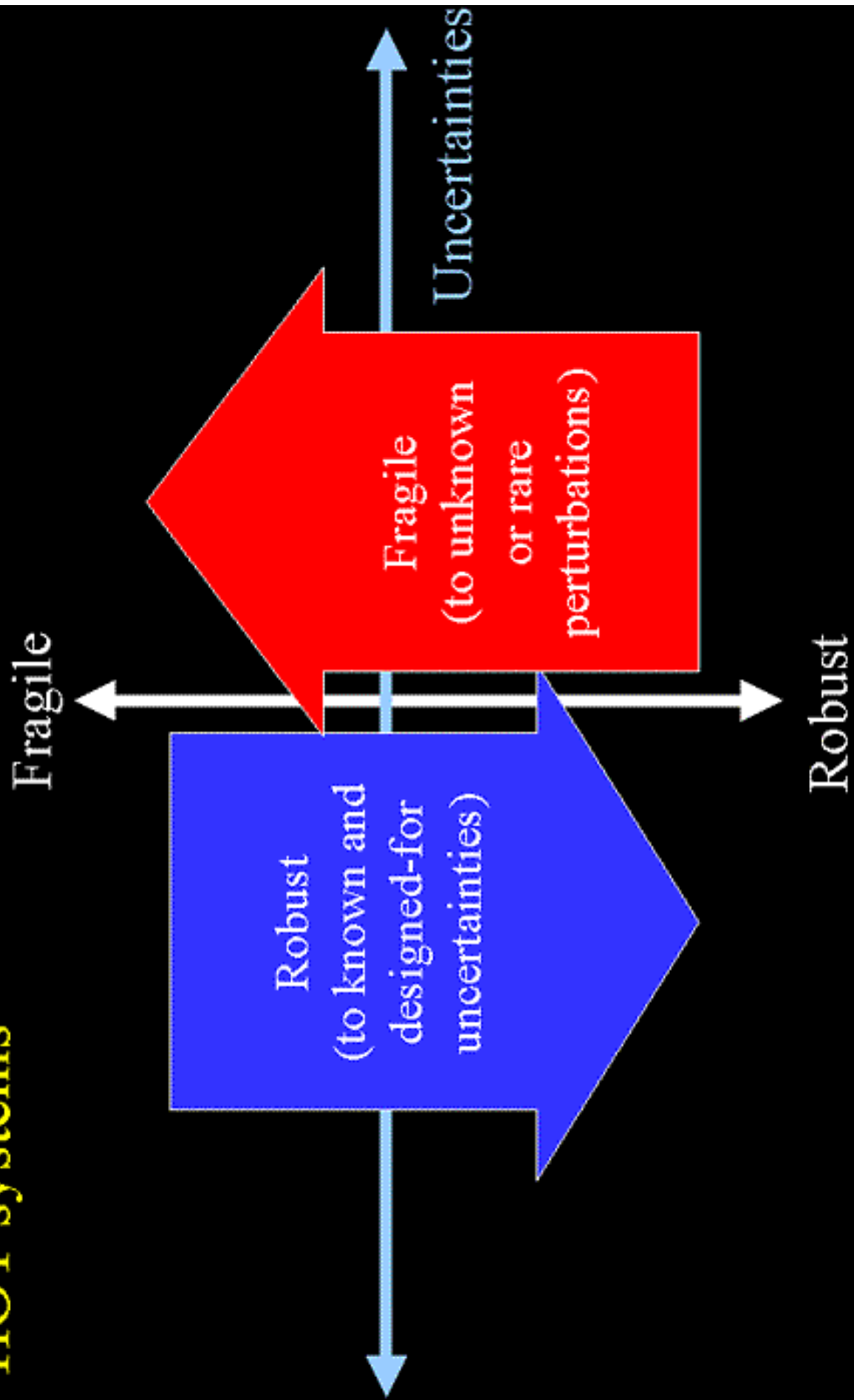


(Bode, ~1940)



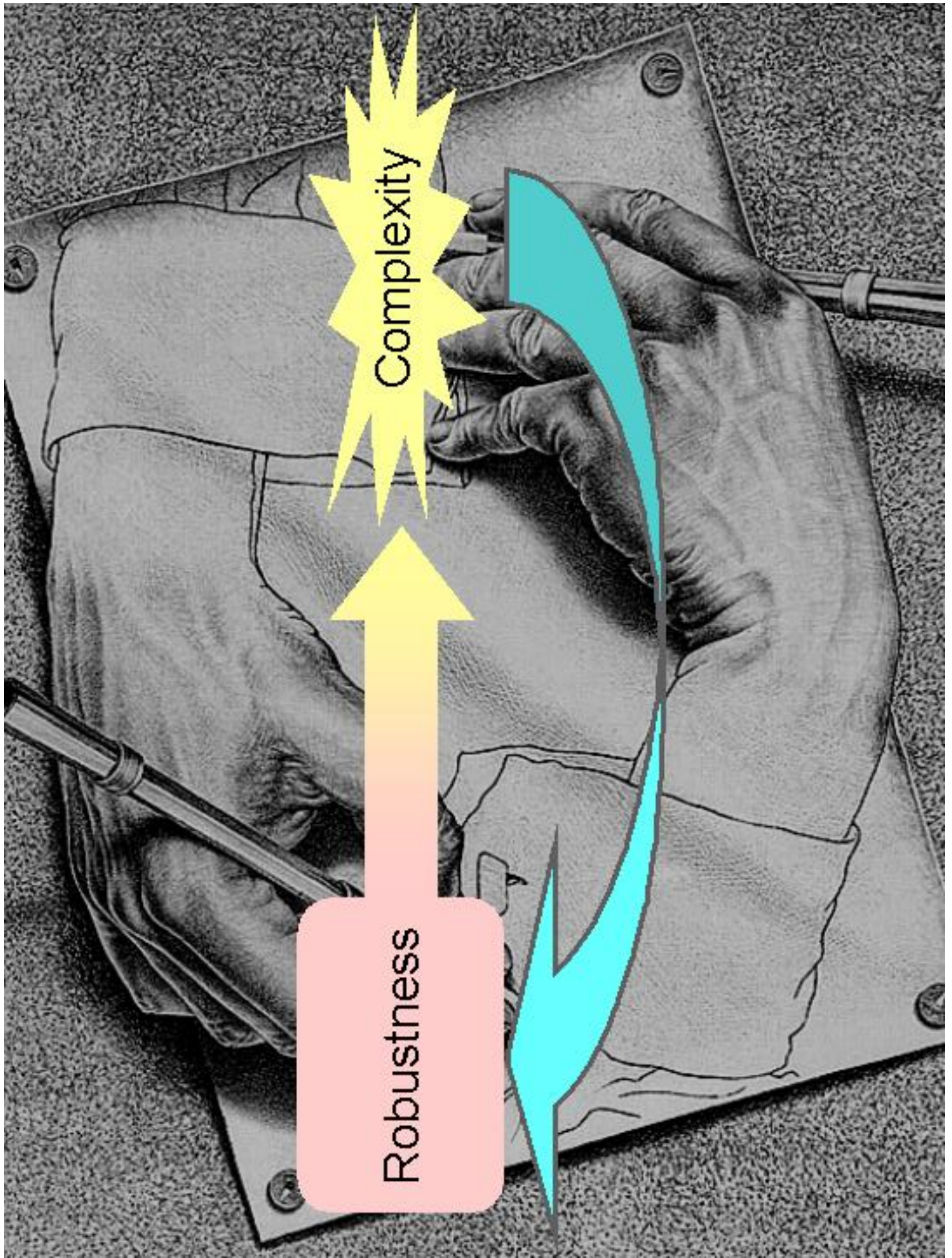


Robustness of HOT systems



Feedback and robustness

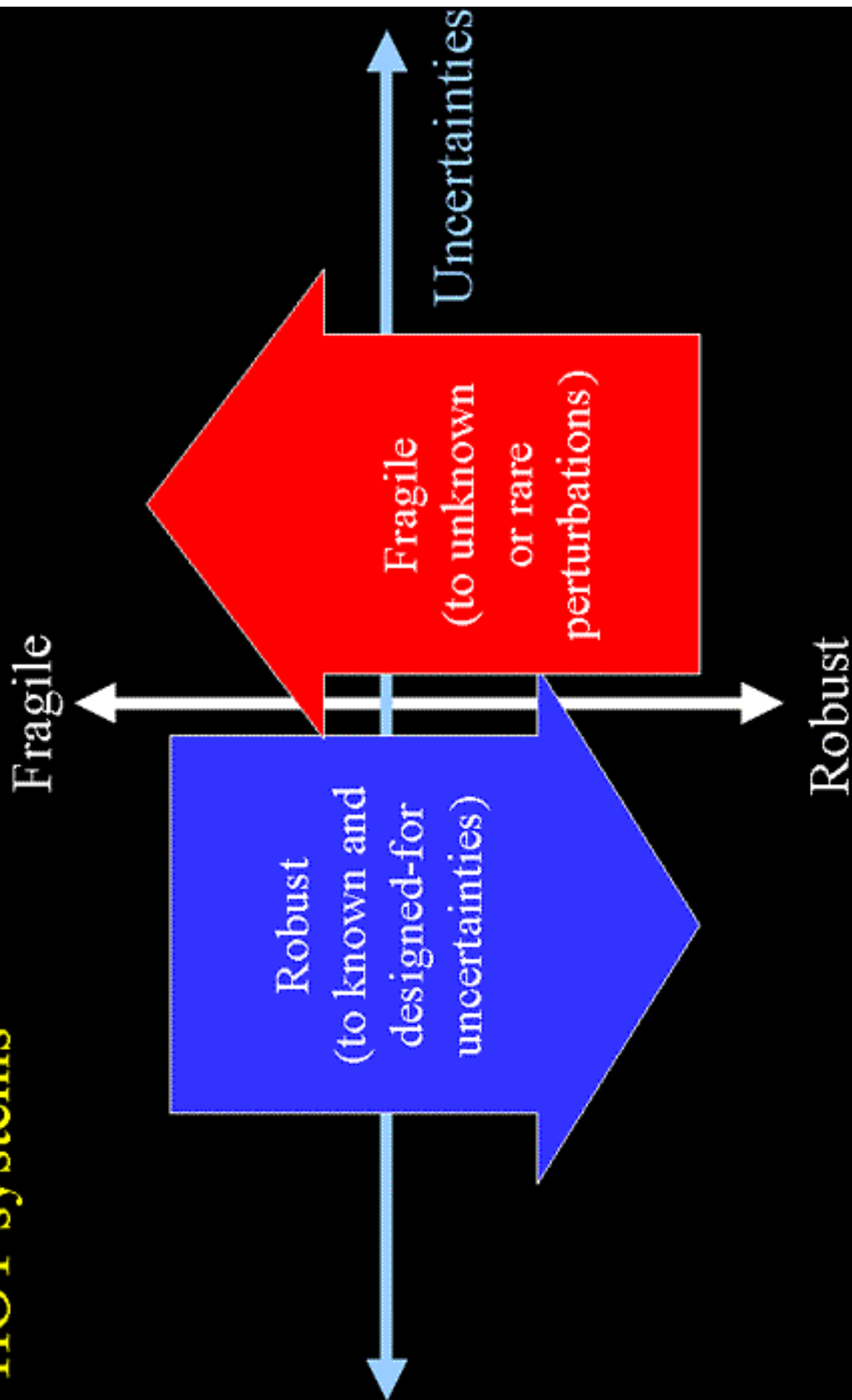
- Negative feedback is both the most powerful and most dangerous mechanism for robustness.
- It is everywhere in engineering, but appears hidden as long as it works.
- Biology seems to use it even more aggressively, but also uses other familiar engineering strategies:
 - Positive feedback to create switches (digital systems)
 - Protocol stacks
 - Feedforward control
 - Randomized strategies
 - Coding



Current research

- So far, this is all undergraduate level material
- Current research involves lots of math not traditionally thought of as “applied”
- New theoretical connections between robustness, evolvability, and verifiability
- Beginnings of a more integrated theory of control, communications and computing
- Both biology and the future of ubiquitous, embedded networking will drive the development of new mathematics.

Robustness of HOT systems



Robustness of HOT systems

Humans

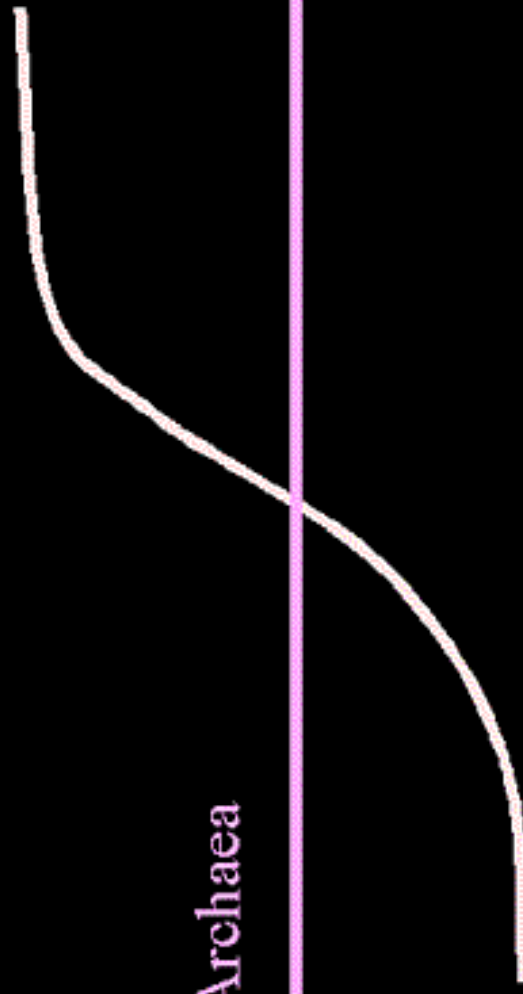
Archaea

Meteors

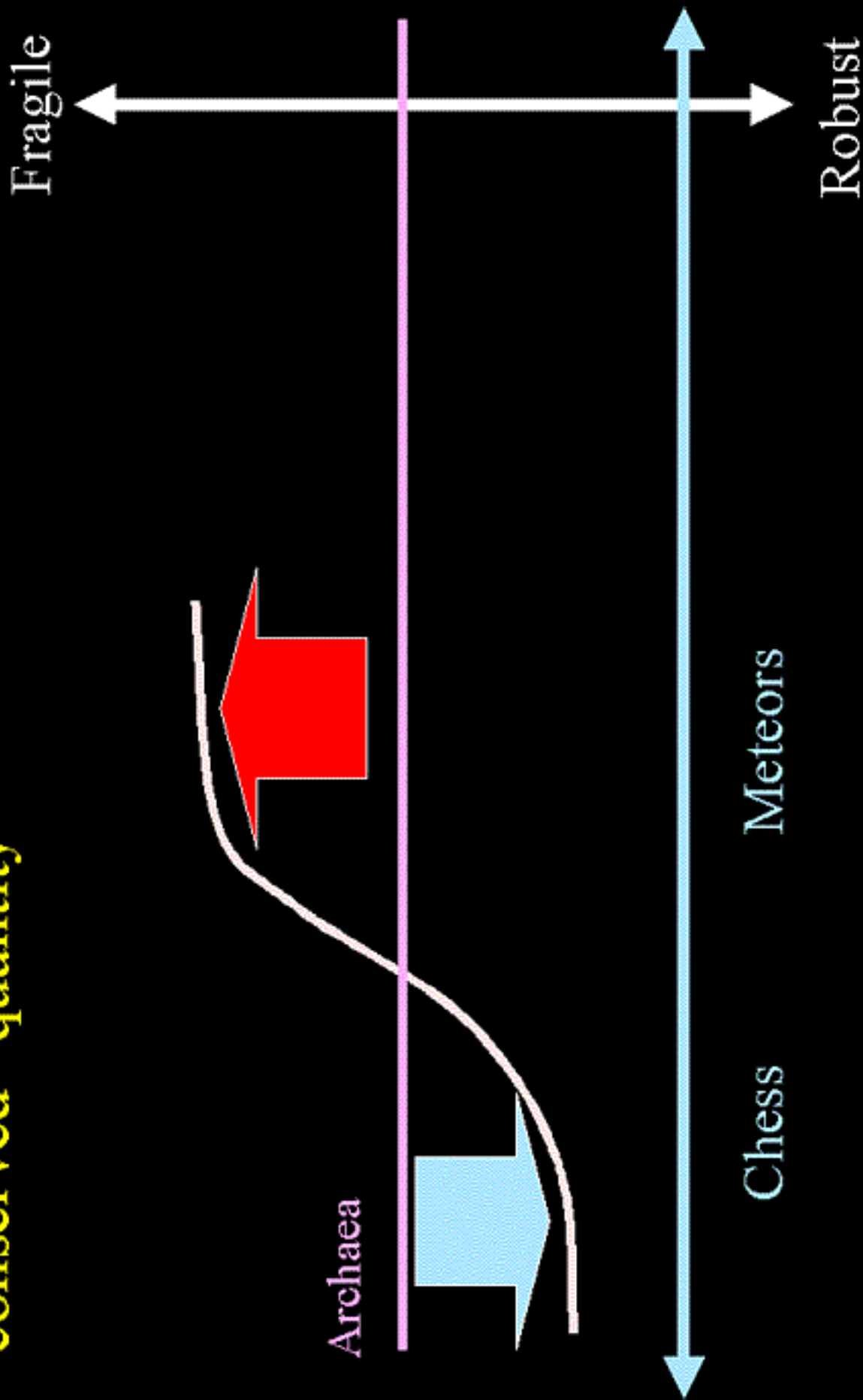
Chess

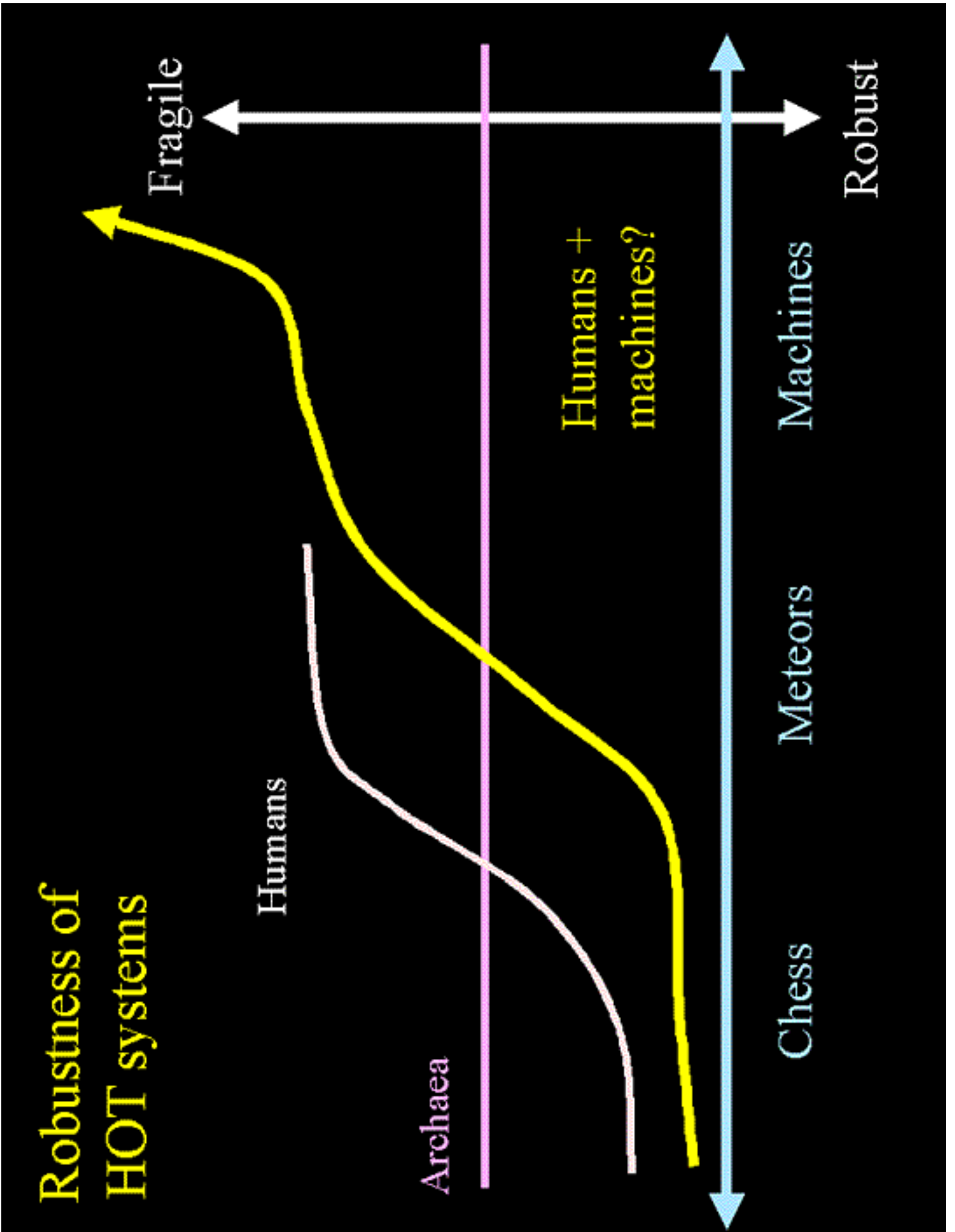
Fragile

Robust



Robustness is a
“conserved” quantity





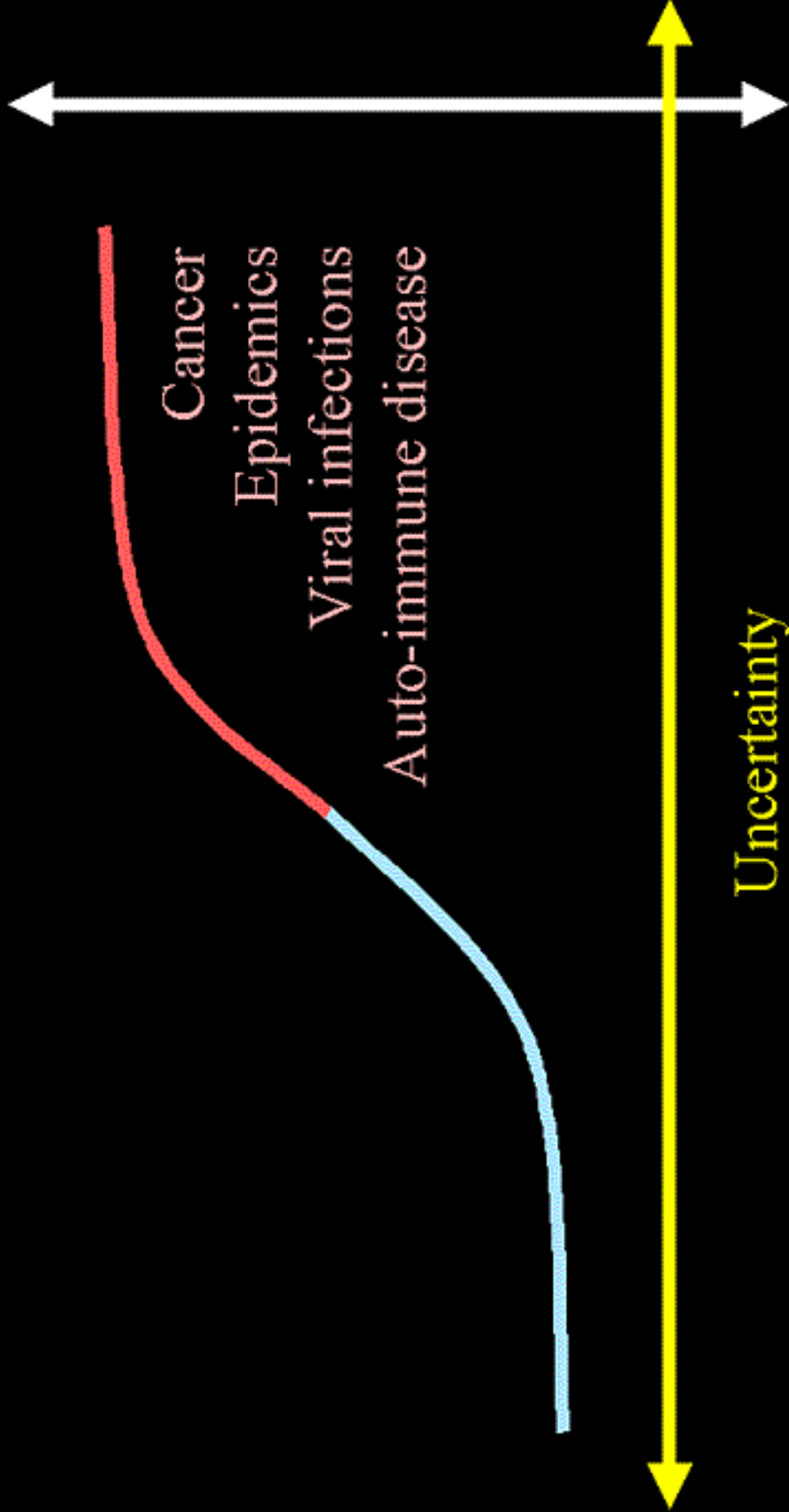
Diseases of complexity

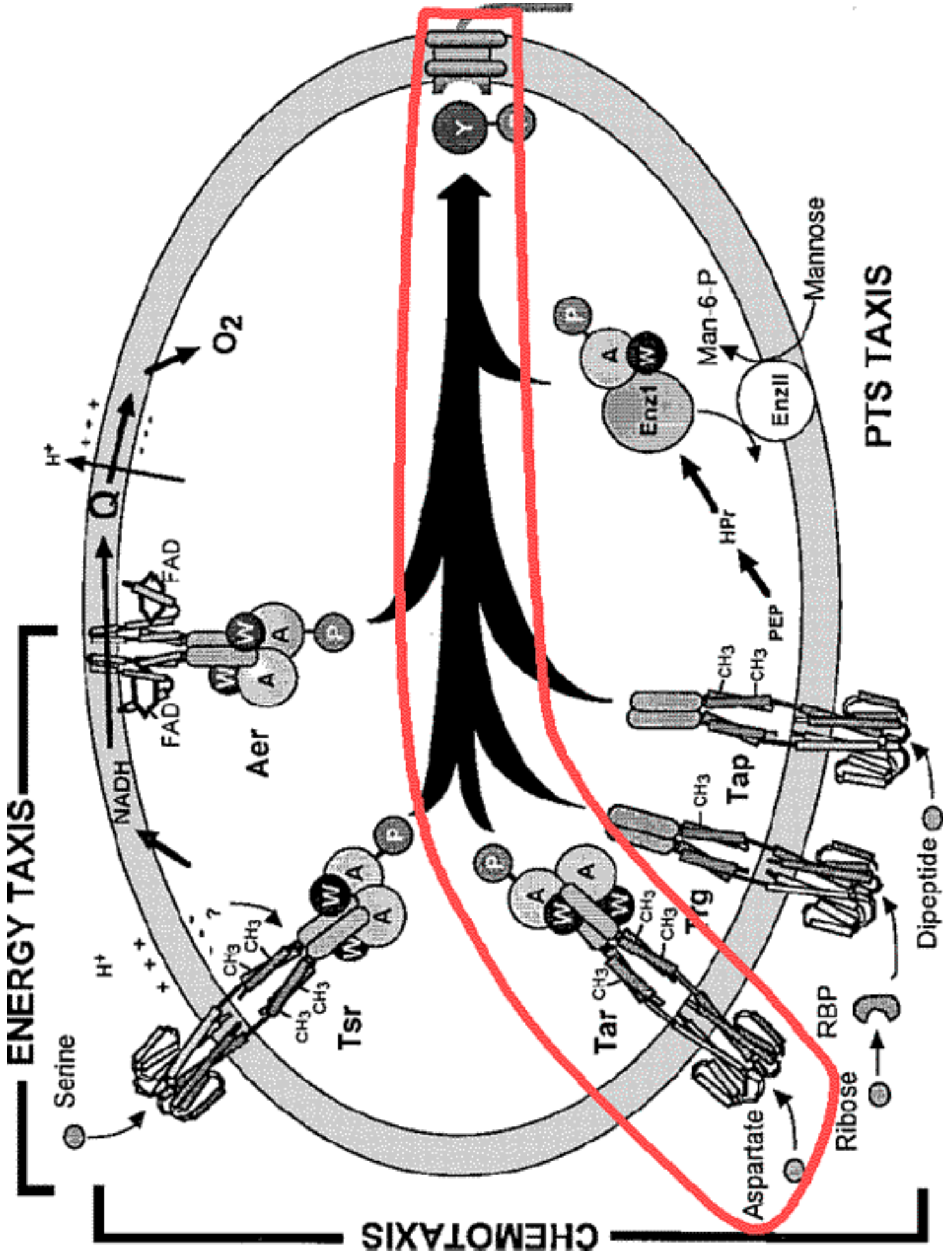
Fragile

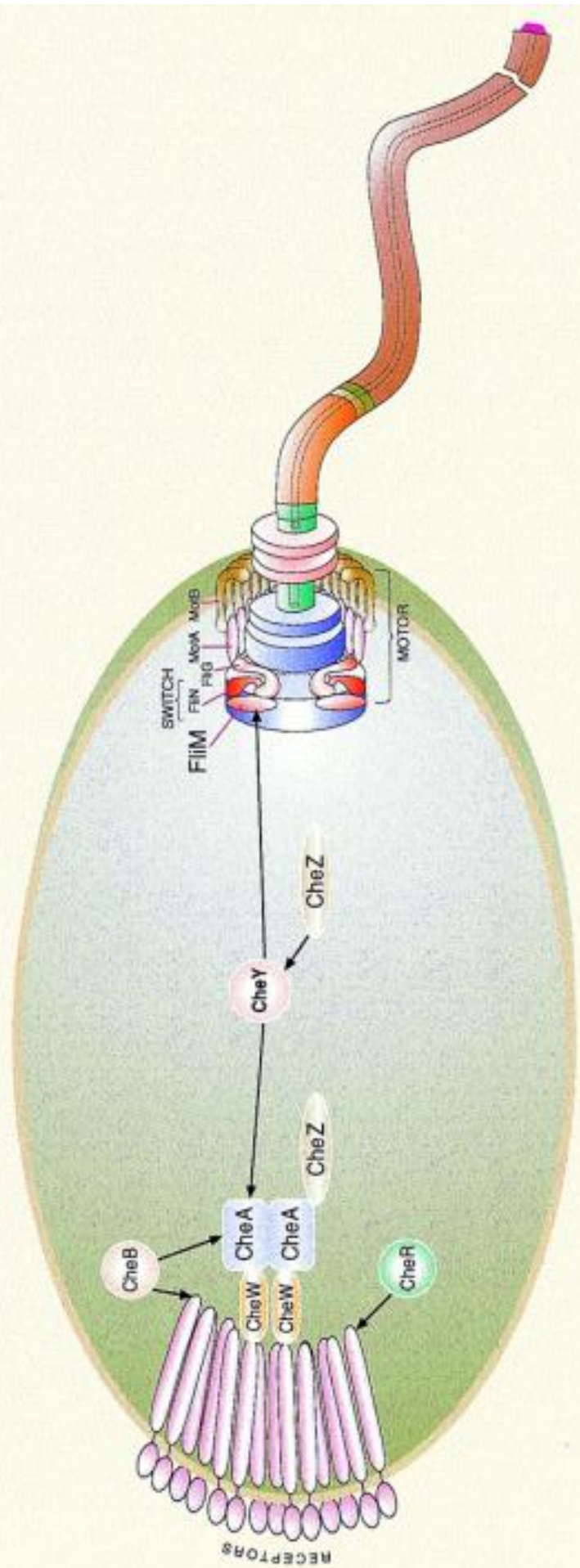
Cancer
Epidemics
Viral infections
Auto-immune disease

Robust

Uncertainty







Bacterial chemotaxis

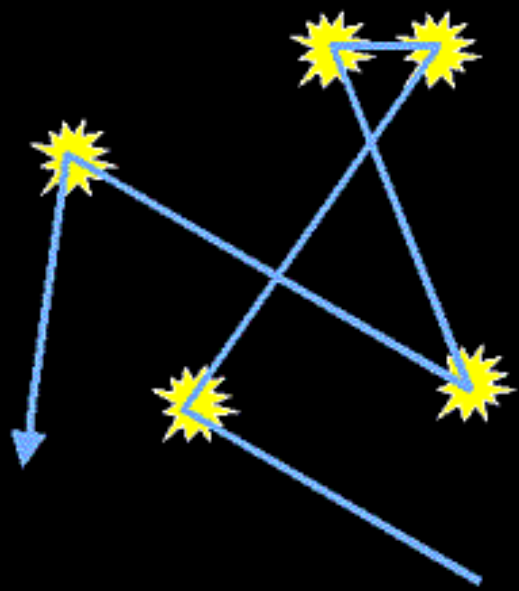
Proc. Natl. Acad. Sci. USA,
Vol. 97, Issue 9, 4649-4653, April 25, 2000

Biophysics

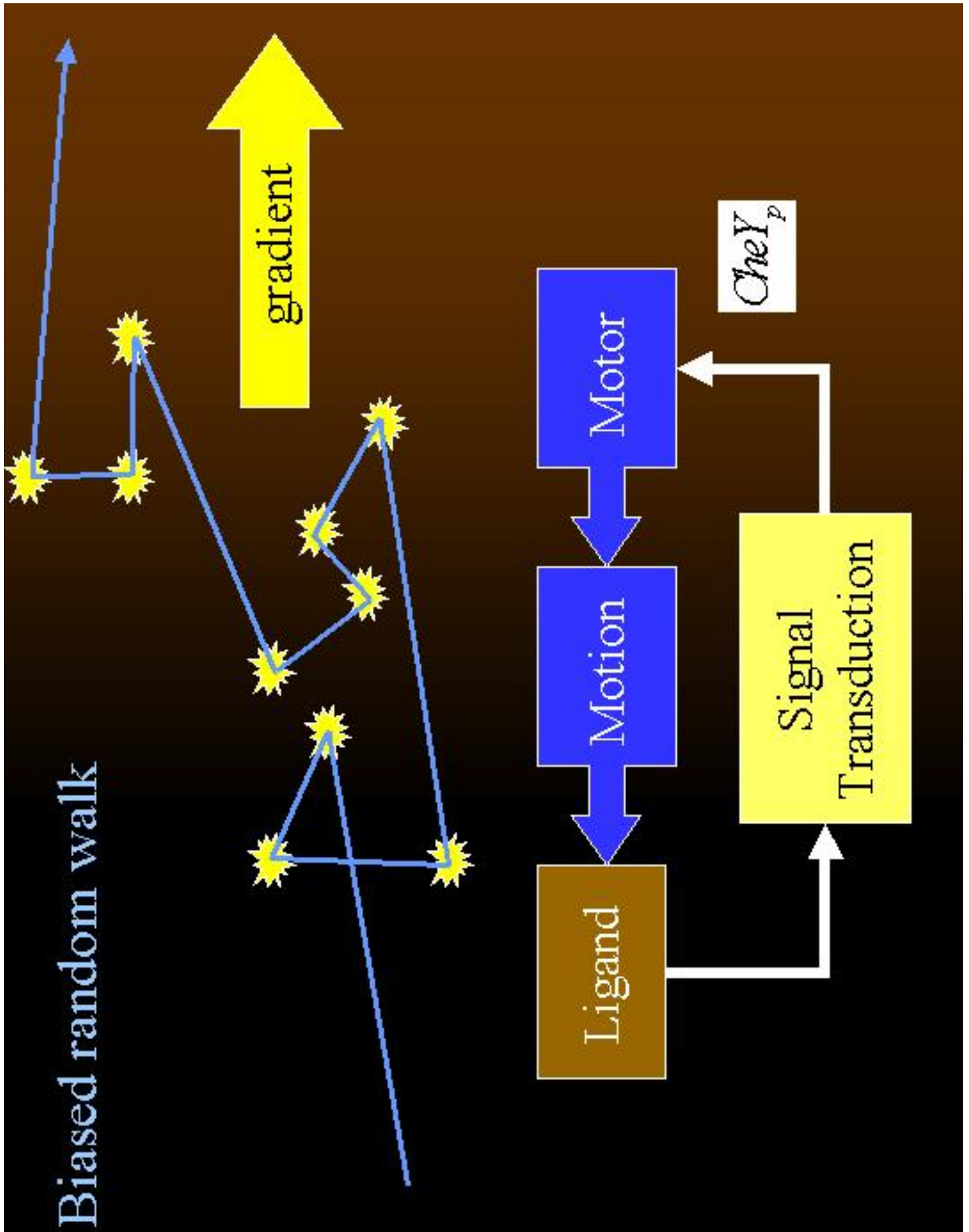
**Robust perfect adaptation in
bacterial chemotaxis through
integral feedback control**

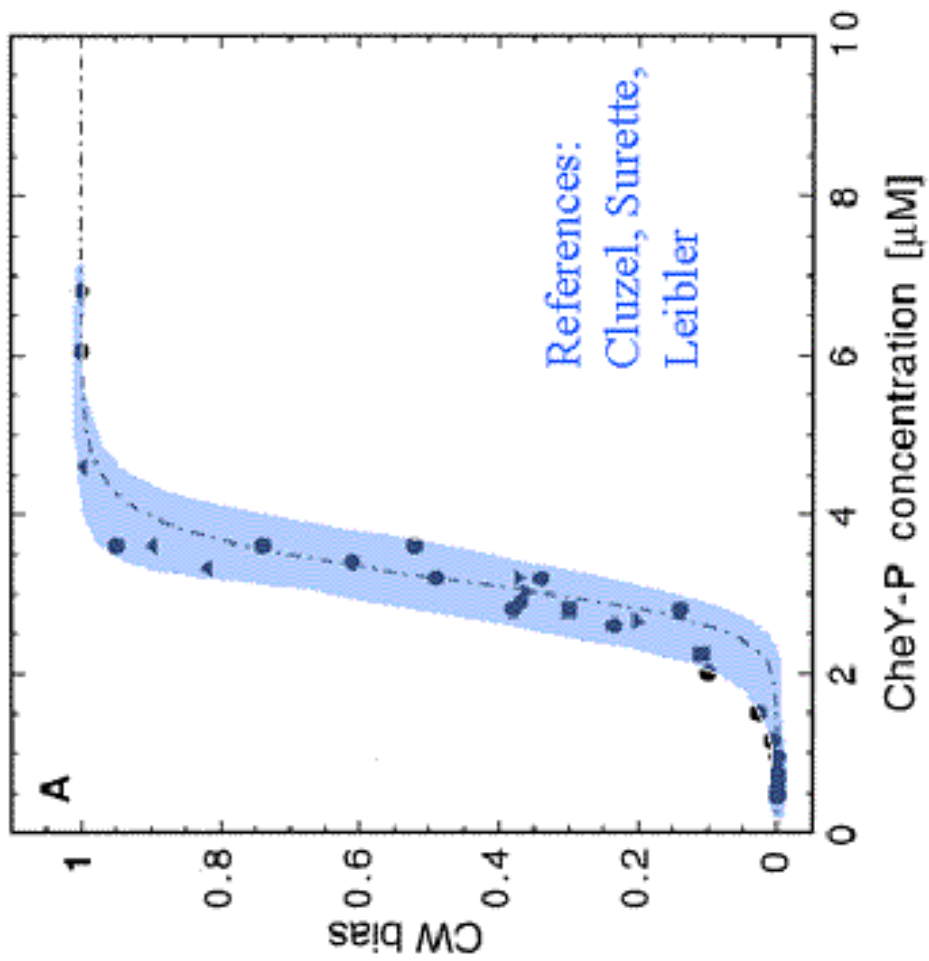
(Yi, Huang, Simon, Doyle)

Random walk



Bacterial chemotaxis (Yi, Huang, Simon, Doyle)





High gain (cooperativity)

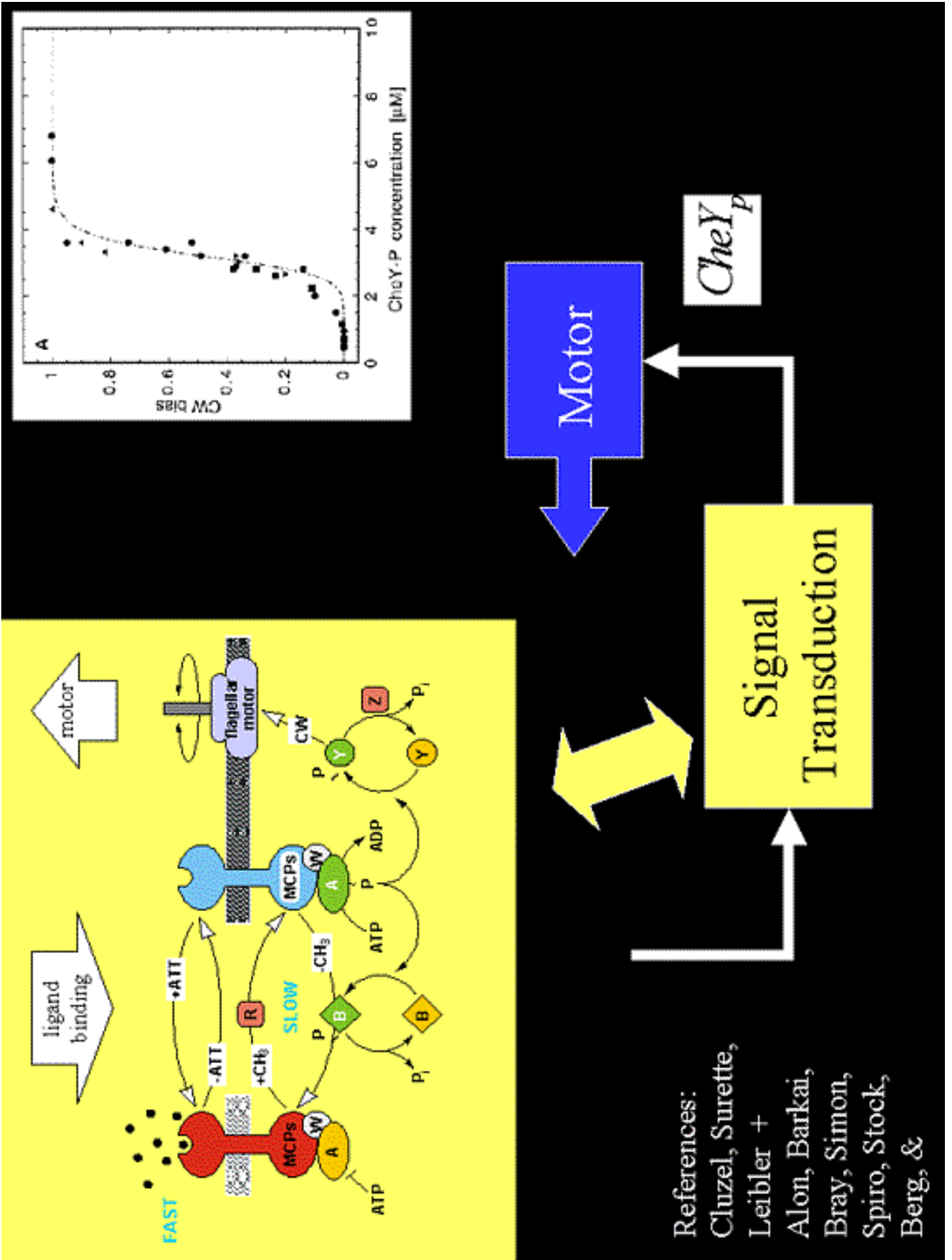
“ultrasensitivity”



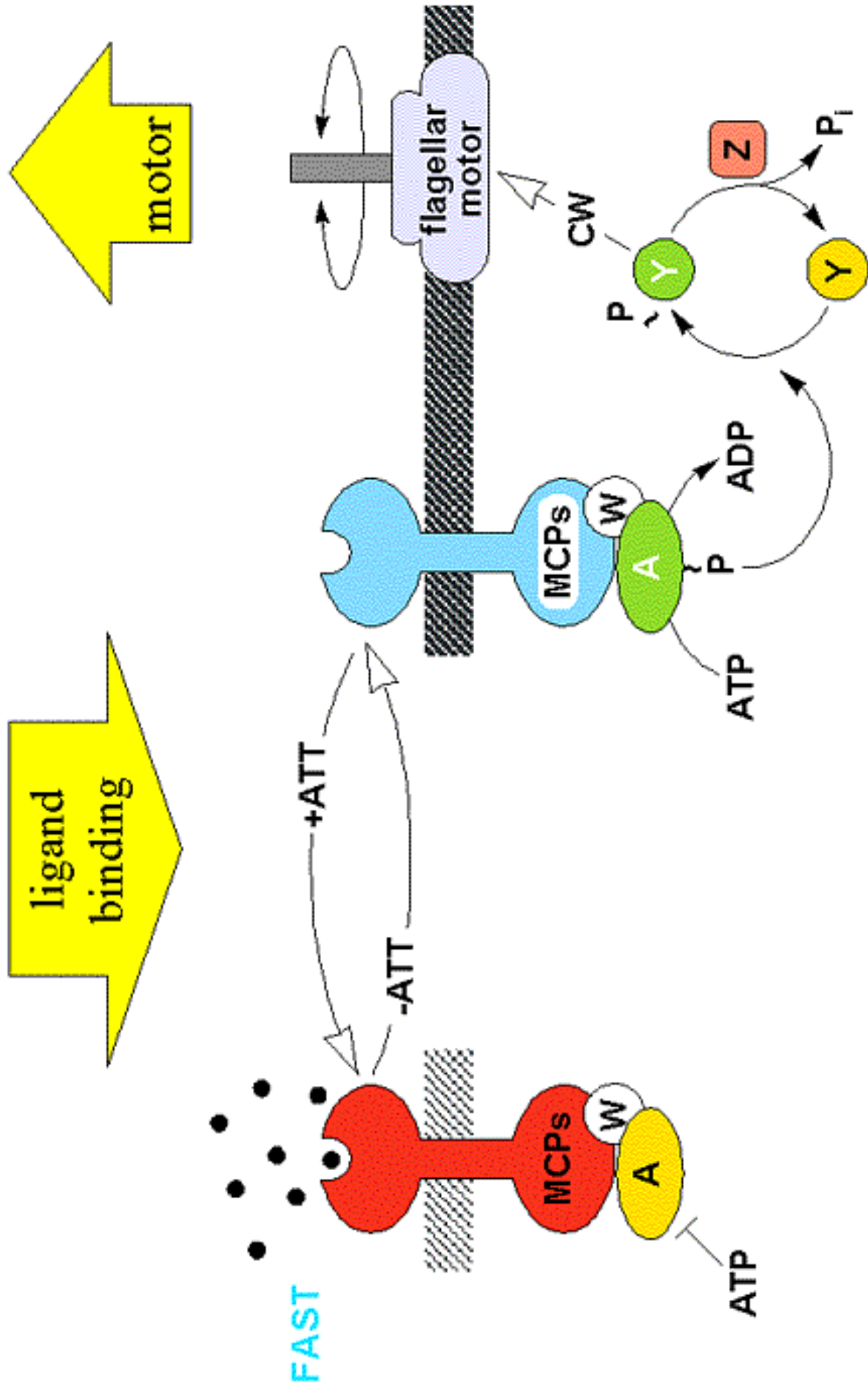
$CheY_P$



on

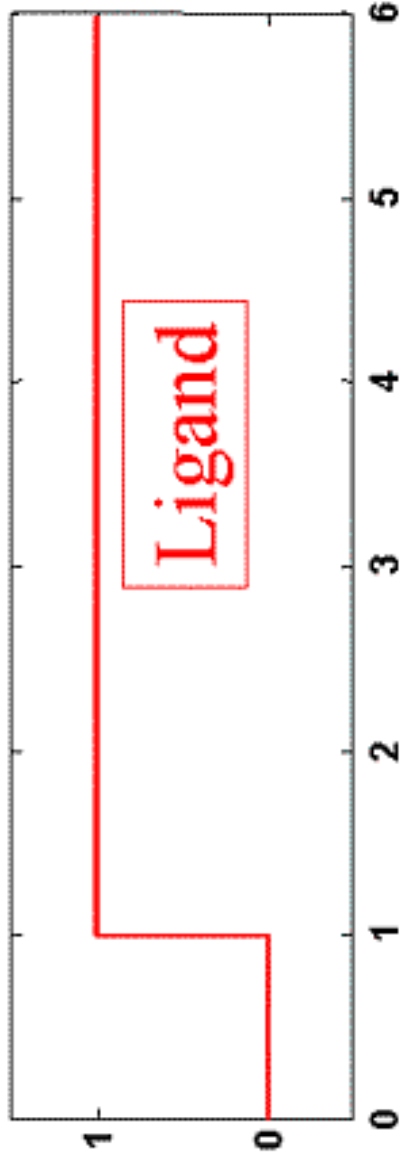


References:
 Cluzel, Surette,
 Leibler +
 Alon, Barkai,
 Bray, Simon,
 Spiro, Stock,
 Berg, &

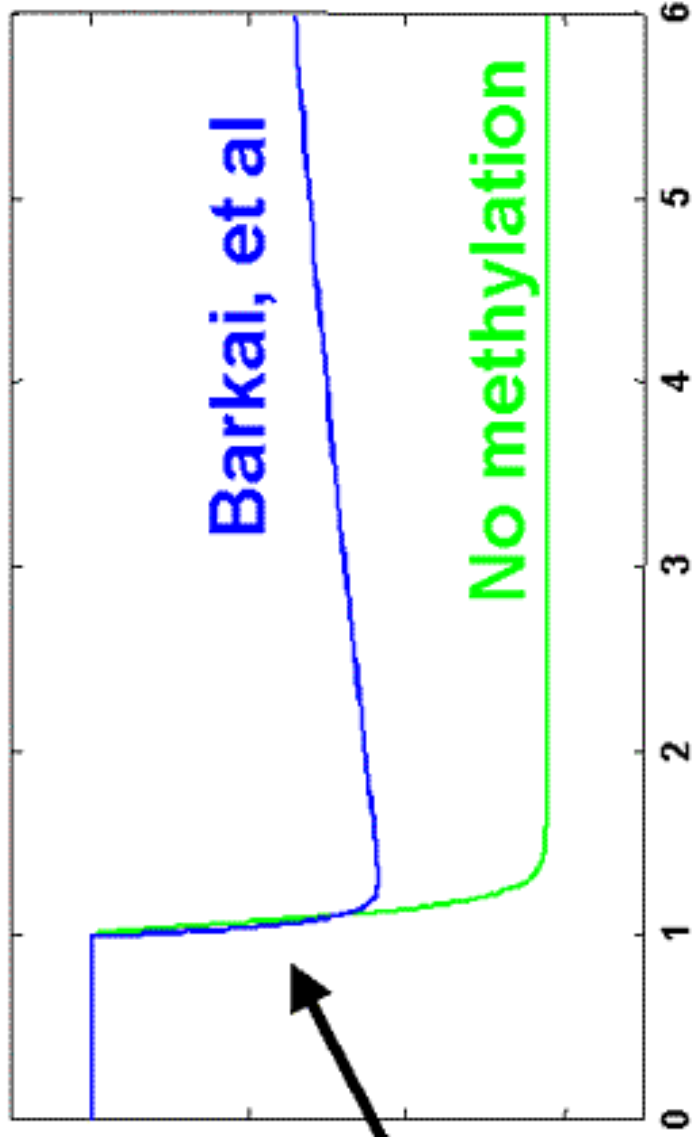


Fast (ligand and phosphorylation)

Short time
 Y_p response



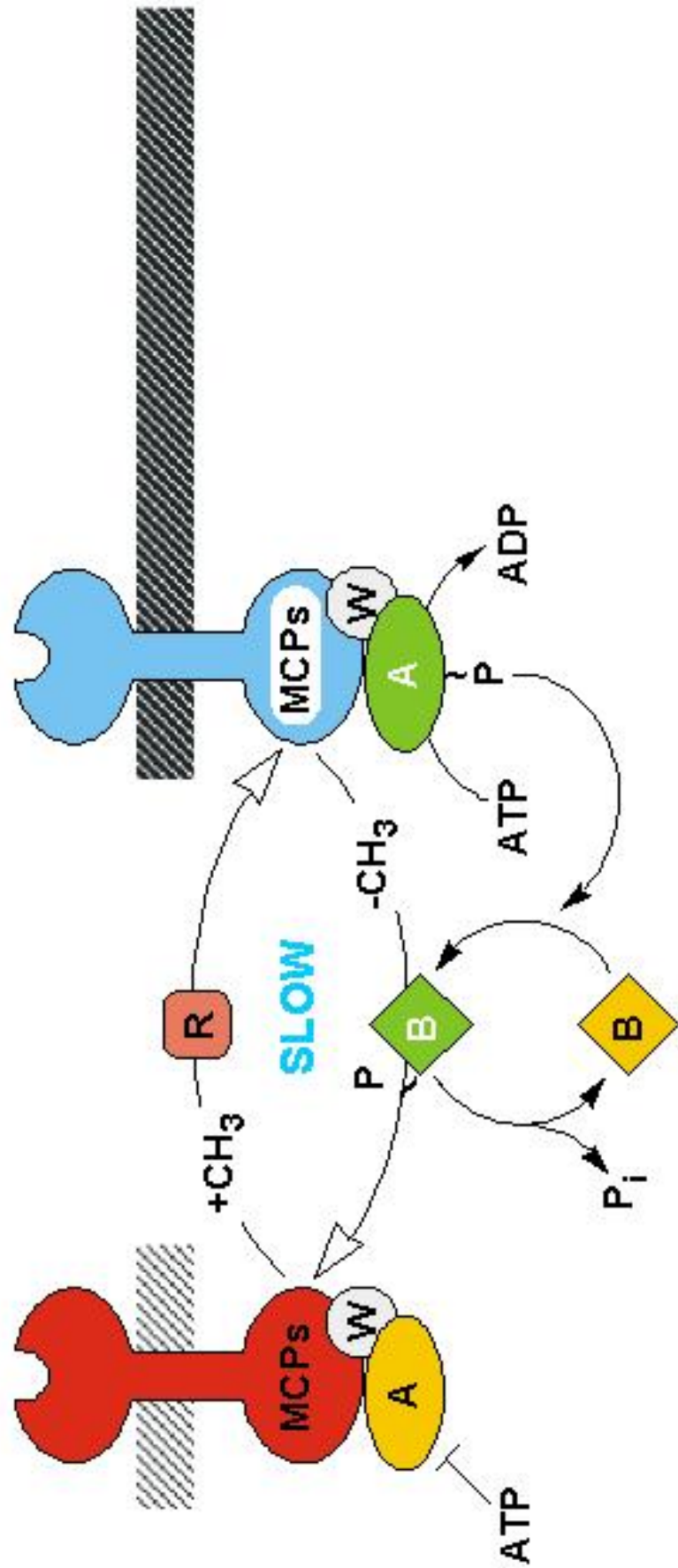
Che Yp

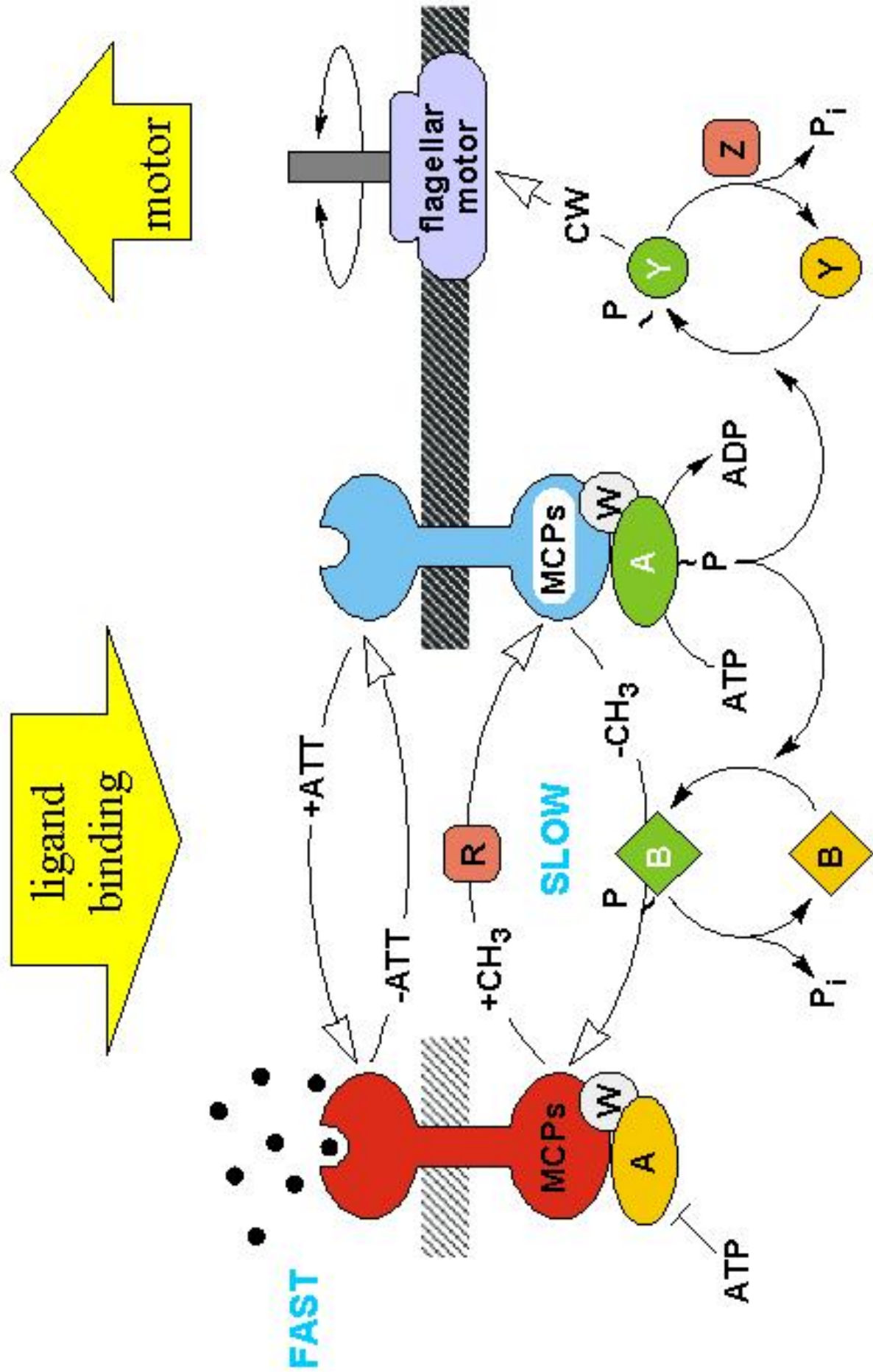


Extend run
(more ligand)

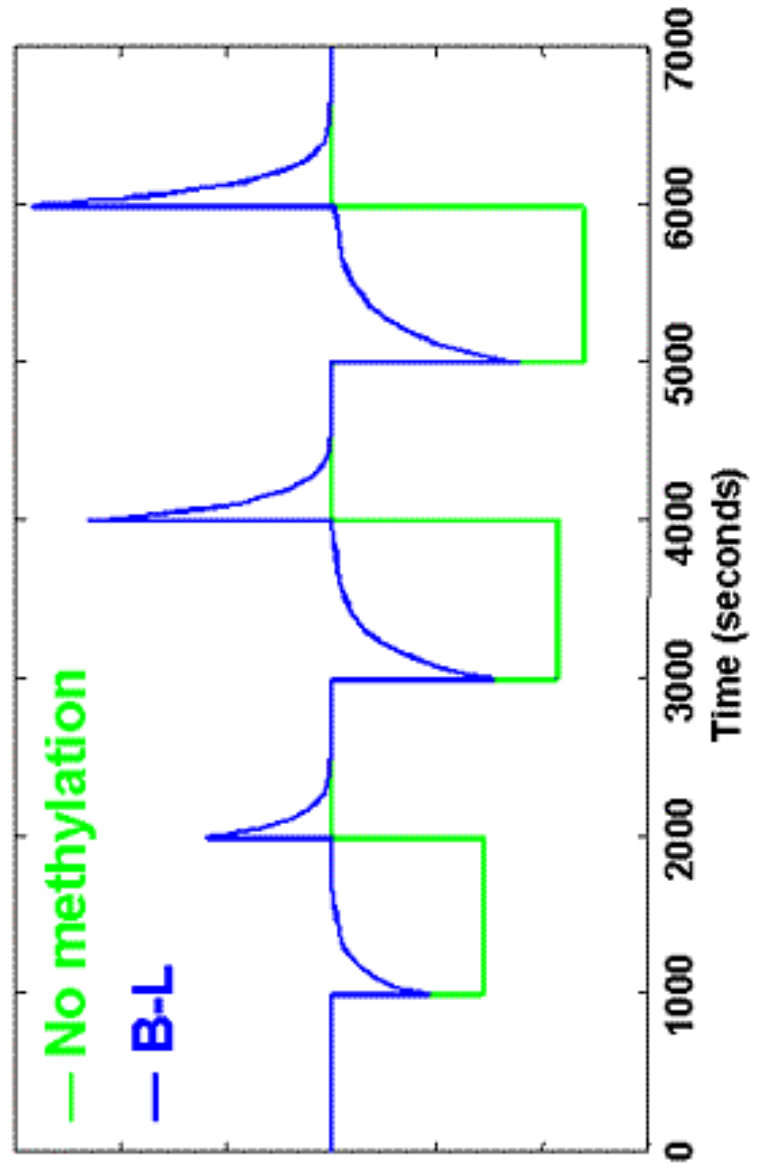
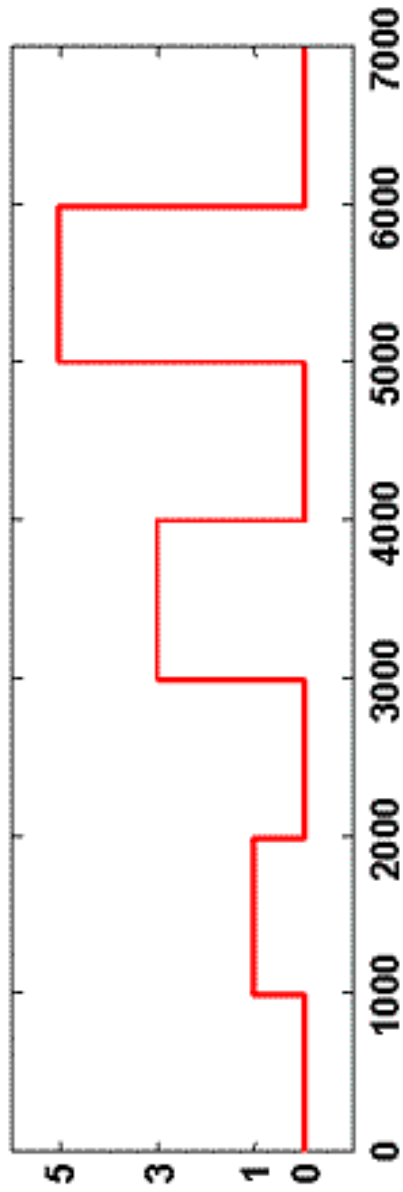
Time (seconds)

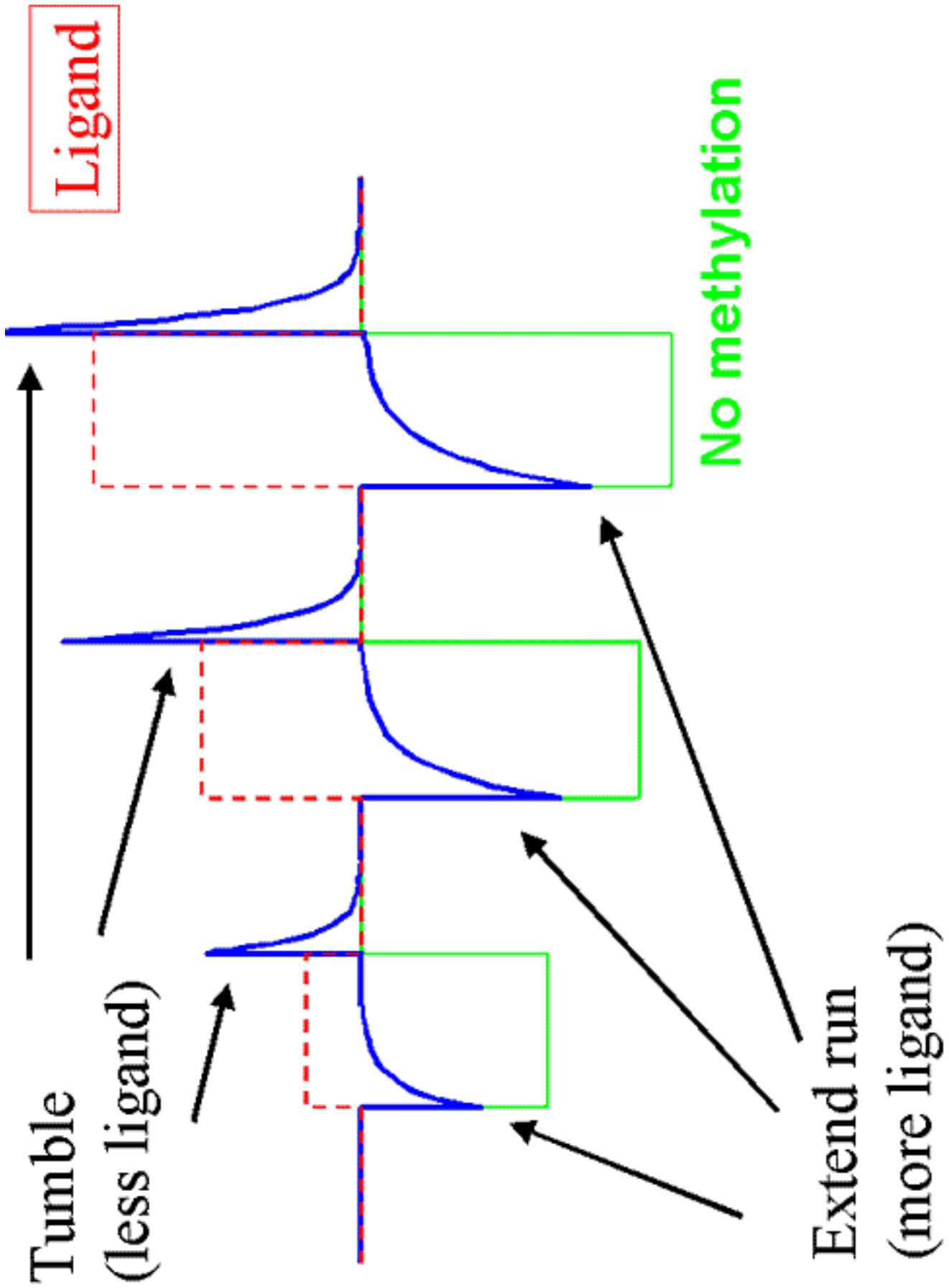
Slow (de-) methylation dynamics



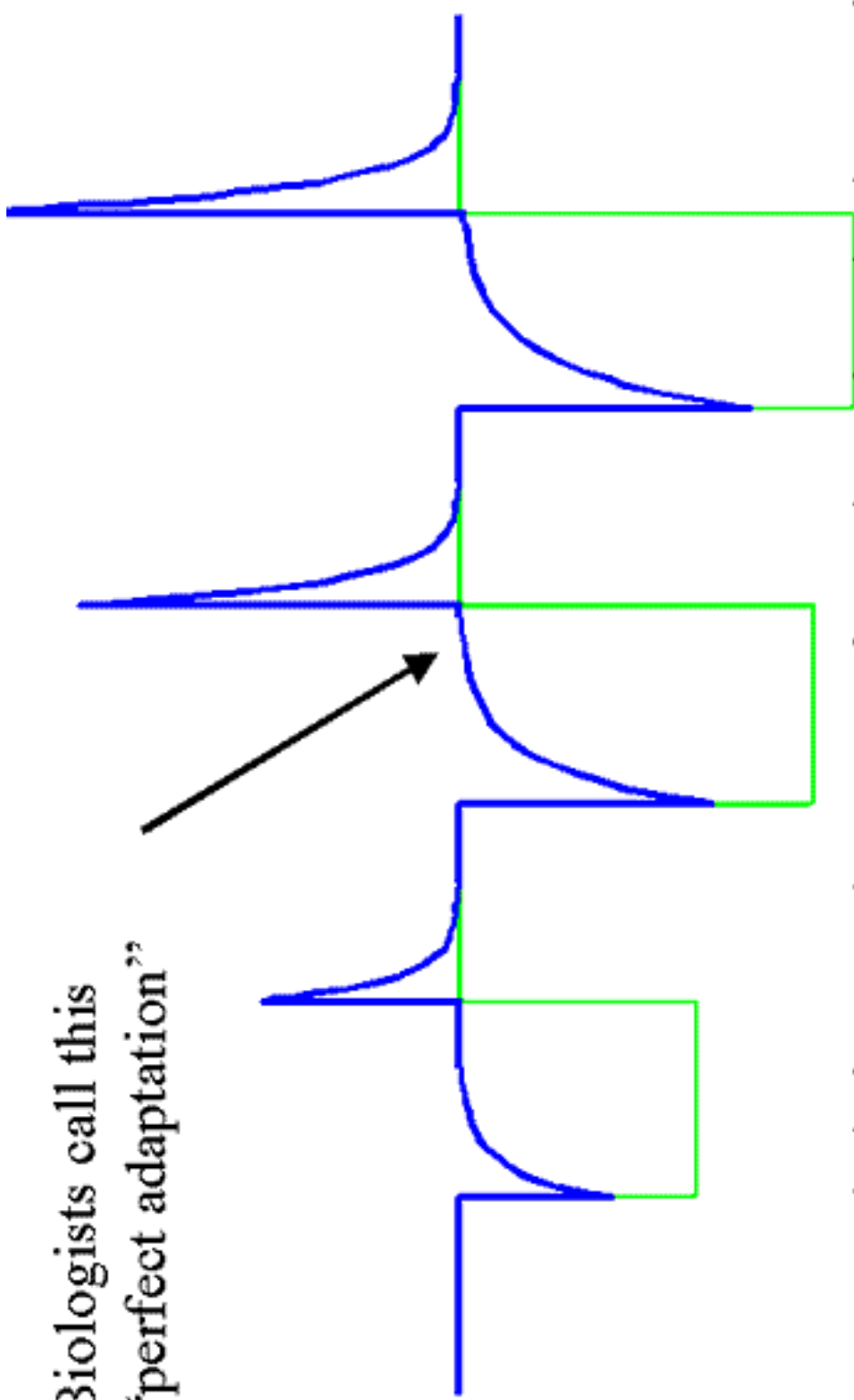


Long time Yp response





Biologists call this
“perfect adaptation”

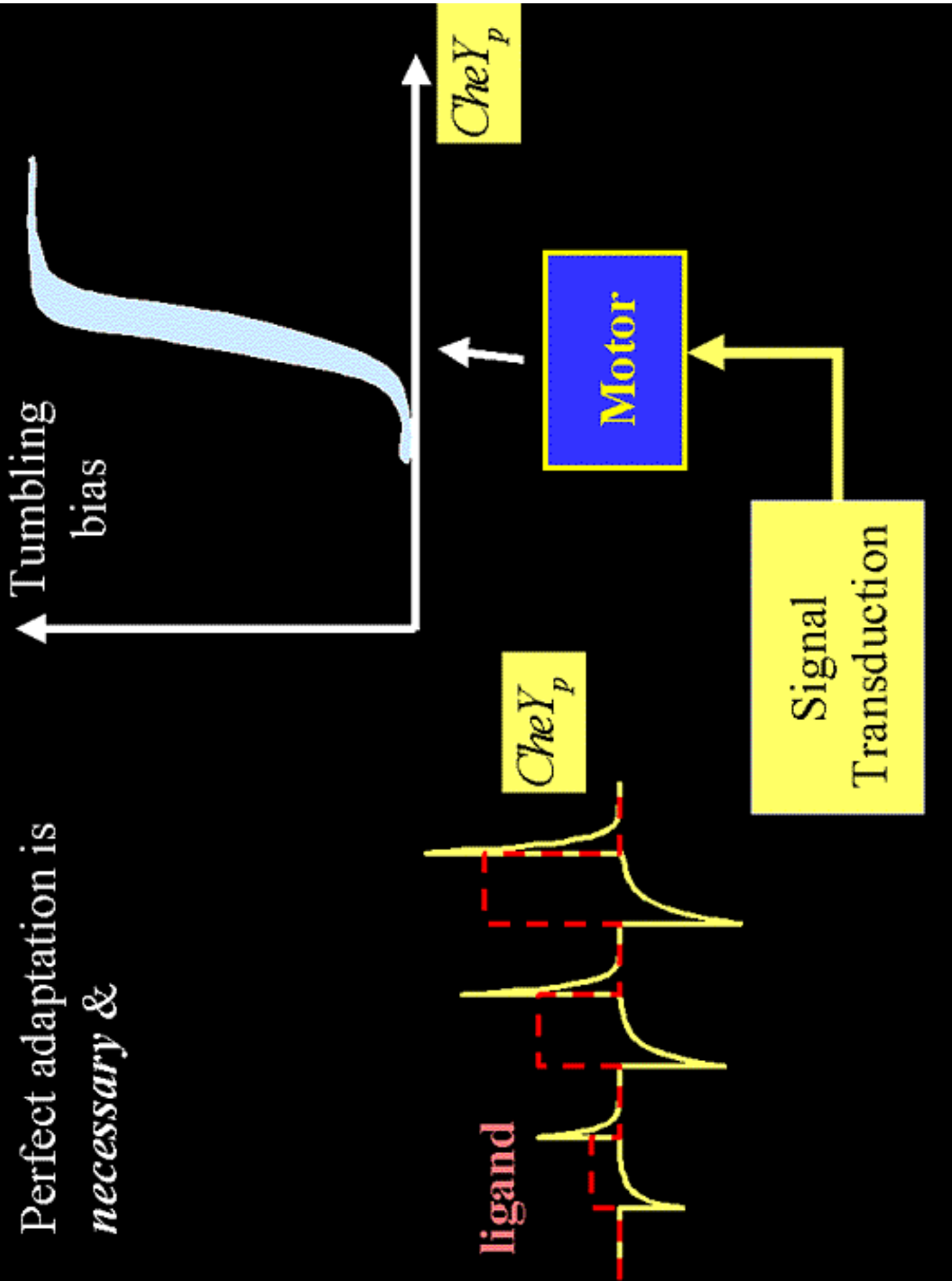


• Methylation produces “perfect adaptation” by *integral*

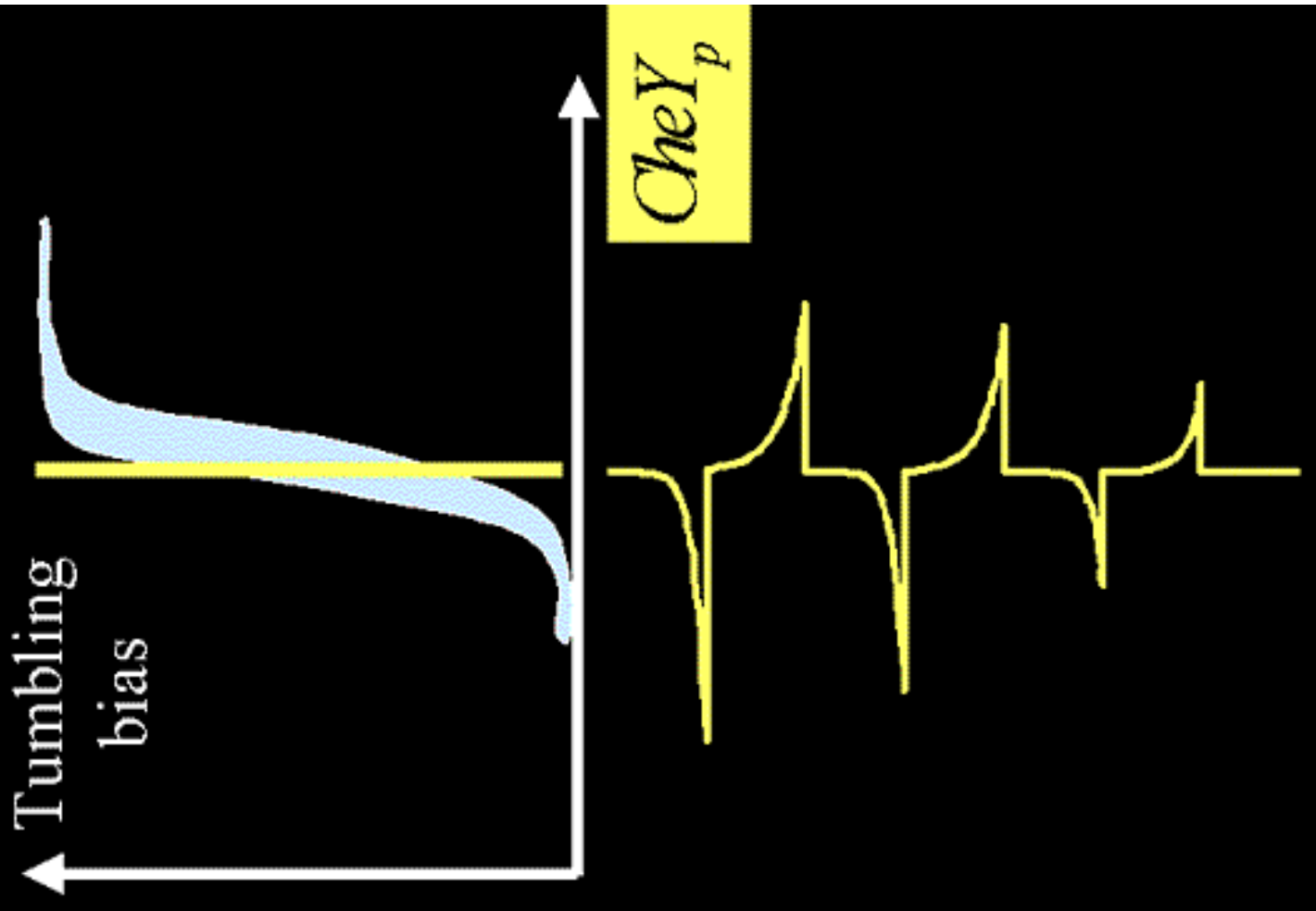
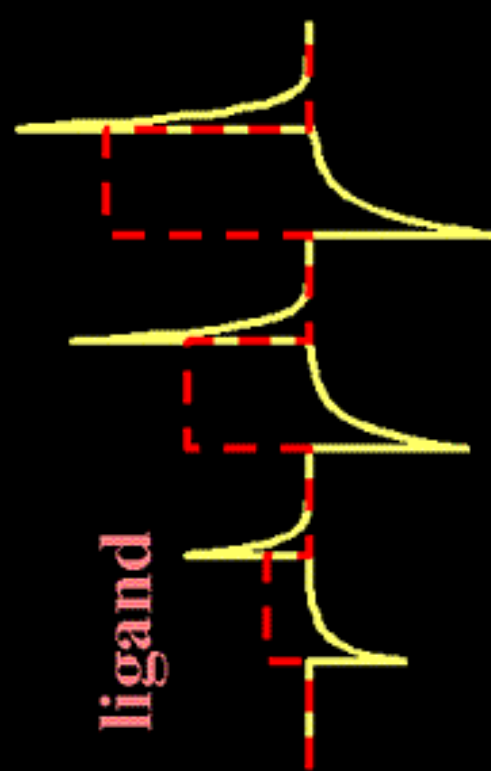
feedback.

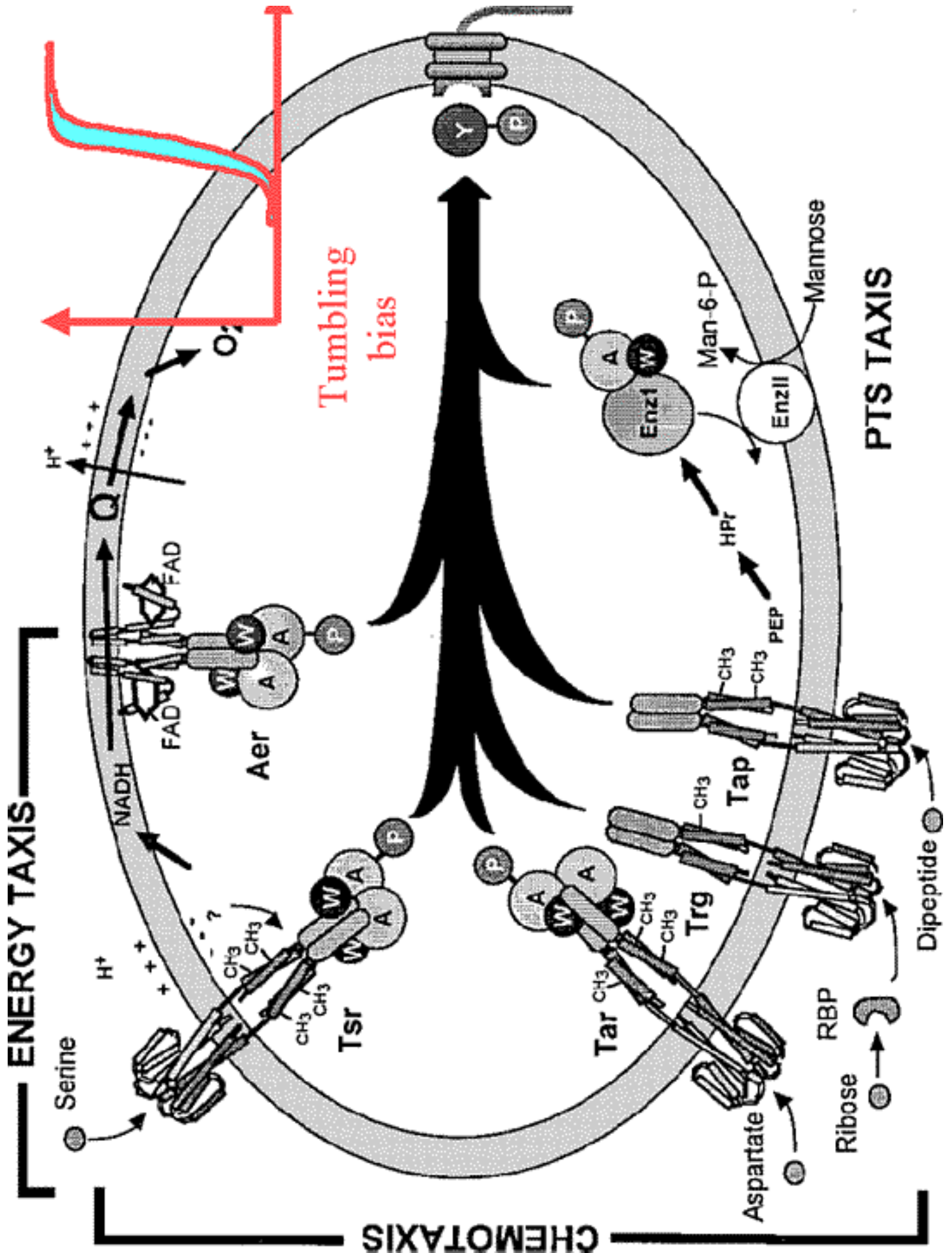
- Integral feedback is ubiquitous in both engineering systems and biological systems.
- Integral feedback is *necessary* for *robust* perfect adaptation.

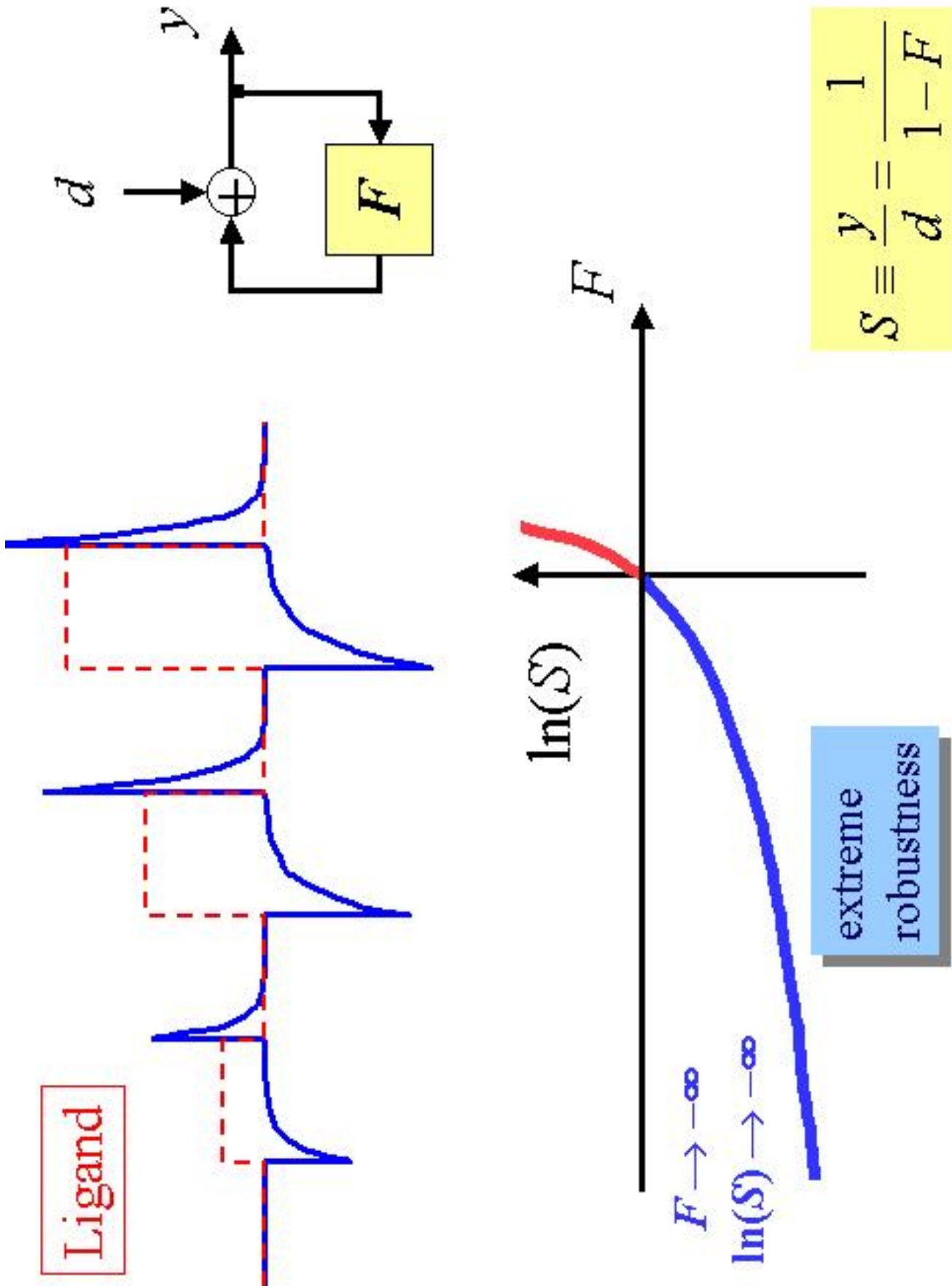
Perfect adaptation is
necessary &



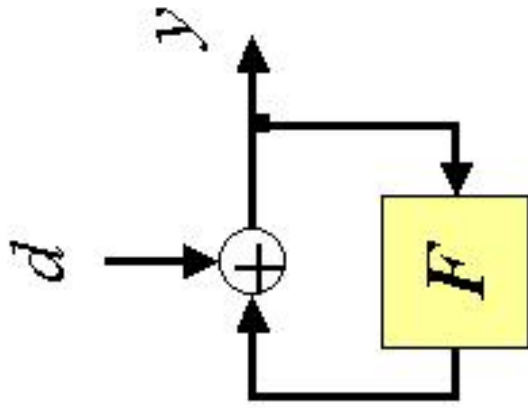
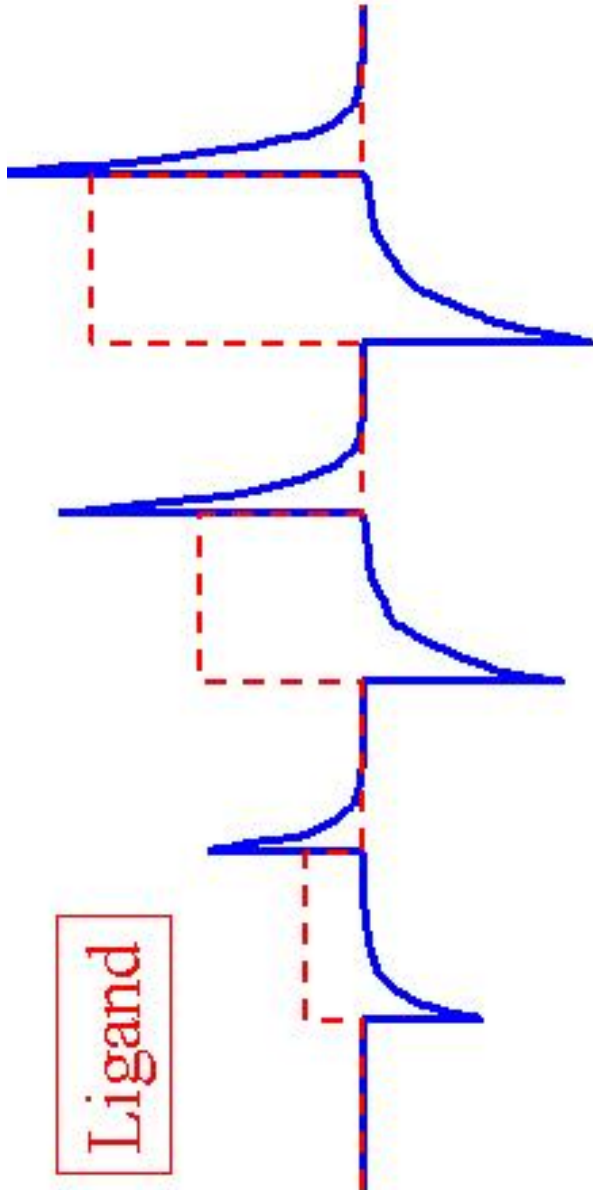
Perfect adaptation is
necessary ...
...to keep CheYp in the
responsive range of the
motor.







Ligand



Integral feedback

$$F(s) = \frac{\hat{F}(s)}{s}, \quad \hat{F}(0) < 0$$

$$\Leftrightarrow F(0) = -\infty$$

$$\Leftrightarrow S(0) = 0$$

$$F \rightarrow -\infty$$

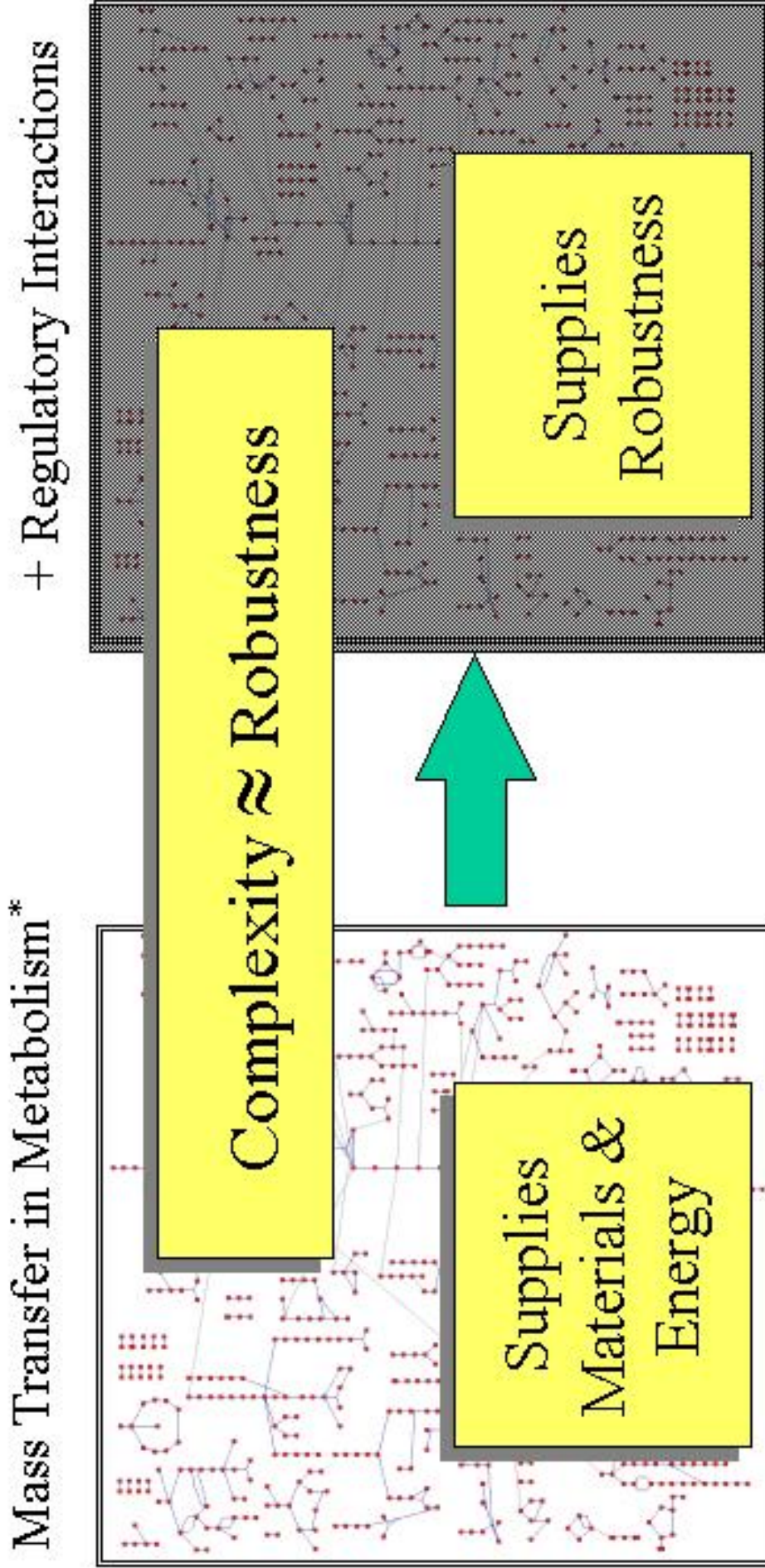
$$\ln(S) \rightarrow -\infty$$

$$S \equiv \frac{y}{d} = \frac{1}{1-F}$$

Fine tuned or robust ?

- Maybe just not the right question.
- Fine tuned *for* robustness...
- ... with resource costs and new fragilities as the price.
- Necessity:
 - How network *must* be (robustness)
 - Price to pay (fragility)

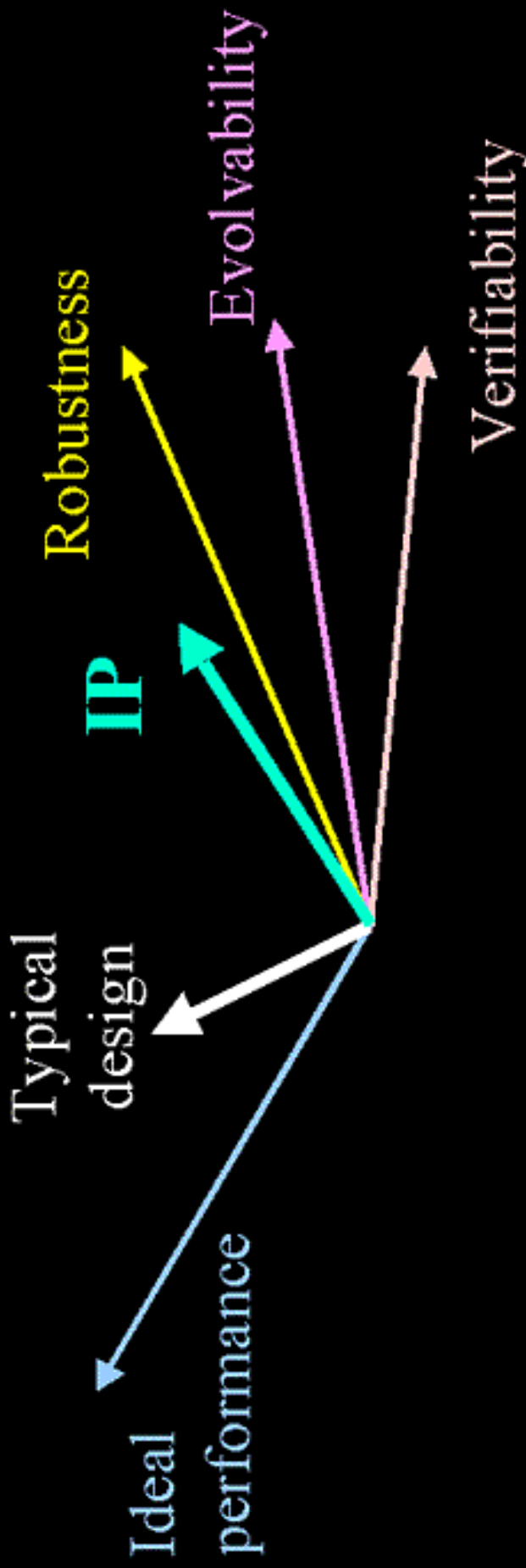
Biochemical Network: *E. Coli* Metabolism



From Adam Arkin

* from: EcoCYC by Peter Karp

Robustness, evolvability/scalability, verifiability



- Theory: Robust control, complexity (P-NP-coNP), convex and stochastic relaxations, semi-algebraic geometry, semidefinite programming.
- Examples from biology and engineering

What about ?

- Information & entropy
 - Fractals & self-similarity
 - Chaos
 - Criticality and power laws
 - Undecidability
 - Fuzzy logic, neural nets, genetic algorithms
 - Emergence
 - Self-organization
 - Complex adaptive systems
 - New science of complexity
- Not really about complexity
 - These concepts themselves are “robust, yet fragile”
 - Powerful in their niche
 - Brittle (break easily) when moved or extended
 - Some are relevant to biology and engineering systems
 - Comfortably reductionist
 - Remarkably useful in getting published

Feedback is very powerful, but there are limitations.

It gives us remarkable robustness, as well as recursion and looping.

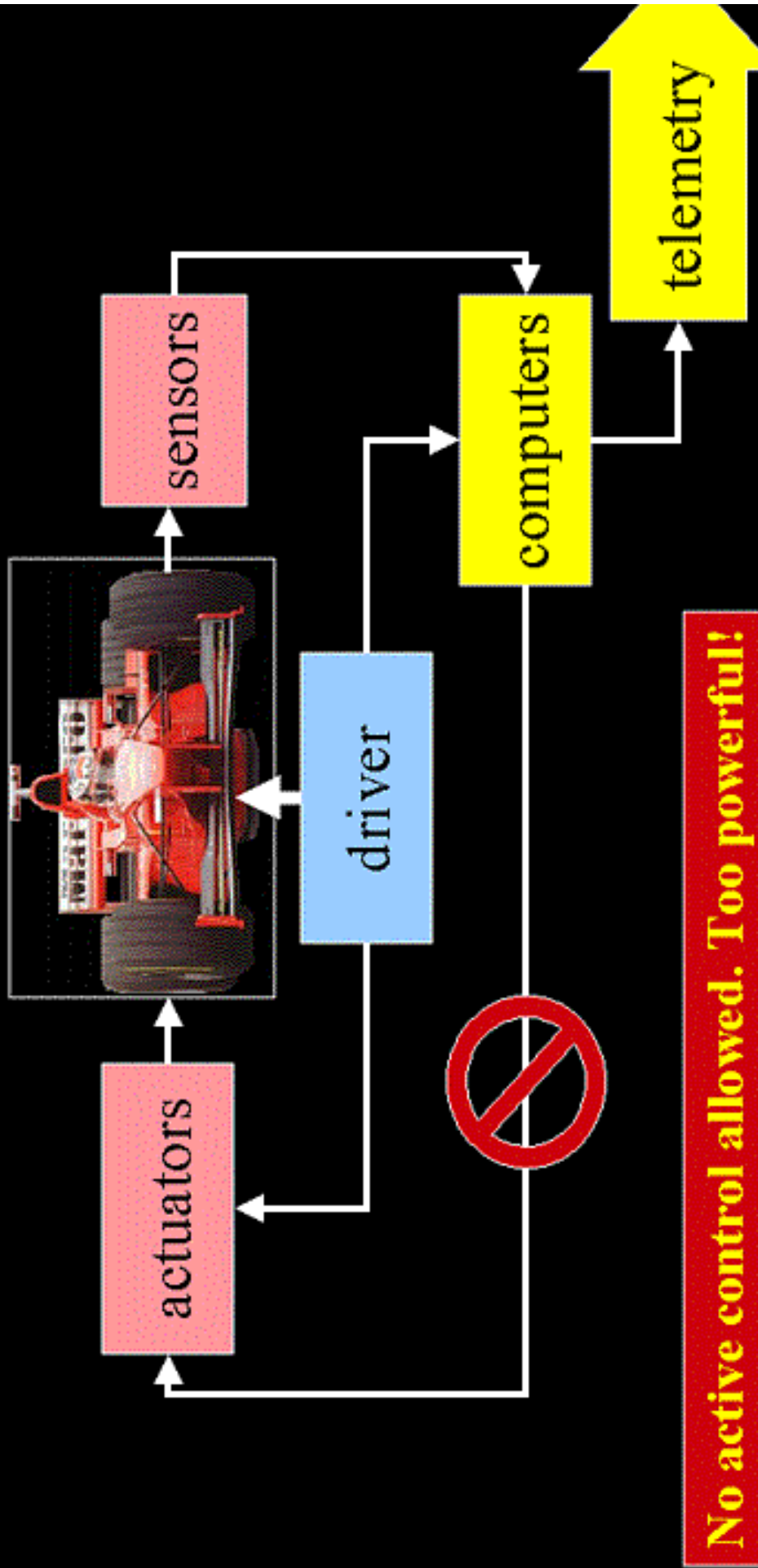
But can lead to instability, chaos, and undecidability.



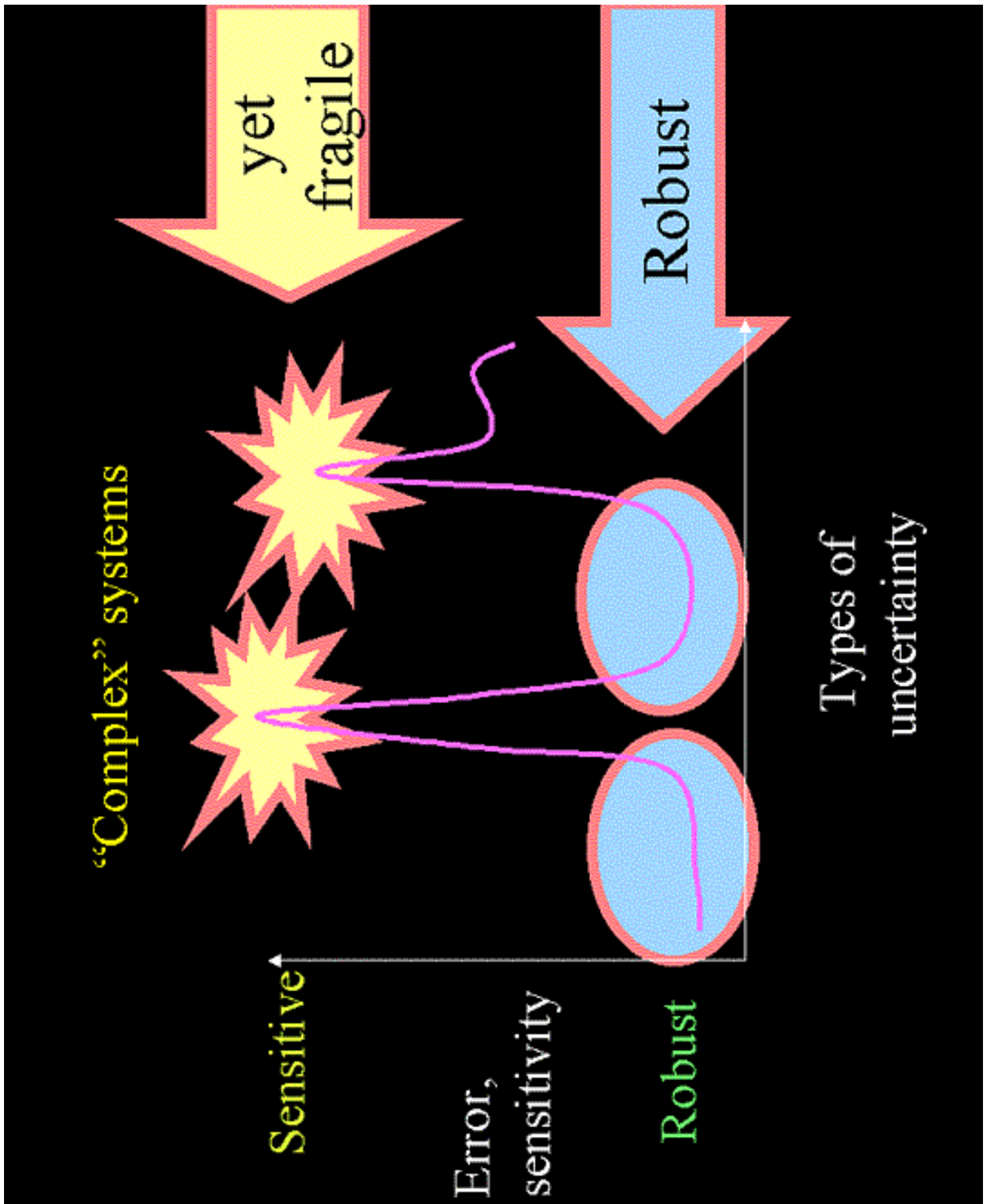
**Formula 1:
The ultimate high
technology sport**

- Electronic fuel injection
- Computers
- Sensors
- Telemetry/Communications
- Power steering

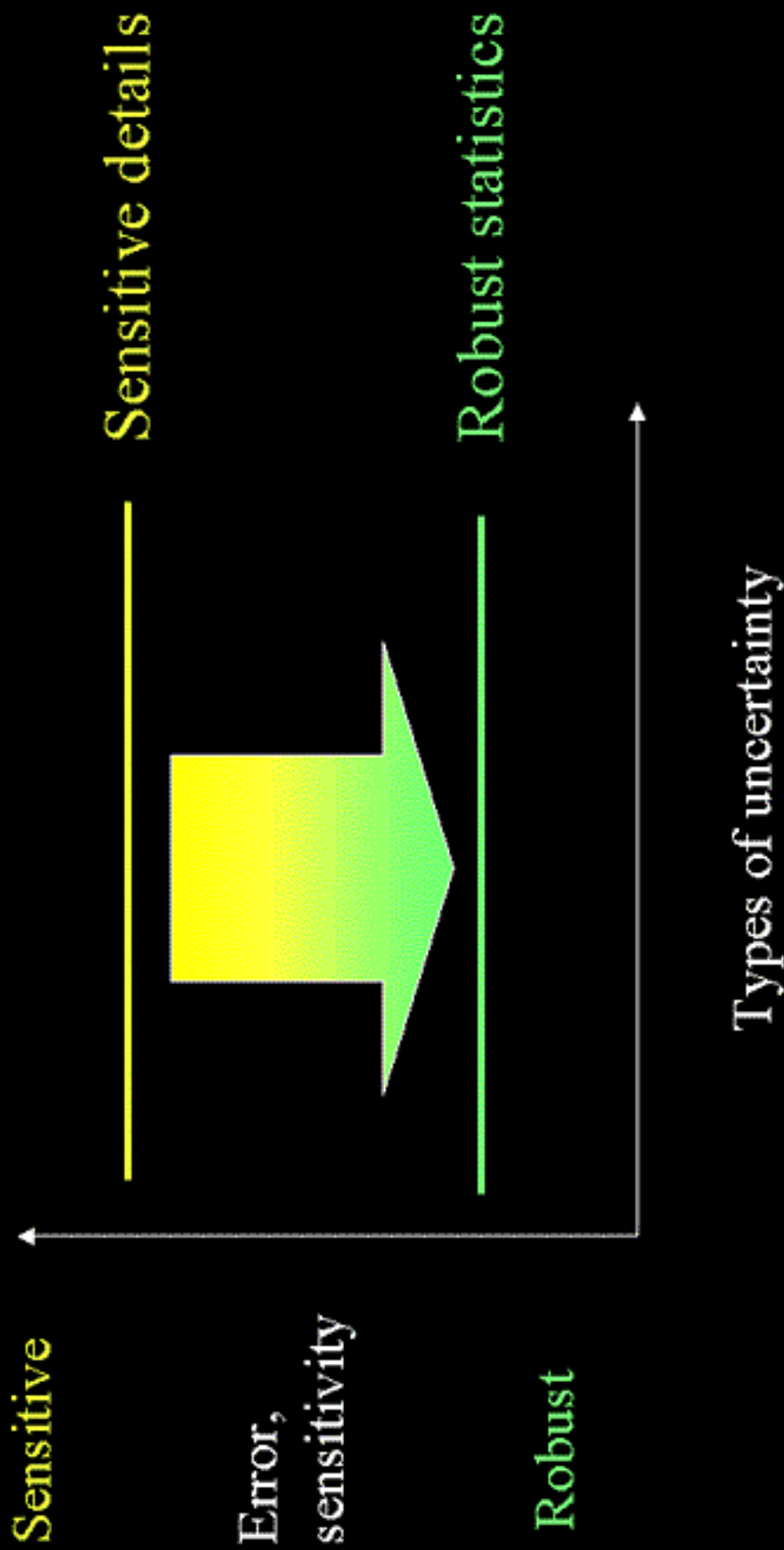
Formula 1 allows:



No active control allowed. Too powerful!



Multiscale modeling: Homogeneous systems



Multiscale, homogeneous example

Gas molecules
in this room

Statistical
Mechanics

Temperature
and pressure

Sensitive details

Robust statistics

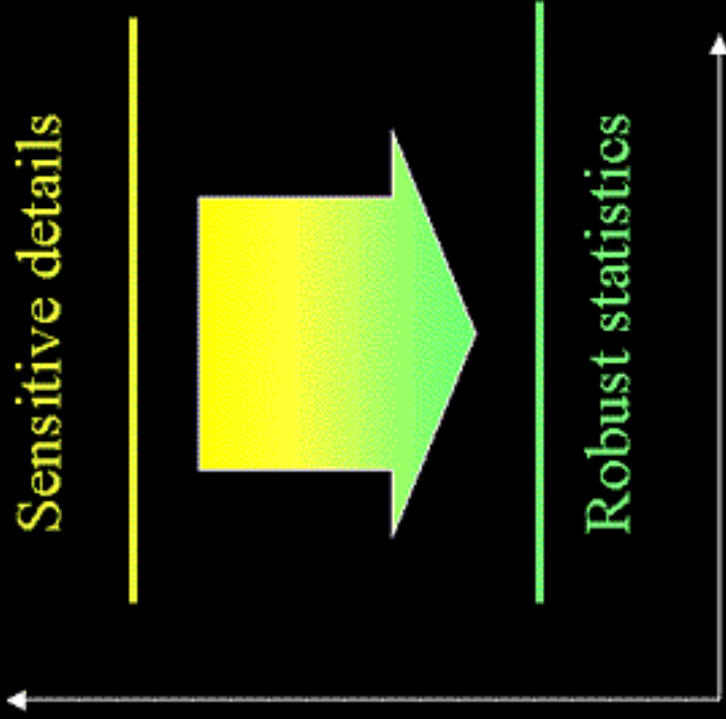
Sensitive

Error,
sensitivity

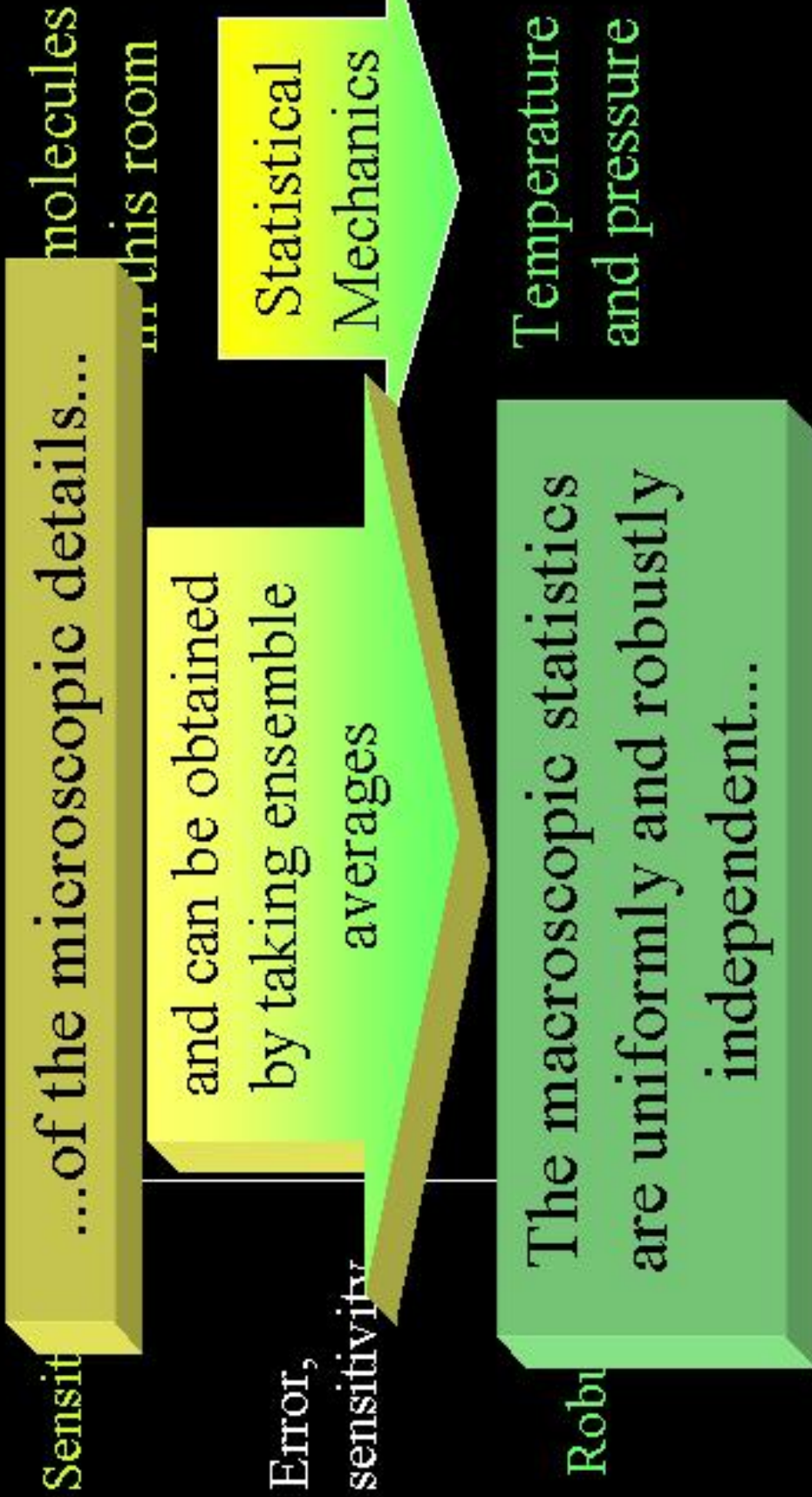
Robust

initial conditions

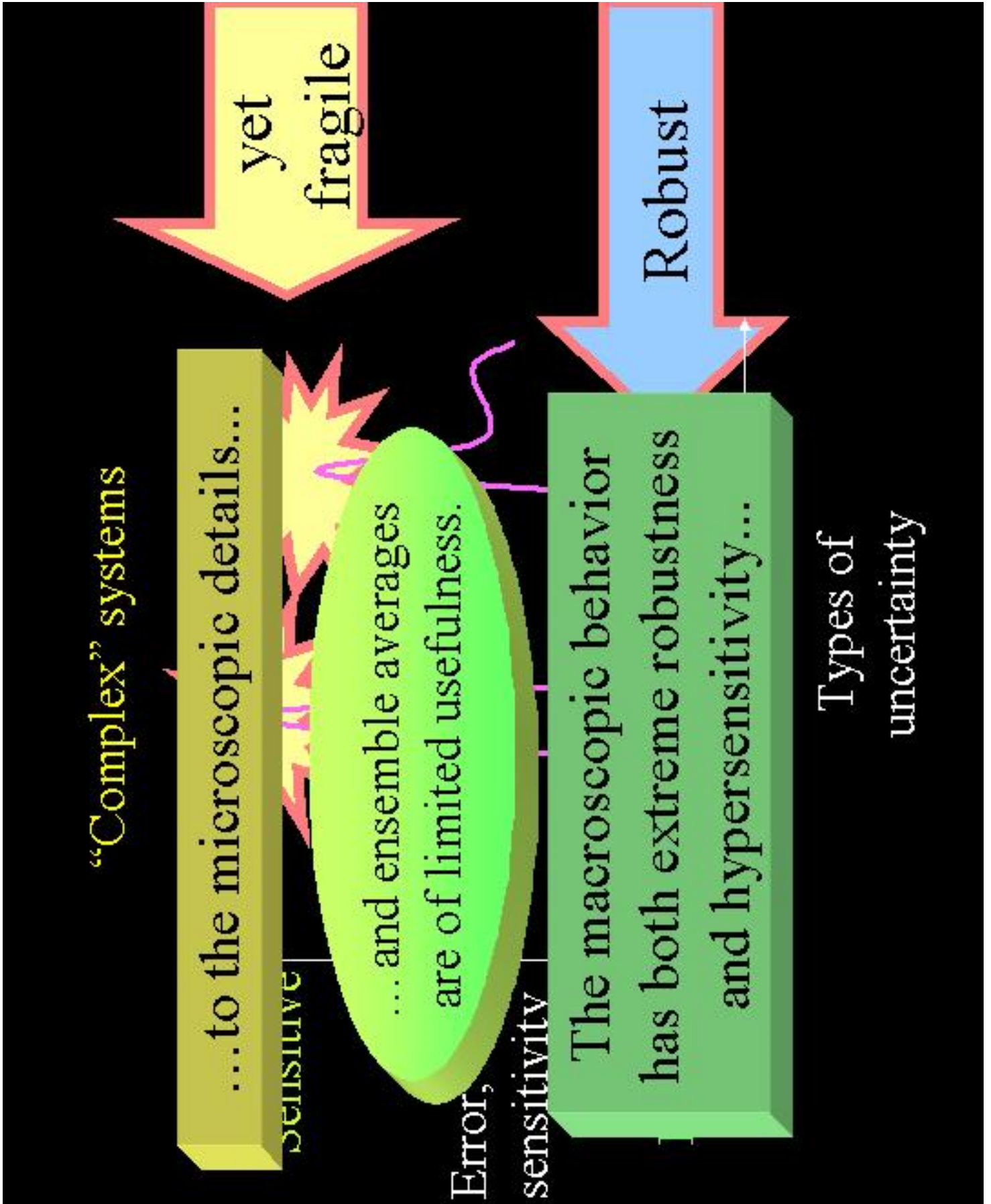
Types of uncertainty



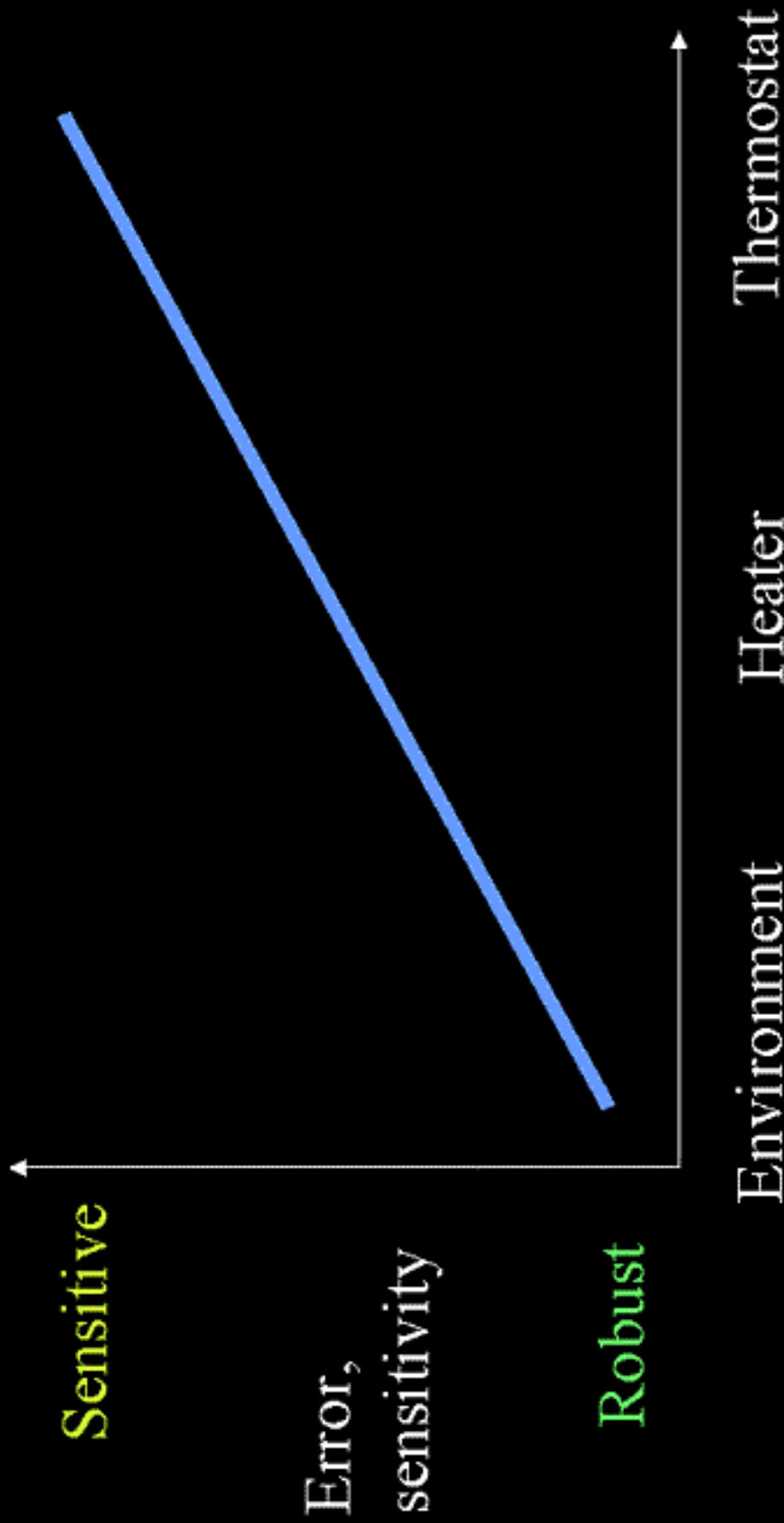
Multiscale, homogeneous example



Types of uncertainty

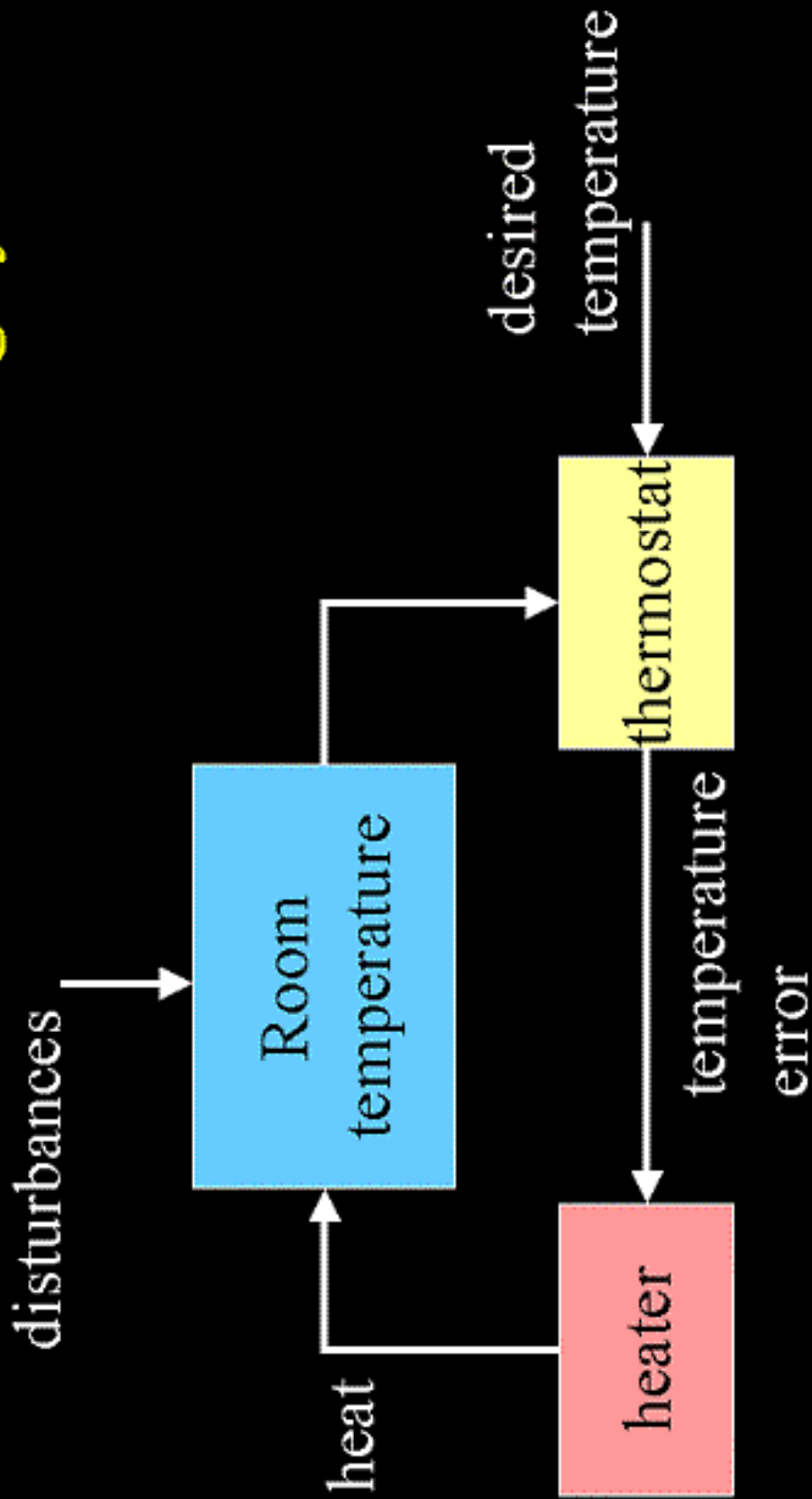


Heating system



Types of uncertainty

Heating system



Uncertainty and robustness in heating system

Environment

Heater

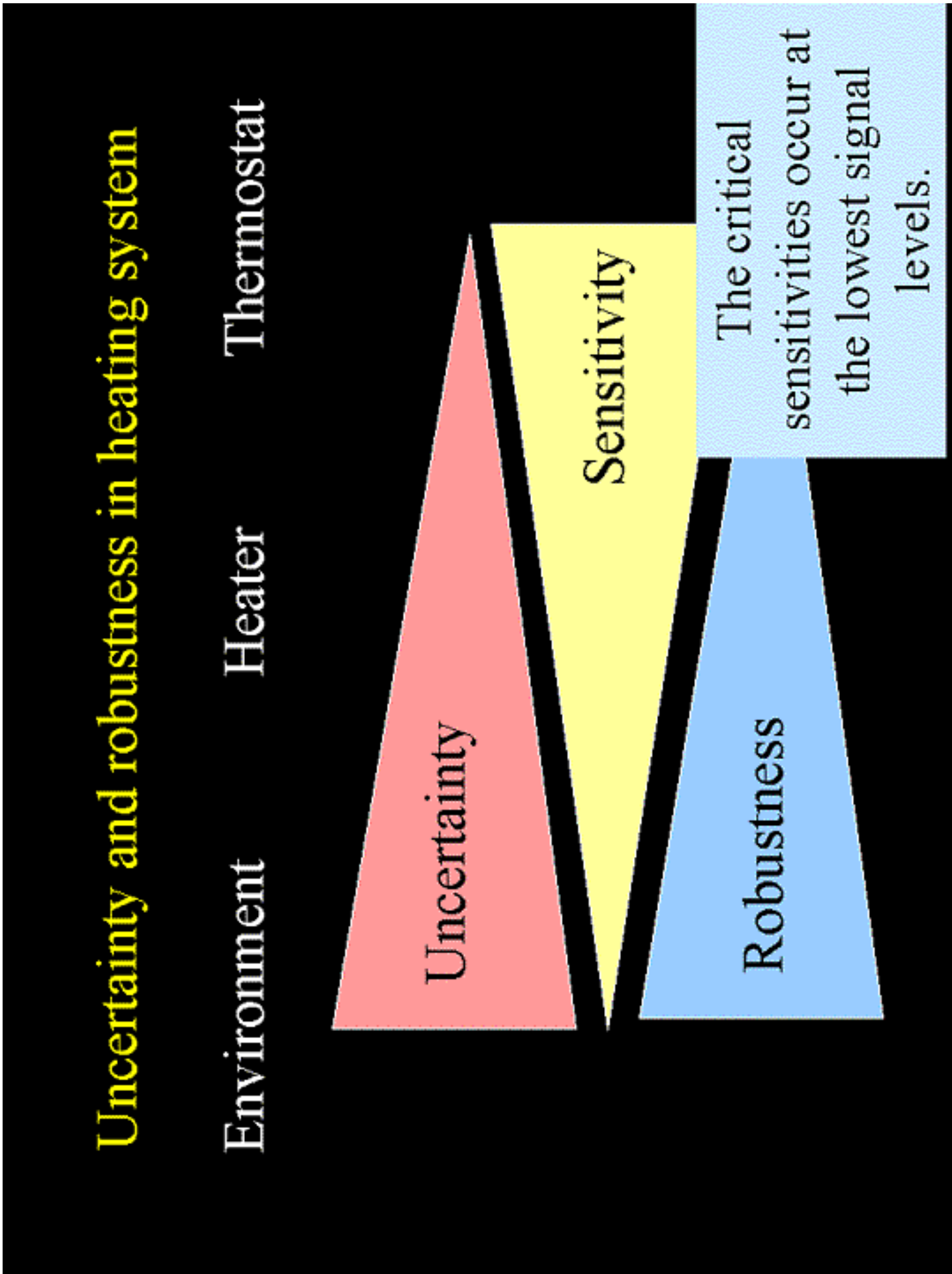
Thermostat

Uncertainty

Sensitivity

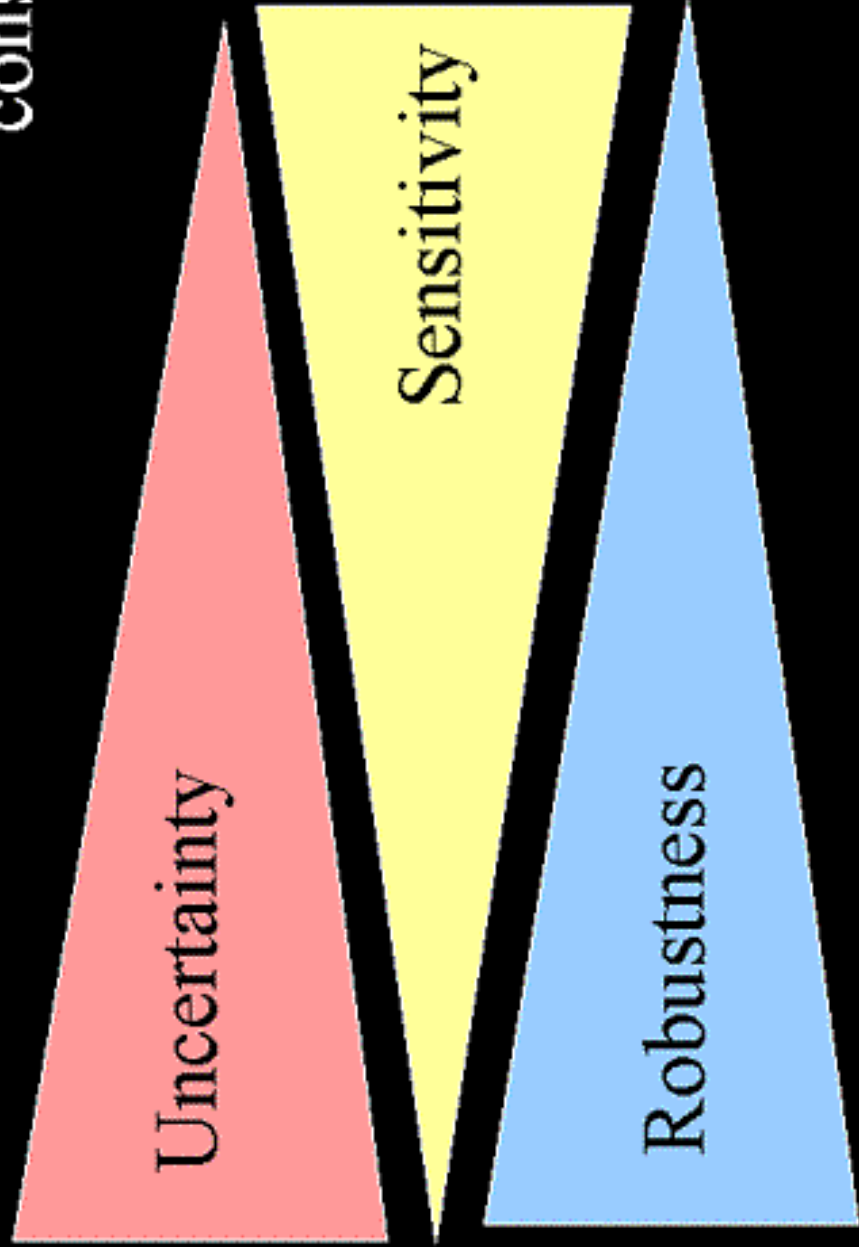
Robustness

The critical sensitivities occur at the lowest signal levels.

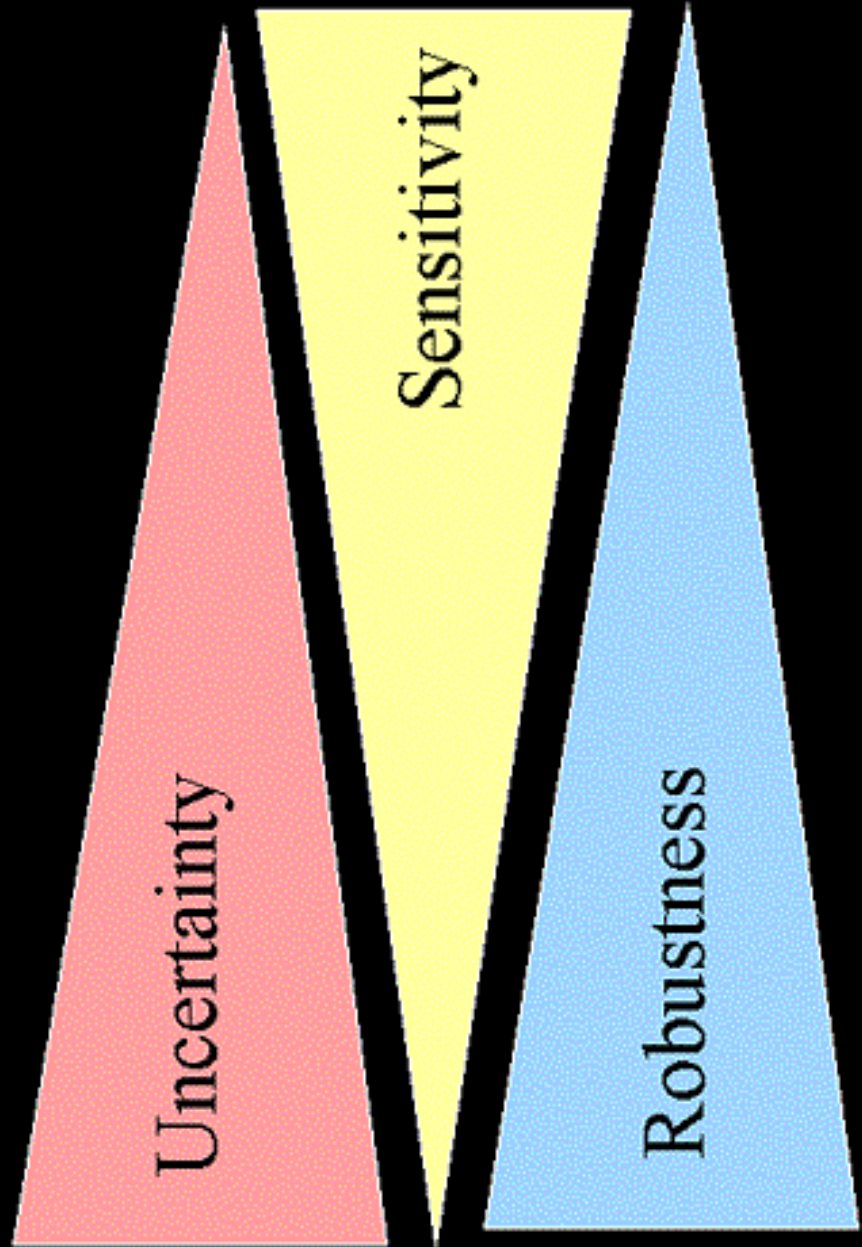


Uncertainty and robustness in chemotaxis

Environment Concentrations Rate constants

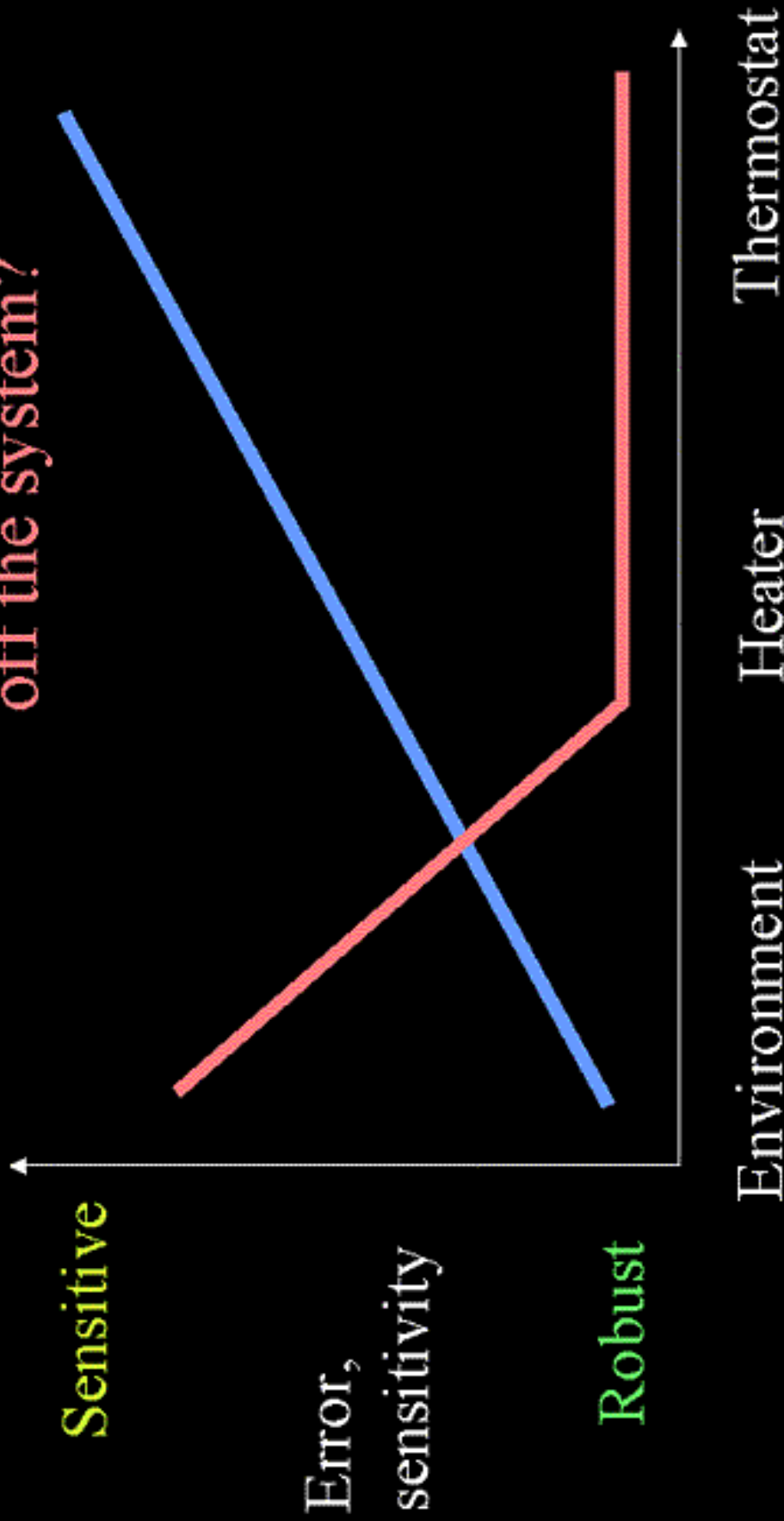


Uncertainty/robustness in complex systems



Heating system

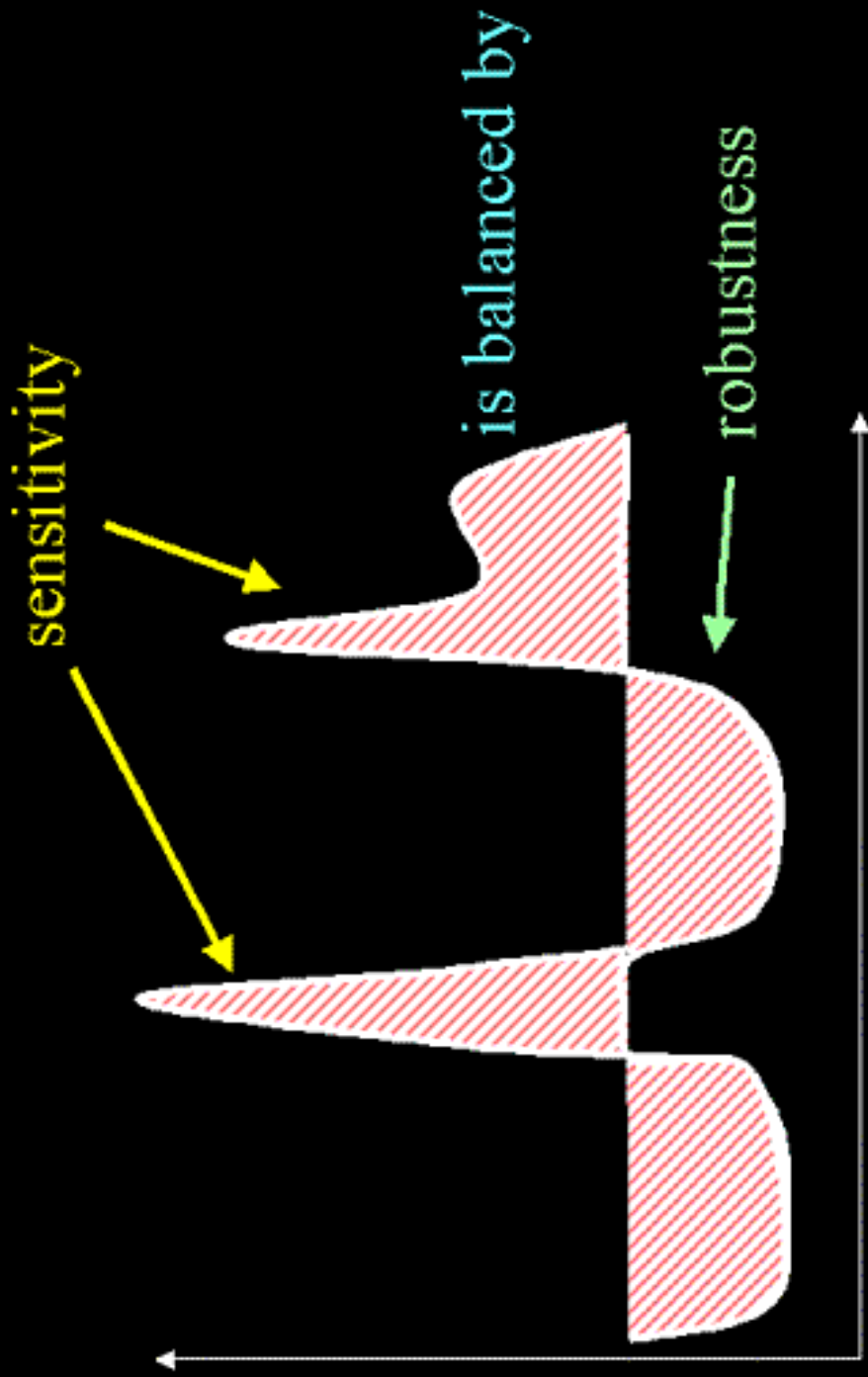
What if we turn off the system?



Types of uncertainty

Robustness “constraint” laws

Error,
sensitivity



“Complex” systems

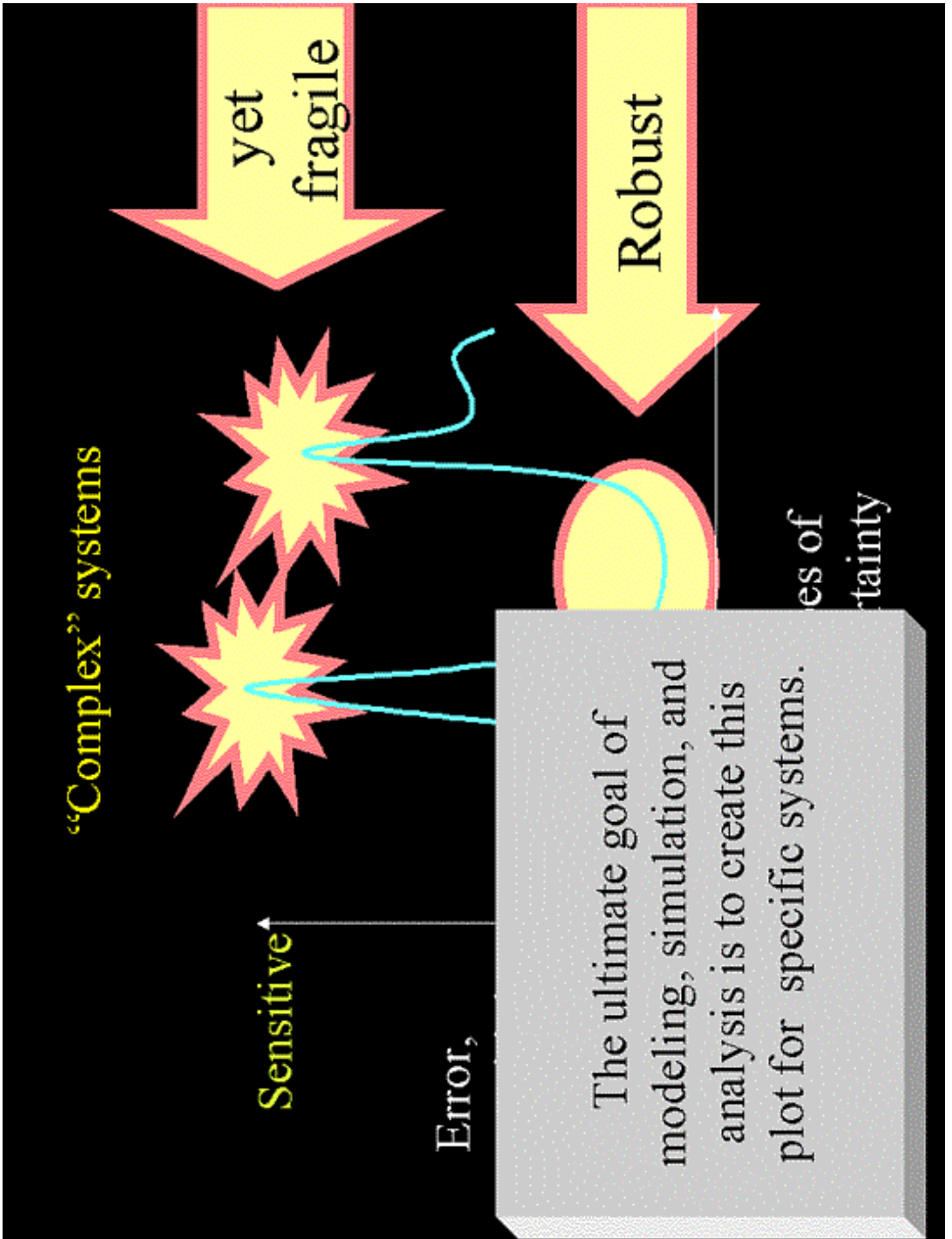
- Thus we are (forced into) redoing statistical physics from scratch for these systems.
- The HOT results are among the first outcomes.
- These are just steps toward...

yet
fragile

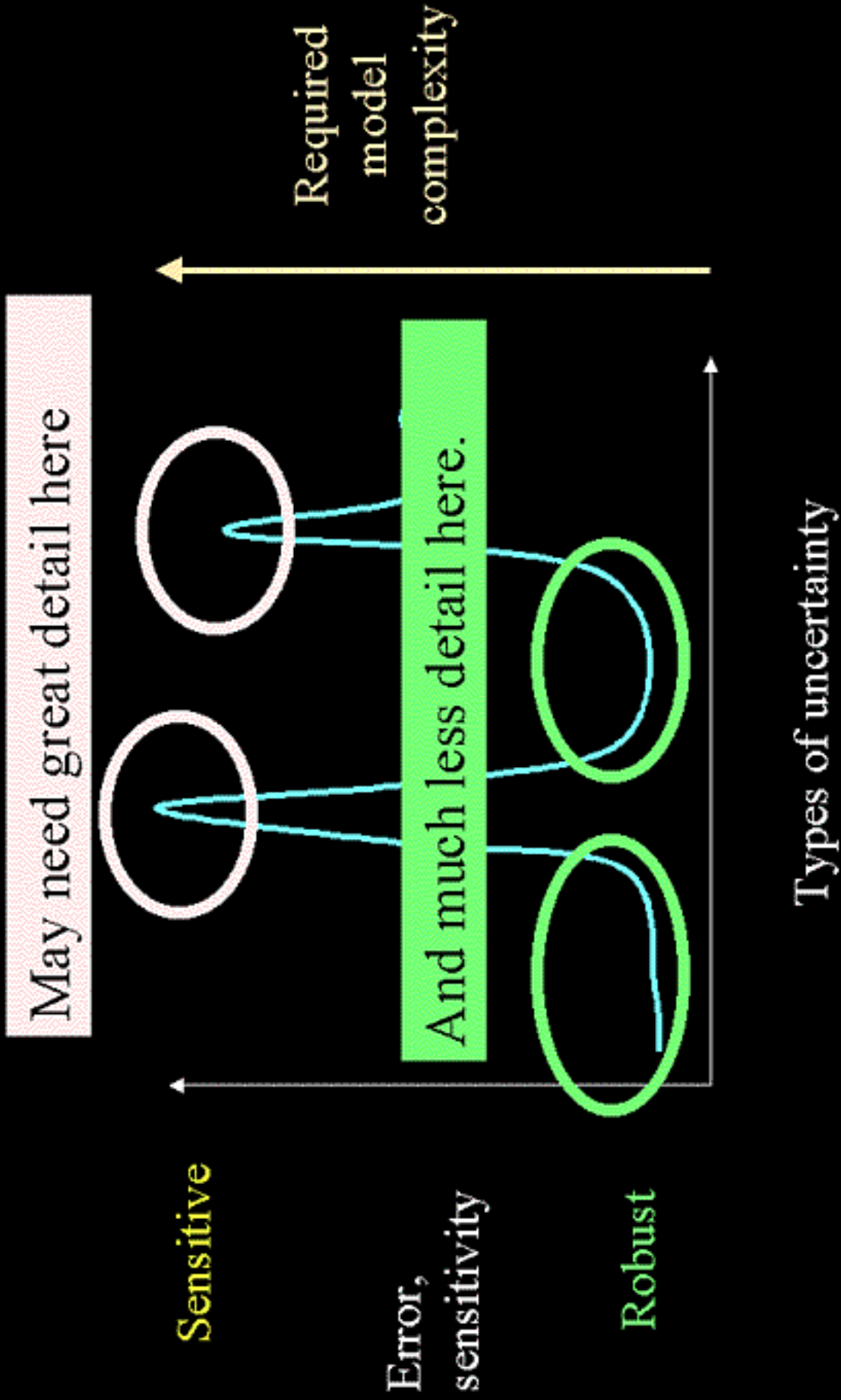
Robust

Types of
uncertainty

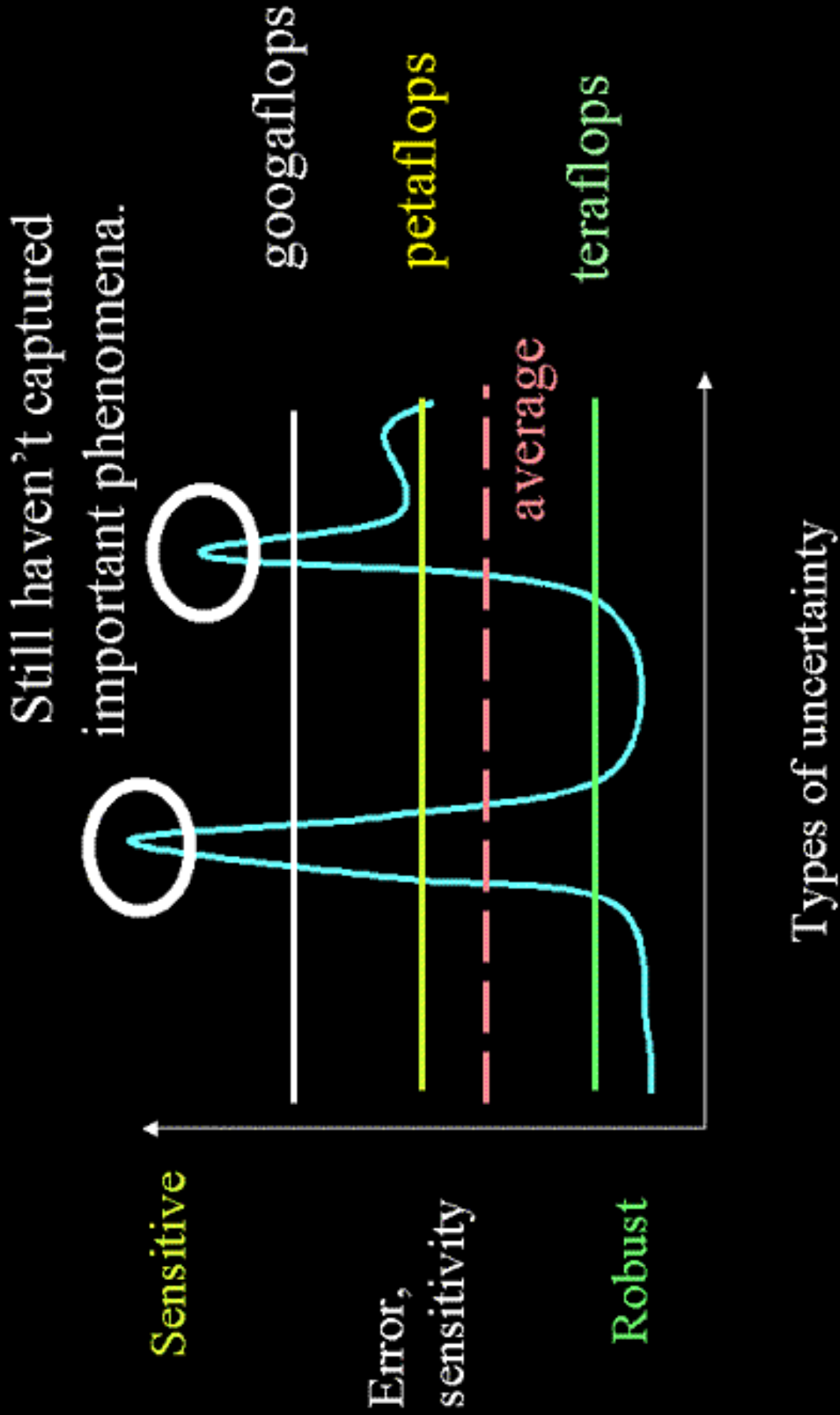
Er
ser



Modeling “complex” systems



Why “scientific supercomputing” has disappointed.



Why “scientific supercomputing” has disappointed.

- Relied too heavily on the homogeneous view of multiscale phenomena from physics.
- Deep misunderstanding of the “robust, yet fragile” character of much complex phenomena.

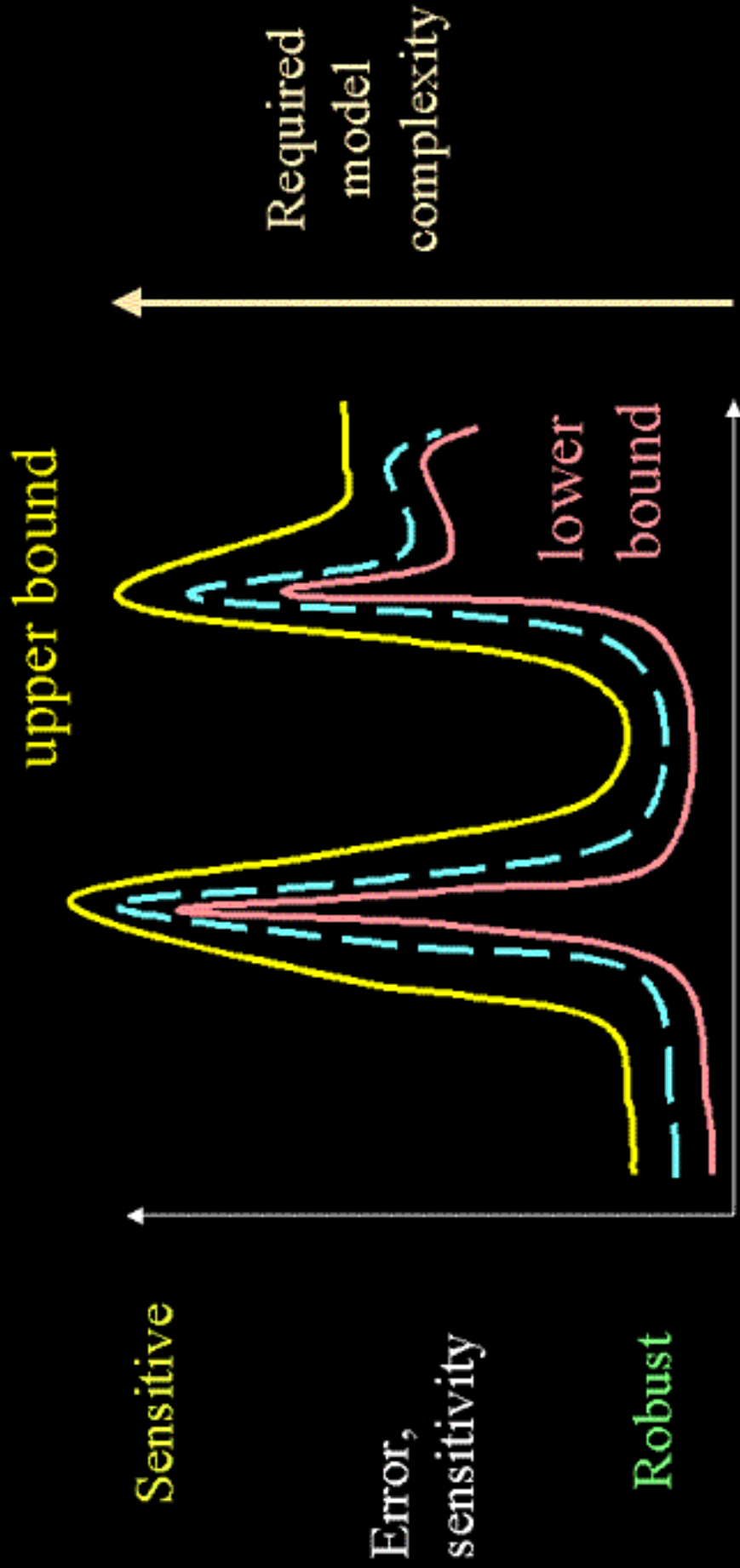
Sens

Error,
sensitivity

Rob

Types of uncertainty

**But we can't know this
"true" system.**



Sensitive

Error,
sensitivity

Robust

Types of uncertainty

Required
model
complexity

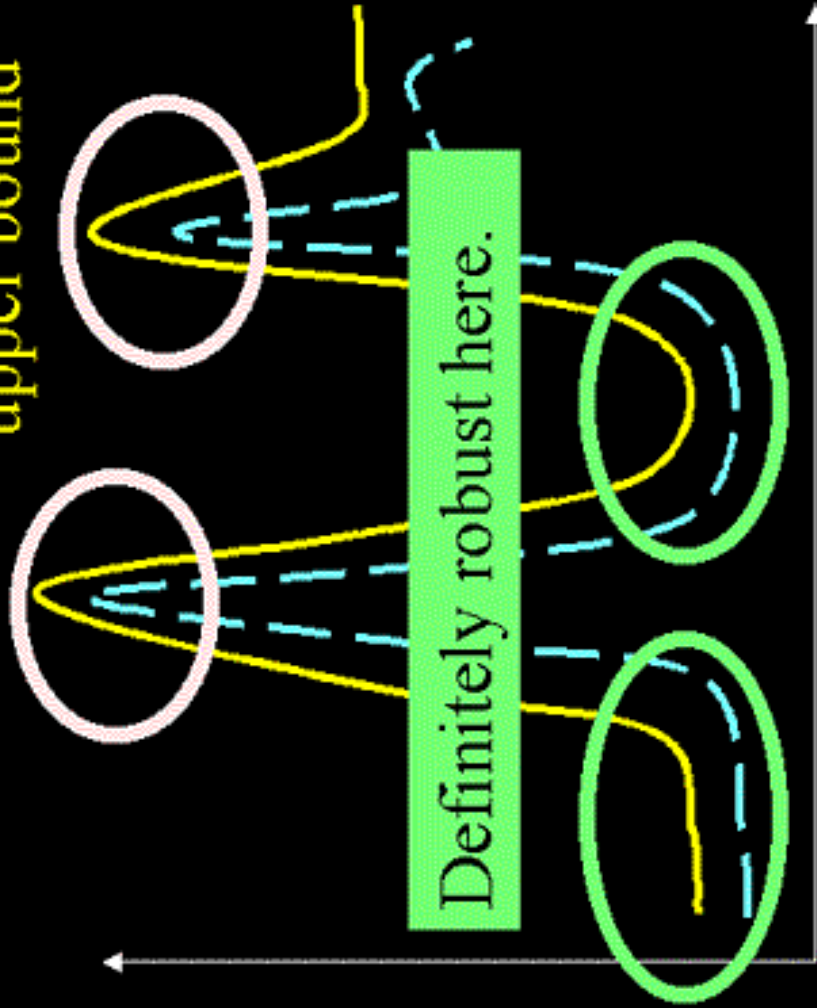
upper bound

lower
bound

Upper bounds

Can I tell here.

upper bound



Sensitive

Error, sensitivity

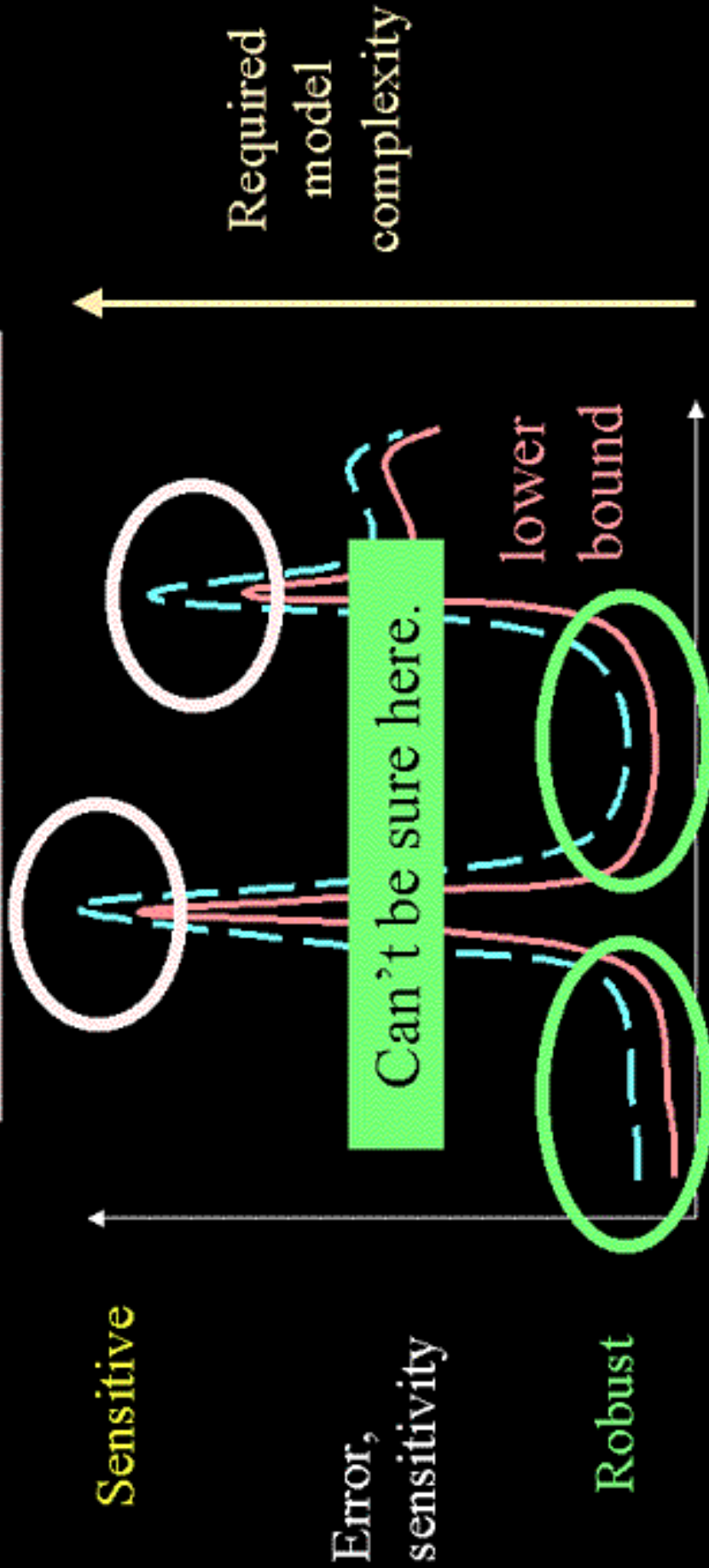
Robust

Required model complexity

Types of uncertainty

Lower bounds.

Definitely sensitive here.



Sensitive

Error,
sensitivity

Robust

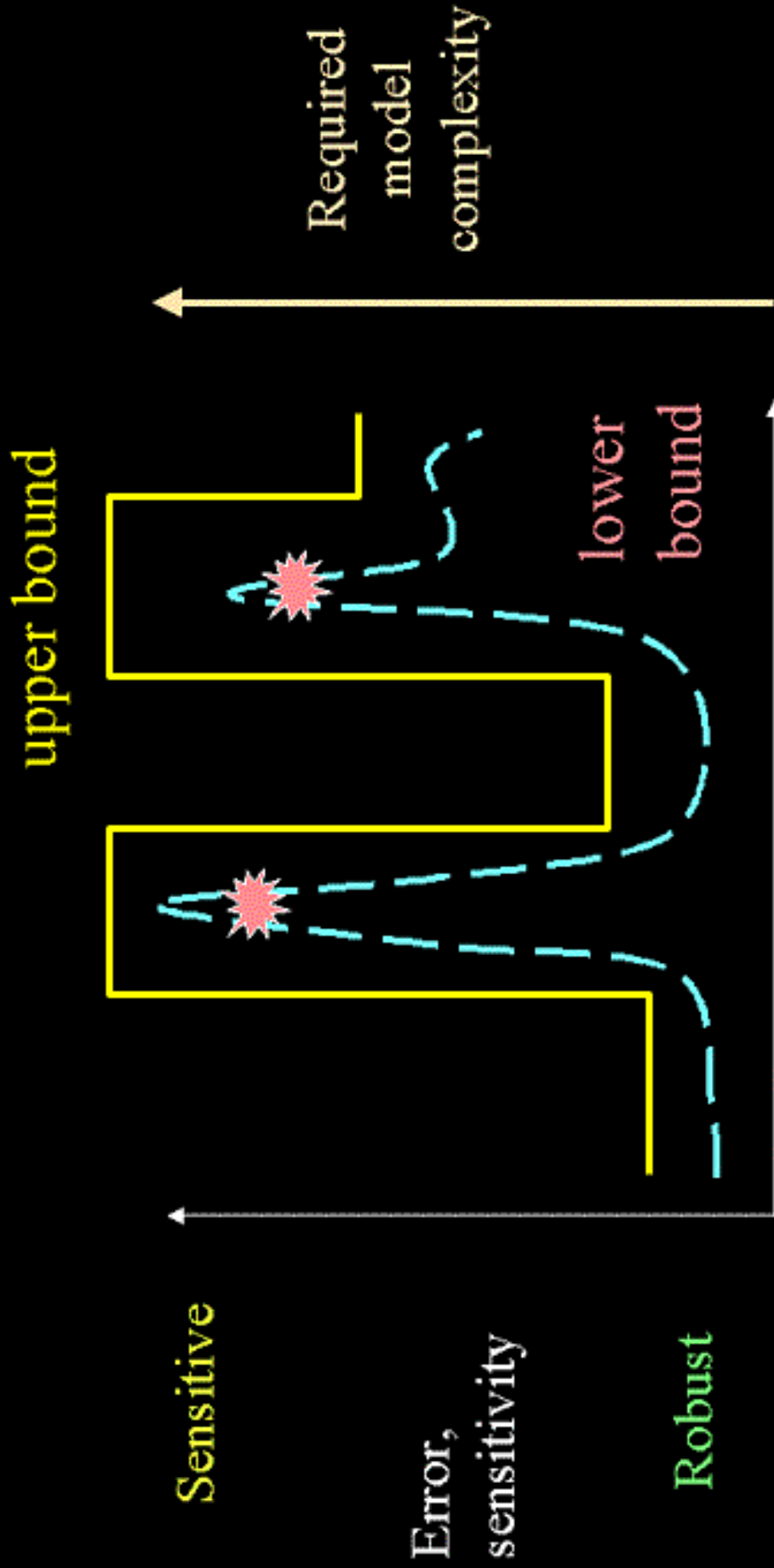
Required
model
complexity

lower
bound

Can't be sure here.

Types of uncertainty

Simplified, structured modeling



Sensitive

Error,
sensitivity

Robust

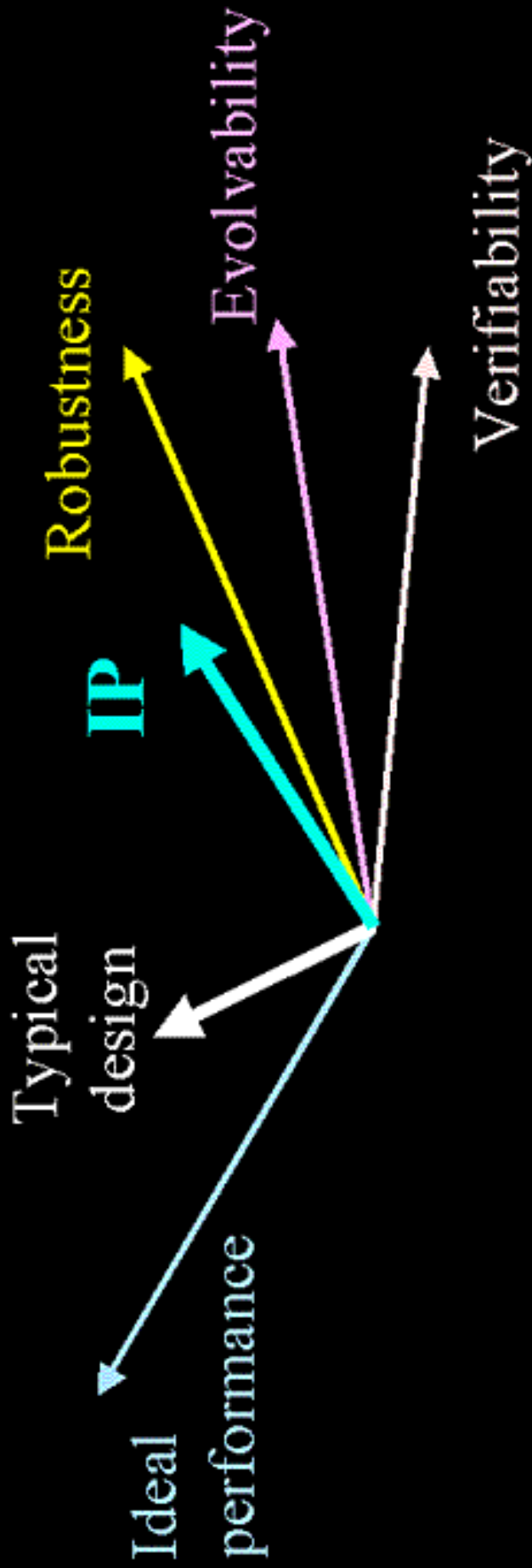
upper bound

lower
bound

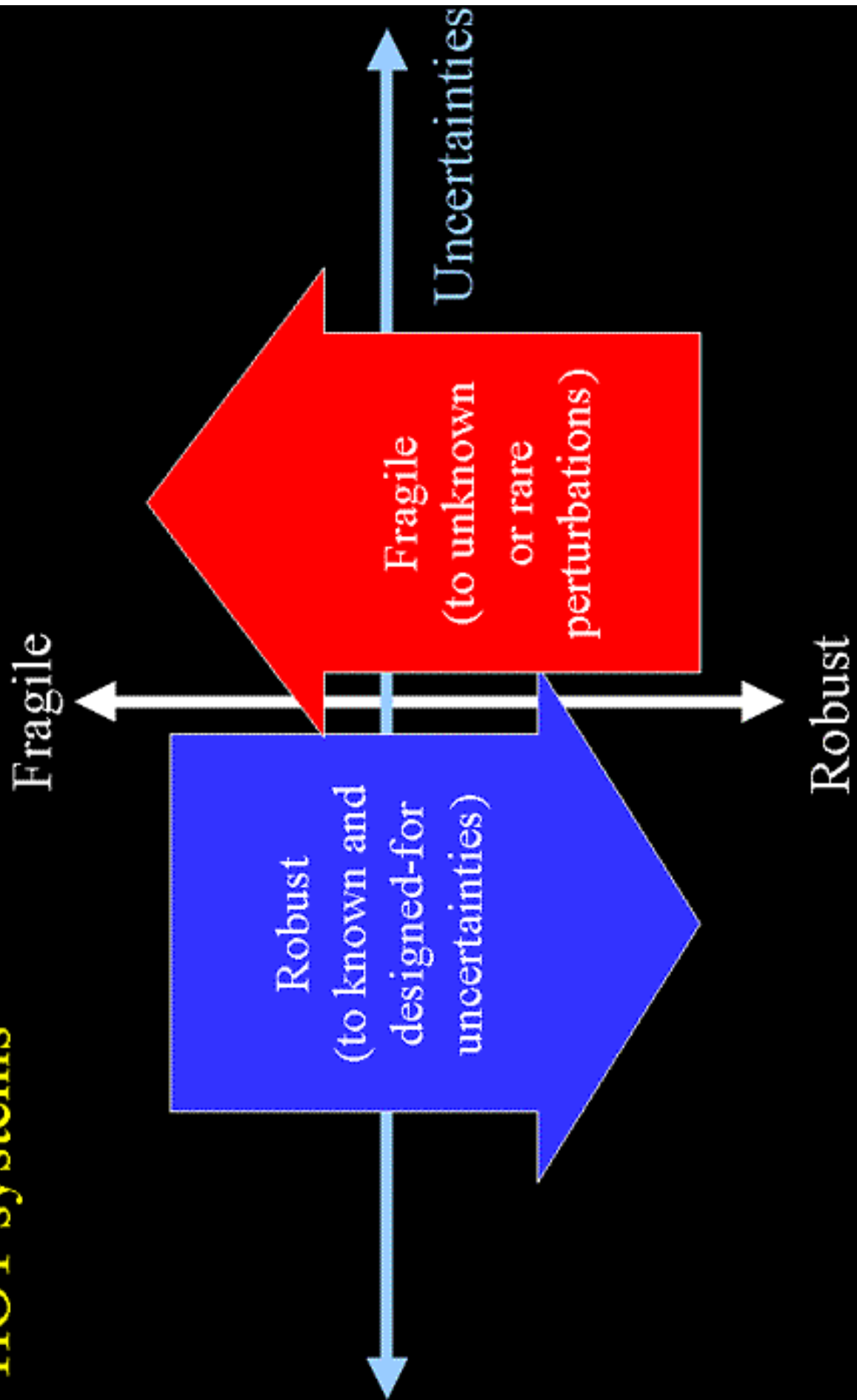
Required
model
complexity

Types of uncertainty

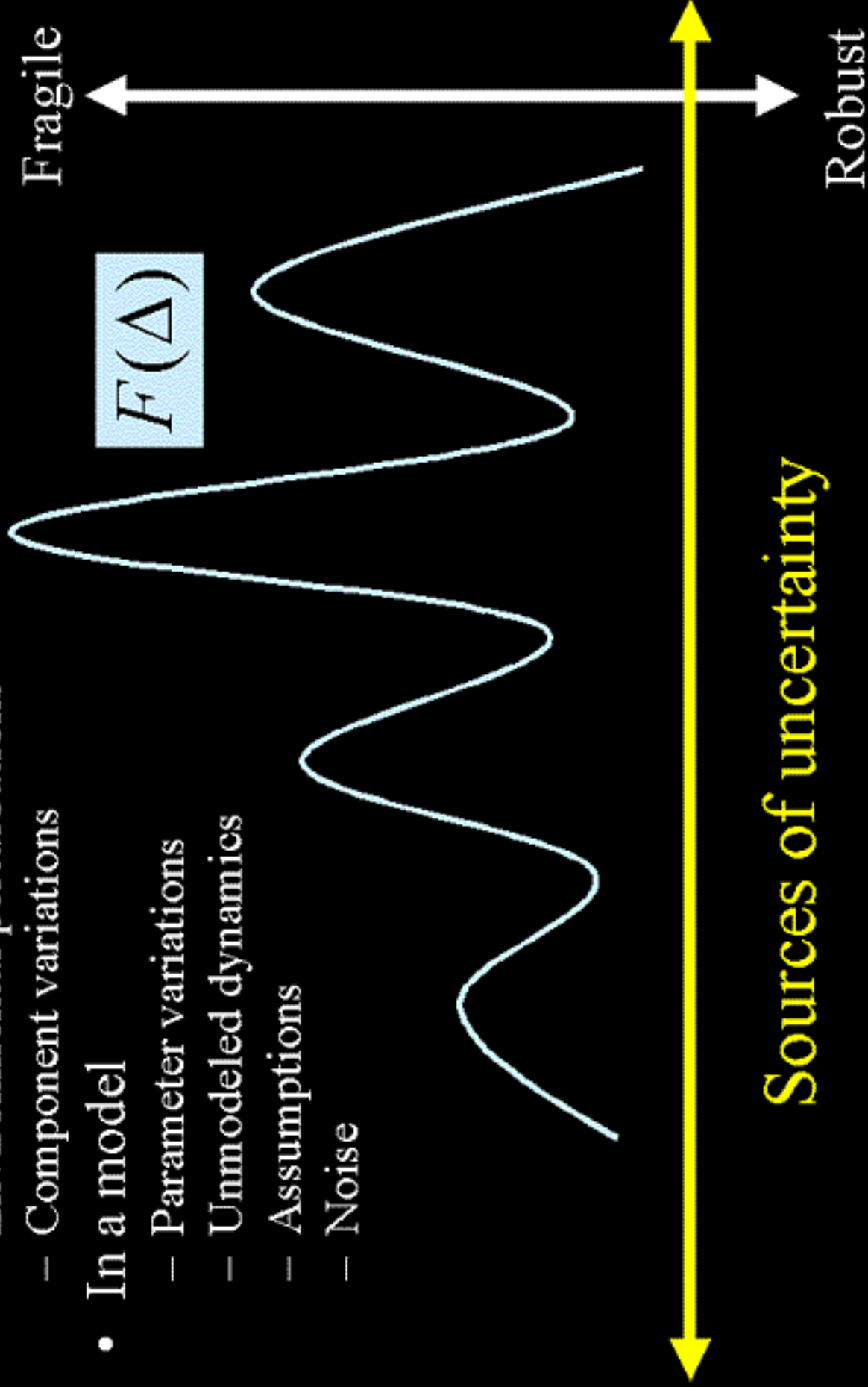
Robustness, evolvability/scalability, verifiability



Robustness of HOT systems

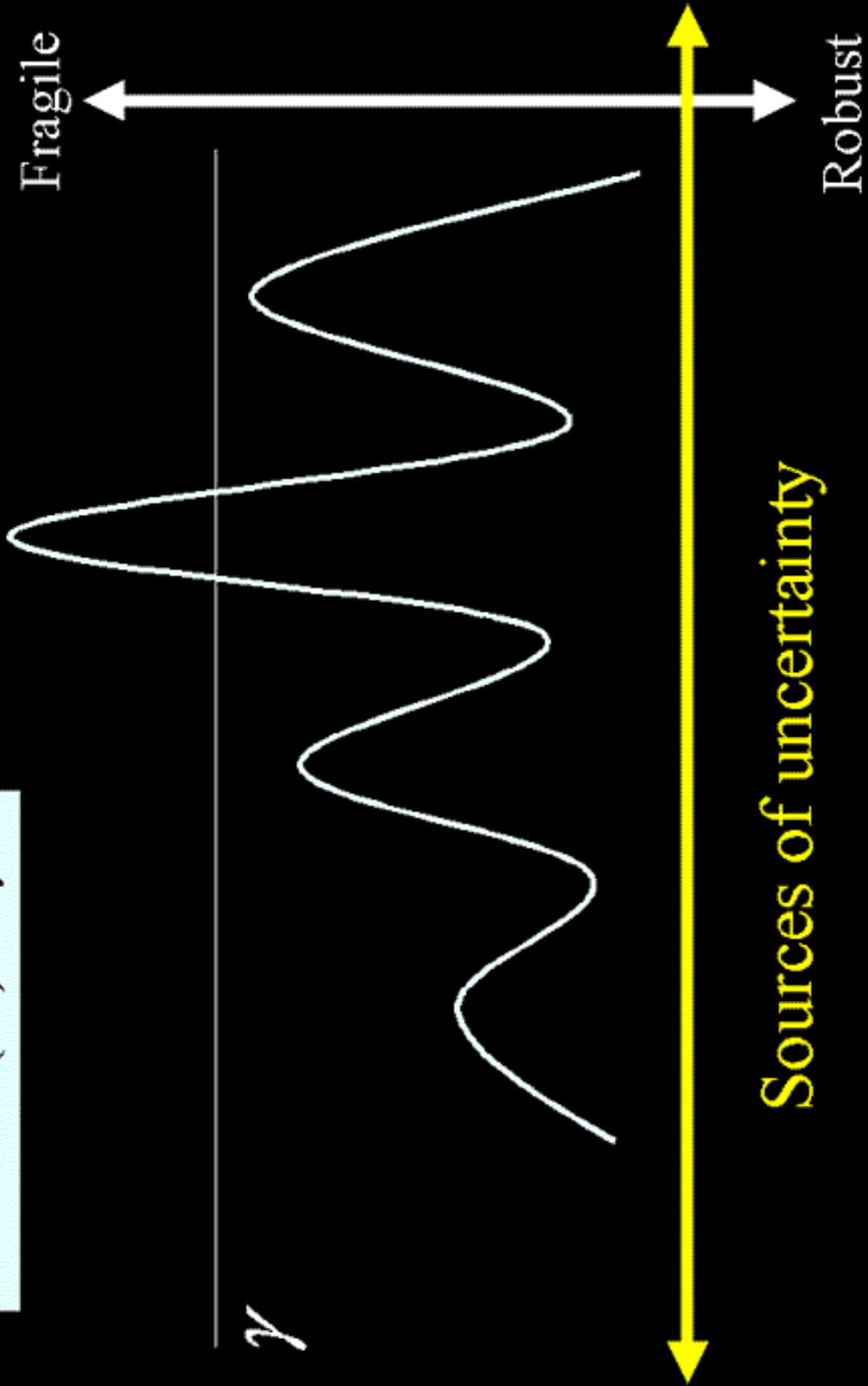


- In a system
 - Environmental perturbations
 - Component variations
- In a model
 - Parameter variations
 - Unmodeled dynamics
 - Assumptions
 - Noise



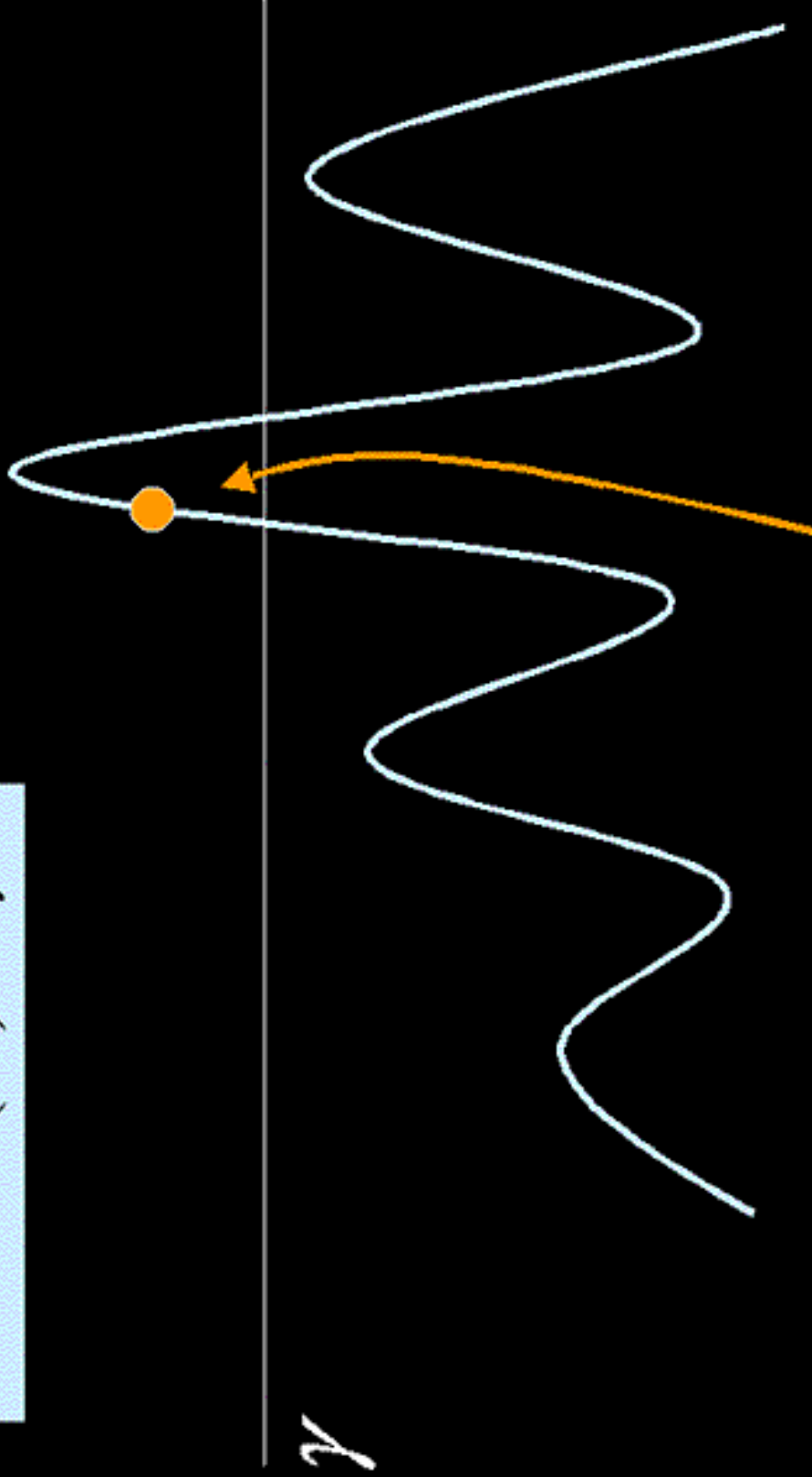
Sources of uncertainty

$$\exists \Delta \ni F(\Delta) > \gamma ?$$



$$\exists \Delta \ni F(\Delta) > \gamma?$$

Typically NP hard.



- If true, there is always a short proof.
- Which may be hard to find.

$$\forall \Delta, F(\Delta) \leq \gamma?$$

Typically coNP hard.

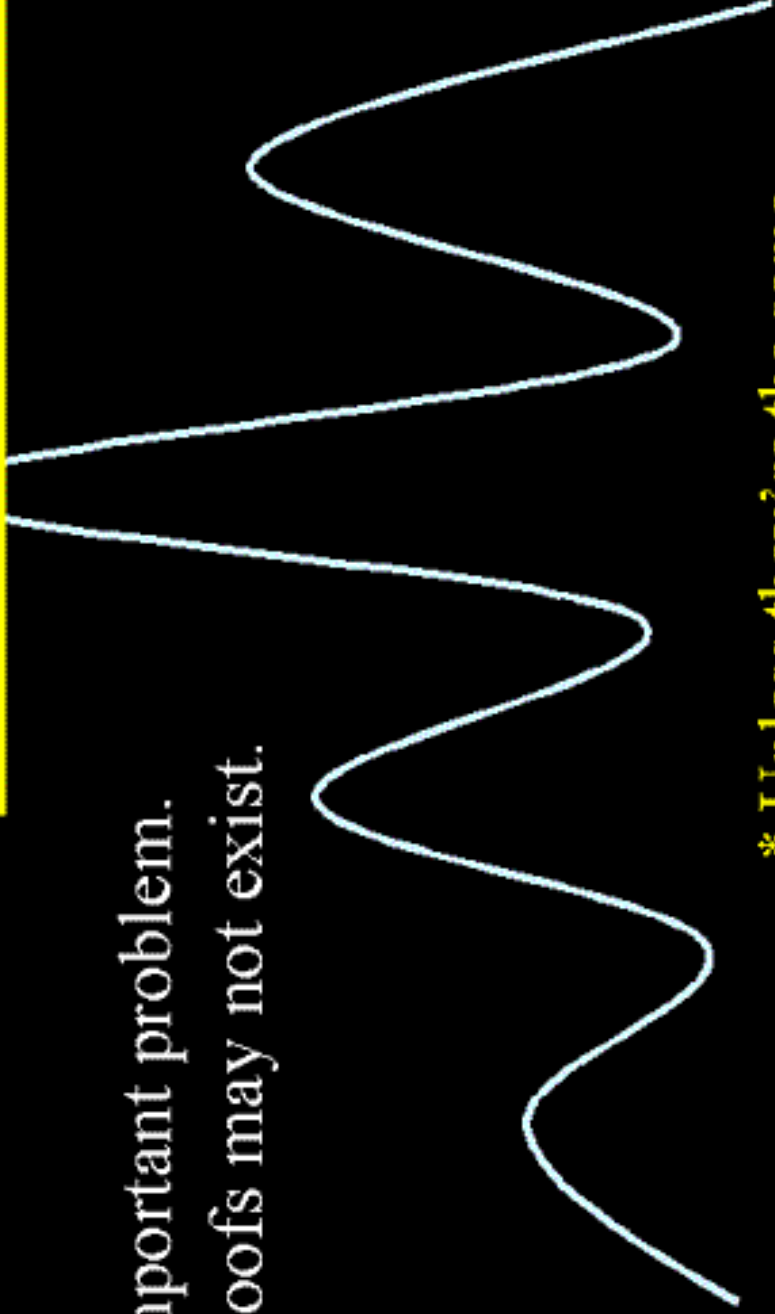
Fundamental asymmetries*

- Between P and NP
- Between NP and coNP

γ

- More important problem.
- Short proofs may not exist.

* Unless they're the same...



How do we *prove* that

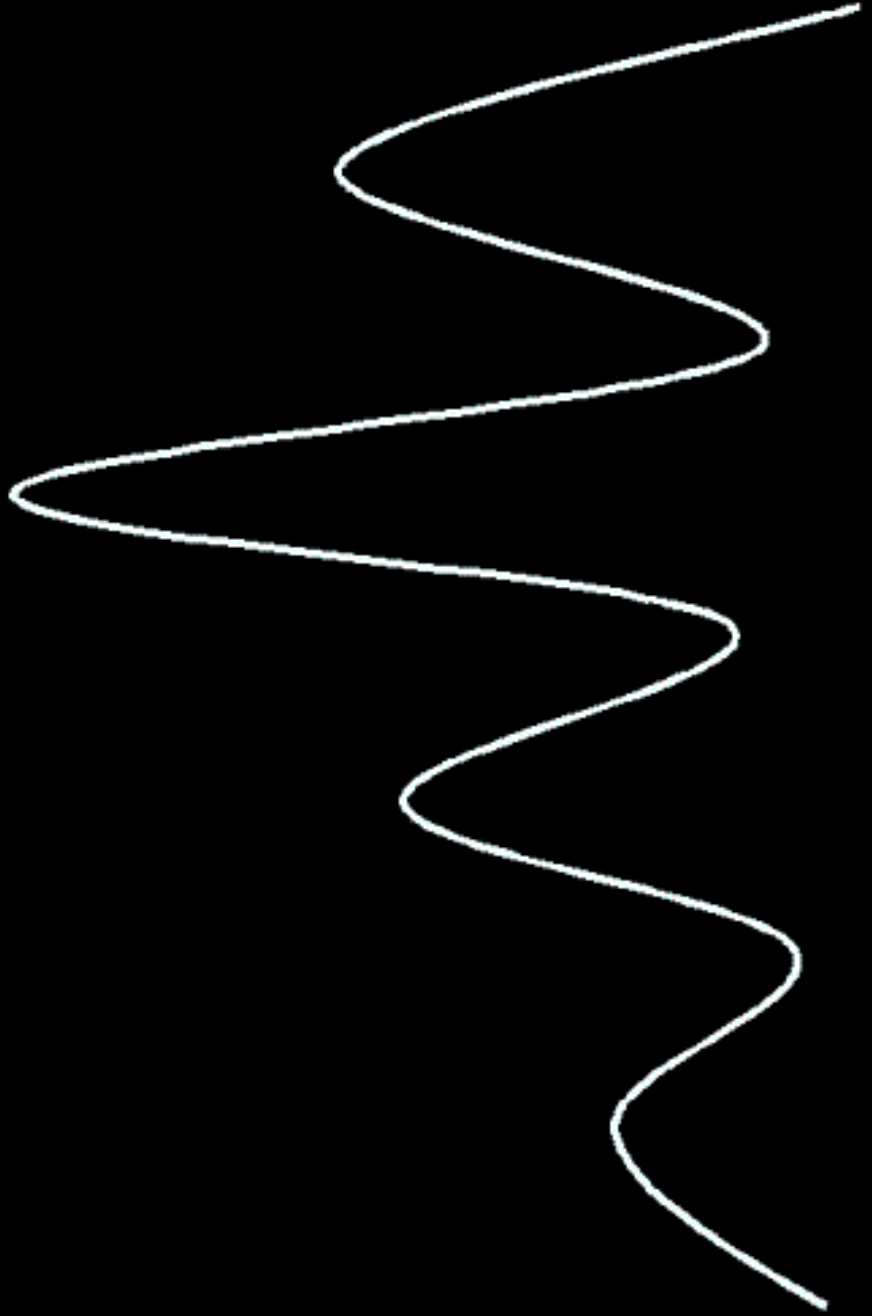
$$\forall \Delta, F(\Delta) \leq \gamma ?$$

- Standard techniques include relaxations, Grobner bases, resultants, numerical homotopy, etc...
- Powerful new method based on real algebraic geometry and semidefinite programming (Parrilo, Shor, ...)
- Nested series of polynomial time relaxations search for polynomial sized certificates
- Exhausts coNP (but no uniform bound)
- Relaxations have both computational and physical interpretations
- Beats gold standard algorithms (eg MAX CUT) handcrafted for special cases
- Completely changes the P/NP/coNP picture

**Robustness, evolvability,
scalability, verifiability**

$$\forall \Delta, F(\Delta) \leq \gamma ?$$

γ



$$\forall \Delta, \bar{F}(\Delta) \leq \gamma?$$



$$\forall \Delta, F(\Delta) \leq \gamma?$$



$$\forall \Delta, \bar{F}(\Delta) \leq \gamma?$$



$$\forall \Delta, F(\Delta) \leq \gamma?$$



$$\forall \Delta, \bar{F}(\Delta) \leq \gamma?$$

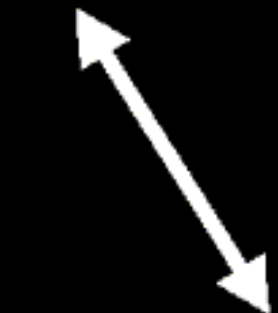


$$\forall \Delta, \bar{F}(\Delta) \leq \gamma?$$

$$\forall \Delta, F(\Delta) \leq \gamma?$$



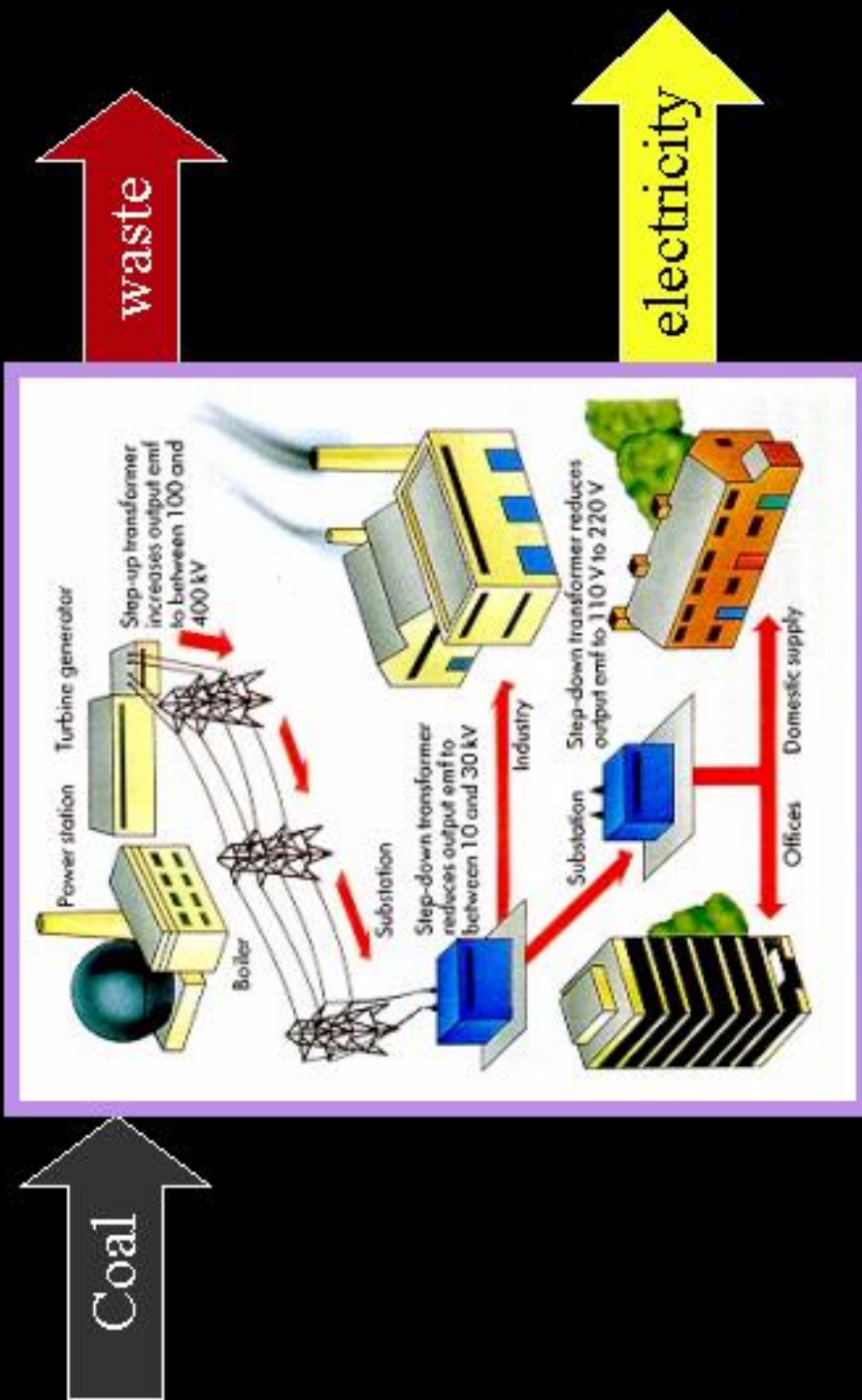
$$\forall \bar{\Delta}, F(\bar{\Delta}) \leq \gamma?$$

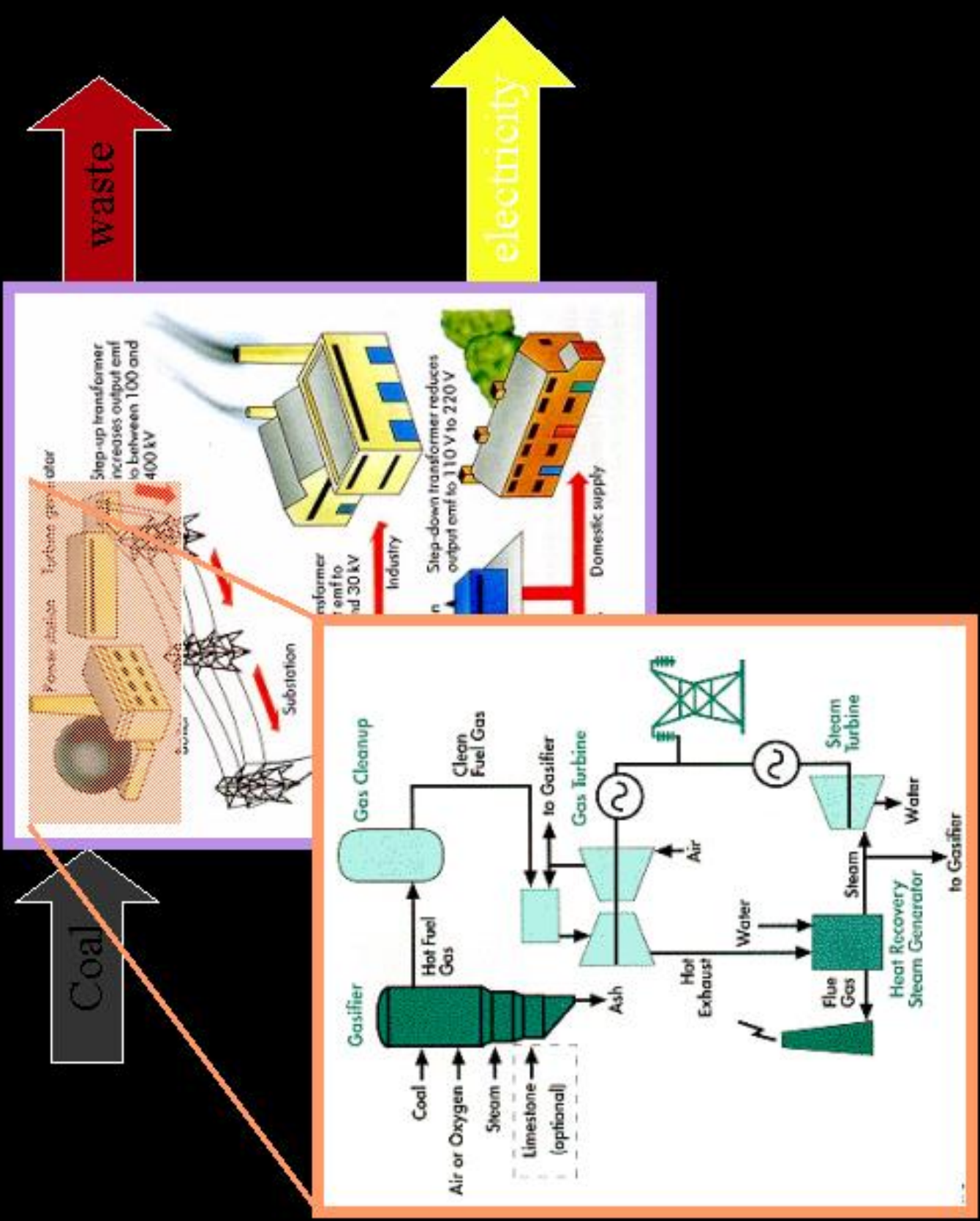


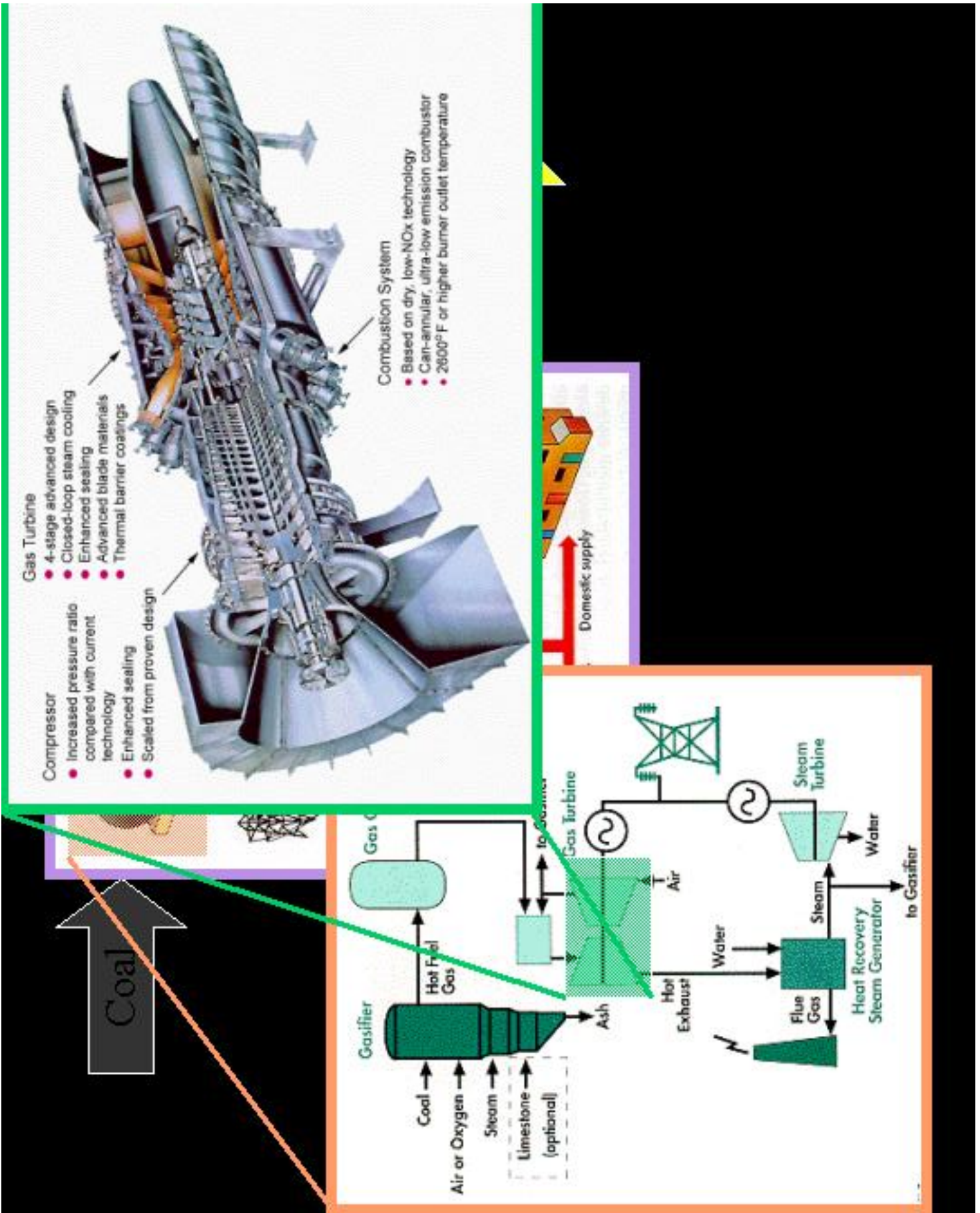
Verifiability (short proofs)

\approx

Extra robustness







Coal