Bethe ansatz
and quantum computing

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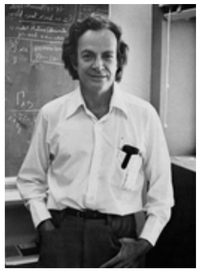
Prologue : Simulation of quantum systems with quantum computers

"Can you do it [simulate a quantum system] with a new kind of computer — a quantum computer?"
- Richard Feynman, 1981



"Feynman was on the right track when he suggested using quantum computers to solve problems in quantum physics and chemistry. That is still the most important application we can clearly foresee…"
-John Preskill, 2021

"This [the simulation of quantum physics and chemistry] was not only Feynman's original vision, but it's probably still, even after all these years, our best shot (at least that we know) at an economically useful application of quantum computers."
-Scott Aaronson, 2022

Prologue : Simulation of quantum systems with quantum computers is a problem worth thinking about

Today's talk is about an effort to partially address this problem

Outline

1. Heisenberg quantum spin chain
2. Brief review of Bethe ansatz
3. Bethe states on a quantum computer

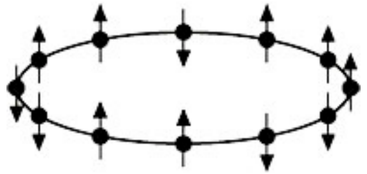
arXiv: 2103.13388	J. Van Dyke, G. Barron, N. Mayhall, E. Barnes, S. Ecomonou
arXiv: 2201.03021	W. Li, M. Okay, RN
arXiv: 2109.05607	J. Van Dyke, E. Barnes, S. Economou, RN

4. Outlook

1. Heisenberg quantum spin chain

Deceptively simple model of magnetism (1928)

- L spin-1/2 spins arranged in a circle
- each spin interacts with its neighbor



Hamiltonian

$$H = -\frac{1}{2}\sum_{n=0}^{L-1}(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - I)$$

$$\vec{\sigma}_L \equiv \vec{\sigma}_0$$

periodic boundary conditions

$$\sum_{j=1}^{3}\sigma_n^j\sigma_{n+1}^j$$

$$\vec{\sigma}_n = \mathbb{I}\otimes\cdots\mathbb{I}\otimes\vec{\sigma}\otimes\mathbb{I}\cdots\otimes\mathbb{I}$$

$0 \qquad n \qquad L-1$

Pauli spin matrices

$$\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$$

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2^L \times 2^L$ matrix

Problem: find eigenvalues (E) and eigenvectors (v) of H

can be realized experimentally;
connections with CFT, AdS/CFT, …

$$H\,v = E\,v$$

Brute force doesn't go very far…

2. Brief review of Bethe ansatz

Coordinate Bethe ansatz (1931)

Eigenvectors are multi-particle ("magnon") states

ground (0-particle) state:

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L} = |\uparrow \cdots \uparrow\rangle$$

$$H|\psi_0\rangle = 0$$

1-particle state:

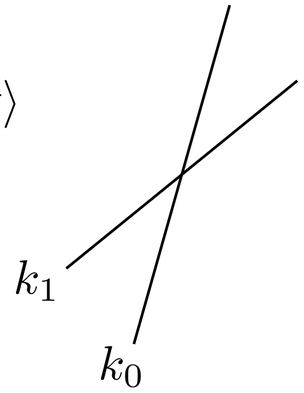
$$|\psi(k)\rangle = \sum_{x=0}^{L-1} e^{ikx} |\uparrow \cdots \underset{x}{\downarrow} \cdots \uparrow\rangle$$

$$H|\psi(k)\rangle = e(k)|\psi(k)\rangle \qquad \boxed{e(k) = 4\sin^2(\tfrac{k}{2})} \qquad \text{1-particle energy}$$

provided $e^{ikL} = 1$

2-particle state:

$$|\psi(k_0,k_1)\rangle = \sum_{0\le x_0<x_1\le L-1} \left[A^{(01)}e^{i(k_0x_0+k_1x_1)} - A^{(10)}e^{i(k_1x_0+k_0x_1)}\right] |\uparrow\cdots\underset{x_0}{\downarrow}\cdots\underset{x_1}{\downarrow}\cdots\uparrow\rangle$$



2-particle S-matrix

$$A^{(10)} = -S(k_1,k_0)A^{(01)} \qquad \boxed{S(k_1,k_0) = \frac{u(k_1)-u(k_0)+i}{u(k_1)-u(k_0)-i}} \qquad u(k) = \tfrac{1}{2}\cot(\tfrac{k}{2})$$

$$H|\psi(k_0,k_1)\rangle = E|\psi(k_0,k_1)\rangle \qquad E = e(k_0)+e(k_1)$$

provided $\begin{cases} e^{ik_0L} = S(k_0,k_1) \\ e^{ik_1L} = S(k_1,k_0) \end{cases}$

M-particle state:

$$|\psi(k_0,\ldots,k_{M-1})\rangle = \sum_{P\in\mathrm{Perm}} \varepsilon_P\, A^{(P)} |k_{P(0)},\ldots,k_{P(M-1)}\rangle$$

$M!$ terms Bethe state

$$\varepsilon_P = \pm 1$$

signature of P

$$|k_0,\ldots,k_{M-1}\rangle = \sum_{0\le x_0<\ldots<x_{M-1}\le L-1} e^{i(k_0x_0+\ldots+k_{M-1}x_{M-1})} |\uparrow\cdots\underset{x_0}{\downarrow}\cdots\underset{x_{M-1}}{\downarrow}\cdots\uparrow\rangle$$

$$A^{(\ldots ij\ldots)} = -S(k_i,k_j)A^{(\ldots ji\ldots)} \quad\Rightarrow\quad \boxed{A^{(P)} = A^{(I)} \times \text{product of 2-particle S-matrices}}$$

$$H|\psi(k_0,\ldots,k_{M-1})\rangle = E|\psi(k_0,\ldots,k_{M-1})\rangle \qquad E = \sum_{j=0}^{M-1} e(k_j)$$

"exact solution"!

provided

$$\boxed{e^{ik_jL} = \prod_{l=0;l\neq j}^{M-1} S(k_j,k_l)\,, \qquad j=0,\ldots,M-1}$$

Bethe equations

SU(2) symmetry $\Rightarrow$ degeneracy $L - 2M + 1$

$$M \leq \frac{L}{2}$$

Example: $L = 4$ $M = 0, 1, 2$

M	k_j	E	degeneracy
0	-	0	5
1	$\frac{1}{2}\pi$	2	3
1	$\frac{3}{2}\pi$	2	3
1	π	4	3
2	$\pm i\infty$	2	1
2	$\pm\frac{2}{3}\pi$	6	1

total: $16 = 2^4 = 2^L$ “complete” ✓

Remarks

- Bethe's solution is possible because the model is *quantum integrable*

  There are infinitely-many such models, corresponding to solutions of Yang-Baxter equation

  - Heisenberg model is prototype

- Bethe equations (BE) are generally hard to solve

  complete set of solutions of BE up to L=14 [Hao, RN, Sommese 2013; Marboe, Volin 2016]

- Significant effort is generally still required to explicitly compute quantities of physical interest

~~"exact solution"~~ ⟶ "exact half-solution" 😉

Can quantum computers help?

3. Bethe states on a quantum computer

"Quantum computer":

- device with n "qubits": 2-state systems
- can initialize each qubit $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- can perform unitary transformations on qubits (decomposed into 1-qubit & 2-qubit unitary "gates")
- can perform projective measurements $\rightarrow$ $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "computational basis"

Currently: $n \sim 10^2$ IBM, Google, … "noisy" - make many errors!

Dream: $n \sim 10^4$ fault-tolerant

In the meantime, can test algorithms ("quantum circuits") using noiseless *simulators*

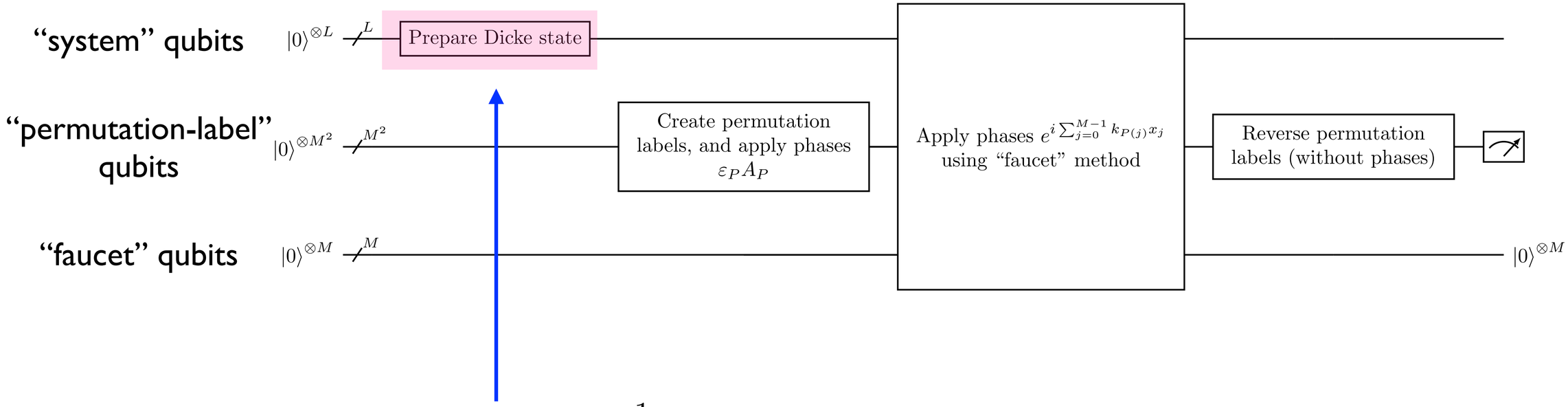
Here: IBM Qiskit simulators $n \sim 30$

idea:

- solve Bethe equations classically for *real* $\{k_0, \ldots, k_{M-1}\}$ (easy!)
- use quantum computer to construct exact eigenstates using coordinate Bethe ansatz (classically hard!)

[Van Dyke et al 2021]

Schematic diagram of quantum circuit :



Prepare "system" qubits in Dicke state $\dfrac{1}{\sqrt{\binom{L}{M}}} \sum_{0 \le x_0 < \ldots < x_{M-1} \le L-1} |\uparrow \cdots \underset{x_0}{\downarrow} \cdots \underset{x_{M-1}}{\downarrow} \cdots \uparrow\rangle$

[Bärtschi, Eidenbenz 2019; Mukherjee et al 2020, …]

[Van Dyke et al 2021]

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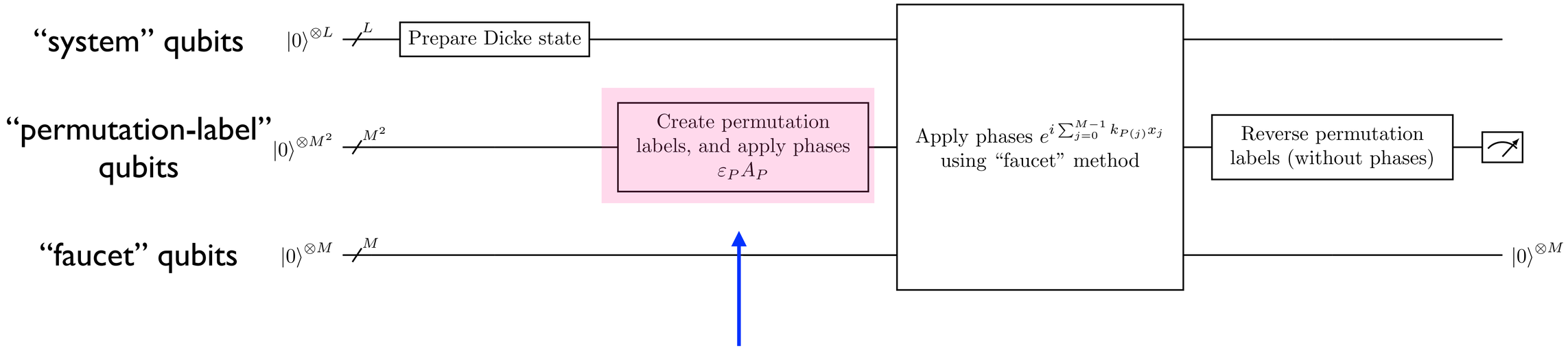

Prepare "permutation-label" qubits in state $\dfrac{1}{\sqrt{M!}} \sum_P \varepsilon_P \, A_P |P\rangle$

$|P\rangle$ encode permutation P

[Van Dyke et al 2021]

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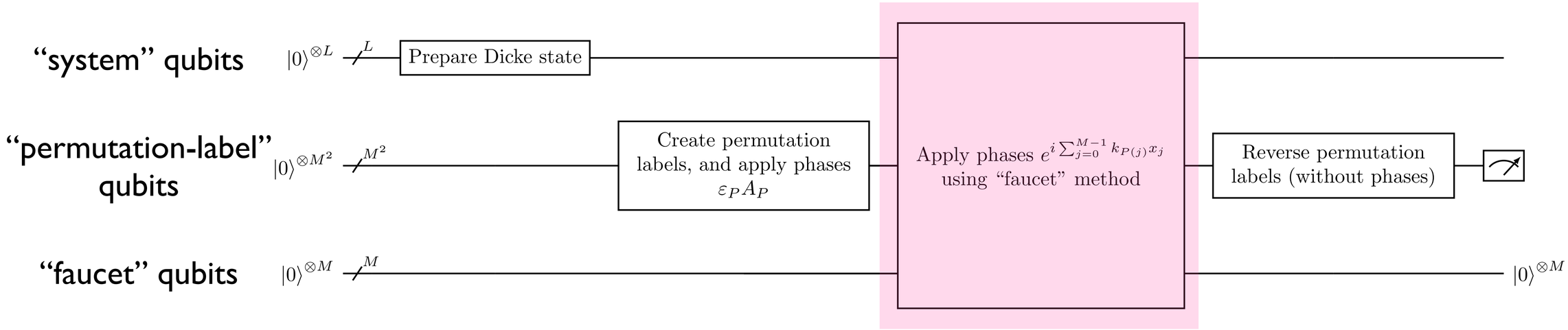

Apply phases $e^{i\sum_j k_{P(j)} x_j}$ to "permutation-label" qubits

[Van Dyke et al 2021]

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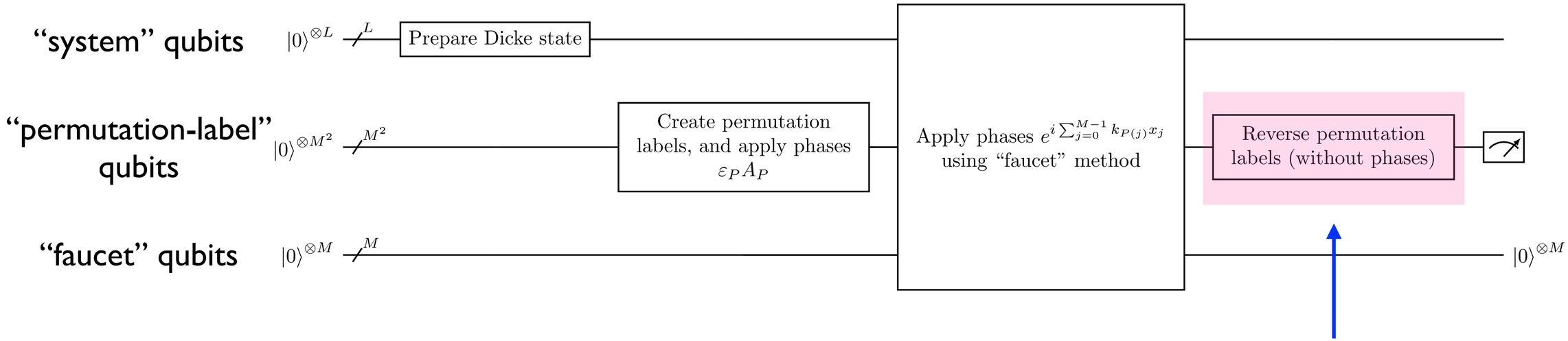
Schematic diagram of quantum circuit :



Reverse of step 2, but without phases

[Van Dyke et al 2021]

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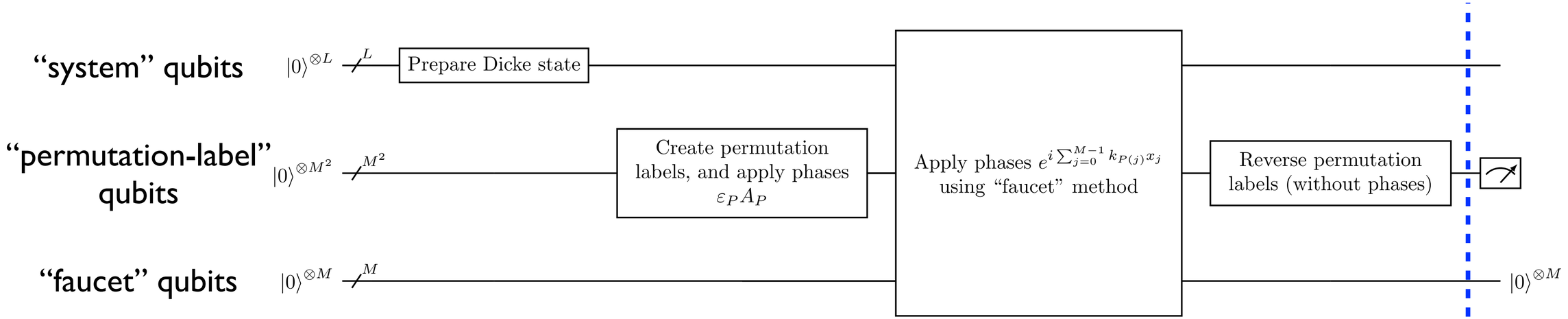

$$|\Psi\rangle = \alpha|00\cdots0\rangle|\tilde{\phi}\rangle + \ldots$$

$|\tilde{\phi}\rangle$ normalized Bethe state

[Van Dyke et al 2021]

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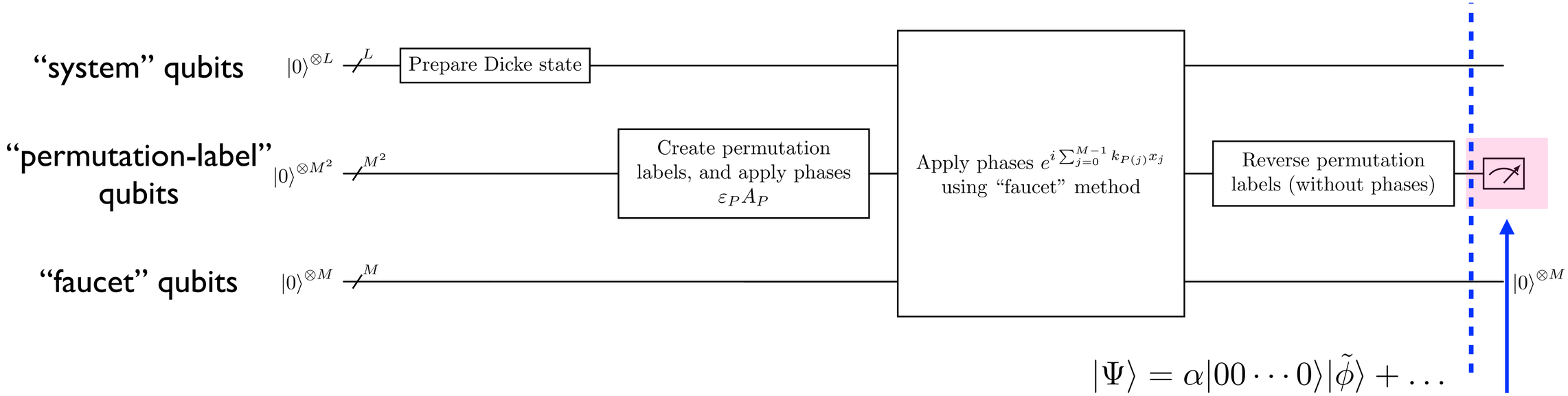
Schematic diagram of quantum circuit :



Measure "permutation-label" qubits

[Van Dyke et al 2021]

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- solve Bethe equations classically for *real* $\{k_0, \ldots, k_{M-1}\}$ (easy!)
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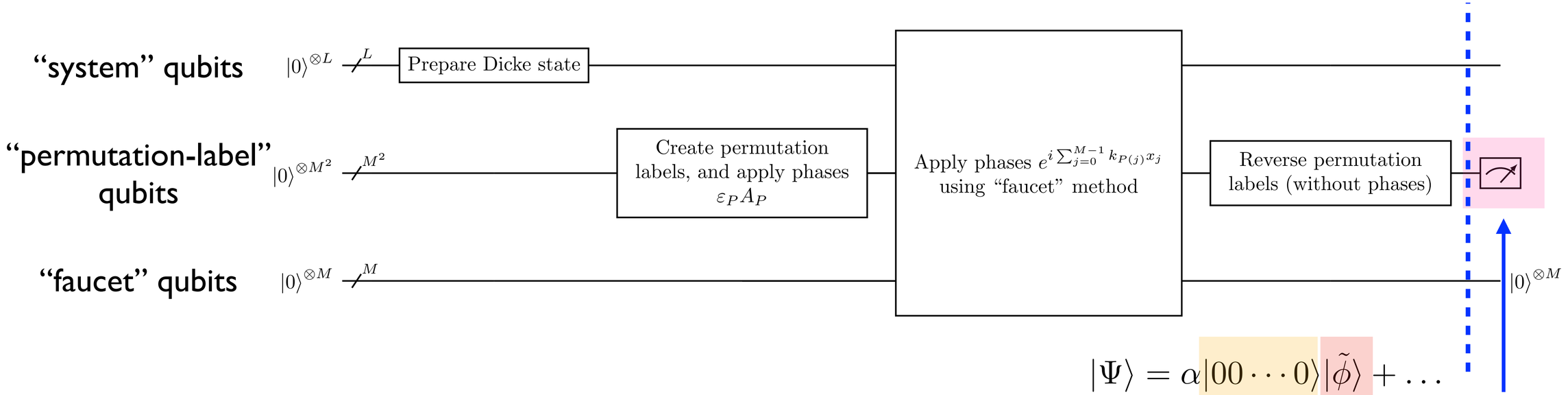


$$|\Psi\rangle = \alpha|00\cdots0\rangle|\tilde{\phi}\rangle + \ldots$$

Measure "permutation-label" qubits

If $|00\cdots0\rangle$ then "system" qubits are in Bethe state $|\tilde{\phi}\rangle$ Success! 😄

Algorithm is *probabilistic* "success probability" $\equiv |\alpha|^2$ Value?

$$|\alpha|^2 = \frac{(L-M)!}{L!\,M!}\det G$$

[Li, Okyay, RN 2022]

G: "Gaudin matrix" $\qquad M \times M$

$$G_{m,n} = \delta_{m,n}\left[L - \sum_l \frac{4\,(1-\cos k_l)}{(n,l)\,(l,n)}\right] + \frac{4\,(1-\cos k_m)}{(n,m)\,(m,n)}$$

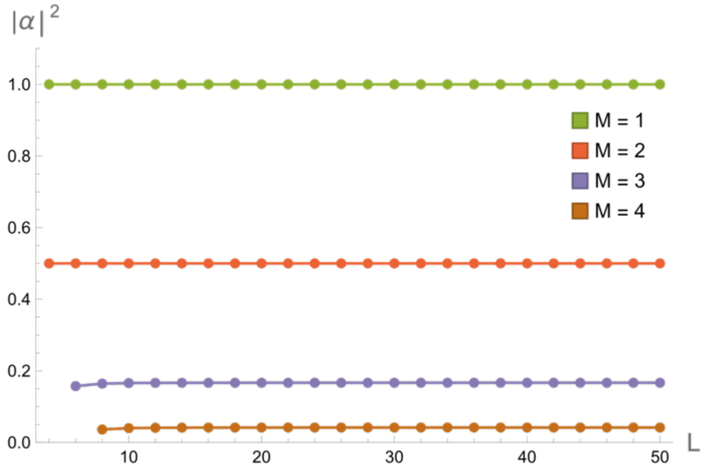
[Gaudin, McCoy, Wu 1981]

[Korepin1982]

$$(j,l) := 2 - e^{-ik_j} - e^{ik_l}$$

Large L (fixed M):

$$\lim_{L\to\infty} |\alpha|^2 = \frac{1}{M!}$$



What could we hope to compute using this quantum circuit?

Spin-spin correlation functions

[Kitanine, Maillet, Terras 1999;
Kitanine, Maillet, Slavnov, Terras 2002;
Lukyanov, Terras 2003;
Caux, Maillet 2005, …]

$$\langle\psi_0|\sigma_0^z\,\sigma_l^z|\psi_0\rangle\,, \qquad l=1,2,\ldots,\frac{L}{2}$$

$|\psi_0\rangle$ normalized *anti-ferromagnetic* ground state (Hamiltonian $-H$) $M=\frac{L}{2}$ real Bethe roots

To compute an expectation value on a quantum computer:

[Li, Okyay, RN 2022]

- perform multiple "shots" How many? For error ϵ , need $N \le N_{\max}\,,$ $\boxed{N_{\max} := \frac{1}{|\alpha|^2\,\epsilon^2}}$
- usually, can use built-in functionality

Not here, since $|\psi_0\rangle$ is produced probabilistically!

Can proceed by measuring all qubits, and then appropriately combining the corresponding probabilities

$$\langle\psi_0|\sigma_0^z\,\sigma_l^z|\psi_0\rangle\,, \qquad l=1,2,\ldots,\frac{L}{2}$$

$$M=\frac{L}{2}$$

	$L=4$		$L=6$		$L=8$		$L=\infty$
l	th	exp $\lvert\alpha\rvert^2=0.5$ $N=2\times10^4$	th	exp $\lvert\alpha\rvert^2=0.157232$ $N=6.4\times10^4$	th	exp $\lvert\alpha\rvert^2=0.0361418$ $N=2.8\times10^5$	th
1	-0.666667	-0.6657 ± 0.0076	-0.622839	-0.6230 ± 0.0072	-0.608516	-0.6091 ± 0.0077	-0.590863
2	0.333333	0.3309 ± 0.0090	0.27735	0.2777 ± 0.0086	0.261037	0.2602 ± 0.0086	0.242719
3	–	–	-0.309022	-0.3102 ± 0.0090	-0.251937	-0.2519 ± 0.0088	-0.200995
4	–	–	–	–	0.198831	0.1988 ± 0.0093	0.138611

experimental (exp): 100 trials of N shots on Qiskit qasm simulator (no noise), with $\epsilon=0.01$

As L increases, $|\alpha|^2$ decreases, so N increases $\quad N\approx\dfrac{1}{|\alpha|^2\,\epsilon^2}$ standard deviations $<\epsilon$ ✓

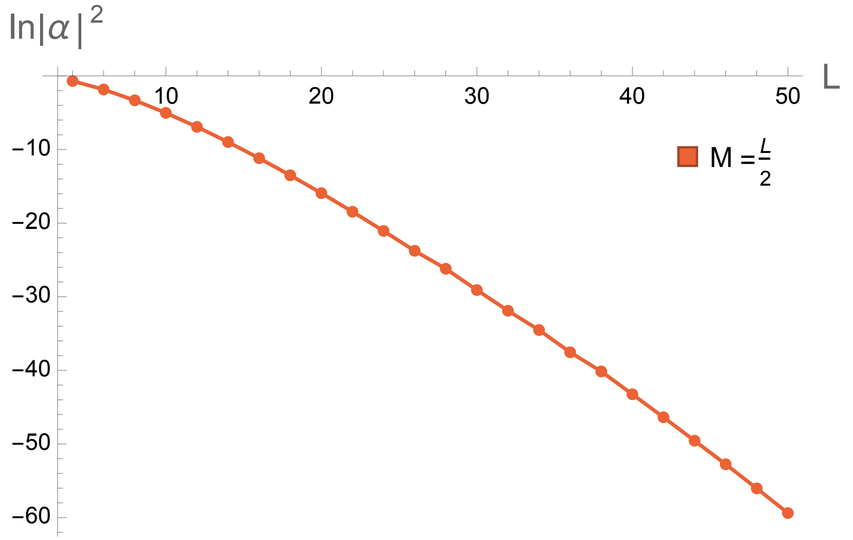
theoretical (th): Mathematica

$L=\infty$ [l=1: Huthén 1938; l=2: Takahashi 1977; l=3: Sakai et al 2003; l=4: Boos, et al. 2005]

Correlation functions for small L values are feasible!

How about for larger L values?

Success probability $|\alpha|^2$ decreases exponentially with L



For L=40: $|\alpha|^2 \sim 5 \times 10^{-20} \quad \Rightarrow \quad N \sim 2 \times 10^{23}$ not feasible ☹

Nevertheless, it should be feasible to use this quantum circuit to study other states with *small* M values, even for moderate L values

Open spin chain (i.e. with boundaries)

[Van Dyke, Barnes, Economou, RN 2021]

non-trivial generalization of formalism for closed spin chains

$$H = -\frac{1}{2}\sum_{n=0}^{L-2}(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - I)$$

$M = 2$:

$$|\psi(k_0,k_1)\rangle = \sum_{0\le x_0<x_1\le L-1}\Big[A(k_0,k_1)e^{i(k_0x_0+k_1x_1)} - A(-k_0,k_1)e^{i(-k_0x_0+k_1x_1)}$$
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[Gaudin '71, Alcaraz et al. '87]

$$|\psi(k_0,\ldots,k_{M-1})\rangle = \sum_{0\le x_1<\ldots<x_{M-1}\le L-1}\sum_{P}\varepsilon_P A(k_0,\ldots,k_{M-1})e^{i(k_0x_0+\ldots+k_{M-1}x_{M-1})}|\uparrow\cdots\underset{x_0}{\downarrow}\cdots\underset{x_{M-1}}{\downarrow}\cdots\uparrow\rangle$$

all permutations and negations of $\{k_0,\ldots,k_{M-1}\}$

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all permutations and negations of $\{k_0,\ldots,k_{M-1}\}$

changes sign at each such "mutation"

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[Van Dyke, Barnes, Economou, RN 2021]

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depends on both bulk and boundary S-matrices

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[Van Dyke, Barnes, Economou, RN 2021]

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Hamiltonian can include boundary terms $h\sigma_0^z + h'\sigma_{L-1}^z$ and anisotropy

$2^M\, M!$ terms

Formulated quantum circuit for preparing Bethe states with real Bethe roots

- M additional qubits for reflections total # qubits = $L + M^2 + 2M$
- *probabilistic*

4. Outlook

- Integrable models are attractive candidates for simulation on quantum computers ("half-solved", benchmarking)
- Quantum circuits for preparing exact eigenstates of XXX models total # qubits $\sim L + M^2$
- *Probabilistic* success probability $\sim \frac{1}{M!}$ $\Rightarrow$ small M is feasible
- Large M ?
- Complex Bethe roots?
- Higher-rank models?
- Other approaches?
    - algebraic Bethe ansatz arXiv: 2202.04673 Sopena, Gordon, García-Martín, Sierra, López
    - variational approaches (no Bethe ansatz) arXiv: 2207.09994 Yu, Zhao, Wei

Thank you for your attention!