KITP program “Non-perturbative effects and dualities in QFT and Integrable systems”, Santa Barbara, August 3, 2011

Solving the AdS/CFT Y-system

Vladimir Kazakov (ENS,Paris)

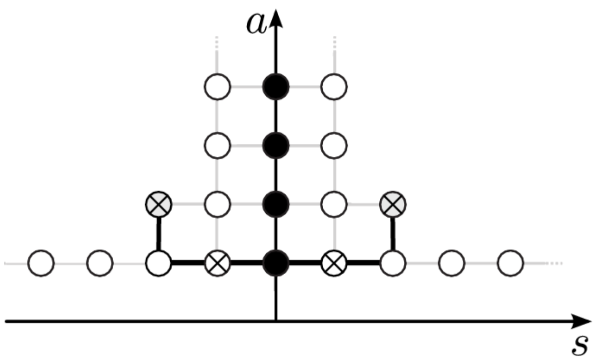
with N.Gromov , S.Leurent, D.Volin
(in preparation)

Integrability in AdS/CFT

- Integrable planar superconformal 4D N=4 SYM and 3D N=8 Chern-Simons... (non-BPS, summing genuine 4D Feynman diagrams!)
- Based on AdS/CFT duality to very special 2D superstring σ-models on AdS-background
- Y-system (for planar AdS_5/CFT_4 , AdS_4/CFT_3 ,...) calculates exact anomalous dimensions of all local operators at any coupling
- Y-system is an infinite set of functional or integral nonlinear eqs.

$$Y_{a,s}\left(u+\frac{i}{2}\right) Y_{a,s}\left(u-\frac{i}{2}\right) = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\left[1+\frac{1}{Y_{a+1,s}}\right]}$$

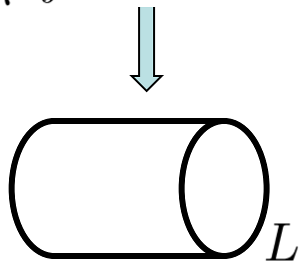
Gromov,V.K.,Vieira



- Problem: how to transform Y-system into a finite system of non-linear integral equations (FiNLIE) using its Hirota discrete integrable dynamics and analyticity properties in spectral parameter?

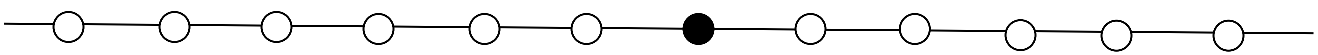
Worm up: SU(2)xSU(2) principal chiral field at finite L

$$\mathcal{A} = \frac{\sqrt{\lambda}}{4\pi} \int d^2x \ \mathrm{Tr} \left(g^{-1} \partial_\mu g(x) \right)^2 , \qquad g \in SU(2).$$



- Y-system

$$Y_s(u + \frac{i}{2})\, Y_s(u - \frac{i}{2}) = \left[1 + Y_{s+1}(u)\right] \ \left[1 + Y_{s-1}(u)\right]$$



- Energy

$$E(Lm) = -\frac{1}{2} \int_{-\infty}^{\infty} du \cosh(\pi u) \log(1 + Y_0(u))$$

- Large volume

$$Y_s(u) \simeq C_s e^{-\delta_{s,0}\, mL \cosh \pi u}, \qquad mL \to \infty$$

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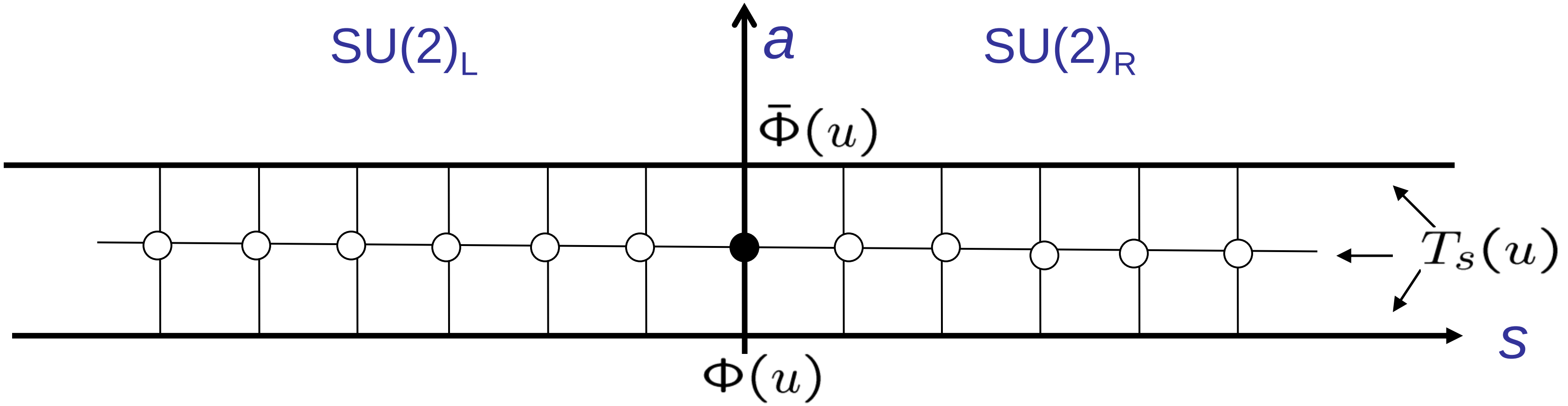
Y-system and Hirota relation

Parametrize: $$Y_s(u) = \frac{T_{s+1}(u)T_{s-1}(u)}{\Phi(u+is/2)\bar{\Phi}(u-is/2)}$$

Hirota equation:

$$T_s(u+i/2)T_s(u-i/2)-T_{s-1}(u)\,T_{s+1}(u) = \Phi(u+is/2)\,\bar{\Phi}(u-is/2)$$



Determinant representation and gauge transformation

- Determinant solution of Hirota eq.

$$T_s(u) = h(u+is/2) \begin{vmatrix} Q(u+i\frac{s+1}{2}) & R(u+i\frac{s+1}{2}) \\ \bar{Q}(u-i\frac{s+1}{2}) & \bar{R}(u-i\frac{s+1}{2}) \end{vmatrix}$$

$$h(u+i) = h(u)$$

$$\Phi(u) = h(u+i/2) \begin{vmatrix} R(u) & Q(u) \\ R(u+i) & Q(u+i) \end{vmatrix}$$

- Gauge transformations

$$\begin{aligned} T_s(u) &\to g\left(u+i\frac{s}{2}\right)\bar{g}\left(u-i\frac{s}{2}\right) T_s(u) \\ \Phi(u) &\to g(u-i/2)g(u+i/2)\Phi(u) \\ Q(u) &\to g(u-i/2)Q(u) \end{aligned}$$

Leaves Y's invariant!

Lax pair and Baxter relation

Krichever,Lipan,Wiegmann, Zabrodin

- Solution: linear Lax pair (discrete integrable dynamics!)

$$\left\{ \begin{array}{c} T_{s+1}(u)Q(u+is/2) - T_s(u-i/2)Q(u+i(s+2)/2) \;=\; \Phi(u+i(s+1)/2)\bar{Q}(u-i(s+2)/2) \\ \text{Complex conjugate} \end{array} \right.$$

- Baxter relations

$$T_1(u) = T_0(u-i/2)\frac{Q(u+i)}{Q(u)} + \Phi(u)\frac{\bar{Q}(u-i)}{Q(u)}$$

$$T_{-1}(u) = T_0(u+i/2)\frac{Q(u)}{Q(u+i)} - \Phi(u)\frac{\bar{Q}(u)}{Q(u+i)}$$

Definition:

$$f(u \pm i/2 \equiv f^{\pm}$$

$$f(u \pm ik/2 \equiv f^{[\pm k]}$$

- Wing exchange symmetry:

$$T_s \leftrightarrow T_{-s} \;,\; \Phi \leftrightarrow -\bar{\Phi} \;,\; Q^+ \leftrightarrow \bar{Q}^- \;,\; \bar{Q}^- \leftrightarrow Q^+$$

- Q(R) are interpreted (in different gauges) as Baxter functions for right (left) wing

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Analyticity and ground state solution Q=1

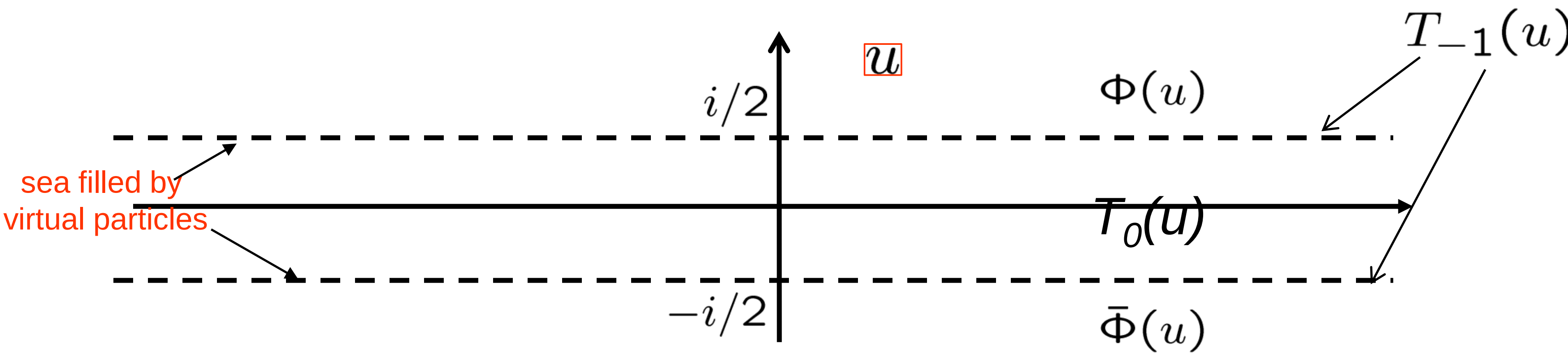
Gromov,V.K.,Vieira

- Solution: we assume $T_0(u)$, $\Phi(u)=T_0(u+i/2+i0)$ *and* $T_{-1}(u)$:

$$T_1(u) = T_0(u-i/2)+T_0(u+i/2) \quad \text{- Baxter eq.}$$
$$T_{-1}(u) = T_0(u+i/2-i0)-T_0(u+i/2+i0) \quad \text{- “Jump” eq.}$$

relates T_0 *and* Φ *to* $T_{-1}(u)$ through analyticity:

$$T_0 = 1 - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dv\, T_{-1}(v)}{u-v-i/2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dv\, T_{-1}(v)}{u-v+i/2}$$

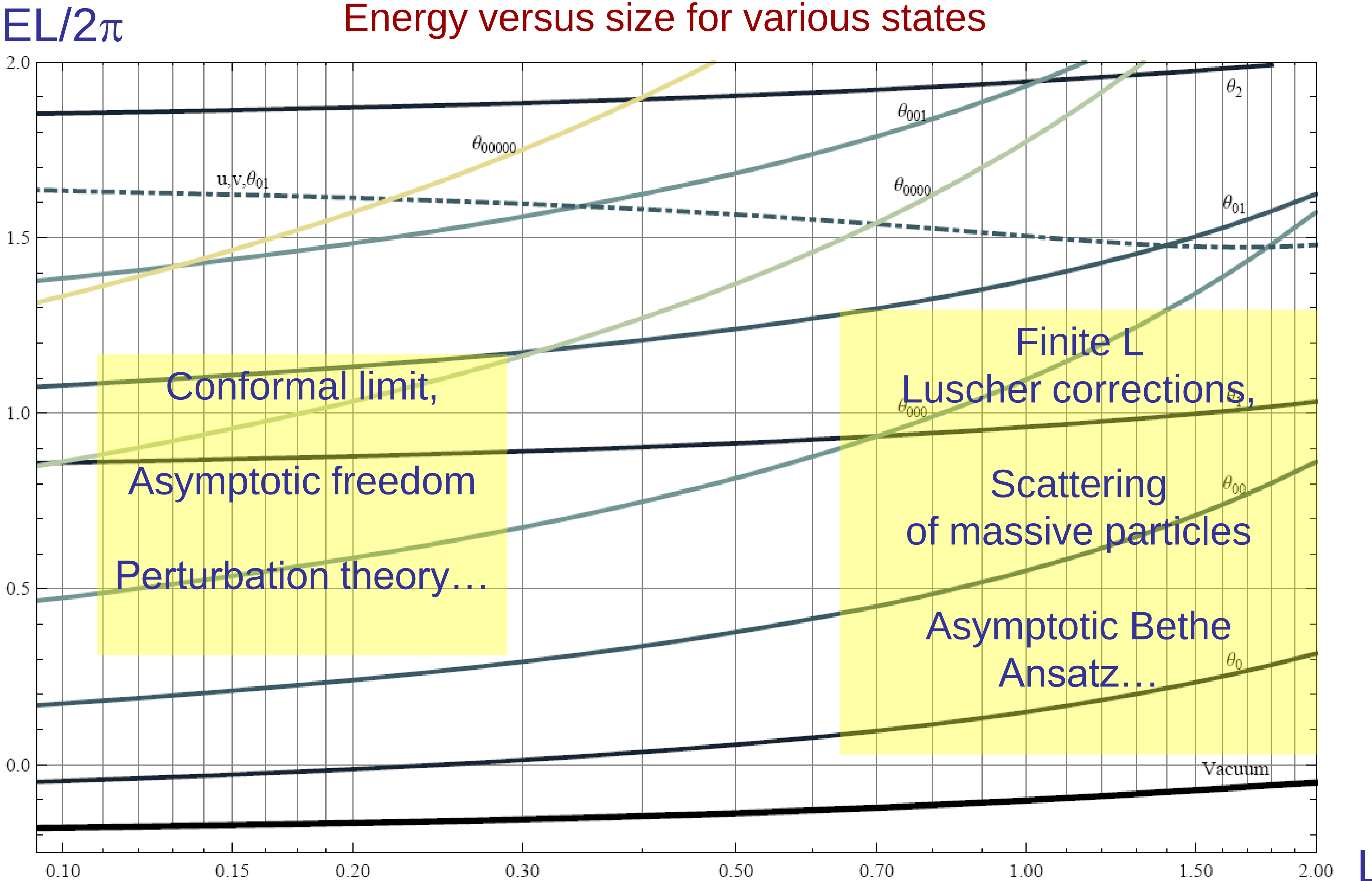


- T_s have no singularities on real axis $\quad T_{s-1} = s + \frac{s}{2\pi}\int_{-\infty}^{\infty}\frac{dv\, T_{-1}(v)}{(u-v)^2+s^2/4}$
- TBA eq. for Y_0 , imposing for vacuum $Y_1 = Y_{-1}$ get non-linear integral eq. for T_{-1}

$$\log Y_0 = -L\cosh\pi u + 2r*\log\left[1+Y_1\right], \qquad r = \frac{1}{2\cosh\pi u}$$

Gromov,V.K.,Vieira’08

SU(2) PCF numerics (using Hirota solution): Energy versus size for various states



For vacuum and mass gap in accordance with Balog,Hegedus’00’02

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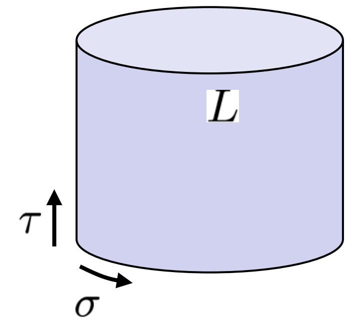
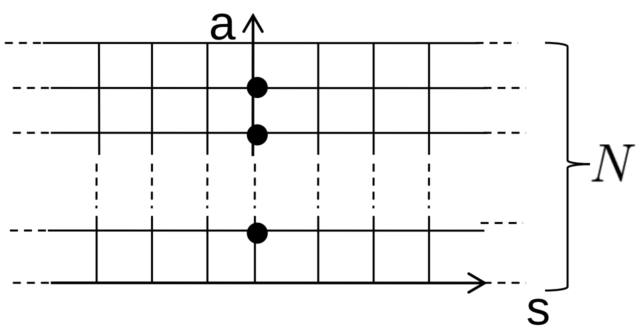
Inspiring example: $SU(N)_L \times SU(N)_R$ principal chiral field

$$\mathcal{S} = \frac{\sqrt{\lambda}}{4\pi} \int d^2x \; \mathrm{Tr}\, \left(g^{-1}\partial_\mu g(x)\right)^2, \qquad g \in SU(N).$$



- Y-system ⇨ ***Hirota*** dynamics in a in (a,s) plane.

$$Y_{a,s}\left(u+\frac{i}{2}\right) Y_{a,s}\left(u-\frac{i}{2}\right) = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\left[1+\frac{1}{Y_{a+1,s}}\right]}$$



- Relation of Y-system to T-system (Hirota equation)

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

$$T_{a,s}(u+\tfrac{i}{2})T_{a,s}(u-\tfrac{i}{2}) = T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)\,T_{a-1,s}(u)$$

- Gauge symmetry

$$T_{a,s} \to g_1^{[+a+s]} g_2^{[+a-s]} g_3^{[-a-s]} g_4^{[-a+s]} T_{a,s}$$

General wronskian solution in a band

- Parametrized through N functions of u: $q_1, q_2, \cdots, q_N$

- Representation in terms of exterior forms:

$$q \equiv q_j \theta^j, \qquad p \equiv p_j \theta^j, \qquad \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^N = 1$$

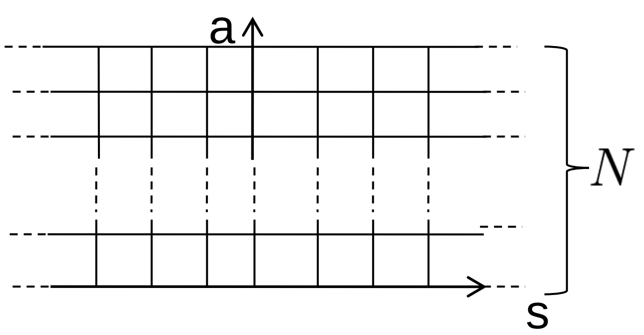
Gromov,V.K.,Leurent,Volin

- k-forms: $q_{(k)} \equiv q^{[k-1]} \wedge q^{[k-3]} \ldots \wedge q^{[1-k]}$

- Solution in N-band:

Krichever,Lipan,
Wiegmann,Zabrodin

$$T_{a,s} = q_{(a)}^{[+s]} \wedge p_{(N-a)}^{[-s]}$$



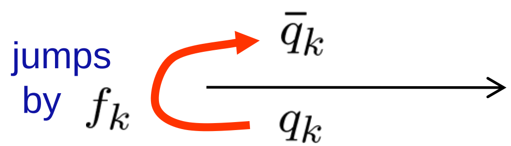
Analyticity properties and solution

V.K.,Leurent

- Finite volume solution: finite system of NLIE:
  parametrization fixing the analytic structure:

$$q_k(u) \;=\; P_k(u) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{f_k(u)}{u-v}\,, \qquad \mathrm{Im}(u) < 0\,,$$

polynomials fixing a state



- From reality: $p_k(u) = \bar{q}_k(u)$

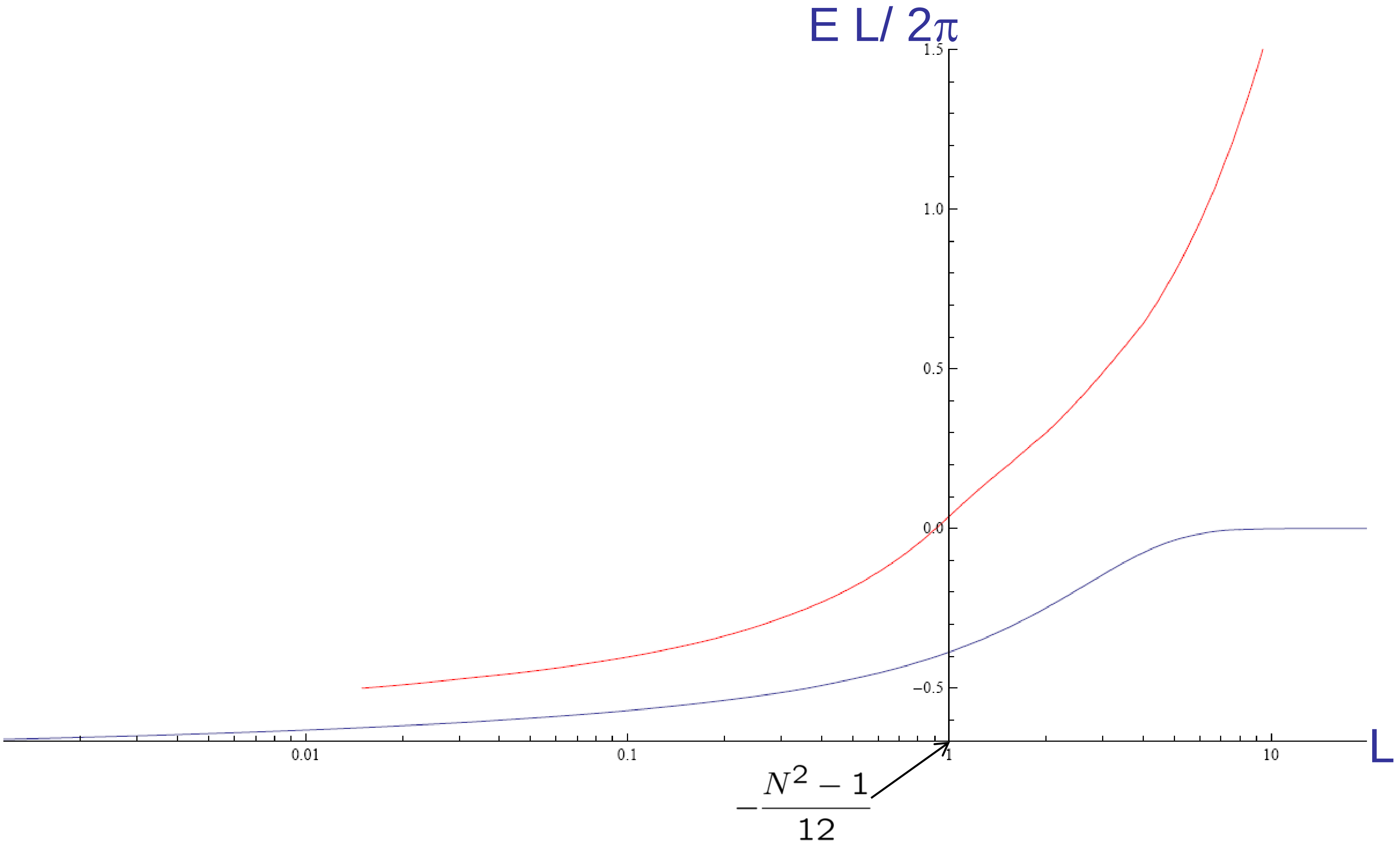
- N-1 spectral densities $f_k(u)$ (for L $\leftrightarrow$ R symmetric states): $Y_{a,s} = Y_{a,-s}$

$$\log Y_{a,0} = -L m_a \sinh(2\pi N u) + K_{a,a'} * \log\left[\frac{(1+Y_{a',1})^2}{(1+Y_{a'+1,0})(1+Y_{a'-1,0})}\right], \qquad \text{where} \quad Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$$

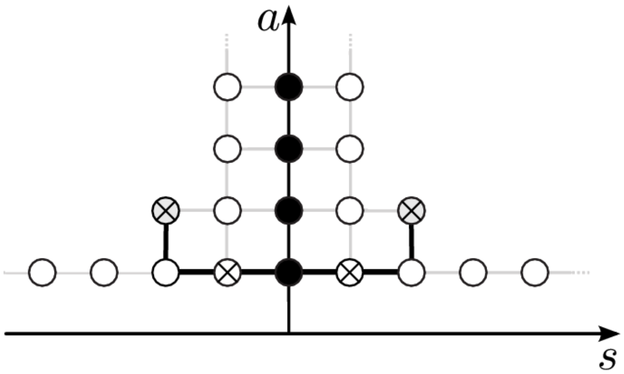
- Solved numerically by iterations

SU(3) PCF numerics: Energy versus size for vacuum and mass gap

V.K.,Leurent'09



AdS/CFT Y-system and asymptotics



$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a+1,s}} = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\left[1+Y_{a+1,s}\right]}$$

- Large L asymptotics:

$$Y_{a,s}(u) \simeq C_{a,s} e^{\delta_{s,0}\, L\, i p_a(u)}$$

- Momentum of elementary excitation $p_a(u) = -i \log \frac{x^{[+a]}}{x^{[-a]}}$ appears as a zero mode of discrete D'Alembert operator in the l.h.s.

  We should fix it and find the corresponding energy as well.

Dispersion relation in physical and crossing channels

- Exact one particle dispersion relation: $\epsilon^2 = 1 + \lambda \sin^2 \frac{p}{2}$

Santambrogio,Zanon
Beisert,Dippel,Staudacher
N.Dorey

- Bound states (fusion) $\epsilon_a^2 = a^2 + \lambda \sin^2 \frac{p_a}{2}$

$$-\mathcal{Z}-\mathcal{Z}-\mathcal{D}-\mathcal{Z}-\mathcal{Z}-\mathcal{D}-\mathcal{Z}-$$
$$u - \frac{i}{2} \qquad\qquad u + \frac{i}{2}$$

- Changing physical L-circle to cross channel R-circle

$$\epsilon_a^2 = a^2 + \lambda \sin^2 \frac{p_a}{2} \quad \Longrightarrow \quad -\tilde{\epsilon}_a^2 = a^2 - \lambda \sinh^2 \frac{\tilde{p}_a}{2}$$

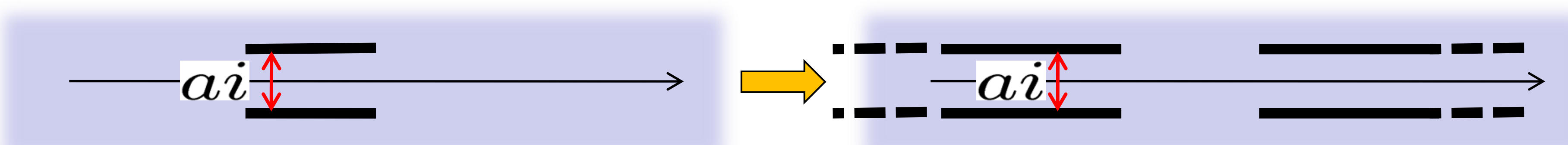
Ambjorn,Janik,Kristjansen
Arutyunov,Frolov

- Parametrization for the dispersion relation by Zhukovsky map: $u = \sqrt{\lambda}\left(z + \frac{1}{z}\right)$

$$\begin{cases} p_a(u) = \frac{1}{i} \log \frac{z(u + ia/2)}{z(u - ia/2)} \\ \epsilon_a(u) = 2i\sqrt{\lambda}\,[z(u - ia/2) - z(u + ia/2)] + 1 \end{cases}$$

- From physical to crossing kinematics: continuation through the cut

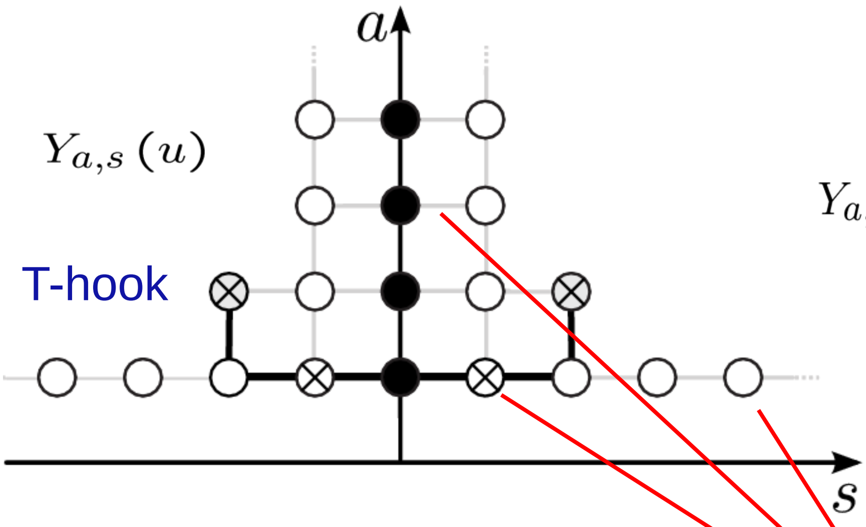
$$z = \frac{1}{2\sqrt{\lambda}}\left(u + \sqrt{u - 2\sqrt{\lambda}}\sqrt{u + 2\sqrt{\lambda}}\right) \quad \Longrightarrow \quad \tilde{z} = \frac{1}{2\sqrt{\lambda}}\left(u + i\sqrt{4\lambda - u^2}\right)$$



cuts in complex u -plane

Y-system for excited states of AdS/CFT at finite size

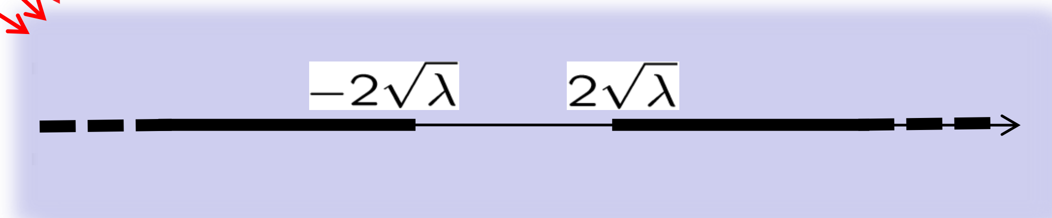
Gromov,V.K.,Vieira



$$Y_{a,s}\left(u+\frac{i}{2}\right) Y_{a,s}\left(u-\frac{i}{2}\right) = \frac{\left[1+Y_{a,s+1}\right]\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\left[1+\frac{1}{Y_{a+1,s}}\right]}$$

- Complicated analyticity structure in u dictated by non-relativistic dispersion

cuts in complex u-plane



- Extra equation (remnant of classical Z_4 monodromy):

$$Y_{2,\pm 2}(u+i0)\cdot Y_{1,\pm 1}(u-i0) = 1$$

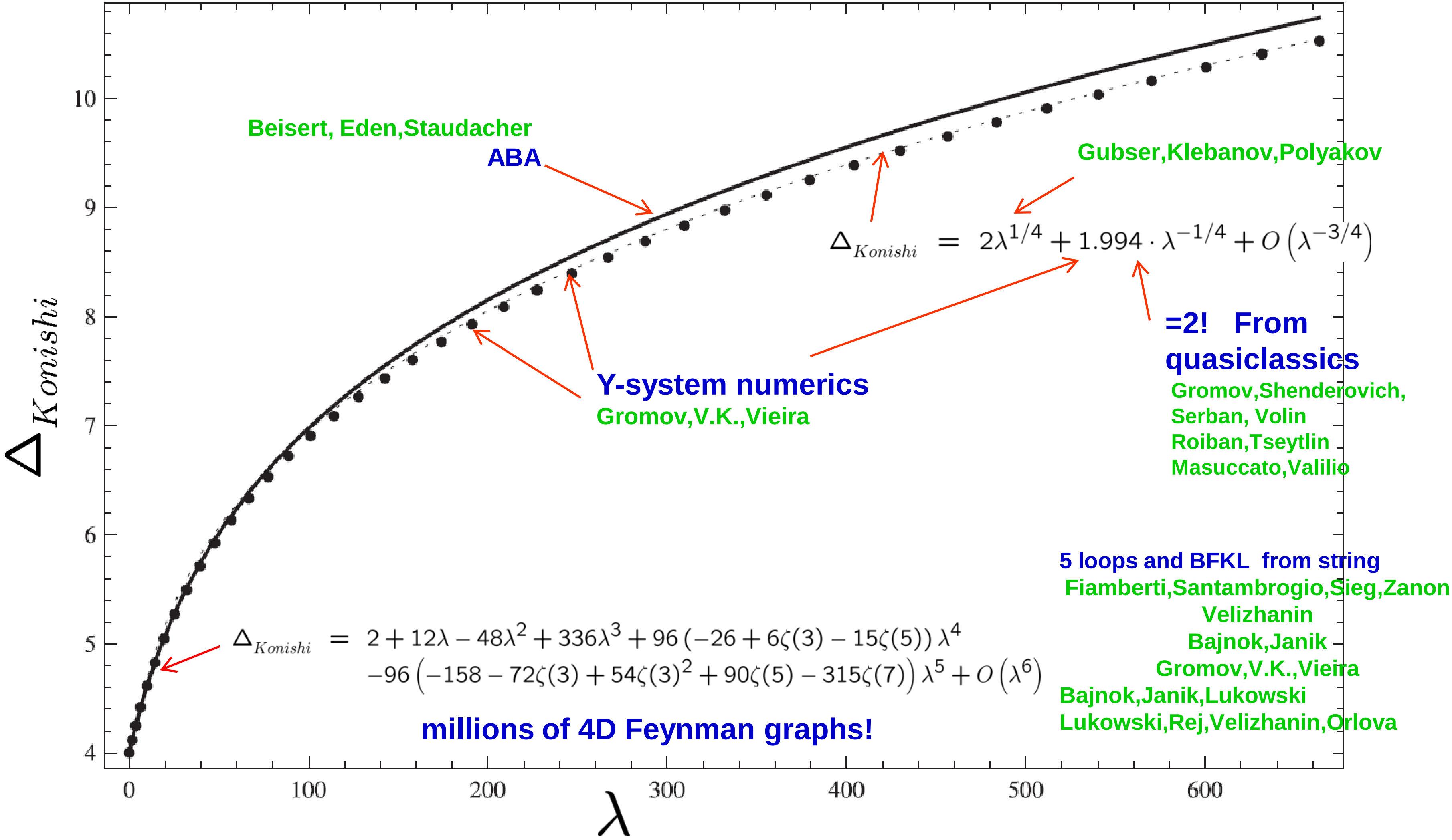
- Energy : (anomalous dimension)

$$\Delta - \Delta_0 = f_{min} = \int \frac{du}{2\pi i}\, \partial_u \tilde{\epsilon}_a \, \log\left(1+Y_{a,0}\right)$$

- u_j obey the exact Bethe eq.: $\quad Y_{1,0}(u_j)+1=0$

New insights into analyticity:
Cavaglia,Fioravanti,Tateo
Hegedus,Balog

Konishi operator $\mathrm{Tr}\,[\mathcal{D}, Z]^2$: numerics from **Y-system**



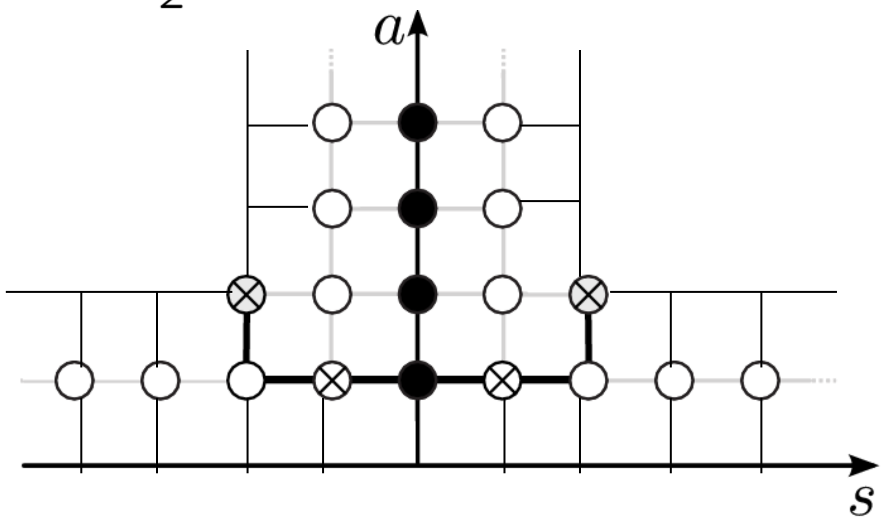
- Y-system passes all known tests

Y-system and Hirota eq.: discrete integrable dynamics

- Relation of Y-system to T-system (Hirota equation)
  (the Master Equation of Integrability!)

$$Y_{a,s} = \frac{T_{a,s+1}\, T_{a,s-1}}{T_{a+1,s}\, T_{a-1,s}}$$

$$T_{a,s}(u+\tfrac{i}{2})T_{a,s}(u-\tfrac{i}{2}) = T_{a,s-1}(u)\, T_{a,s+1}(u)+T_{a+1,s}(u)\, T_{a-1,s}(u)$$



Hirota eq. in T-hook for AdS/CFT

Gromov, V.K., Vieira

Discrete classical integrable dynamics!

Quasiclassical solution of AdS/CFT Y-system

- Classical limit: highly excited long strings/operators, strong coupling:

$$L \sim \sqrt{\lambda} \sim u \to \infty$$

- Explicit u-shift in Hirota eq. dropped (only slow parametric dependence)

$$T_{a,s}\left(u+\cancel{\frac{i}{2\sqrt{\lambda}}}\right) T_{a,s}\left(u-\cancel{\frac{i}{2\sqrt{\lambda}}}\right) = T_{a,s-1}(u)\, T_{a,s+1}(u) + T_{a+1,s}(u) T_{a-1,s}(u)$$

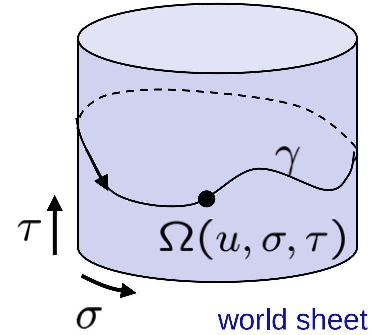
Gromov,V.K.,Tsuboi

- (Quasi)classical solution - psu(2,2|4) character of classical monodromy matrix in Metsaev-Tseytlin superstring sigma-model

$$T_{a,s} = \mathrm{Tr}_{a,s}\, \Omega$$

$$\Omega(u) = P \exp \oint_\gamma \mathcal{A}(u) \quad \in PSU(2,2|4)$$

Zakharov,Mikhailov
Bena,Roiban,Polchinski



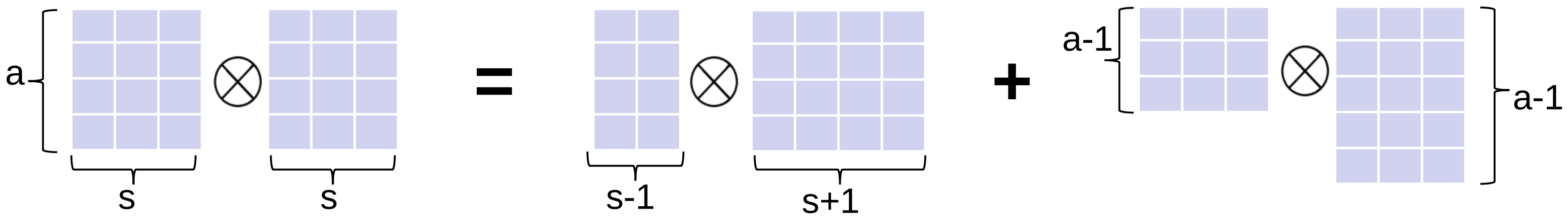
- Its eigenvalues (quasi-momenta) encode conservation lows

- An important property: $\mathbb{Z}_4$ symmetry of the coset and related monodromy for the eigenvalues (quasimomenta)

V.K.,Marshakov,Minahan,Zarembo
Beisert,V.K.,Sakai,Zarembo

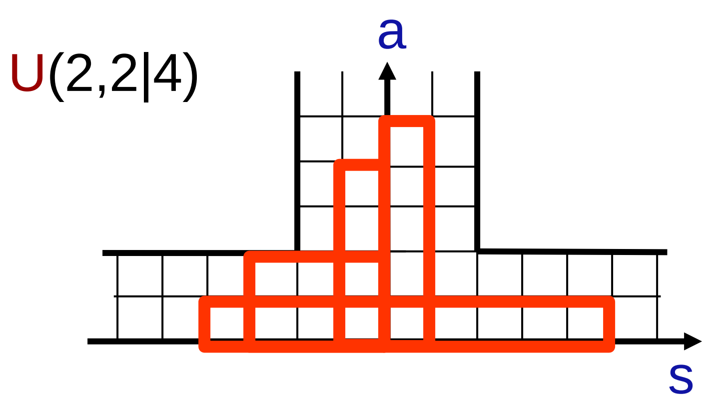
(Super-)group theoretical origins

- A curious property of gl(N|M) representations with rectangular Young tableaux:



- For characters – simplified Hirota eq.: $$T^2_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$

- Boundary conditions for Hirota eq.:
  ∞ - dim. unitary highest weight representations of u(2,2|4) in "T-hook" !



Kwon
Cheng,Lam,Zhang

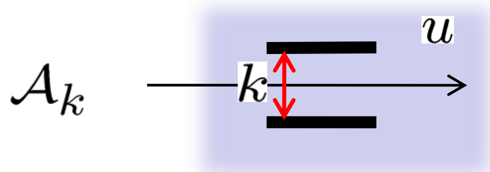
Gromov, V.K., Tsuboi

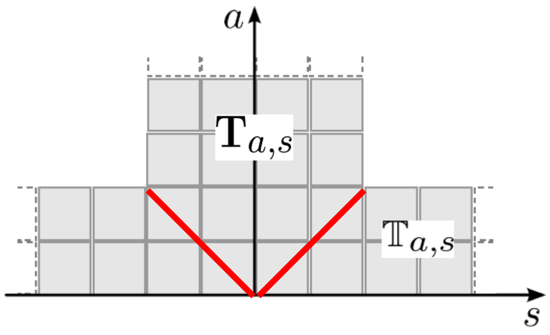
- Solution of Hirota for any irrep: Jacobi-Trudi formula for GL(K|M) characters:

$$T_{a,s}[g] = \det_{1\le i,j\le a} T_{1,s-i+j}[g]\,, \qquad g\in GL(K|M).$$

Construction of solution for AdS/CFT T-system

- There are three different analytically friendly gauges for T-functions for right, left and upper bands.
- They have certain analyticity strips:





- Right (left) band

only two cuts

Gromov,V.K.,Leurent,Volin (in progress)

$$\mathbb{T}_{0,\pm s} = 1 \quad , \quad \mathbb{T}_{1,\pm s} \in \mathcal{A}_s \quad , \quad \mathbb{T}_{2,\pm s} \in \mathcal{A}_{s-1}$$

- Upper band

$$\mathbf{T}_{a,0} \in \mathcal{A}_{a+1} \quad , \quad \mathbf{T}_{a,\pm 1} \in \mathcal{A}_a \quad , \quad \mathbf{T}_{a,\pm 2} \in \mathcal{A}_{a-1}$$

$$\mathbf{T}_{n,2} = \mathbf{T}_{2,n}$$ - from representation theory

$$\mathbf{T}^{+}_{0,0} = \mathbf{T}^{-}_{0,0}$$ - from quantum unimodularity of monodromy matrix

- Relating right (left) and upper bands

- $\mathbb{Z}_4$ symmetry of coset: $\mathbf{T}_{a,s}$ can be analytically continued in labels a,s

$$\hat{\mathbf{T}}_{a,s} = (-1)^s \hat{\mathbf{T}}_{-a,s} \,, \qquad \hat{\mathbb{T}}_{a,s} = (-1)^a \hat{\mathbb{T}}_{a,-s} \,.$$

Wronskian solution for the right (left) band

- Specifying it for N=2 we get:

$$\hat{\mathbb{T}}_{0,s} = 1 \;\;,\;\; \hat{\mathbb{T}}_{1,s} = q^{[+s]} \wedge q^{[-s]} \;\;,\;\; \hat{\mathbb{T}}_{2,s} = \hat{\mathbb{T}}_{1,1}^{[+s]} \hat{\mathbb{T}}_{1,1}^{[-s]} \;.$$

- Using the $\mathbb{Z}_4$ symmetry and after a certain gauge transformation

$$\hat{\mathbb{T}}_{1,s} = \hat{h}^{[+s]} \hat{h}^{[-s]} \hat{\mathcal{T}}_{1,s} \;\;,\;\; \hat{\mathbb{T}}_{2,s} = \hat{h}^{[+s+1]} \hat{h}^{[+s-1]} \hat{h}^{[-s+1]} \hat{h}^{[-s-1]} \hat{\mathcal{T}}_{2,s}$$

we find a representation similar to SU(2) PCF (but with two short cuts)

$$\begin{aligned} \hat{\mathcal{T}}_{0,s} &= 1 \,, \\ \hat{\mathcal{T}}_{1,s} &= q^{[+s]} - q^{[-s]}, \\ \hat{\mathcal{T}}_{2,s} &= \hat{\mathcal{T}}_{1,1}^{[+s]} \hat{\mathcal{T}}_{1,1}^{[-s]} \end{aligned} \qquad q = -iu + \frac{1}{2\pi i} \int_{-2g}^{2g} \frac{\rho(v)dv}{u-v},$$

$$q_1 = q, \quad q_2 = 1$$

Wronskian solution for the upper band

- Specifying it for N=2 we get: $\hat{\mathbf{T}}_{a,s} = \mathbf{q}^{[+a]}_{(2-s)} \wedge \mathbf{p}^{[-a]}_{(2+s)}\,.$

$$\mathbf{q}_{(2)} = \frac{\mathbf{q}^{+} \wedge \mathbf{q}^{-}}{\mathbf{q}_{\emptyset}}, \quad \mathbf{q}_{(3)} = \frac{\mathbf{q}^{++} \wedge \mathbf{q} \wedge \mathbf{q}^{--}}{\mathbf{q}^{+}_{\emptyset}\mathbf{q}^{-}_{\emptyset}}, \quad \mathbf{q}_{(4)} = \frac{\mathbf{q}^{[+3]} \wedge \mathbf{q}^{+} \wedge \mathbf{q}^{-} \wedge \mathbf{q}^{[-3]}}{\mathbf{q}^{++}_{\emptyset}\mathbf{q}_{\emptyset}\mathbf{q}^{--}_{\emptyset}}$$

- From reality, $\mathbb{Z}_4$ symmetry and asymptotic properties at large L and considering only left-right symmetric states $\hat{\mathbf{T}}_{a,-s} = \hat{\mathbf{T}}_{a,s}$ we find

$$\begin{aligned} \hat{\mathbf{T}}_{a,\pm 1} &= \mathbf{q}_1^{[+a]}\bar{\mathbf{q}}_2^{[-a]} + \mathbf{q}_2^{[+a]}\bar{\mathbf{q}}_1^{[-a]} + \mathbf{q}_3^{[+a]}\bar{\mathbf{q}}_4^{[-a]} + \mathbf{q}_4^{[+a]}\bar{\mathbf{q}}_3^{[-a]}\,, \\ \hat{\mathbf{T}}_{a,0} &= \mathbf{q}_{12}^{[+a]}\bar{\mathbf{q}}_{12}^{[-a]} + \mathbf{q}_{34}^{[+a]}\bar{\mathbf{q}}_{34}^{[-a]} - \mathbf{q}_{14}^{[+a]}\bar{\mathbf{q}}_{14}^{[-a]} - \mathbf{q}_{23}^{[+a]}\bar{\mathbf{q}}_{23}^{[-a]} - \mathbf{q}_{13}^{[+a]}\bar{\mathbf{q}}_{24}^{[-a]} - \mathbf{q}_{24}^{[+a]}\bar{\mathbf{q}}_{13}^{[-a]} \end{aligned}$$

- We choose to parameterize T's through $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_{12}$

- Remarkably, if $\mathbf{q}_1, \mathbf{q}_2$ are analytic in the upper half plane, and $\mathbf{q}_{12}$ analytic above $-i/2 + \mathbb{R}$ then all T-functions have the right analyticity strips.

- We can parameterize parameterize $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_{12}$ in terms of 2 spectral densities ρ_2, ρ_3 and find from analyticity of a closed system of equations on them - FiNLIE

Closing FiNLIE: spectral densities and sawing together 3 bands

Gromov,V.K.,Leurent,Volin
(in progress)

- We can parameterize parameterize $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_{12}$
  in terms of 2 spectral densities ρ_2, ρ_3

polynomial encoding Bethe roots

$$\mathbf{q}_1 = \sqrt{U} f^+ f^-, \quad \mathbf{q}_2 = \sqrt{U} f^+ f^- W, \quad \mathbf{q}_{12} = f^2 V \ ,$$

Example:

$$U = x^\mu(u) \int_{-\infty}^{+\infty} \frac{dv}{2\pi i} \frac{\rho_3(v)}{u-v} \ , \quad \text{Im } u > 0$$

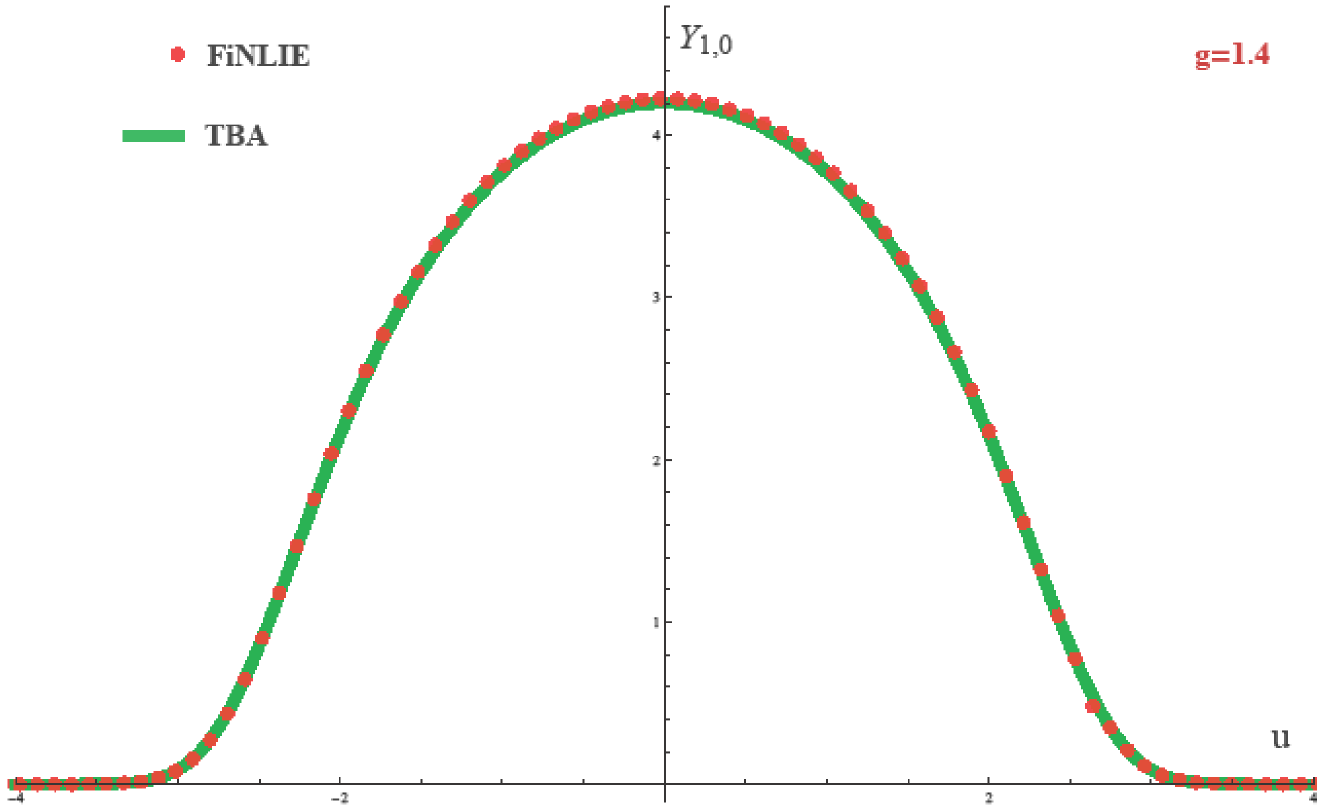
- From analyticity - closed system of equations on densities ρ, ρ_2, ρ_3

FiNLIE!

Numerical solution of FiNLIE

- Writing our FiNLIE as $X = f(X), \qquad X = (\rho, \rho_2, \rho_3, \ldots)$

  we attempted to solve it on Mathematica by iterations: $X_{n+1} = f(X_n)$



- The coincidence with earlier results from the infinite Y-system (TBA) is very satisfactory!

Comments

- The Bethe roots characterizing a state are encoded into zeres of some q-functions (in particular $\mathbf{q}_{12}$)

- The energy of a state can be extracted from the asymptotics

$$\mathbf{T}_{0,0}(u) \sim u^{2E}, \quad u \to \infty$$

Conclusions

- Y-system obeys integrable Hirota dynamics – can be reduced to a finite system of non-linear integral eqs (FiNLIE).
- Our FiNLIE is based on a few natural, or even physical analyticity asumptions. Not based on TBA but proven to have the same solution
- Certainly our FiNLIE is still perfectible.
- Hirota dynamics provides a general method of solving quantum σ-models on a finite space circle.
- Possible mathematical subject: "sigma model"-like solution of Hirota equations and the associated Riemann-Hilbert problems for general (a,s) boundary conditions.

END