String Theory of The Omega Deformation

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Outline

1. *Gauge theories and the gauge-Bethe correspondence*
   - Effective twisted superpotential in two dimensions
2. *The Ω background*
3. *A String Theory construction*
4. *Conclusion*

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2. *The Ω background*
3. *A String Theory construction*
4. *Conclusion*

Motivation

- The Gauge–Bethe correspondence – in its simplest manifestation – is the equivalence of the ground states of a supersymmetric gauge theory and the spectrum in a sector of a spin chain.
- There are years of experience in the study of both sides of the correspondence.
- We can translate problems from one side to the other
- Fresh perspective on existing problems
- New natural questions arise, too

Today's talk

- Today I will try to understand this correspondence in the context of String Theory.
- I will describe a String Theory (D-brane) construction.
- There is a simple brane construction that reproduces the gauge theory action precisely, except for the twisted mass terms for the adjoint chiral multiplets.
- We construct the twisted mass deformation in terms of an exact solution of the closed string background of superstring theory involving curvatures, fluxes and dilaton gradients.

The message

- The gauge–Bethe correspondence relates the vacuum sectors of a set of supersymmetric gauge theories to states of a single spin chain, and to each other
- Spin chains have symmetries that relate different sectors
- String theory provides a framework in which these different gauge theories can be treated in a unified way and the spin chain symmetry understood as symmetry enhancement for coincident 5-branes (D or NS).

Basics of $\mathcal{N} = (2,2)$ field theories: field content

- Field theories in $1+1$ dimensions with two (real) positive and two (real) negative chirality supercharges.
- A chiral superfield satisfies $\overline{\mathcal{D}}_\pm \Phi = 0$. The θ-expansion of the chiral superfield is given by

$$\Phi = \varphi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm),$$

- A twisted chiral superfield satisfies $\overline{\mathcal{D}}_+ \Sigma = \mathcal{D}_- \Sigma = 0$.
- The super field strength $\Sigma = \frac{1}{2}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\}$ is a twisted chiral superfield and its θ-expansion is given by

$$\Sigma = \sigma(\tilde{y}^\pm) + i\theta^+\overline{\lambda}_+(\tilde{y}^\pm) - i\bar{\theta}^-\lambda_-(\tilde{y}^\pm) + \theta^+\bar{\theta}^-[D(\tilde{y}^\pm) - iA_{01}(\tilde{y}^\pm)] + \ldots$$

Basics of $\mathcal{N} = (2,2)$ field theories: action

- The kinetic term of the Lagrangian is

$$L_{\text{kin}} = \int d^4\theta \left(\sum_k X_k^\dagger e^V X_k - \frac{1}{2e^2} \mathrm{Tr}(\Sigma^\dagger \Sigma) \right),$$

additional terms:

- The twisted masses: $L_{\text{tw}} = \int d^4\theta \, (X^\dagger e^{\theta^- \bar{\theta}^+ \widetilde{m}_X + \text{h.c.}} X)$,
- Fayet-Iliopoulos (FI) and theta-term: $L_{\text{FI},\vartheta} = -\frac{\imath}{2} \tau \int d\bar{\theta}^- d\theta^+ \,\mathrm{Tr}\,\Sigma + \text{ h.c.}$,

Effective theory in the Coulomb branch

- Main objective: describe the Coulomb branch of the theory.
- Consider the low energy effective theory obtained for slowly varying σ fields after integrating out the massive matter fields.
- In this way, we obtain an effective twisted superpotential $\widetilde{W}_{\text{eff}}(\Sigma)$

The vacua of the theory are the solutions of the equation

$$\exp\left[2\pi \frac{\partial \widetilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i}\right] = 1\,.$$

Gaussian integration

- By supersymmetry, the effective action must have the form

$$S_{\text{eff}}(\Sigma) = -\int d^4\theta\, K_{\text{eff}}(\Sigma, \overline{\Sigma}) + \frac{1}{2}\int d^2\theta\, \widetilde{W}_{\text{eff}}(\Sigma) + \text{h.c.}\,.$$

- In the absence of an F-term, the action $S(\Sigma, X)$ is quadratic in the matter fields X, and the effective action can be evaluated exactly via a one-loop calculation:

$$e^{\imath S_{\text{eff}}(\Sigma)} = \int \mathcal{D}X\, e^{\imath S(\Sigma, X)}\,.$$

Quiver gauge theories: effective action

Contributions to the effective twisted superpotential:

- For each fundamental field Q_k with twisted mass $\widetilde{m}^{\mathrm{f}}_k$:

$$\widetilde{W}^{\mathrm{f}}_{\mathrm{eff}} = \frac{1}{2\pi} \sum_{i=1}^{N} \left(\sigma_i - \widetilde{m}^{\mathrm{f}}_k \right) \left(\log(\sigma_i - \widetilde{m}^{\mathrm{f}}_k) - 1 \right) .$$

- For each anti-fundamental field $\overline{Q}_k$ with twisted mass $\widetilde{m}^{\bar{\mathrm{f}}}_k$:

$$\widetilde{W}^{\bar{\mathrm{f}}}_{\mathrm{eff}} = \frac{1}{2\pi} \sum_{i=1}^{N} \left(- \sigma_i - \widetilde{m}^{\bar{\mathrm{f}}}_k \right) \left(\log(- \sigma_i - \widetilde{m}^{\bar{\mathrm{f}}}_k) - 1 \right) .$$

- For each adjoint field Φ with twisted mass $\widetilde{m}^{\mathrm{adj}}$:

$$\widetilde{W}^{\mathrm{adj}}_{\mathrm{eff}} = \frac{1}{2\pi} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \left(\sigma_i - \sigma_j - \widetilde{m}^{\mathrm{adj}} \right) \left(\log(\sigma_i - \sigma_j - \widetilde{m}^{\mathrm{adj}}) - 1 \right) .$$

Special example

- Consider a $U(N)$ gauge theory with L flavors and one adjoint
- Using the rules above, the effective twisted superpotential reads:

$$\widetilde{W}_{\text{eff}}(\sigma) = \frac{L}{2\pi}\sum_{i=1}^{N}\left(\sigma_i - \widetilde{m}^{\text{f}}\right)\left(\log(\sigma_i - \widetilde{m}^{\text{f}}) - 1\right)$$
$$+\frac{L}{2\pi}\sum_{i=1}^{N}\left(-\sigma_i - \widetilde{m}^{\bar{\text{f}}}\right)\left(\log(-\sigma_i - \widetilde{m}^{\bar{\text{f}}}) - 1\right)$$
$$+\frac{1}{2\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\left(\sigma_i - \sigma_j - \widetilde{m}^{\text{adj}}\right)\left(\log(\sigma_i - \sigma_j - \widetilde{m}^{\text{adj}}) - 1\right)$$

SUSY vacua

- The vacua of the theory are obtained from

$$\exp\left[2\pi \frac{\partial \widetilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i}\right] = 1\,.$$

- Explicitly

$$\boxed{\left(\frac{\sigma_i - \widetilde{m}^{\text{f}}}{\sigma_i + \widetilde{m}^{\bar{\text{f}}}}\right)^L = \prod_{\substack{j=1\\ i\neq j}}^{N} \frac{\sigma_i - \sigma_j - \widetilde{m}^{\text{adj}}}{\sigma_i - \sigma_j + \widetilde{m}^{\text{adj}}} \qquad \forall i = 1,\ldots,N}$$

This is precisely the same equation describing the Bethe ansatz for the XXX spin chain

$$\widetilde{m}^{\text{f}} = \frac{\imath}{2} \qquad \widetilde{m}^{\bar{\text{f}}} = \frac{\imath}{2} \qquad \widetilde{m}^{\text{adj}} = \imath$$

Brief summary of the gauge/Bethe correspondence

- The Hilbert space for a spin chain decomposes into (magnon) sectors
- The Algebraic Bethe Ansatz provides the spectrum of the transfer matrix (and the Hamiltonian)
- The very same equations describe the vacua of a two-dimensional $(2, 2)$ gauge system with a precise choice of twisted masses.
- The spin chain is both bigger and smaller than any given gauge theory: we solve the spectrum sector by sector, different values of N correspond to different magnon sectors; however the full Hilbert space of the spin-chain corresponds only to the vacuum sectors of the gauge theories.

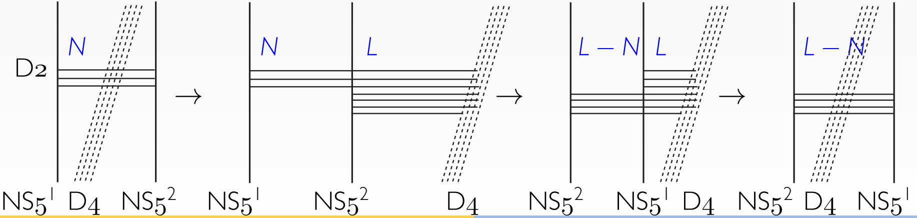
The dictionary

gauge theory			integrable model
number of nodes in the quiver	r	r	rank of the symmetry group
gauge group at a-th node	$U(N_a)$	N_a	number of particles of species a
effective twisted superpotential	$\widetilde{W}_{\rm eff}(\sigma)$	$Y(\lambda)$	Yang-Yang function
equation for the vacua	$e^{2\pi \, d\widetilde{W}_{\rm eff}} = 1$	$e^{2\pi \imath dY} = 1$	Bethe ansatz equation
flavor group at node a	$U(L_a)$	L_a	effective length for the species a
lowest component of the twisted chiral superfield	$\sigma_i^{(a)}$	$\lambda_i^{(a)}$	rapidity
twisted mass of the fundamental field	$\widetilde{m}^{{\rm f}(a)}_k$	$\frac{\imath}{2}\Lambda^a_k + \nu^{(a)}_k$	highest weight of the representation and inhomogeneity
twisted mass of the adjoint	$\widetilde{m}^{{\rm adj}(a)}$	$\frac{\imath}{2}C^{aa}$	diagonal of the Cartan matrix
twisted mass of the bifundamental field	$\widetilde{m}^{{\rm b}(ab)}$	$\frac{\imath}{2}C^{ab}$	non-diagonal of the Cartan matrix
FI-term for $U(N_a)$	τ_a	$\hat{\vartheta}^a$	boundary twist parameter

D–brane configurations

	0	1	2	3	4	5	6	7	8	9
$NS5^{1,2}$	×	×	×	×	×	×				
D2	×	×					×			
D4	×	×						×	×	×

Spin-flip symmetry in the spin chain equals charge-conjugation symmetry in the gauge theory equals the Hanany-Witten effect in the brane construction:

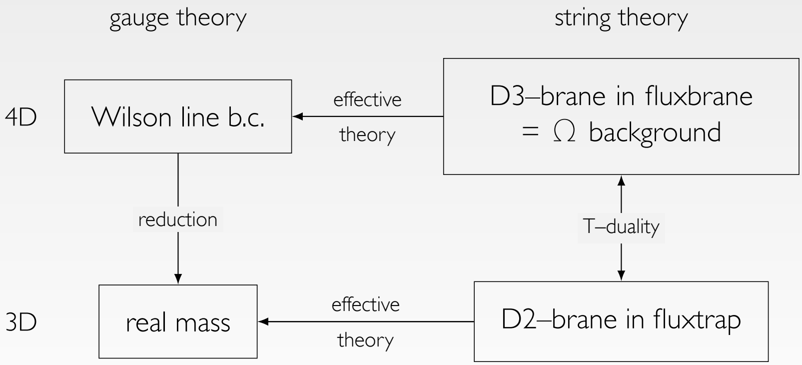


Outline

Parameters from the String Theory

- In order to obtain a faithful representation of our gauge theories from String Theory we need to reproduce all the parameters
- A fundamental but not standard ingredient is given by the twisted masses
- Consider the simplified case of no NS5 branes
- It is convenient to lift the 2d theory to three dimensions: the twisted masses become real masses
- Lifting to four dimensions: the real masses correspond to Wilson line boundary conditions for the compactification
- These boundary conditions can be obtained in String Theory in terms of a fluxbrane [Melvin, Strominger, Gutperle, Takayanagi]

Real masses, monodrofolds, and String Theory



Fluxbrane background

- The fluxbrane background is obtained starting from flat space in ten dimensions and imposing identifications.
- The D2 brane comes from a D3 brane, extended in the directions 0128
- we give a complex structure to the remaining six

$$w_1 = y_1 + \imath y_2 \,, \qquad w_2 = y_3 + \imath y_4 \,, \qquad w_3 = y_5 + \imath y_6 \,,$$

- we impose the identification

$$\tilde{x}^8 \simeq \tilde{x}^8 + 2\pi \tilde{R}, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \simeq \begin{pmatrix} e^{2\pi \imath m \tilde{R}} & 0 \\ 0 & e^{-2\pi \imath m \tilde{R}} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \,.$$

The Ω background

- In an appropriate coordinate system

$$\widetilde{ds^2} = d\vec{x}^2_{0\ldots3} + d\rho_1^2 + \rho_1^2 d\varphi_1^2 + d\rho_2^2 + \rho_2^2 d\varphi_2^2$$
$$+2m\tilde{R}\left(\rho_1^2 d\varphi_1 - \rho_2^2 d\varphi_2\right) d\tilde{u} + \tilde{R}^2\left(1 + m^2\left(\rho_1^2 + \rho_2^2\right)\right) d\tilde{u}^2 + dx_9^2 \,,$$

- or, in rectilinear coordinates

$$x_4 + \imath x_5 \equiv \rho_1 e^{\imath\varphi_1}\,, \qquad x_6 + \imath x_7 \equiv \rho_2 e^{\imath\varphi_2}\,, \qquad x_8 \equiv \tilde{R}\tilde{u}\,,$$

the metric becomes the standard Ω–deformation of flat space

$$d\vec{x}^2_{0\ldots3} + \sum_{i=4}^{7}\left(dx_i + mV^i dx_8\right)^2 + dx_8^2 + dx_9^2 \,,$$

where $V^i\partial_i$ is the Killing vector

$$V^i\partial_i = -x^5\partial_{x_4} + x^4\partial_{x_5} + x^7\partial_{x_6} - x^6\partial_{x_7} = \partial_{\varphi_1} - \partial_{\varphi_2}$$

The fluxtrap background

- To get the real mass we need to T–dualize in x_8, in order to get a D2 brane
- The resulting background is a fluxtrap:

$$
\begin{aligned}
ds^2 &= d\vec{x}_{0\dots3}^2 + d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\varphi_1^2 + \rho_2^2 d\varphi_2^2 \\
&\quad + \frac{-m^2\left(\rho_1^2 d\varphi_1 - \rho_2^2 d\varphi_2\right)^2 + dx_8^2}{1 + m^2\left(\rho_1^2 + \rho_2^2\right)} + dx_9^2 \,, \\
B &= m\,\frac{\rho_1^2 d\varphi_1 - \rho_2^2 d\varphi_2}{1 + m^2\left(\rho_1^2 + \rho_2^2\right)} \wedge dx_8 \,, \\
e^{-\Phi} &= \frac{\sqrt{1 + m^2\left(\rho_1^2 + \rho_2^2\right)}}{g_3^2\sqrt{\alpha'}} \,.
\end{aligned}
$$

- The effective theory for a D2 brane in this background acquires a real mass term m.

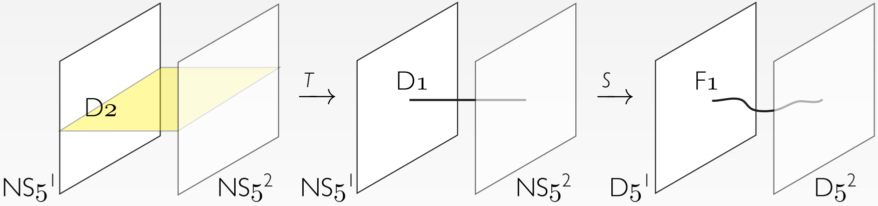
Outline

Brane creation and annihilation?

- The Gauge–Bethe correspondence maps (ground states of) gauge theories to sectors of spin chains
- The spin chains have symmetries that mix different sectors
- It is natural to look for an interpretation of operators that change the gauge group $U(N) \to U(N+1)$.
- In a D–brane interpretation like the one of the configurations we have seen before this amounts to changing the number of D2 branes
- One is tempted to speak of creation/annihilation operators for D–branes.

U duality

- The conceptual problem we need to overcome is that we need to consider states with different boundary conditions at infinity (N and $N+1$ D2 branes)
- Such states should never superpose due to the superselection principle.
- Compactify in the x_1 direction and use a chain of dualities to go to a more tractable configuration: $(\mathrm{NS5}, \mathrm{D2}) \xrightarrow{T} (\mathrm{NS5}, \mathrm{D1}) \xrightarrow{S} (\mathrm{D5}, \mathrm{F1})$



- In terms of this last configuration it is easy to understand where the sl_2 action comes from: once the D5 branes coincide the fundamental strings are charged under the enhanced symmetry.

Real string theory and the Ω-deformation

- Notice that we have not mentioned topological string theory anywhere.
- The states responsible for the correspondence do not show up in any obvious way in the topological sector of the closed or open Ω-background.
- At finite coupling g_3 these states are not massless Cartan gauge bosons, but massive *W*-bosons.
- Note: It is not just the vaccum states but the entire spectrum that is organized under the $SU(k)$ symmetry at strong coupling.

Real string theory and the Ω-deformation

- Key question: are there any non-vacuum states that survive at weak coupling?
- Naive expectation: all states go to mass scale set by the scale where the coupling becomes strong.
- This is the scale where excitations lie in generic 2D gauge theories.
- Too naive! There are states that lie parametrically below the strong coupling scale, due to BPS protection!
- Now we will discuss the BPS protection in more detail.

Real mass terms and the SUSY algebra

- Real (twisted) mass terms in 2D are neither superpotential nor twisted superpotential terms.
- They cannot be viewed as perturbations of the action symmetric under a fixed superalgebra or integrals of covariant terms over superspace.
- The twisted mass terms are associated with a deformation of the superalgebra itself.
- A real twisted mass in 2D can be understood by lifting to an $\mathcal{N} = 2$ theory in 3 dimensions.
- The lifted deformation (="real mass") expresses itself in terms of a central extension of the undeformed theory:

$$\{Q_\alpha, \bar{Q}_\beta\} = \Gamma^{\mu}_{\alpha\beta} P_\mu + \delta_{\alpha\beta} Z ,$$

where $Z \equiv m^I q_I$ is a linear combination of non-R global symmetries q_I of the theory.

Real mass terms and the SUSY algebra

- This leads to a simple, general and superspace-free construction of the twisted mass deformation in the general case:
- Couple a fictitious (nondynamical) vector multiplet $A_{0,1,2}$, σ, λ_α to the theory, for each Abelian global symmetry.
- Give the real (appropriately normalized) fields σ^I fixed values m^I.
- This preserves $\mathcal{N} = 2$ SUSY in 3 dimensions, but adds the central extension Z.
- This construction of real/twisted masses is universal and superspace-free as well as giving a simple sufficient condition for real mass deformations of theories with superpotentials.

Real mass terms and the SUSY algebra

- For certain 3D theories there's an even simpler description: those that lift to 4D.
- This assumes the global symmetries are exact in 4D as well.
- Then the σ field lifts to $A_{\tilde{3}}$ where $\tilde{3}$ is the fourth dimension we're lifting to and A is again the nondynamical Abelian gauge field.
- Then the real mass term is realized as a compactification with monodromy $\tilde{R}_3 \cdot m_I q^I$ - this is the integral of the gauge connection around the fourth dimension.

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- Then the fields have generalized momenta $Z = m_I q^I$ and thus Kaluza-Klein masses $Z = |m_I q^I$.

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Real mass terms and the SUSY algebra

- For maximally supersymmetric $3D$ gauge theory ($\mathcal{N} = 8$ in 3D language), the theory lifts to $\mathcal{N} = 4$ in $4D$.
- The only global symmetries are the group $SO(6)$. There are no non-R symmetries under the full $\mathcal{N} = 4$ but under an $\mathcal{N} = 1$ subalgebra, there is an $SU(3)$ non-R symmetry group.
- There are two Cartan generators q^I here. Taking a general combination breaks the SUSY to $calN = 2$ in $3D$ and $(2, 2)$ in $2D$.
- There is a special combination that lies in $SU(2) \subset SU(3)$ that preserves $\mathcal{N} = 4$ in 3D and $(4, 4)$ in 2D.
- This corresponds to the monodromy generated by $(m\tilde{R}_3, -m\tilde{R}_3, 0)$ in complex coordinates.
- This is the Ω deformation of flat space, with $\varepsilon_1 = -\varepsilon_2$.

Outline

The message

- The gauge–Bethe correspondence relates supersymmetric gauge theories to sectors of spin chains.
- Interesting symmetry relating different gauge theories to each other!
- Natural, explicit embedding in string theory – the fluxtrap is the string theory of the Ω deformation.
- This is "physical string" not the topological string.
- Can go beyond the vacuum sector – BPS, near-BPS and non-BPS states can be studied in this framework, and arrange themselves under the enhanced symmetry.
- Important question: What do these states map to in the spin-chain language? Hint: NOT states in the Hilbert space of the spin chain!
- Work in progress!

Thank you for your attention