INTEGRABLE DEFORMATIONS AND TWISTS

BASED ON 2112.12025 AND 2207.14748

WITH SIBYLLE DRIEZEN AND J. LUIS MIRAMONTES
J. MIGUEL NIETO, LEANDER WYSS

TALKING INTEGRABILITY: SPINS, FIELDS AND STRINGS
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"la Caixa" Foundation

RICCARDO BORSATO

IGFAE, SANTIAGO DE COMPOSTELA

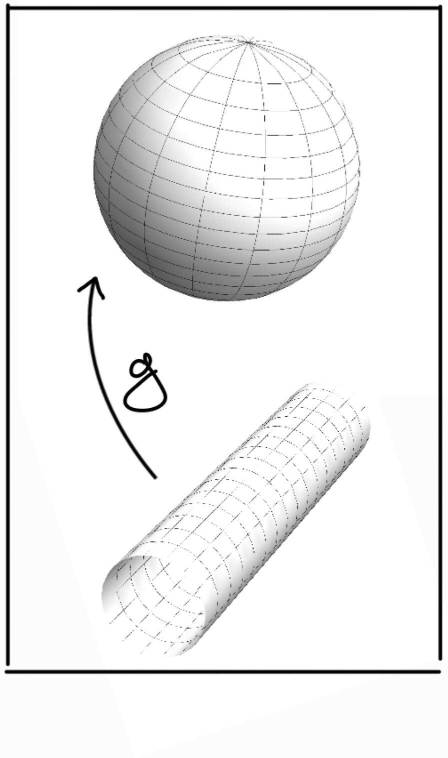
IGFAE Instituto Galego de Física de Altas Enerxías

USC UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

WHAT...

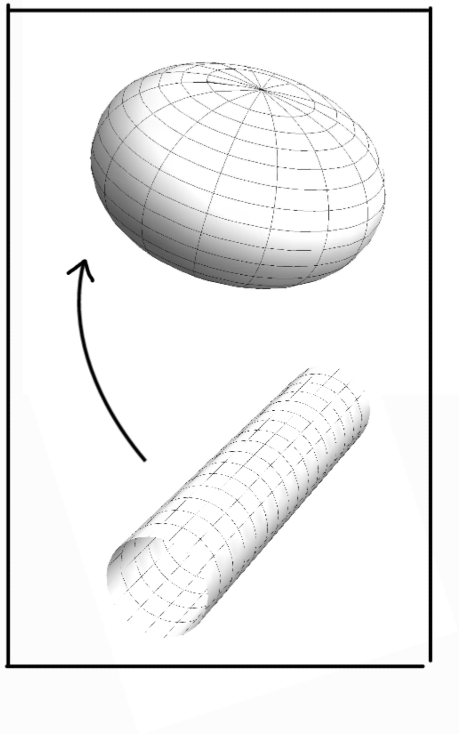
$\tau \in \mathbb{R}$, $\sigma \in [0, 2\pi]$, $g : (\tau, \sigma) \longrightarrow G$ LIE GROUP $\begin{cases} \text{PCM} \\ \text{(SUPER)COSETS} \\ \ldots \end{cases}$

UNDEFORMED & PERIODIC $g(2\pi) = g(0)$



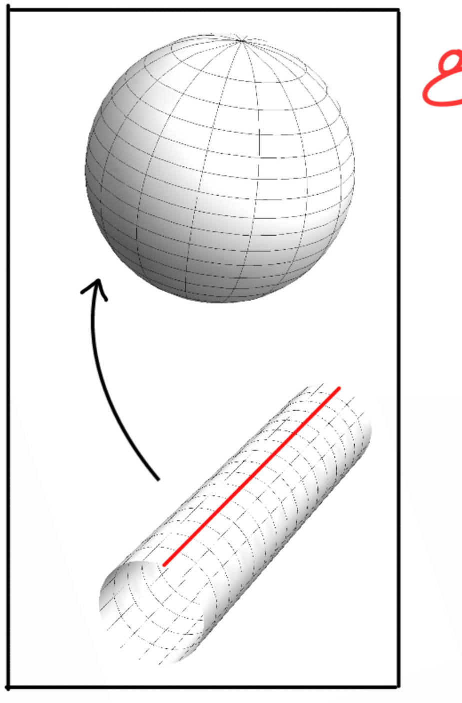
YANG-BAXTER DEF.

DEFORMED & PERIODIC



UNDEFORMED YET TWISTED

$g(2\pi) \neq g(0)$



EQUIVALENT

WHY...

- → INTEGRABLE MODELS ARE PRECIOUS AND RARE
- → BREAK (SUPER) ISOMETRIES
- → UNDERSTAND ROLE OF HIDDEN SYMMETRIES
- → AdS/CFT CORRESPONDENCE

$\boxed{AdS_5 \times S^5 \text{ STRING}} \longleftrightarrow \boxed{\mathcal{N}=4 \text{ SYM}}$

DEFORMATIONS: NEW SUGRA SOLUTIONS

DEFORMATIONS: NON-COMMUTATIVE GAUGE TH'S

- LAX CONNECTION
- WORLDSHEET SCATTERING

- SPIN-CHAIN
- LONG-RANGE INTERACTIONS

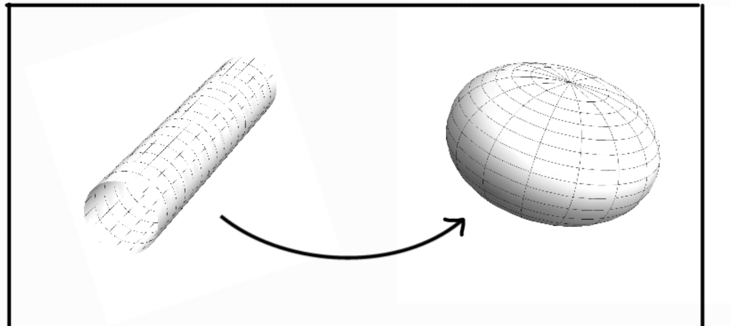
- ALL-LOOP S-MATRIX
- BETHE-YANG EQUATIONS
- TBA / QSC

TsT DEFORMATIONS ⊃ [LUNIN, MALDACENA '05]

SEE [FROLOV '05, FROLOV, ROIBAN, TSEYTLIN '05, ALDAY, ARUTYUNOV, FROLOV '05]

2 COMMUTING ISOMETRIES ACTING AS SHIFTS OF X^1, X^2

T-DUALITY: $X^1 \to T(X^1)$, SHIFT: $X^2 \to X^2 - \eta T(X^1)$, T-DUALITY: $T(X^1) \to X^1$



DEFORMED σ-MODEL

WITH FIELDS X^1, X^2

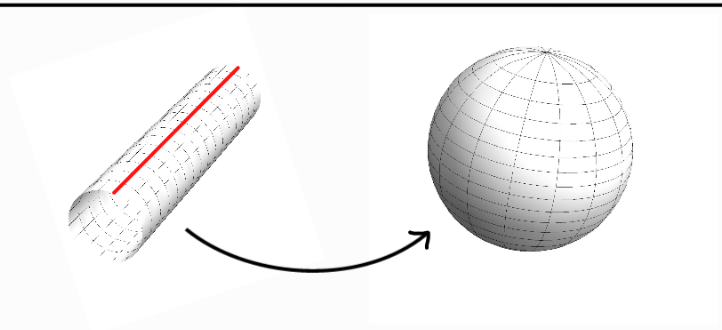
PERIODIC BOUND. COND.

$$X^i(2\pi) = X^i(0)$$

ON-SHELL

$\longleftrightarrow$

EQUIVALENT



UNDEFORMED σ-MODEL

WITH FIELDS $\tilde{X}^1, \tilde{X}^2$

TWISTED BOUND. COND.

$$\tilde{X}^i(2\pi) = \tilde{X}^i(0) + \eta\, \varepsilon^{ij} Q_j$$

$$\partial_\alpha X^i = \partial_\alpha \tilde{X}^i + \eta\, \varepsilon^{ij} \varepsilon_{\alpha\beta} J_j^\beta, \qquad (\partial_\alpha J_i^\alpha = 0)$$

YANG-BAXTER DEFORMATIONS: NON-LINEAR RELATIONS FOR TWISTED B.C.

TWISTED PCM

$g(\tau,\sigma) \in G$ LIE GROUP $\quad \mathfrak{g} = \mathrm{Lie}(G) \quad \sigma^{\pm} = \tau \pm \sigma$

$$\boxed{S_{PCM} = -\int d^2\sigma \, \mathrm{Tr}(J_+ J_-)} \, , \qquad J = g^{-1} dg$$

EQ'S OF MOTION FROM $\delta_g S_{PCM} \Rightarrow \partial_\alpha J^\alpha \simeq 0$

VANISHING BOUNDARY TERM: $\mathrm{Tr}(g^{-1}\delta g J_\sigma)$ PERIODIC IN $\sigma \in [0, 2\pi]$

W CONSTANT $\in G$ $\quad \boxed{g(2\pi) = W g(0)}$ $\quad$ LEFT TWIST (HERE NO RIGHT)

FIELD REDEFINITION $\quad g' = hg \, , \quad h$ CONSTANT $\in G \quad W' = h W h^{-1}$

LEFT TWISTS CLASSIFIED BY ADJOINT ORBITS OF G

G CONNECTED AND COMPACT $\Rightarrow$ MAXIMAL ABELIAN SUBALGEBRA / WEYL GROUP

YANG-BAXTER DEF'S $\approx$ W WRITTEN IN TERMS OF CONSERVED CHARGES

YANG-BAXTER DEFORMATION (OF PCM)

[KLIMČÍK '03, DELDUC, MAGRO, VICEDO '13, KAWAGUCHI, MATSUMOTO, YOSHIDA '14, VAN TONGEREN '15...]

SEE [HOARE '21] FOR A REVIEW

$$\boxed{S_{YB} = -\int d^2\sigma \, \mathrm{Tr}\left(\partial_+ g g^{-1} \, \mathcal{O}^{-1}(\partial_- g g^{-1})\right)}$$

$$\mathcal{O} : \mathfrak{g} \longrightarrow \mathfrak{g} \qquad \boxed{\mathcal{O} = \mathbb{1} - \eta R}$$

$$R : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad t_a \in \mathfrak{g}, \qquad R(t_a) = R_a{}^b \, t_b$$

G_R UNBROKEN, G_L BROKEN TO $H \subset G_L$, $\mathrm{Ad}_h^{-1} \circ R \circ \mathrm{Ad}_h = R$

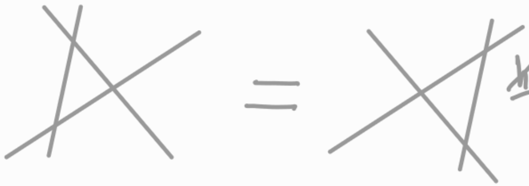
ANTI-SYMMETRY $\quad \mathrm{Tr}(x \, R(y)) = -\mathrm{Tr}(R(x) \, y)$

AND ...

CLASSICAL YANG-BAXTER EQUATION (CYBE)

$$\boxed{[R(x), R(y)] - R\big([R(x), y] + [x, R(y)]\big) = 0} \qquad \forall x, y \in \mathfrak{g}$$

⚠ RHS = 0 ⇝ "HOMOGENEOUS" YB DEF.



$$\xrightarrow{\hbar \sim 0} \quad [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

$$\boxed{r = R^{ab}\, T_b \wedge T_a \in \mathfrak{g} \wedge \mathfrak{g}}$$

EX: $r = T_1 \wedge T_2$

ABELIAN $\quad [T_1, T_2] = 0 \longrightarrow$ TsT [OSTEN, VAN TONGEREN '16]

NON-ABELIAN $\quad [T_1, T_2] = T_2 \rightsquigarrow$ JORDANIAN DEF.

2-COCYCLE CONDITION (2CC)

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = 0$$

$\mathfrak{f} = \mathrm{Im}(R)$ IS SUBALGEBRA OF $\mathfrak{g}$

$$\mathfrak{g} = \mathrm{Ker}(R) \oplus \mathfrak{f}^*$$

$$\boxed{\begin{array}{c} R : \mathfrak{f}^* \longrightarrow \mathfrak{f} \text{ SOLVES CYBE} \\ \Downarrow \\ \omega = R^{-1} : \mathfrak{f} \longrightarrow \mathfrak{f}^* \text{ SOLVES 2CC} \end{array}}$$

$\hat{\omega} : \mathfrak{f} \wedge \mathfrak{f} \longrightarrow \mathbb{R}$, $\qquad \hat{\omega}(x, y) = \mathrm{Tr}(x \ \omega(y))$

$$\boxed{\hat{\omega}(x, [y, z]) + \hat{\omega}(y, [z, x]) + \hat{\omega}(z, [x, y]) = 0}$$

ω IS <u>COBOUNDARY</u> IF $\exists\, x \in \mathfrak{g}$ s.t. $\omega = \mathrm{ad}_x$ $\quad (\omega(y) = [x, y])$

CLASSICAL INTEGRABILITY

$$[\partial_+ + \mathcal{L}_+ , \partial_- + \mathcal{L}_-] = 0$$

PCM

$\tilde{g} \in G$, $\quad \tilde{J}_\pm = \tilde{g}^{-1} \partial_\pm \tilde{g}$

① $\partial_+ \tilde{J}_- + \partial_- \tilde{J}_+ \simeq 0$

② $\partial_+ \tilde{J}_- - \partial_- \tilde{J}_+ + [\tilde{J}_+ , \tilde{J}_-] = 0$

$$\mathcal{L}_\pm = \frac{\tilde{J}_\pm}{1 \mp z}$$

YB

$g \in G$, $\quad A_\pm = \dfrac{1}{1 \pm \eta \, \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g} \left(g^{-1} \partial_\pm g \right)$

① $\partial_+ A_- + \partial_- A_+ \simeq 0$

② $\partial_+ A_- - \partial_- A_+ + [A_+ , A_-] \simeq 0$

$R^T = -R$
AND
CYBE

$$\mathcal{L}_\pm = \frac{A_\pm}{1 \mp z}$$

ON-SHELL EQUIVALENCE

$$\tilde{J}_\pm = A_\pm$$

THE TWIST

SEE [VICEDO '15] AND [VAN TONGEREN '18]

$$\underset{\text{YB}}{g} = \underset{\text{TWIST FIELD}}{\mathcal{F}}\,\underset{\text{PCM}}{\tilde{g}}$$

$$\tilde{J}_\pm = A_\pm \quad \Rightarrow \quad \partial_\pm \mathcal{F} = V_\pm \mathcal{F}, \qquad V_\pm = \pm \eta R\left(g A_\pm g^{-1}\right)$$

$$\boxed{\mathcal{F}(\tau,\sigma) = \mathcal{P}\exp\left(\int_{(0,0)}^{(\tau,\sigma)} d\xi^\alpha\, V_\alpha\right)\mathcal{F}(0,0)}$$

PERIODIC YB

$$g(2\pi) = g(0)$$

$$\boxed{W \xrightarrow{\eta\to 0} 1}$$

TWISTED PCM

$$\tilde{g}(2\pi) = \mathcal{F}^{-1}(2\pi)\, g(2\pi)$$

$$= \underbrace{\mathcal{F}^{-1}(2\pi)\,\mathcal{F}(0)}_{W}\,\tilde{g}(0)$$

W TWIST

A NEW TWIST

[RB, DRIEZEN, MIRAMONTES '21]

$$\partial_\pm \mathcal{F}\mathcal{F}^{-1} = \pm \eta R\left(\mathcal{F}\,\partial_\pm \tilde{g}\tilde{g}^{-1}\mathcal{F}^{-1}\right)$$

$$\downarrow$$

$$\boxed{\eta^{-1} P^T\left(\mathcal{F}^{-1}\omega(\partial_\pm \mathcal{F}\mathcal{F}^{-1})\mathcal{F}\right) = P^T\left(\partial_\pm \tilde{g}\tilde{g}^{-1}\right)} \qquad \text{(s.o.v.)}$$

$$\mathcal{F} = \exp(RX) \in F$$

$$\boxed{Y = \frac{1}{\eta} P^T \frac{1-e^{-\mathrm{ad}_{RX}}}{\mathrm{ad}_{RX}} X}$$

$$e^{-z}de^{z} = \frac{1-e^{-\mathrm{ad}_z}}{\mathrm{ad}_z}dz$$

$$\partial_\pm Y = \eta^{-1} P^T\left(\mathcal{F}^{-1}\omega(\partial_\pm \mathcal{F}\mathcal{F}^{-1})\mathcal{F}\right) \rightsquigarrow Y = \int P^T\left(* d\tilde{g}\tilde{g}^{-1}\right)$$

$$\partial_\pm W = 0, \qquad W \in F \qquad W = \exp(-\eta RQ)$$

JORDANIAN DEFORMATION

$$sl(2,\mathbb{R}) \subset \mathfrak{g} \qquad [T_0, T_\pm] = \pm T_\pm \;, \quad [T_+, T_-] = 2T_0$$

$$\boxed{r = T_+ \wedge T_0}$$

$$X = X_0 T^0 + X_+ T^+ \qquad Y = Y_0 T^0 + Y_+ T^+$$

$$\boxed{Y = \frac{1}{\eta} P^T \frac{1 - e^{-ad_{RX}}}{ad_{RX}} X}$$

$$\boxed{\begin{aligned} X_+ &= -\lg(1 - \eta Y_+) \\ X_0 &= -\frac{Y_0}{Y_+} \lg(1 - \eta Y_+) \end{aligned}}$$

$$\boxed{W = \exp[Q(T_0 - q T_+)]}$$

$$Q = \lg \frac{1 - \eta Y_+(0)}{1 - \eta Y_+(2\pi)} \qquad q = \frac{Y_0(2\pi) - Y_0(0)}{Y_+(2\pi) - Y_+(0)}$$

MONODROMY MATRIX AND CSC

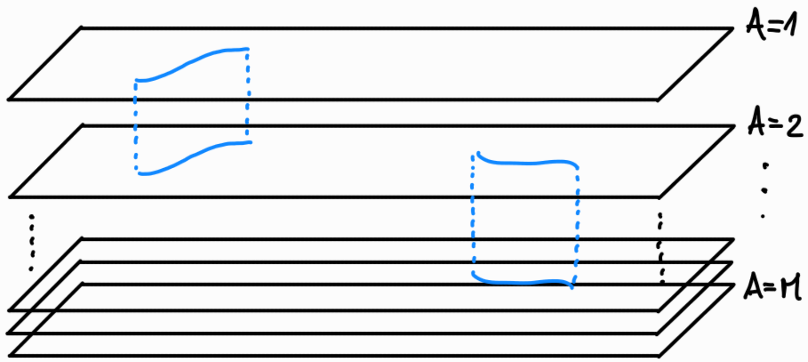
$$\Omega(z,\tau) = \mathcal{P}\exp\left(-\oint \mathcal{L}_\sigma\right), \qquad \mathcal{L}_\pm = \frac{\tilde{J}_\pm}{1 \mp z}$$

$z \sim 0$ $$\mathcal{L}' = \tilde{g}(\mathcal{L} + d)\tilde{g}^{-1}$$

$$\boxed{\Omega = \tilde{g}(2\pi)^{-1}\,\mathcal{P}\exp\left(-\oint \mathcal{L}'_\sigma\right)\tilde{g}(0) = \tilde{g}(0)^{-1}\, W^{-1}\left(1 - z\int_0^{2\pi} d\sigma\, \partial_\tau \tilde{g}\,\tilde{g}^{-1} + \dots\right)\tilde{g}(0)}$$

EIGENVALUES $\lambda_A = e^{ip_A}$ "POLARIZED" BY W

$$p_A \overset{z\sim\pm1}{\sim} \pm\frac{q_A^\pm}{1\mp z}$$



$$p(z) \overset{z\sim 0}{\sim} p(0) + z\,p'(0) + \dots$$

$$p(z) \overset{z\sim\infty}{\sim} \frac{p_\infty}{z} + \dots$$

$$p_A(z+i\epsilon) - p_B(z-i\epsilon) = 2\pi\, n_{(A,B)} \qquad z \in C_{(A,B)}$$

JORDANIAN DEFORMATION OF $AdS_5 \times S^5$

[RB, DRIEZEN, NIETO GARCIA, WYSS '22]

$32 \to$ 12 SUPERISOMETRIES

AdS QUASIMOMENTA

$$\begin{cases} \hat{p}_1 = -\frac{2\pi}{x}(E+S) + \mathcal{O}(1/x^2) \\ \hat{p}_2 = -\frac{i}{2} Q + \frac{2\pi}{x} S + \mathcal{O}(1/x^2) \\ \hat{p}_3 = +\frac{i}{2} Q + \frac{2\pi}{x} S + \mathcal{O}(1/x^2) \\ \hat{p}_4 = +\frac{2\pi}{x}(E-S) + \mathcal{O}(1/x^2) \end{cases}$$

$$\hat{p}_1 \sim \frac{2\pi}{x}(+E-S_1+S_2)$$
$$\hat{p}_2 \sim \frac{2\pi}{x}(+E+S_1-S_2)$$
$$\hat{p}_3 \sim \frac{2\pi}{x}(-E-S_1-S_2)$$
$$\hat{p}_4 \sim \frac{2\pi}{x}(-E+S_1+S_2)$$

GLOBAL-TIME TRASL. GENERATED BY $p^0 - p^1 + \frac{1}{2}(K^0 + K^1)$ $\neq$ BMN $p^0 + K^0$

S = ANGULAR MOMENTUM IN AdS (J_{23})

SEMICLASSICAL CORRECTION TO CSC

SEMICLASSICAL QUANTIZATION À LA [GROMOV AND VIEIRA '07]

$$p \longrightarrow p + \delta p$$

NEW POLES FOR "MICRO CUTS"

δp FIXED BY CONSISTENCY CONDITIONS (GLUING AT CUTS AND ASYMPTOTICS)

BMN-LIKE SOLUTION: $E = a\,,\ S = 0\,,\ Q = 4\pi a \beta$

$$\hat{p}_1 = -\hat{p}_4 = \frac{2\pi a\sqrt{x^2-\beta^2}}{1-x^2}\,, \qquad \hat{p}_2 = -\hat{p}_3 = \frac{2\pi a x\sqrt{1-\beta^2 x^2}}{1-x^2}$$

$$\delta E = \frac{a}{2}\left[2\beta(1-\beta) - (3-\beta^2)\log(1-\beta^2) - (1+\beta^2)\log(1+\beta^2) + (1+\beta^2)\log\frac{1-\beta}{1+\beta}\right]$$

$$\delta S = 0 \qquad\qquad \delta Q = 0$$

SUMMARY, OUTLOOK, AND (SOME OF) WHAT I DIDN'T SAY

- EASIER TO WORK WITH TWISTED MODEL THAN DEF. ONE

- TWIST ($\Rightarrow p$) HIGHLY NON-LOCAL IN TERMS OF $g \in YB$
  $\hookrightarrow$ HIDDEN SYMMETRIES IMPORTANT

- DEFORMATION / TWIST & EXPANSION OF p DON'T COMMUTE
  $p \sim \ldots + \sqrt{\mathbf{Q}^2 + \#/x^2 + \cdots}$ NO DEF. OF BETHE EQ'S ?

- $\delta \mathbf{Q} = 0$, ACCIDENT OF THIS SOLUTION ?

- FACTORIZATION OF THE JORDANIAN TWIST $W = e^{q T_+} e^{\mathbf{Q} T_0} e^{-q T_+}$

- UNIMODULAR (= SUGRA) VS NON-UNIMODULAR (= NON-SUGRA) JORD. DEF'S
  SPECTRAL EQUIVALENCE (?)

- YB ≈ INTERPOLATION BETWEEN MODEL AND NON-ABELIAN T-DUALITY
[HOARE, TSEYTLIN '16, RB WULFF '16]

- POISSON-LIE SYMMETRIES & POISSON-LIE DUALITY TRANSF'S [KLIMČÍK, SEVERA '95]

- RELATION TO 4D CHERN-SIMONS THEORY [COSTELLO, WITTEN, YAMAZAKI '17, ...]

- RELATION TO GENERALISED GEOMETRY AND DOUBLE FIELD THEORY [...]

- DEF'S BREAKING BMN ISOMETRIES
  ↪ ALTERNATIVES TO BMN SOLUTION ?
  - MASSIVE SPECTRUM ?
  - SYMMETRIES FOR S-MATRIX ?
  - ...

- DRINFELD TWISTS $\left(\Delta_F = F^{-1} \circ \Delta \circ F\right)$ FOR THE QUANTUM MODELS ?
  SEE [VAN TONGEREN '15 '16]

- NON-DIAGONALISABLE TWISTS ? [GUICA, LEVKOVICH-MASLYUK, ZAREMBO '17]

- TWISTED SPIN-CHAINS ?

- DEFORMATIONS OF $\mathcal{N}=4$ SYM ?

- ...

Thank
you !