

Arithmetic and physics of Higgs moduli spaces

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Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; fix group GL_n

$$\mathcal{M}_{\text{Dol}}^d := \left\{ \begin{array}{l} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, \text{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_{\text{B}}^d := \{A_1, B_1, \dots, A_g, B_g \in GL_n \mid \prod_{i=1}^g A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} // PGL_n$$

when $(d, n) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\text{Dol}}^d \stackrel{\text{diff}}{\cong} \mathcal{M}_{\text{B}}^d$
(Hitchin, Donaldson, Corlette, Simpson)
- $g = 1 \rightsquigarrow$ Stone-von Neumann \rightsquigarrow
 $\mathcal{M}_{\text{B}}^d \cong (\mathbb{C}^*)^2 \cong T^*\text{Jac}(C) \cong \mathcal{M}_{\text{Dol}}^d$
- Problem: what is Poincaré polynomial
 $P(\mathcal{M}_{\text{Dol}}^d; t) = P(\mathcal{M}_{\text{B}}^d; t)?$

Mixed Hodge polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$ for any complex algebraic variety X
- $H(X; q, t) = \sum \dim(W_i/W_{i-1}(H^k(X))) t^k q^{\frac{i}{2}}$, *mixed Hodge polynomial*
- $P(X; t) = H(X; 1, t)$, *Poincaré polynomial*
- $E(X; q) = q^d H(1/q, -1)$, *E-polynomial of X .*

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over \mathbb{Z} and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q , then $E(M; q) = E(q)$.

Mixed Hodge polynomials

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^d; q) = |\mathcal{M}_B^d(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$$

- $\leadsto E(\mathcal{M}_B^d; q) = E(\mathcal{M}_B^{d'}; q)$ when $(d, n) = (d', n) = 1$
- \mathcal{M}_B^d and $\mathcal{M}_B^{d'}$ Galois conjugate $\Rightarrow H(\mathcal{M}_B^d; q, t) = H(\mathcal{M}_B^{d'}; q, t)$

Conjecture (Hausel-Villegas, 2008)

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left(\sum_{n,k} \frac{H(\mathcal{M}_B^d; w^{2k}, -(zw)^{-2k}) (zw)^{dn}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{nk}}{k} \right)$$

- when $g = 1$ $\mathcal{M}_B^d = (\mathbb{C}^*)^2$ by Stone-von Neumann $\overset{HV}{\leadsto}$

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^2}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left(\sum_{k \geq 1} \frac{(z^k - w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)} \frac{T^k}{k} \right)$$

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^d; q) = |\mathcal{M}_B^d(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$$

- we find $E(\mathcal{M}_B^d; q) = q^{d_n} E(\mathcal{M}_B^d; 1/q)$ palindromic
by *Alvis-Curtis duality*

$$q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))$$

- \rightsquigarrow Curious Hard Lefschetz Conjecture (theorem when $n = 2$):

$$L^l : \underset{X}{Gr_{d_n-2l}^W(H^{i-l}(\mathcal{M}_B^d))} \rightarrow \underset{X \cup \alpha^l}{Gr_{d_n+2l}^W(H^{i+l}(\mathcal{M}_B^d))},$$

where $\alpha \in W_4 H^2(\mathcal{M}_B^d)$

- The implied functional equation on the conjectured $H(\mathcal{M}_B^d; q, t) = (qt)^{d_n} H(\mathcal{M}_B^d; \frac{1}{qt^2}, t)$ holds

Perverse filtration

- $f : X \rightarrow Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration* $P_0 \subset \dots \subset P_i \subset \dots \subset P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves
- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when X smooth and Y affine:
take $Y_0 \subset \dots \subset Y_i \subset \dots \subset Y_d = Y$
s.t. Y_i generic with $\dim(Y_i) = i$ then

$$P_{k-i-1}H^k(X) = \ker(H^k(X) \rightarrow H^k(f^{-1}(Y_i)))$$

- the Relative Hard Lefschetz Theorem holds:

$$L^l : \begin{array}{ccc} Gr_{d-l}^P(H^*(X)) & \rightarrow & Gr_{d+l}^P H^{*+2l}(X) \\ x & \mapsto & x \cup \alpha^l \end{array}$$

where $\alpha \in H^2(X)$ is a relative ample class

Main conjecture

- recall Hitchin map $\chi : \mathcal{M}_{\text{Dol}}^d \rightarrow \mathbb{A} := \bigoplus_{i=1}^n H^0(C; K^i)$
 $(E, \phi) \mapsto \text{charpol}(\phi)$
- (Hitchin 1987) \rightarrow completely integrable Hamiltonian system and *proper*

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\text{Dol}}^d) \cong W_{2k}(\mathcal{M}_{\text{B}}^d)$ under the isomorphism

$H^*(\mathcal{M}_{\text{Dol}}^d) \cong H^*(\mathcal{M}_{\text{B}}^d)$ from non-Abelian Hodge theory.

In particular $\text{CHL} \Leftrightarrow \text{RHL}$

Theorem (de Cataldo-Hausel-Migliorini 2010)

$P = W$ for $n = 2$.

- proof mirroring Ngô's proof of the fundamental lemma
- evidence for $n > 2$?

Refined Gopakumar-Vafa conjecture for local curves

- we follow (Chuang-Diaconescu-Pan 2011)
- Y total space of $\mathcal{O}_C \oplus K_C$ over C
- Y CY 3-fold , "local curve"
- conjectural "quantum" Pandharipande-Thomas invariants

$$Z_{PT}^{ref} := \sum_{\beta \in H_2(Y)} \sum_{n \in \mathbb{Z}} T^\beta q^n E_{virt}(\mathcal{P}(Y, \beta, e); y)$$

- Gopakumar-Vafa generating function of refined BPS invariants:

$$F_{GV}^{ref} := \sum_{\substack{k \geq 1 \\ \beta \in H_2(Y)}} \sum_{j_L, l \geq 0} \frac{T^{k\beta}}{k} (-1)^{j_L+l} N_\beta^{(j_L, l)} \frac{(q^{-kj_L} + \dots + q^{kj_L}) q^{-k} y^l}{(1-(qy)^{-k})(1-(q/y)^{-k})}$$

Conjecture ("refined BPS", Chuang-Diaconescu-Pan 2011)

$$Z_{PT}^{ref} = \exp(F_{GV}^{ref})$$

- Gopakumar-Vafa's BPS invariants $N_{\beta}^{j_L, j_R}$ heuristically arise from decomposing the cohomology $H^*(\mathcal{M}_{\beta}^e)$ of the space of D-branes via a putative action of $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$
- for local curve Y , (Chuang-Diaconescu-Pan 2011) argue that $\mathcal{M}_{\beta}^e \cong \mathcal{M}_{Dol}^d$ where $\beta = n[C]$ and $d = e + n(g - 1)$
- recall $\chi : \mathcal{M}_{Dol}^d \rightarrow \mathbb{A}^1$ induces perverse filtration P on $H^*(\mathcal{M}_{Dol}^d)$ with RHL
- RHL on $\mathrm{Gr}^P(H^*(\mathcal{M}_{Dol}^d)) \rightsquigarrow (\mathfrak{sl}_2)_L$ action on $\mathrm{Gr}^P(H^*(\mathcal{M}_{Dol}^d))$
- the corresponding primitive decomposition $H^m(\mathcal{M}_{Dol}^d) \cong \bigoplus_{i,j} Q^{i,j;m}$ gives at least a $(\mathfrak{gl}_1)_R$ action
- Chuang-Diaconescu-Pan define $N_{\beta}^{j_L; l} := \dim(Q^{j_L, 0; l})$

Z_{PT}^{ref} via geometric engineering

- geometric engineering $\leadsto Z_{PT}^{ref} = Z_{gauge}$
 Z_{gauge} partition function of certain gauge theory
- Chuang-Diaconescu-Pan argue that it should be a $U(1)$ -gauge theory on $X = \mathbb{R}^4 \cong \mathbb{C}^2$

- $Z_{gauge} = \sum_{k \geq 0} Q^k \chi_{\mathcal{Y}'}^{\mathbb{T}^2} \left(\det(\mathcal{V}_k)^{1-g} \otimes (T^*X^{[k]})^{\oplus g} \right)$

\mathcal{V}_k is the tautological bundle on the Hilbert scheme $X^{[k]}$

\mathbb{T}^2 acts on X and so on $X^{[k]}$ with isolated fixed points

$\leadsto Z_{gauge}$ is defined by localizing to the fixed points

- after changes of variables we have

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} \stackrel{g.e.}{=} Z_{PT}^{ref} \stackrel{CDP}{=} \exp(F_{GV}) \Big|_{P=W}$$

$$\stackrel{HV}{=} \exp \left(\sum_{n,k} \frac{H(\mathcal{M}_B^d; w^{2k}, -(zw)^{-2k})(zw)^{dn}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{nk}}{k} \right)$$

- conclusion: $HV, CDP + g.e. \Rightarrow P = W$

- when $g = 1$ $\mathcal{M}_B^d = (\mathbb{C}^\times)^2$ by Stone-von Neumann $\overset{HV}{\rightsquigarrow}$

$$\sum_{\lambda} \prod \frac{(z^{2l+1} - w^{2a+1})^2}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} = \exp \left(\sum_{k \geq 1} \frac{(z^k - w^k)^2}{(z^{2k} - 1)(1 - w^{2k})(1 - T^k)} \frac{T^k}{k} \right)$$

- after geometric engineering this formula becomes

$$\sum_n \chi_y^{\mathbb{T}^2}(X^{[n]}) T^n = \sum_n \chi_{y, st}^{\mathbb{T}^2}(X^n / S_n) T^n$$

(Waelder, 2008) proves geometrically a more general DMVV formula for equivariant elliptic genus \rightsquigarrow our $g = 1$ formula follows!

- (Chuang-Diaconescu-Pan 2011) refined BPS conjecture \Leftrightarrow (Hausel-Villegas 2008) conjecture on $H(\mathcal{M}_B^d; q, t)$ provided $P = W$ of (de Cataldo-Hausel-Migliorini 2010)
- studying wall-crossing for the stability condition for $Z_{PT}^{ref}(Y) \rightsquigarrow$ recursive formulas (Chuang-Diaconescu-Pan 2010) for $P_t(\mathcal{M}_{Dol}^d)$ studied by (Mozgovoy 2011)
- mathematically geometric engineering proposes a deep connection between $K_{\mathbb{T}^2}((\mathbb{C}^2)^{[n]})$ and $H^*(\mathcal{M}_{Dol}^d)$
- may lead to connections between (Haiman 2002) and (Hausel-Letellier-Villegas 2008) explaining the appearance of Macdonald polynomials in both
- DAHA acts on $K_{\mathbb{T}^2}((\mathbb{C}^2)^{[n]})$ by (Gordon-Stafford 2004) it is expected that DAHA acts on $H^*(\mathcal{M}_{Dol}^d)$ from (Yun 2009)
- are these DAHA actions related by geometric engineering?