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Integrable Systems, $N=2$ gauge theories + 2d/4d gauge theories

reference:

KM-Taki, 1006.4505

Bonelli-KM-Tanzini, 1104.4015

- Gauge theory and integrable system
- 2d/4d (AGT) correspondence
- Integrable system via AGT
- " via matrix model
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Gauge theory and integrable system

$N=2$ susy gauge theories (with vanishing beta functions)

low energy -- Seiberg-Witten theory

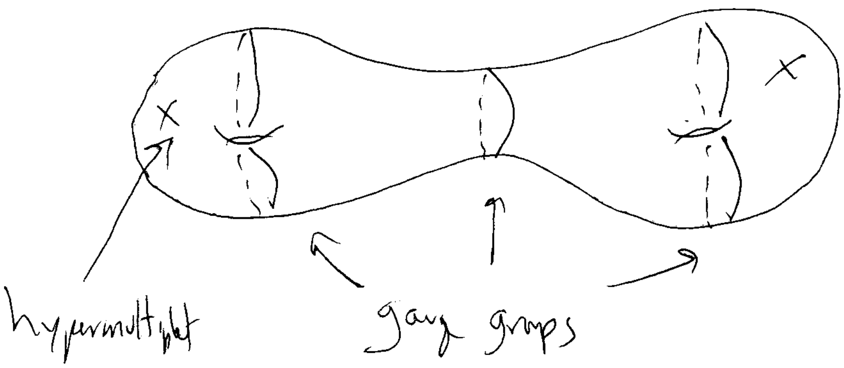
curve Σ, λ_{SW}

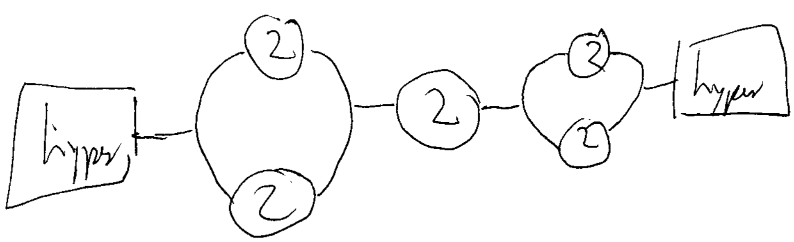
$$\begin{cases} a_i = \oint_{A_i} \lambda_{SW} \\ \dfrac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} \lambda_{SW} \end{cases} \qquad m_f = \int_{C_f} \lambda_{SW}$$

6d (2,0) theory on $\mathbb{R}^{1,3} \times C_{g,n}$

$C_{g,n}$: Riemann surface with punctures

We specify A_1 case (2 M5-branes) which leads to the case where all the gauge groups are SU(2).





$$\Rightarrow \quad \Sigma : \; x^2 = \phi_2(z) \qquad z : \text{coordinate on } C_{g,n}$$

$$\lambda_{SW} = x\,dz$$

$$\phi_2(z) \sim \frac{m_q^2}{(z-q)^2} + \frac{f(a, m_q)}{(z-q)} + \cdots$$

The associated integrable system is

Hitchin system on $C_{g,n}$

[Donagi–Witten, Gaiotto–Moore–Neitzke]

$$\Sigma_H: \det(x\mathbb{1} - \varphi(z)) = 0$$

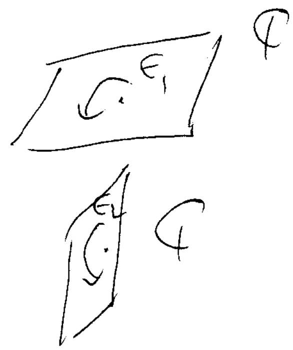
$\varphi(z)$: Higgs field

<u>quantization of Hitchin system</u>

gauge theory on $\mathbb{R}^4 \to \mathbb{R}^4_{\epsilon_1, \epsilon_2 = 0}$

$\downarrow$

2d $N=(2,2)$ theory



(matrix model)

→ Using AGT relation, we consider quantization of integrable system

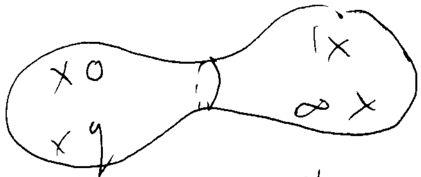
• <u>AGT</u>

$N=2$ gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ $\longleftrightarrow$ 2d Virasoro/W algebra

<u>ex</u> $U(2)$ gauge theory with 4 hypermultiplets with masses m_f ($f=1,\ldots,4$)

$$Z_{Nek} = \sum_{k=0}^{\infty} q^k Z_k(a, m_f, \epsilon_1, \epsilon_2) \qquad q = e^{2\pi i \tau}$$

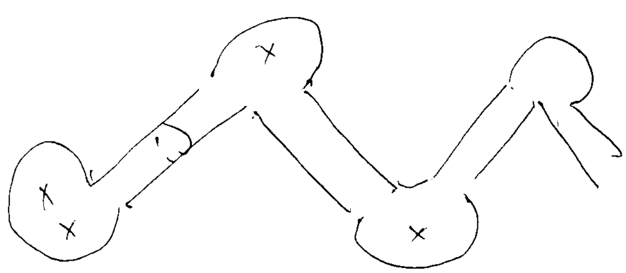
$$\boxed{c = 1 + 6\left(b + \frac{1}{b}\right)^2}$$



$$\mathcal{B} = \left\langle \prod_{i=1}^{4} V_{\alpha_i^{ext}}(z_i) \right\rangle = \sum q^k \mathcal{B}_k(\alpha_i^{ext}, \alpha^{int}, b)$$

$$Z_{Nek} = \mathcal{B} \times Z_{U(1)} \quad \text{with} \quad \frac{\epsilon_1}{\sqrt{\epsilon_1 \epsilon_2}} = b, \quad \frac{\epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}} = \frac{1}{b}$$

$$\frac{a \, \cancel{\alpha^{int}}}{\sqrt{\epsilon_1 \epsilon_2}} \simeq \alpha^{int}$$



$\boxed{\text{hyper}}$ — ② — ② — ② —

• Seiberg–Witten curve

$$\left\langle T(z)\prod_{i=1}^{4} V_{\alpha_i^{ext}}(z_i)\right\rangle \xrightarrow[\alpha^{ext},\,\alpha^{int} \gg \frac{1}{b},\, b]{} \frac{\phi_2(z)}{\epsilon_1\epsilon_2}\left\langle \prod_{i=1}^{4} V_{\alpha_i^{ext}}(z_i)\right\rangle$$

• Integrable system via AGT

Insertion of the degenerate field $V_{-\frac{1}{2b}}(z)$

$$\left(b^2 L_{-1}^2 + L_{-2}\right)V_{-\frac{1}{2b}} = 0 \qquad L_{-1}^2 \leftarrow \partial_z^2$$

$$\mathcal{B}^{deg} = \left\langle V_{\frac{1}{2b}}(z)\prod_{i=1}^{4} V_{\alpha_i^{ext}}(z_i)\right\rangle$$

$\epsilon_2 \to 0$.

$q_1 = 0,\ q_2 = q$
$q_3 = 1,\ q_4 = \infty$

$$\left[\epsilon_1^2\frac{\partial^2}{\partial z^2} + \Phi(z)\right]\mathcal{B}^{deg}(z) = 0.$$

$$\Phi(z) = \sum_{i=1}^{3}\left[\frac{\mu_f(\mu_f+\epsilon_1)}{(z-q_f)^2} + \frac{H_f}{z-q_f} + \frac{c_f}{z-q_f}\right]$$

$$H_f = \sum_{f'\neq f}\frac{\mu_f\mu_{f'}}{q_f-q_{f'}}, \qquad c_f \simeq \epsilon_1\epsilon_2\frac{\partial}{\partial \ln q}$$

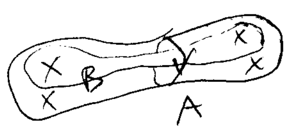
$\mathcal{B}^{deg}(z)$ behaves that

$$\mathcal{B}^{deg}(z) = \exp\left(\frac{\mathcal{F}(\epsilon_1)}{\epsilon_1 \epsilon_2} + \frac{1}{\epsilon_1} \int^z \lambda(z', \epsilon_1) dz' + \mathcal{O}(\epsilon_2) \right)$$

$$\lambda^2 + \lambda' \epsilon_1 + \sum_{f=1}^{3} \frac{\mu_f(\mu_f + \epsilon)}{z - q_f} + \frac{H_f}{z - q_f} - \frac{\partial \mathcal{F}}{\partial \ln q} = 0$$

Ω deformed SW relation

In the CFT, we calculate the monodromies of $\mathcal{B}^{deg}(z)$.



$$\mathcal{B}^{deg}_{(a,z)} \xrightarrow[A]{} \mathcal{B}^{deg}_{(a,z)} e^{2\pi i \frac{a}{\epsilon_1}}$$

$$\xrightarrow[B]{} \mathcal{B}^{deg}(a + \epsilon_2, z)$$

In the limit $\epsilon_2 \to 0$

$$\oint_A \lambda(\epsilon_1) = 2\pi i a, \quad \oint_B \lambda(\epsilon_1) = \frac{\partial \mathcal{F}(\epsilon_1)}{\partial a}$$

$$\oint_A \lambda_{SW}(\epsilon_1, u) = a(\epsilon_1, u)$$

$$\oint_B \lambda_{SW}(\epsilon_1, u) = \frac{\partial \mathcal{F}}{\partial a}(\epsilon_1, u)$$

$$\left[\epsilon_1^2 \frac{\partial^2}{\partial z^2} + \phi_{SW}(z) \right] \Psi(z) = 0.$$

change $\frac{\partial \mathcal{F}}{\partial \ln q}$ to u in above diff. eqn.

$$u = \langle \mathrm{Tr}\,\varphi^2 \rangle \xrightarrow[\text{class}]{} 2a^2$$

<u>matrix model</u> [Dijkgraaf - Vafa, 2009]

The matrix model corresponding to the conformal block on $C_{0,4}$

$$Z = \int \prod_{I=1}^{N} dx_I \prod_{I<J} (x_I - x_J)^{2\beta} \exp\left(\frac{b}{g_s} \sum_{J=1}^{N} W(x_J)\right)$$

$$\beta = -b^2$$

$$W(x) = \sum_{i=1}^{3} 2 m_i \log(x - q_i) \quad \text{with} \quad \sum_{i=1}^{4} m_i = b g_s N$$

<u>free field rep.</u>

$$\left\langle \prod_{i=1}^{4} e^{-2m_i \varphi(q_i)} \left(\int dz\, e^{2b\varphi(z)} \right)^N \right\rangle$$

$$\varphi(z)\varphi(0) \sim -\frac{1}{2} \log z$$

$$\epsilon_1 = b g_s, \quad \epsilon_2 = g_s / b$$

$$\frac{\left\langle e^{-\frac{1}{b}\varphi(z)} \cdots \right\rangle}{Z} = e^{-\frac{W(z)}{2bg_s}} \left\langle \prod_{I=1}^{N} (z - x_I) \right\rangle$$

matrix model

conn

$$W'(x_i) + 2\epsilon_1 \sum \frac{1}{x_i - x_j} = 0$$

This implies e.g. $\langle f(x) g(x) \rangle \Rightarrow \langle f(x) \rangle \langle g(x) \rangle$

$$(\epsilon_2 \to 0) \quad \exp\left(\frac{1}{\epsilon_1} \int^{z} \Bigg(\overbrace{-\frac{W'(z')}{2} + \underbrace{\left\langle \sum_{I=1}^{N} \frac{1}{z' - x_I} \right\rangle}_{R(z')}}^{\lambda(z')} \Bigg) dz' \right)$$

loop eq $\langle R(z)^2 \rangle + (\epsilon_1 + \epsilon_2) \langle R(z)' \rangle$

$+ \langle R(z) \rangle w(z') + f(z) = 0$

$\xrightarrow[\epsilon_2 \to 0]{} \quad \lambda^2(z) + \epsilon_1 \lambda(z)' - T(z) = 0.$