17 August 2011

D. Nadler

(joint with D. Ben-Zvi)

0. Overview
1. Character theory
   χ_G 3d TFT
2. Loop character
   χ_{LG} 3d TFT

---

0. $\mathcal{N}=4$ gauge theory in dim 4

Compactify on S^1

$\rightsquigarrow$ diml reduction $\rightsquigarrow \chi_{LG}$

$\rightsquigarrow$ "S^1-invariant" diml reduction $\rightsquigarrow \chi_G$

$G \supset B$ Borel $\qquad$ W = Weyl group

$$\begin{tikzcd} G \supset B \\ H = B/U \arrow[u, hook] \end{tikzcd}$$

$H = B/U$ max torus

Features

1) parameter $\lambda \in H^\vee / W$

2) S-duality

$$G \longleftrightarrow G^\vee$$

already mathematically established

Goals ~~for today~~

Calculate X_G, X_{LG}

Today: $X_G(S^1)$, $X_{LG}(S^1)$

surface ops.

Find S dual consequences

Starting point

1) $X_{LG}(pt)$

"Tamely ramified Geometric Langlands, dimensionally reduced"

"2-category of $\overset{\wedge}{\text{'highest weight'}}$ LG-module categories"

Concretely, $\mathcal{H}_G^{aff} = D\text{-mod}(I \backslash LG / I)$ ~~$I = \ker(L_+G$~~

$X_G(pt) = \mathcal{H}_G^{aff}$ - modules

Alg. $LG = G((t))$ "matrices with Laurent series entries"

I = Iwahori $\subset G[[t]] \subset LG$

"matrices with power series entries with const. term in B"

Top $LG = Maps(S^1, G)$

$I = \{ f \in Maps(D, G) / f(0) \in B \}$.

LG/I = affine flag manifold

$I \circlearrowright LG/I$ = Schubert geometry

D-modules: X = space

D-mod(X) = A-branes in $T^*(X)$

$\mathcal{H}^{aff}_{G}$ = monoidal category of A-branes in $T^*(I \backslash LG/I)$

S-duality

Frenkel-Gaitsgory

Bezrukavnikov

~~2~~. $X_G(pt) = \mathcal{H}_G$-module categori

1.
$$\mathcal{H}_G = D\text{-mod}(B\backslash G/B)$$
$$= A\text{-brane}(T^*(B\backslash G/B))$$

"S^1-invariant part of $X_{LG}(pt)$"

<u>Goal</u> Calculate $X_G(S^1)$.

$X_G(pt) = \mathcal{H}_G$-module

<u>Expect</u>: $X_G(S^1)$ = "characters of $\mathcal{H}_G$-modules"

char : $X_G(pt) \longrightarrow X_G(S^1)$

⇝ Springer theory, character sheaves.

<u>What is Springer theory?</u> ← Grothendieck-Springer resolution

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "e"] & \tilde{\mathfrak{g}} \arrow[l] \arrow[d] \\ \mathfrak{h}/W & \mathfrak{h} \arrow[l] \end{tikzcd}$$

$\tilde{\mathfrak{g}} = \{X \in \mathfrak{g},\ \mathfrak{b} \subset \mathfrak{g}\ \text{Borel},\ X \in \mathfrak{b}\}$

$\mathfrak{h}/W$: unordered eigenvalues

$\mathfrak{h}$: ordered eigenvalues

$$\begin{tikzcd}
\text{nilp. cone} = \mathcal{N} \arrow[d, hook] & \tilde{\mathcal{N}} = T^*(G/B) \arrow[l] \arrow[d, hook] \\
\mathfrak{g} \arrow[d] & \tilde{\mathfrak{g}} \arrow[l] \arrow[d] \\
\mathfrak{h}/W & \mathfrak{h} \arrow[l]
\end{tikzcd}$$

Note: $\mathfrak{g}^{reg} \xleftarrow{\mu} \tilde{\mathfrak{g}}^{reg}$ is a W-covering

Springer theory

symmetries of H^* of fibers of μ

$R\mu_* \mathbb{C}_{\tilde{\mathfrak{g}}} = S_{\mathfrak{g}}$ assigns H^* of inverse images of μ

sheaf on $\mathfrak{g}$ (A-brane of $T^*\mathfrak{g}$)

Thm $\mathrm{End}(S_{\mathfrak{g}}) = \mathbb{C}[W]$

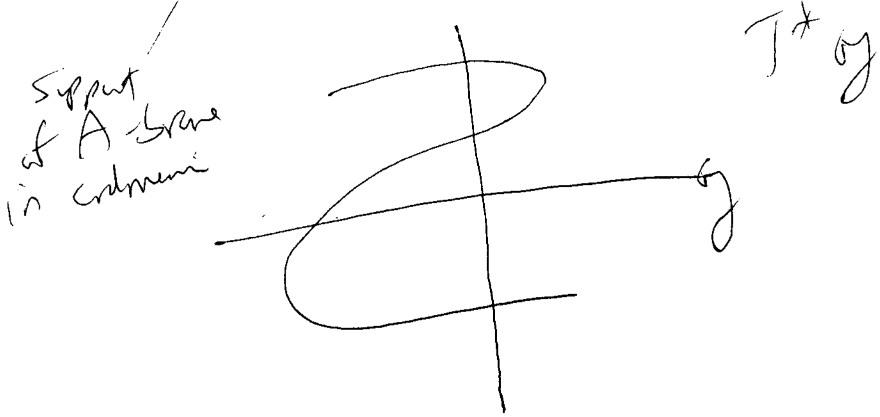
What are character sheaves?

Two special properties of $S_{\mathfrak{g}}$

1) adjoint-equivariant

2) Singular support of $S_{\mathfrak{g}}$ is nilpotent

$\mathfrak{g}^* \supset \mathcal{N}^*$



Def Ch_G = char. sheaves on G

are sheaves $\mathcal{F}$ s.t.

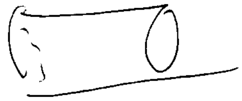
1) $\mathcal{F}$ is adjoint equivariant

2) $ss(\mathcal{F})$ is nilpotent

Thm (Ben-Zvi, Nadler) $\mathcal{X}_G(S^1) = \mathrm{Ch}_G$.

"character sheaves are the characters of $\mathcal{H}_G$-modules"

S_G = group-theoretic Springer sheaf = char(regular $\mathcal{H}_G$-module)



chir: $\mathcal{X}_G(pt) \to \mathcal{X}_G(S^1)$

---

2. $\mathcal{X}_{LG}(pt) = \mathcal{H}^{aff}_G$-modules

$$\mathcal{H}^{aff}_G = D\text{-mod}(I \backslash LG / I)$$
$$= A\text{-branes}(T^*(I \backslash LG / I))$$

⇝ Expect Geometric Langlands on $S^1 \times S^1$

$$\mathcal{X}_{LG}(S^1) = D\text{-mod}(Bun_G(E)) \quad ?$$
$$= A\text{-brane}(Hitch_G(E))$$

↗ TFT

↑ CFT

Def $Ch_{G,E}$ = elliptic char-sheaves

sheaves on $Bun_G(E)$ = A-brane on $Hitch_G(E)$

with nilpotent singular support

= supported on 0-fiber of Hitchin Hamiltonian

Thm in progress (Ben-Zvi, Nadler)

$$\chi_{{}^LG}(S^1) = Ch_{G,E}$$

Cor S-duality for Geometric Langlands on E.
(topological)

Def Ell. Springer sheaf $S_{G,E}$ = Char(regular $\mathcal{H}^{aff}_G$-module)

Facts $S_{G,E}$ is supp. on

$$Bun_G^{ss}(E) \subset Bun_G(E)$$

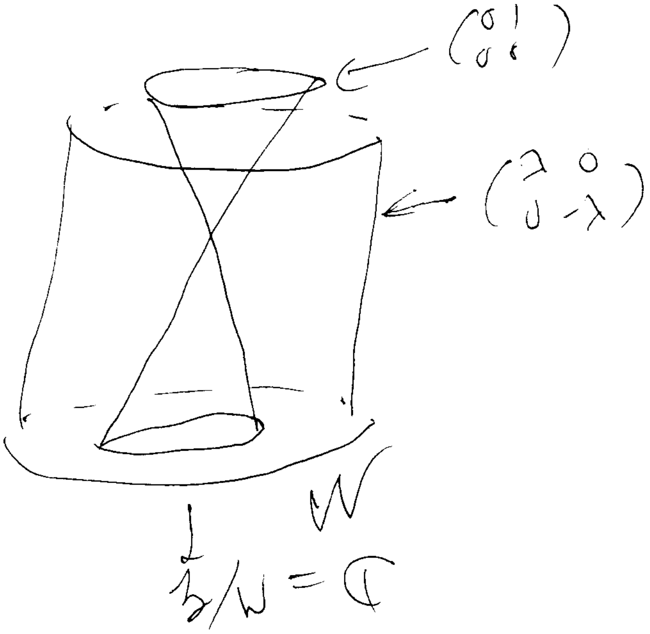
$\uparrow$

local geometry $\rightsquigarrow G$

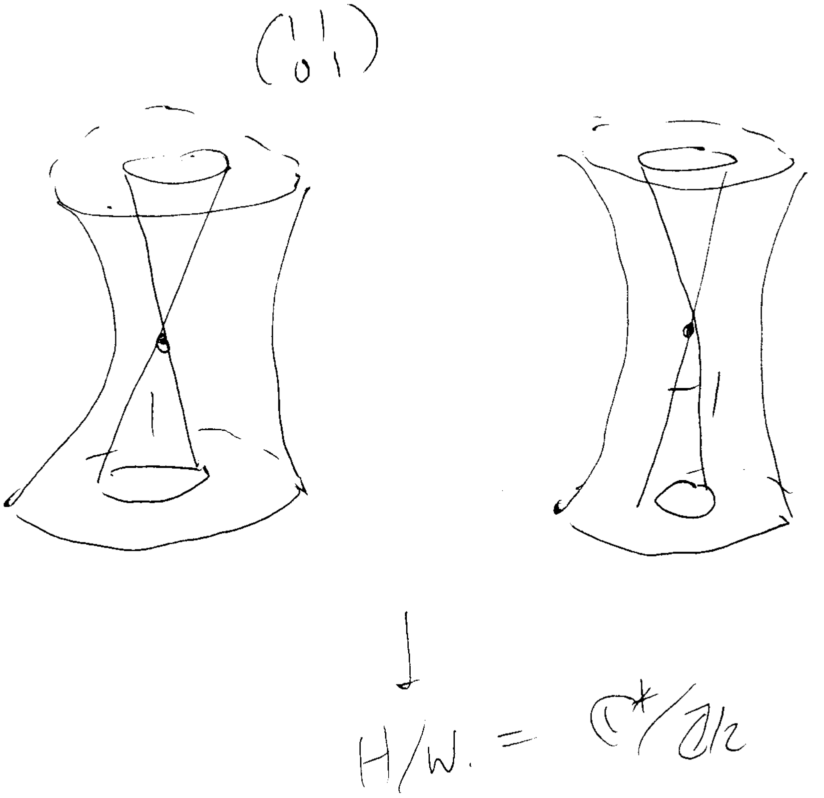
fibers of $S_{G,E}$ look like H^* of affine Springer fibers.

Thm in progress $\mathrm{End}(S_{G,E}) = \mathbb{C}[W_{aff}]$.

$\mathfrak{g} = \mathfrak{sl}_2$

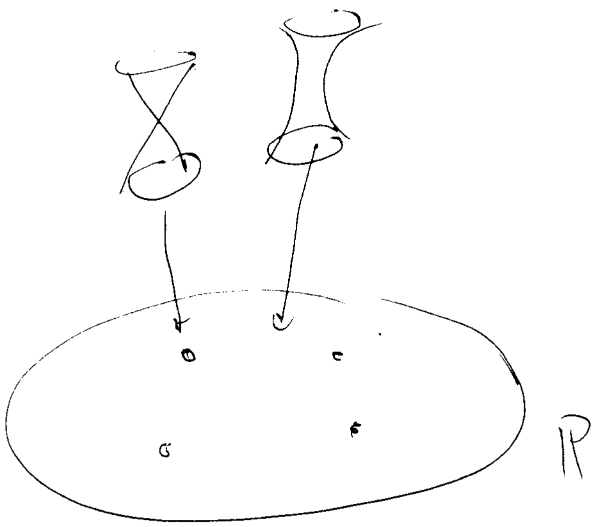


$G = SL_2$



Elliptic version

Elliptic Version



A) $\mathcal{H}^{aff}_G \xrightarrow[\sim]{Bez.}$ B) B-branes (Steinberg variety of $G^\vee$)

monoidal category

Lurie: proves Cobordism hypothesis:

Z TFT $\longleftrightarrow$ $Z(pt)$ suff. dualizable

Ben-Zvi – Nadler, in progress: $\mathcal{H}^{aff}_G$-modules is a sufficiently dualizable 2-category

$\rightsquigarrow X^A = X_{Z_G}$ with a 3d TFT

$\rightsquigarrow X^B$ exists

Today we calculate $\chi^A(S^1) = Ch_{G,E}$ — char. sheaves on $\mathrm{Bun}_G(E)$

Thm (BZ - Francis - N)

$$\chi^B(S^1) = \mathcal{O}\text{-mod}(\mathrm{Loc}_{G^\vee}(E))$$ — char. condition