The relevance of Being Irrelevant

Based on works with G. Camilo, T. Fleury, M. Léncsés and A. Zamolodchikov [2106.11999]
L. Cordova and F. Schaposnik [2110.14666]
and work in progress

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Consider a theory near an RG fixed point (in $D = 2$ dimensions)

$$\mathscr{A} = \left[\mathscr{A}_{\mathrm{CFT}} + \mu \int d^2x \, \Phi_\Delta(x) \right] + \sum_i \alpha_i \int d^2x \, O_{\delta_i}(x)$$

Here Φ_Δ is a relevant operator ($d = 2\Delta < 2$) while
O_{δ_i} are irrelevant operators ($d_i = 2\delta_i > 2$).
No marginal operators for simplicity

- In square brackets is a UV complete theory (consistent at all scales)
- Irrelevant operators shatter UV completeness: theory is *effective*
- Perturbation expansion in α_i leads to accumulation of UV divergencies
- Theory is non-renormalizable $\Longrightarrow$ no predictive power

Can we say something more?

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Can we say something more?

Consider the space Σ of *quasi-local field theories* Wilson & Kogut, '74

$$\Sigma = \left\{ \mathscr{A}_\Lambda[\Phi] \;\middle|\; \mathscr{A}_\Lambda[\Phi] = \int_\Lambda d^2x\, \mathscr{L}[\Phi(x), \partial_\mu\Phi(x), \partial_\mu\partial_\nu\Phi(x), \ldots] \right\}$$

Points are labelled by actions equipped with a UV cut-off Λ

Quasi-local = non-locality range $< \epsilon \equiv \Lambda^{-1}$

Each describe a QFT up to a characteristic length scale ϵ

The RG group acts on Σ as a flow

$$\frac{d}{d\ell}\mathscr{A} = B\{\mathscr{A}\} \;, \quad B\{\mathscr{A}\} \in T\Sigma\Big|_{\mathscr{A}} \;, \quad \ell \propto \log(\epsilon)$$

A QFT is an integral curve of the above flow

$d\ell > 0 \Longrightarrow$ large-scale properties (IR); no problem expected

$d\ell < 0 \Longrightarrow$ short-scale properties (UV); pathology expected!

$\exists \ell_*$ such that $\mathscr{A}_\ell \notin \Sigma\,, \forall \ell < -\ell_*$

$\Longrightarrow \exists$ intrinsic UV scale $\Lambda_* = M e^{\ell_*}$, e.g. Landau Scale of QED

$\Sigma_{\ell_*=\infty}$ sub-space of UV-complete QFT; cut-off can be removed consistently


Σ
Here Be Dragons
Here Be Dragons
Here Be Dragons
CFT
CFT
?
Φ_Δ
O_{δ_i}
$T\overline{T}$
$T\overline{T}$
$T\overline{T}$
$-\ell_*$

The $T\overline{T}$ operator is defined as Smirnov & Zamolodchikov, '16

$$T\overline{T}(x) = -\lim_{x'\to x} T(x,x')\ , \quad T(x,x') = \frac{1}{2} e_{\mu\rho} e_{\nu\sigma} T^{\mu\nu}(x) T^{\rho\sigma}(x')$$

Its expectation value is a constant

$$\frac{\partial}{\partial x^\mu} \langle T(x,x') \rangle = -\frac{\partial}{\partial x'^\mu} \langle T(x,x') \rangle = 0$$

and factorizes (Ward Identities + spectral decomposition)

$$\left\langle T\overline{T}(x) \right\rangle = -\det_{\mu\nu} \langle T^{\mu\nu}(x) \rangle$$

The singularities in the collision limit are under full control

$$T(x,x') \simeq -T\overline{T}(x) + \delta(x-x') T^\mu_\mu(x) + \sum_a C^a_\lambda(x-x') \frac{\partial}{\partial x^\lambda} O_a(x)$$

$$\Longrightarrow \langle T(x,x') \rangle = -\left\langle T\overline{T}(x) \right\rangle + \text{contact term}$$

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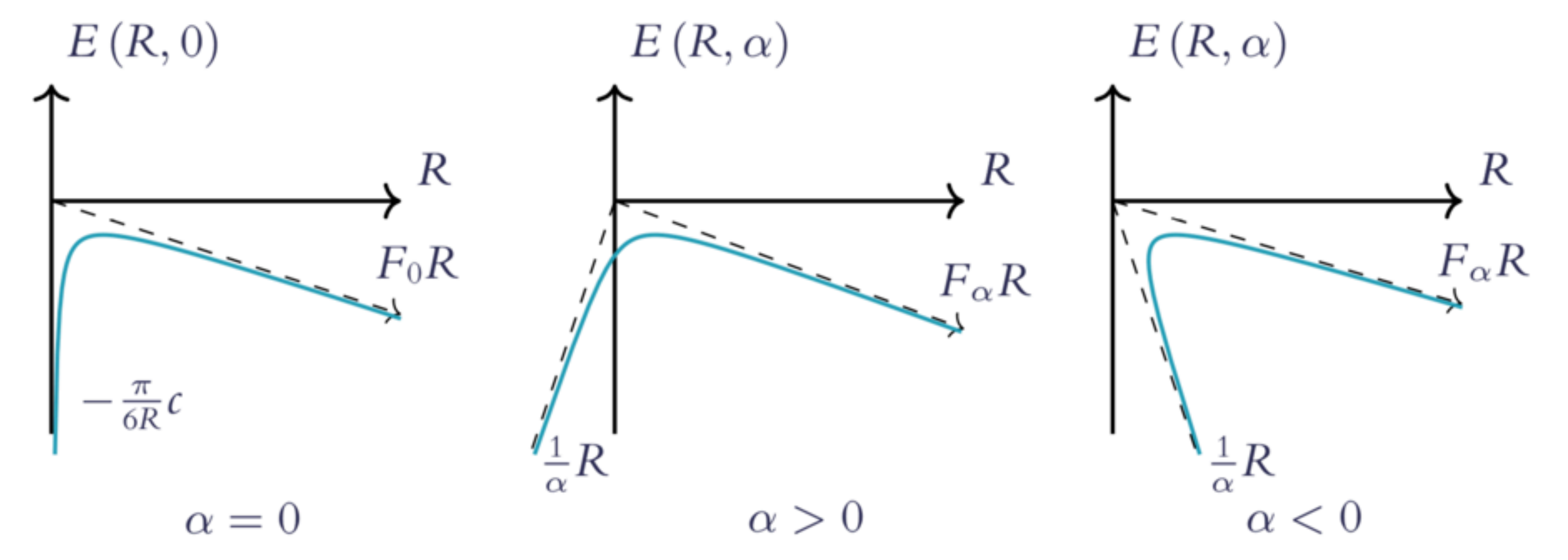
Finite-size (cylinder) spectrum obeys the Burgers equation

$$\frac{\partial}{\partial\alpha}E_n(R,\alpha) + E_n(R,\alpha)\frac{\partial}{\partial R}E_n(R,\alpha) + \frac{1}{R}P_n(R)^2 = 0$$



Use factorization and the standard identifications

$$\left\langle n\,\middle|\,T^{xx}\,\middle|\,n\right\rangle = -\frac{1}{R}E_n(R)\,, \quad \left\langle n\,\middle|\,T^{yy}\,\middle|\,n\right\rangle = -\frac{d}{dR}E_n(R)$$

$$\left\langle n\,\middle|\,T^{xy}\,\middle|\,n\right\rangle = \frac{i}{R}P_n(R)$$

Functional form (in zero momentum sector) Cavaglia, SN, Szecsényi, Tateo, '16

$$E(R,\alpha) = E(R - \alpha E(R,\alpha),0)$$

From the CFT behaviour $E(R,0) \sim -\frac{\pi}{6}\frac{c}{R}$ one extracts

$$E(R,\alpha) \sim \frac{R}{2\alpha}\left(1 - \sqrt{1 + \frac{2\pi c}{3R^2}\alpha}\right)$$

For $\alpha > 0$ there is a finite $R \to 0$ limit: $E(R,\alpha) \to -\sqrt{\frac{\pi c}{6\alpha}}$

Entropy density is finite in vanishing volume $s(R=0,\alpha) \propto \sqrt{c/\alpha}$

For $\alpha < 0$ there is a Hagedorn temperature $1/T_H = R_H = \sqrt{2/3\,\pi c\,|\alpha|}$

Entropy density diverges at R_H as $s(R,-|\alpha|) \sim c/6\,(R^2 - R_H^2)^{-1/2}$

Hagedorn-type high energy spectrum

$$\mathcal{N}(E) \sim e^{R_H E}$$

e.g. Barbon & Rabinovici, '20

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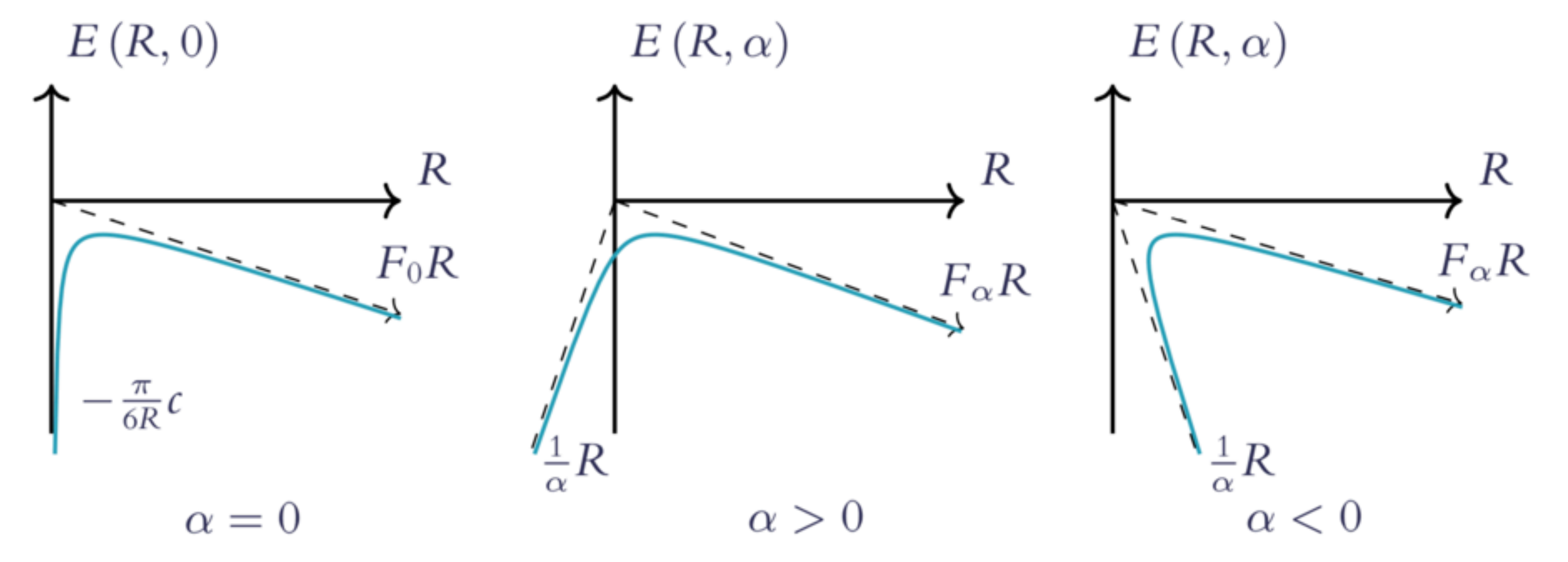

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Why study $\mathrm{T}\overline{\mathrm{T}}$ deformations and its generalisations (stay tuned)?

Main practical reasons:

- They allow a high degree of control: they are *solvable*
- They preserve existing symmetries (e.g. integrable structures)
- $\mathrm{T}\overline{\mathrm{T}}$ is universal: (almost) any $\mathscr{A}_0$ will do
- The family of generalisations span the subspace Σ_{Int} of integrable QFTs

Some important motivations

- Non-Wilsonian UV behaviour (Hagedorn spectrum, non-locality, etc...)
- Robust features $\Longrightarrow$ sensible extension of Wilsonian QFT paradigm
- Intriguing relations to String Theory and Quantum Gravity
- Irrelevant operators control the sub-leading corrections to critical behaviour

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Consider the scaling limit of, say, a lattice system

Tuning the parameters appropriately, the continuum description is a CFT

First-order corrections are given by relevant operators Φ_Δ

Subleading corrections are controlled by irrelevant operators

$$F \underset{T\to T_c}{\sim} F_0 + a\left(T-T_c\right)^{2\nu} + a'\left(T-T_c\right)^{\omega} + \cdots$$

$$R_c^{-1} = M \underset{T\to T_c}{\sim} b\left(T-T_c\right)^{\nu} + b'\left(T-T_c\right)^{\tau} + \cdots$$

Suppose $T\bar{T}$ is the irrelevant operator of lowest dimension ($d_{T\bar{T}} = 4$)

Properties of $T\bar{T}$ constrain the exponents and coefficients

$$\omega = d_{T\bar{T}}\nu = 4\nu\ , \quad \tau = (d_{T\bar{T}}-1)\nu = 3\nu\ , \quad \frac{b'}{a'} = \frac{b}{a}$$

E.g. Ueda, Oshikawa '21 | Ghaemi, Vishwanath, Sentil, '05

Important applications:

- AdS/CFT correspondence McGough, Mezei, Verlinde '16 … …
- (little) String Theory Giveon, Itzhaki, Kutasov '17 … …
- Quantum Gravity (JT dilaton) Dubovski, Gorbenko, Hernandez-Chifflet '18 … …
- dS/dS correspondence Gorbenko, Silverstein, Torroba '18
- Confining/Effective string Chen, Dubovski, Hernandez-Chifflet '18 | Beratto, Billò, Caselle '19
- Generalised hydrodynamics
- Out-of-equilibrium systems Medenjak, Policastro, Yoshimura '20
- Quantum phases Griguolo, Panerai, Papalini, Seminara '21
- Long-range deformations of spin chains

  Pozsgay, Jiang, Takáks '19 | Marchetto, Sfondrini, Yang '19

  Bargheer, Beisert, Loebbert '09

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S. Dubovsky, V. Gorbenko and M. Mirbabayi '17

The $T\bar{T}$ implies the following deformation for the S-matrix

$$\frac{\delta S_{N\to M}\left(\{p_i\},\{q_k\},\alpha\right)}{S_{N\to M}\left(\{p_i\},\{q_k\},\alpha\right)} = \frac{i}{2}\delta\alpha\left[\sum_{p_i<p_j}\vec{p}_i\wedge\vec{p}_j + \sum_{q_k<q_l}\vec{q}_k\wedge\vec{q}_l\right]$$

For Integrable theories, the scattering factorizes in sequence of 2-body processes

The deformed 2-body S-matrix reads (here $\vec{p} = \left(m\cosh\theta, m\sinh\theta\right)$)

$$S_{2\to2}(\theta,\alpha) = e^{i\alpha m^2\sinh\theta}S_{2\to2}(\theta,0) \quad (*)$$

$\exp\left[i\alpha m^2\sinh\theta\right]$ is an *exponential CDD factor*

I.e. it automatically satisfies unitarity, crossing, analyticity and macro-causality

$(*)$ can be taken as a definition of the $T\bar{T}$ deformation

- Action flow can be recovered via the TBA/NLIE   Cavaglia, SN, Szecsényi, Tateo, '16
- Gravitational phase shift $\Delta t = -\alpha E$ means

  $\alpha < 0$: healthy theory (probably no local observables)

  $\alpha > 0$: superluminal propagation, still S-matrix well defined

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Define a family of S-matrix deformations: the *CDD deformations*

$$S_\Phi(\theta) = \Phi(\theta)S_0(\theta)$$

$\Phi(\theta)$ is a CDD factor: a scalar function of the form

$$\Phi(\theta) = \Phi^N_{\text{rat}}(\theta)\Phi_{\text{exp}}(\theta)$$

$$\Phi_{\text{exp}}(\theta) = \exp\left[-i\sum_{s\in\mathbb{S}}a_s\sinh(s\theta)\right]$$

$$\Phi^N_{\text{rat}}(\theta) = \prod_{j=1}^{N}\frac{\sinh\theta_j + \sinh\theta}{\sinh\theta_j - \sinh\theta}$$

$\Phi_{\text{exp}}(\theta)$ is an entire function (series in exponent converges $\forall\theta$)

θ_j restricted: $\text{Im}(\theta_j) \in \left[-\pi,0\right] \mod 2\pi$

- $\text{Re}(\theta_j) \neq 0$: *resonances* of complex mass $m_j = 2m\cos(\theta_j/2)$
- $\text{Re}(\theta_j) = 0$: *virtual states*; no clear interpretation

The S-matrix gives access to the finite-size spectrum via the TBA

$$E_0(R) = -\, m \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta) \log\left[1 + e^{-\varepsilon(\theta)}\right]$$

$$\varepsilon(\theta) = mR\cosh\theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \log\left[1 + e^{-\varepsilon(\theta')}\right] \quad (*)$$

$$\varphi(\theta) = -\, i \frac{d}{d\theta} \log\left[S(\theta)\right]$$

Another important observable: the effective central charge

$$\tilde{c}(R) = -\, 6R/\pi E_0(R)$$

Its $R \to \infty$ and $R \to 0$ limits determine the IR and UV central charges

$$\tilde{c}(R) \underset{R\to\infty}{\sim} 3/\pi\sqrt{2mR}e^{-mR} \to 0$$

$$\lim_{R\to 0} \tilde{c}(R) = c_{\text{UV}} - 12\left(\Delta_{\text{min}} + \overline{\Delta}_{\text{min}}\right)$$

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Exponential CDDs deform the driving term. E.g. for T$\overline{\text{T}}$ $R \to R + a_1 E_0(R)/m^2$

For rational CDD, no exact result: numerics is needed

Asymptotic $R \to \infty$ analysis of $(*)$ shows the following possible behaviours

1 - $\varepsilon \sim mR\cosh\theta$: this is the standard asymptotic, always possible

2 - $\varepsilon \sim A/r\cosh\theta$: this is only possible in the T$\overline{\text{T}}$ case

3 - $\varepsilon \sim -\, rf(\theta)$, with $f(\theta) > 0$, $\theta \in \Theta \subset \mathbb{R}$

this situation is possible only if $|\varphi|_1 \doteq \int_{-\infty}^{\infty} d\theta\, \varphi(\theta)/(2\pi) > 1$

Note that $|\varphi|_1$ measures the difference between bound states and resonances

Whenever $|\varphi|_1 > 1$, we expect the TBA solution to be at least 2-valued

Suspect is confirmed by numerical instability for R sufficiently small

We need a numerical procedure able to handle singular points (e.g. branch points)

Idea: employ methods of numerical analysis used in bifurcation theory

In particular: *the (pseudo)-arc-length continuation method*

Simple, though extremely powerful. Handles bifurcation and turning points

Parametrize solutions $\varepsilon(\theta, r)$ in terms of auxiliary parameter ς, as pairs
$\left\{\varepsilon\left(\theta, r(\varsigma)\right), r(\varsigma)\right\}$

Starting point is a known solution (e.g. obtained by standard iterations at large R)

Follow the solution curve by varying ς by a small step $\Delta\varsigma$

In this way, the solution curve is single-valued (as a function of ς)

Instabilities are resolved and we can move past the branch (turning) point R_*

We slightly altered the method to accomodate complex solutions

This allowed us to follow TBA solutions in the region $R < R_*$, all the way to $R \sim 0$

and extract the limit $\lim_{R\to 0} \tilde{c}(R)$

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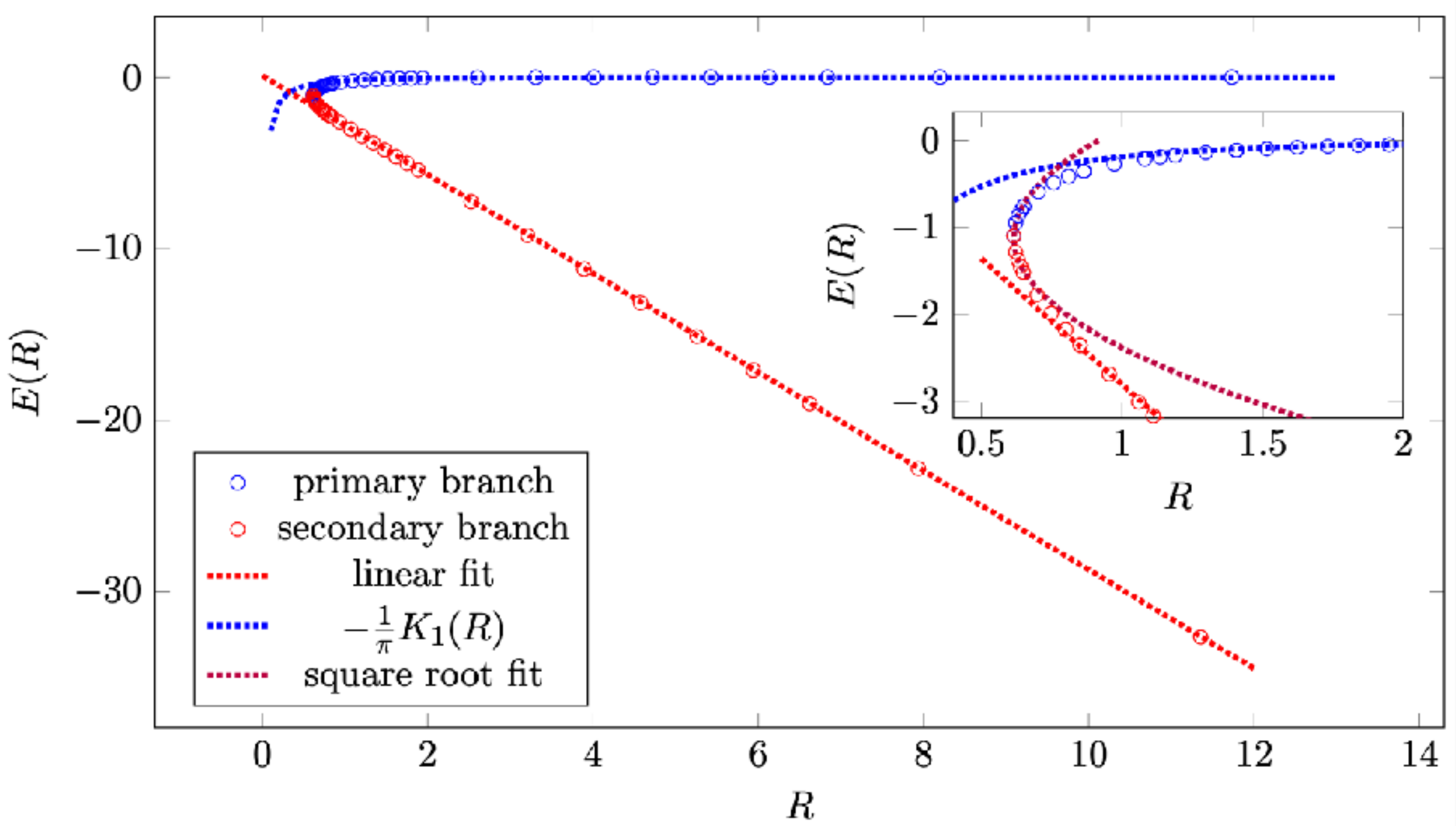
In all the examples that we dealt with, we verified the presence of a branch point, whenever $|\varphi|_1 > 1$



We considered a theory with 2-resonances S-matrix

$$S(\theta) = \frac{i\sin u_1 + \sinh\theta}{i\sin u_1 - \sinh\theta}\frac{i\sin u_2 + \sinh\theta}{i\sin u_2 - \sinh\theta}$$

for various ranges of the poles u_1, u_2. Most interesting is

$u_2 \to -\pi/2 + i\infty$: this is a peculiar case, in which the S-matrix becomes the

"bosonic" version of a 1-resonance S-matrix:

$$S(\theta) = \frac{i\sin u_1 + \sinh\theta}{i\sin u_1 - \sinh\theta}$$

If $u_1 \in [-\pi,0]$ this is the "bosonic counterpart" of sinh-Gordon

If $u_1 = -\pi/2 + i\theta_0$ this is the "bosonic counterpart" of the "staircase"

Both these theories display a branch point at some value $R = R_*$

They are effectively 2-resonance models

NUMERICAL RESULTS: THE 2CDD MODELS

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In all the various choices of parameters u_1, u_2 we witnessed the same qualitative behaviour:

- A "standard" branch with usual $R \to \infty$ asymptotic
- A "second" branch with $\varepsilon(\theta) \sim -rf(\theta)$ as $R \to \infty$, and

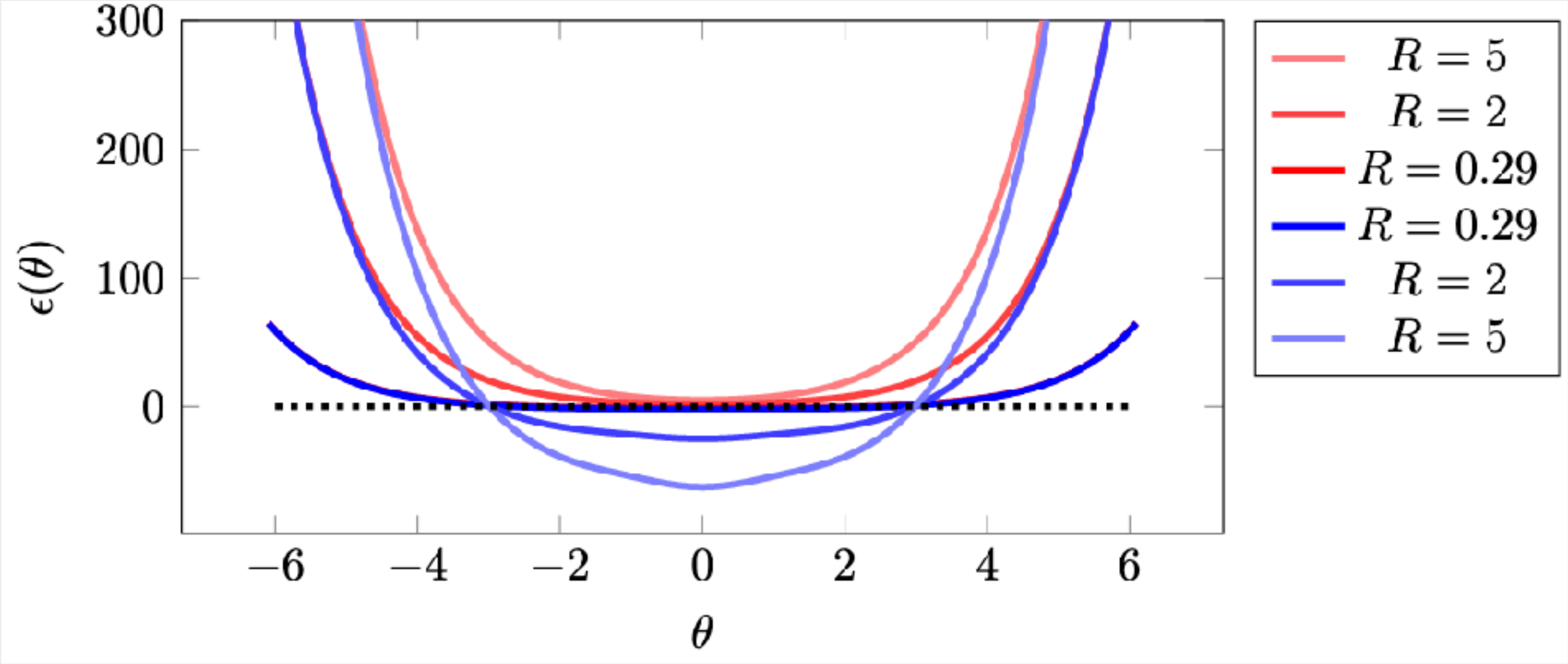
$$f(\theta) = -\cosh\theta + \int_{-B}^{B}\frac{d\theta'}{2\pi}\varphi(\theta-\theta')f(\theta')\,, \quad B > 0$$

- Sub-leading contributions to the energy of order $\sim R^{-3}$:

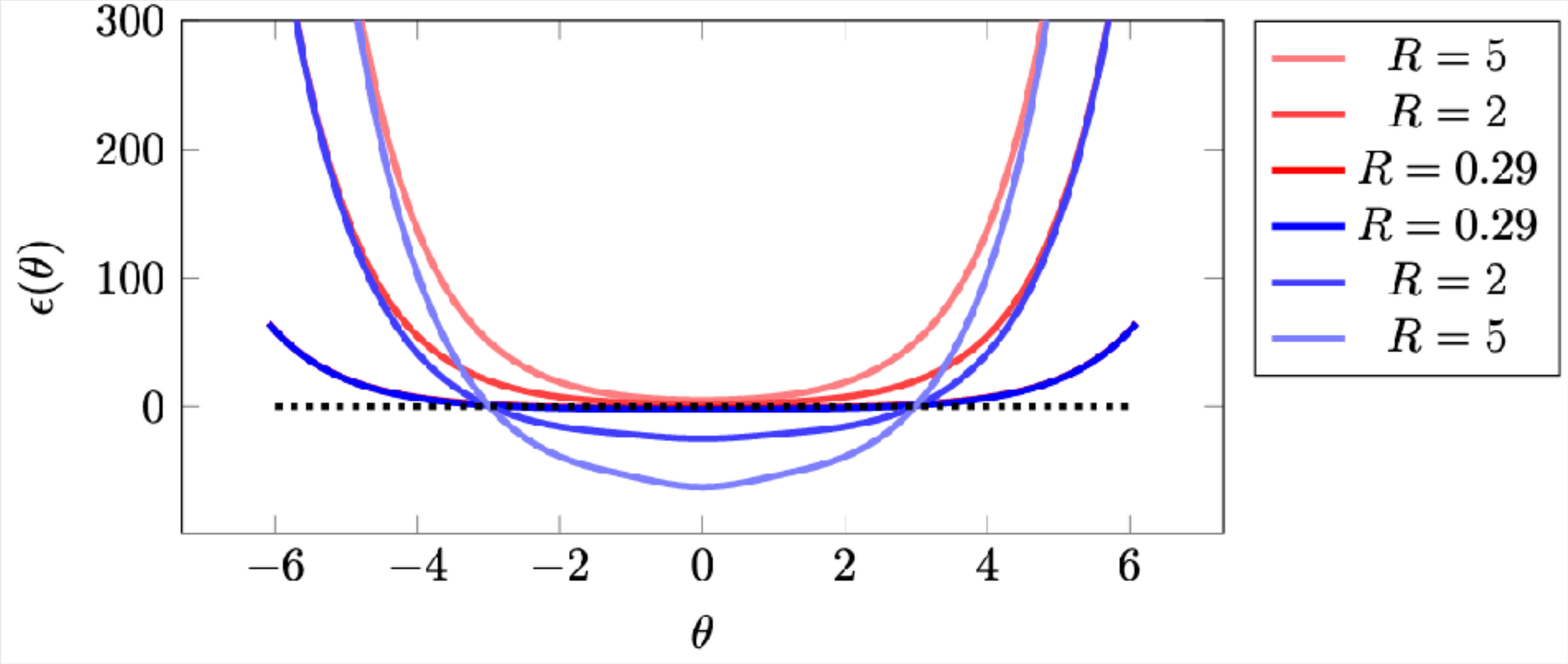
$$E(R) \sim -R\int_{-B}^{B}\frac{d\theta}{2\pi}\cosh\theta f(\theta) - \frac{1}{2\pi R^3}E^{(-3)}(B) + \cdots$$

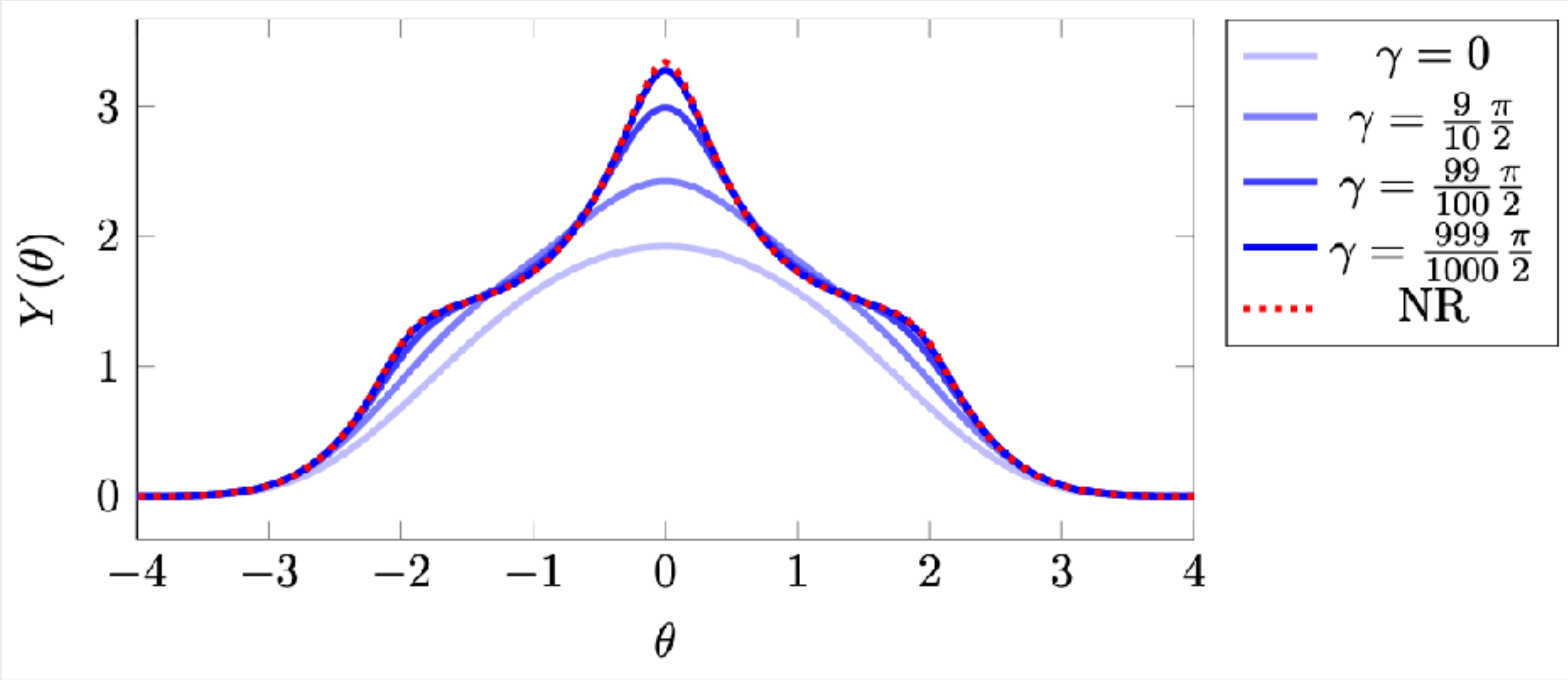
- A branch point *in* R *only*, for $R = R_* > 0$ (i.e. $R_* \not\propto \theta$)

$$\varepsilon(\theta) = \varepsilon_0(\theta) + \sqrt{R - R_*}\,\varepsilon_{1/2}(\theta) + \cdots$$



Pseudo-energy for $u_2 = u_1^*$, $u_1 = -\pi/10 + 2i$





Pseudo-energy for $u_2 = u_1^*$, $u_1 = -\pi/10 + 2i$

In the case $u_2 = u_1^*$, $u_1 = \gamma - \pi/2 + i\theta_0$, in the limit $\gamma \to \pi/2$ the TBA equation reduces to the Narrow Resonance Equation

$$Y(\theta) = e^{-R\cosh\theta}\left[1 + Y(\theta + \theta_0)\right]\left[1 + Y(\theta - \theta_0)\right]$$

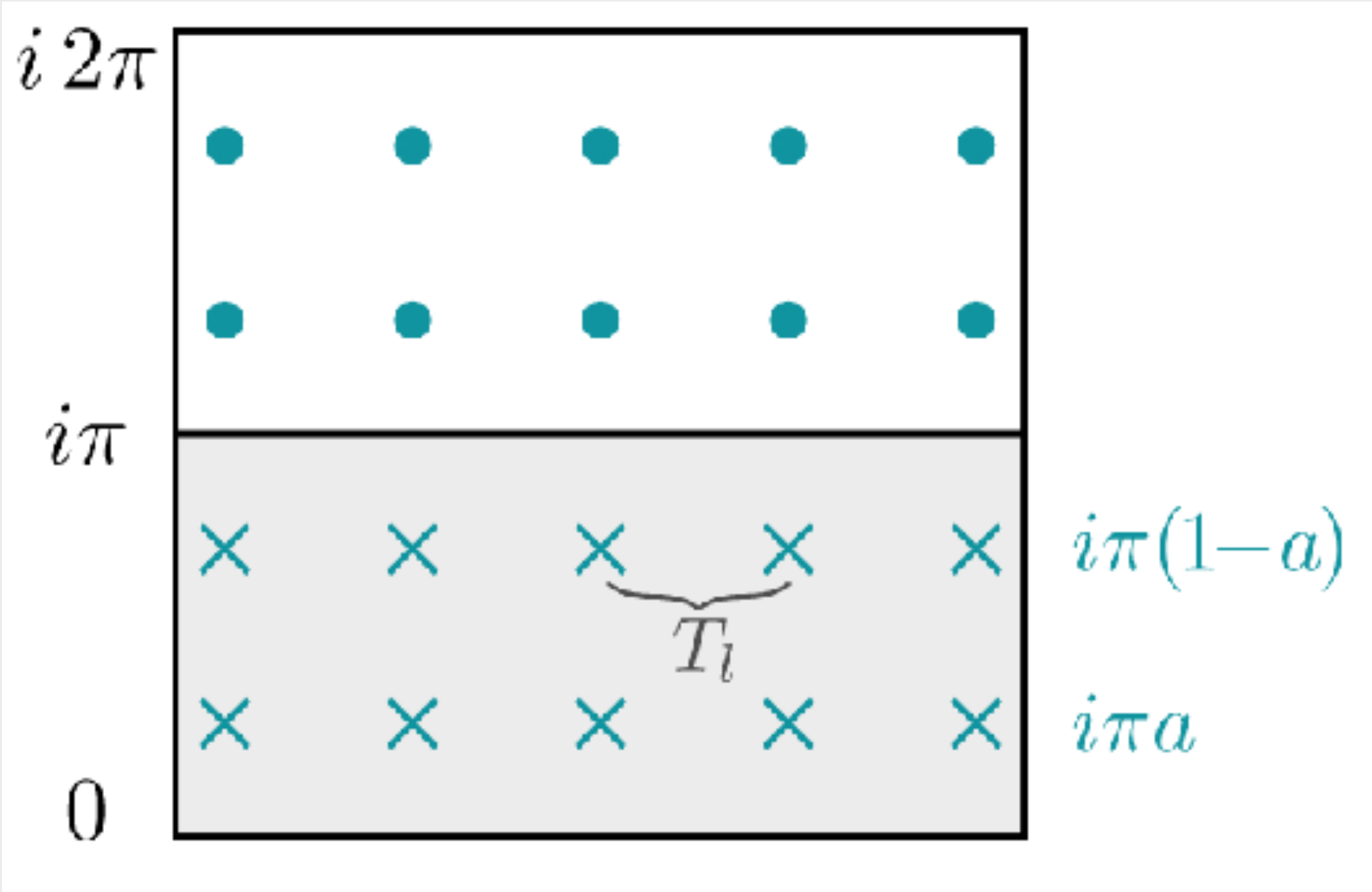
$$Y(\theta) = e^{-\varepsilon(\theta)}$$

Theory determined by the ∞-resonance S-matrix

$$S(\theta) = \frac{\mathrm{sn}_l\left(2iK_l\theta/\pi\right) + \mathrm{sn}_l\left(2K_l a\right)}{\mathrm{sn}_l\left(2iK_l\theta/\pi\right) - \mathrm{sn}_l\left(2K_l a\right)}$$

Here $\mathrm{sn}_l(x)$ is Jacobi elliptic sine function of modulus l

K_l is the complete elliptic integral and $a \in [0,1/2]$ is the coupling constant



$T_l = \pi K_{\sqrt{1-l^2}}/K_l$; crosses (dots) are simple zeroes (poles)

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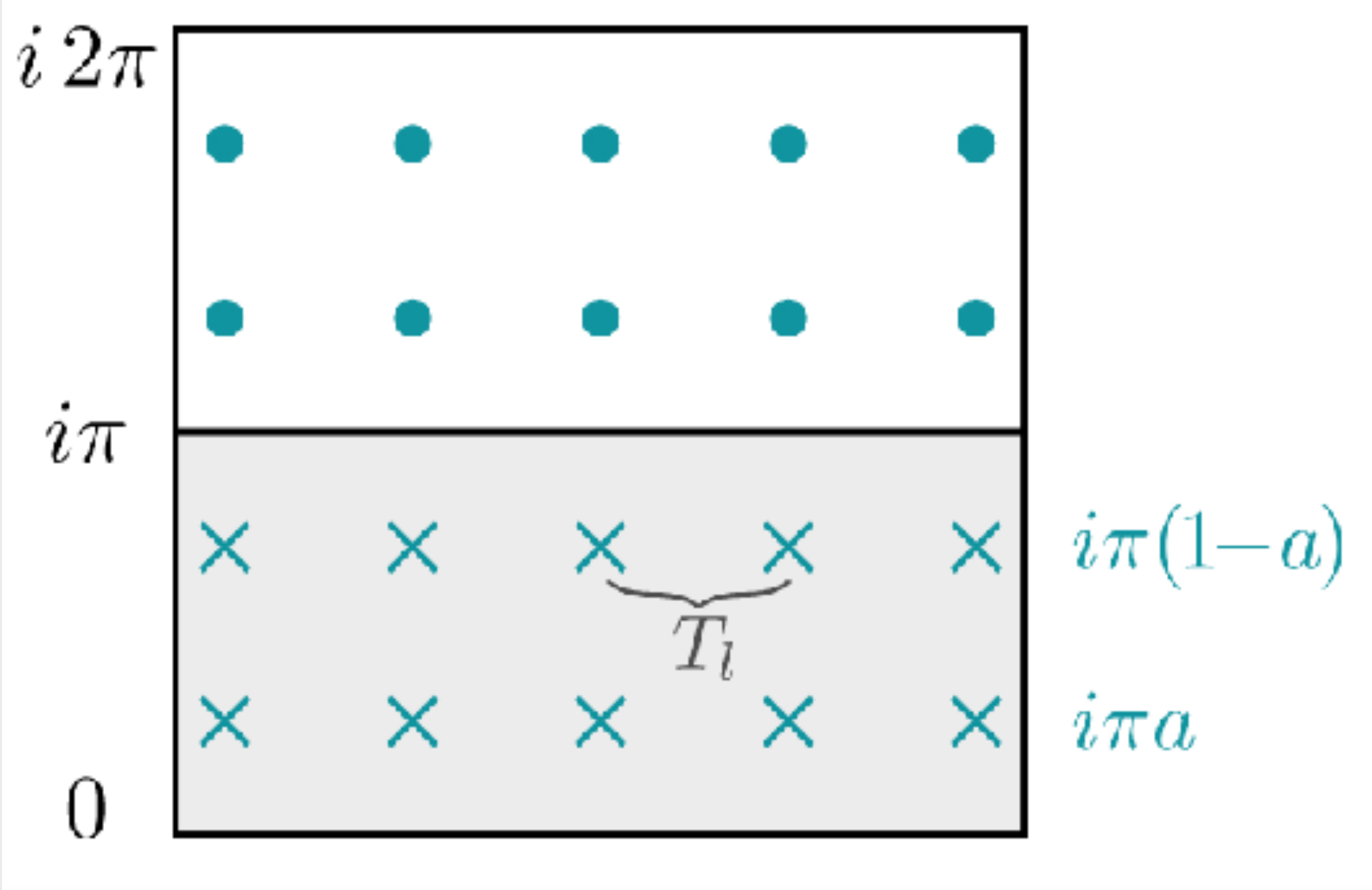
Here $\mathrm{sn}_l(x)$ is Jacobi elliptic sine function of modulus l

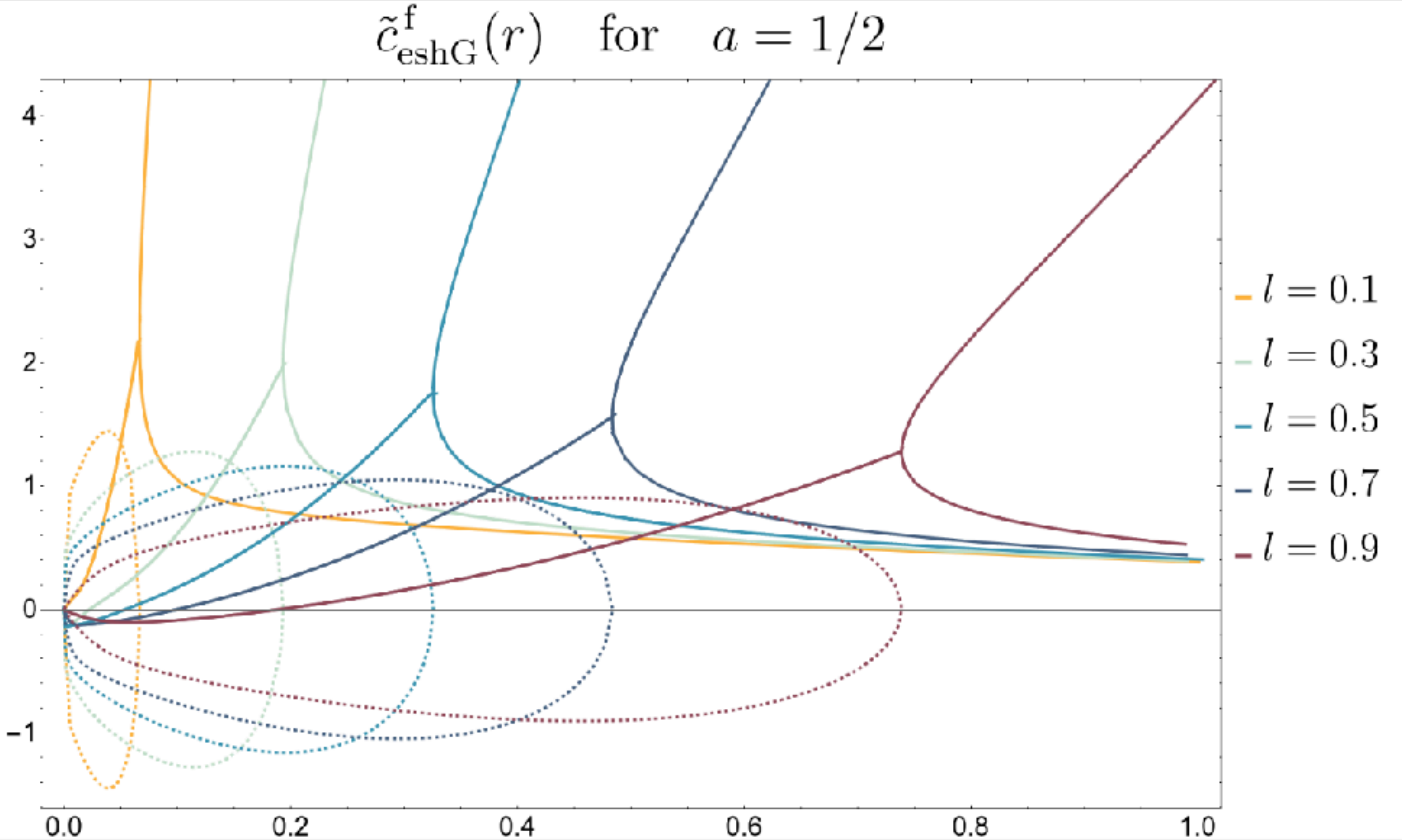
K_l is the complete elliptic integral and $a \in [0,1/2]$ is the coupling constant



$T_l = \pi K_{\sqrt{1-l^2}}/K_l$; crosses (dots) are simple zeroes (poles)

This theory does not have a known action

It represent a toy example of S-matrices with infinite resonances ubiquitous in the S-matrix bootstrap (e.g. $O(N)$ Yang-Baxter model)



The $R \to 0$ limit of the effective central charge $\lim_{R\to 0} \tilde{c}(R) = 0$

can actually be derived analytically via the "dilogarithm trick"

$$\lim_{R\to 0} \tilde{c}(R) = \frac{6}{\pi^2}\left[\mathrm{Li}_2\left(\frac{y_0}{1+y_0}\right) + \frac{1}{2}\log\left(\frac{y_0}{1+y_0}\right)\log\left(\frac{1}{1+y_0}\right)\right]$$

where $y_0 = \exp(-\varepsilon_0)$ is a constant solution of the TBA at $R = 0$

$$\varepsilon_0 = -\,|\,\varphi\,|_1 \log\left[1 + e^{-\varepsilon_0}\right] \implies y_0 = \left(1 + y_0\right)^{|\varphi|_1}$$

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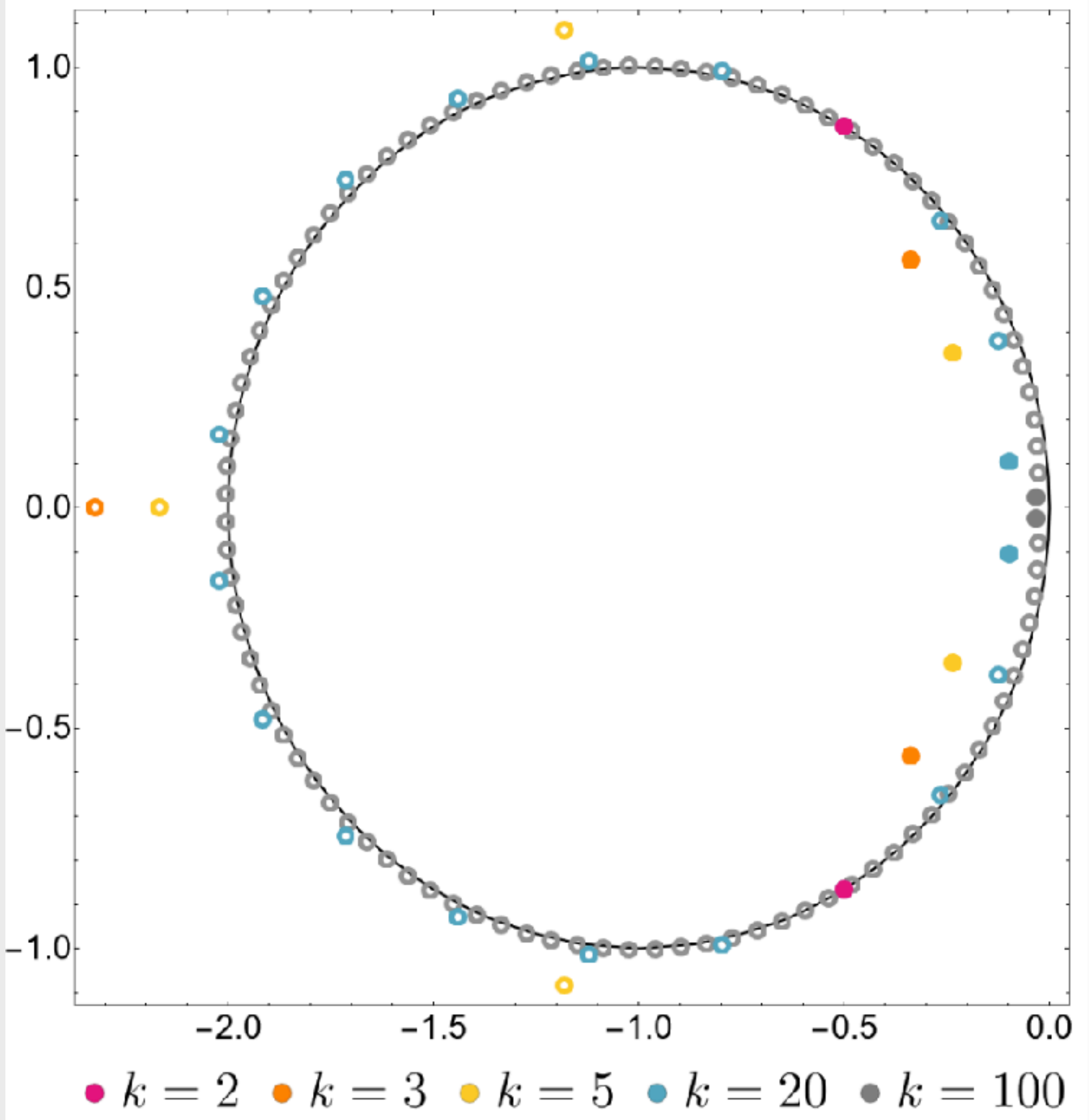
As $|\varphi|_1$ grows, the solutions condense on a unit circle centred around -1

The correct solutions are the ones that minimise $|\tilde{c}(0)|$.

These tend to zero as $|\varphi|_1 \to \infty$



Consider the $\Phi_{1,3}$ deformation of the non-unitary minimal models $\mathscr{M}_{2,2n+3}$, $n > 1$ deformed by the simplest possible CDD factor:

$$\Phi(\theta) = \lim_{u\to 0} \frac{i\sin u + \sinh\theta}{i\sin u - \sinh\theta} = -1$$

The models obtained are the "bosonic counterparts", with S-matrices

$$S_{11}(\theta) = \mathrm{th}_{\frac{2}{2n+1}}(\theta)\ , \quad S_{ab}(\theta) = \mathrm{th}_{\frac{|a-b|}{2n+1}}(\theta)\mathrm{th}_{\frac{a+b}{2n+1}}(\theta) \prod_{k=1}^{\min(a,b)-1} \left[\mathrm{th}_{\frac{|a-b|+2k}{2n+1}}(\theta)\right]^2$$

$$\mathrm{th}_x(\theta) = \frac{\sinh\theta + i\sin(\pi x)}{\sinh\theta - i\sin(\pi x)}$$

These have a spectrum of $n > 1$ particles with masses $m_a = \sin(a\pi/(2n+1))$

and just one added resonance

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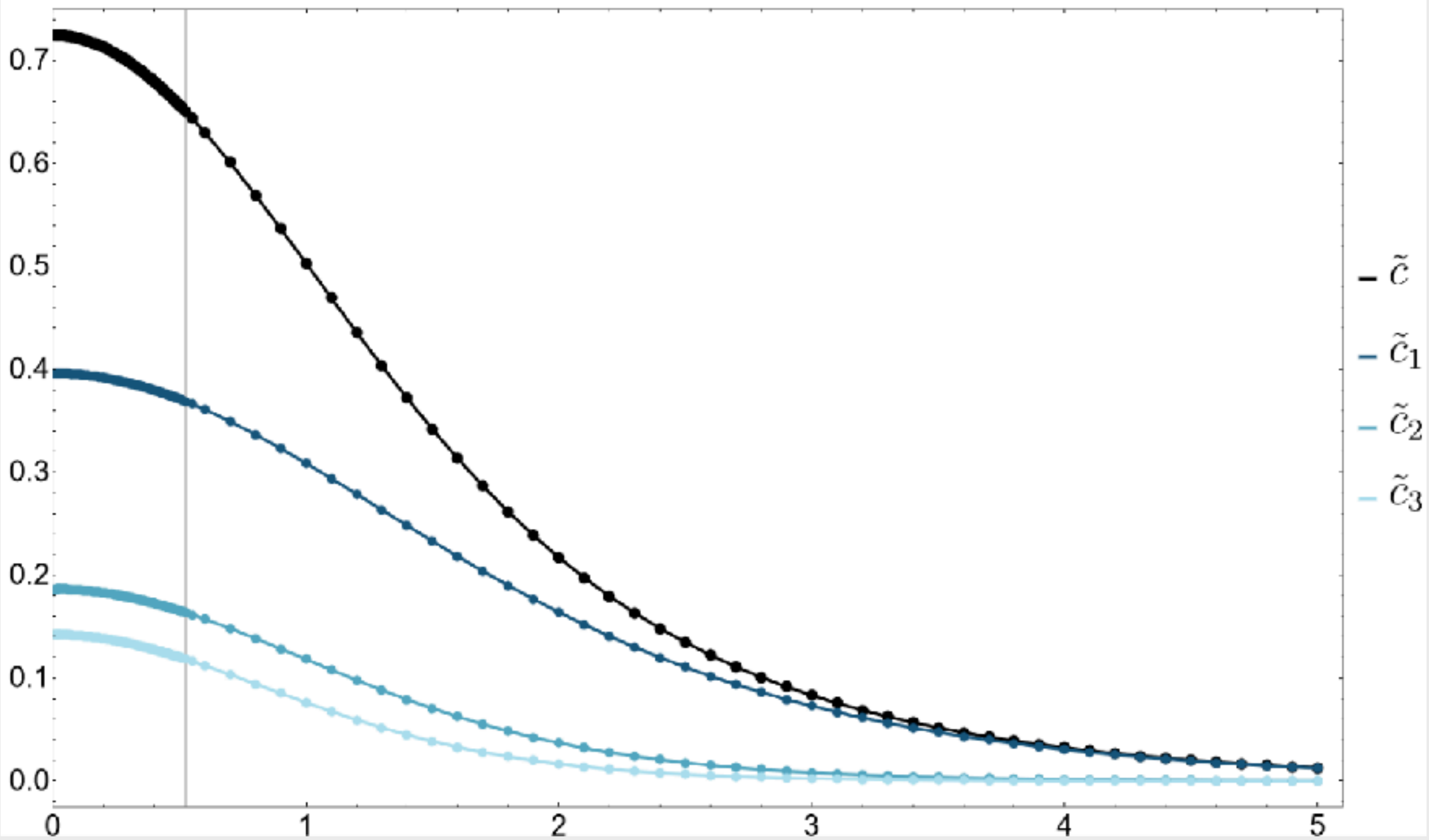
These have a spectrum of $n > 1$ particles with masses $m_a = \sin(a\pi/(2n+1))$ and just one added resonance

We expect them to have a perfectly well-defined UV behaviour

Numerics confirm this expectation

The UV central charges appear not to be rational

$\tilde{c}(0) = 0.641304, 0.724253, 0.778979, 0.817083$ for $n = 2,3,4,5$



Example: $n = 3$. $\tilde{c}_a$ stand for single particle contribution to the total $\tilde{c}$

We begun exploring the vast space of generalised $T\bar{T}$ deformations

The scattering perspective makes contact with the S-matrix bootstrap

$\Longrightarrow$ exploration of the space of consistent, factorisable S-matrices

It appears that the majority of S-matrices in not derivable from local QFTs

We found a condition on the spectrum for the presence of a standard UV

This appears to be generalisable to non-integrable theories

Introduced an improved numerical technique to deal with singularities in the TBA

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Intriguing to think of complex effective central charges in non-UV-complete as belonging to complex CFTs

Explore this point by looking at subleading behaviour and excited state TBA

Consider general CDD deformations of models with bound states

Expect to find many UV complete systems

Particularly interesting: CDD deformed, $\Phi_{1,3}$ unitary minimal models

Keep investigating the TBA for $R < R_{*}$: complex solutions likely signal an instability of the ground state

Against what kind of decay? In case, what are its products?

Thank you