Spectral Networks

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w/ Gaiotto, Moore, in progress

Introduce a new structure:

- $\mathfrak{g} = A_{K-1}$ Lie alg
- smooth compact complex curve C
- tuple $(\varphi_2, \ldots, \varphi_K)$ φ_i meromorphic section of $K_C^{\otimes i}$
- $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$

$\Downarrow$

Spectral network $\leftarrow$ $\mathcal{N}=2$ theory in $d=4$ $S[C, \mathfrak{g}]$ and a point of its Coulomb branch

6d $(2,0)$ theory on C

- Integers $\Omega(\gamma)$
  = 4d BPS degeneracies
  = generalized DT invts for a local CY 3-fold
  (counting special Lagrangian submanifold)
  [Diaconescu, Donagi, Pantev]

- cluster coordinate system
  on space of flat G-connections on C
  (w/ singularities e poles of φ_i)
  = vevs of line operators on $\mathbb{R}^3 \times S^1$

- framed 2d-4d wall-crossing

Spectral trajectories

Consider spectral cover

$$\begin{array}{l} \Sigma = \{ \lambda^k - \sum_{i=2}^{k} \lambda^{k-1} \varphi_i = 0 \} \subset T^*C \\ \big\downarrow \quad K:1 \quad \text{branched cover} \\ C \end{array}$$

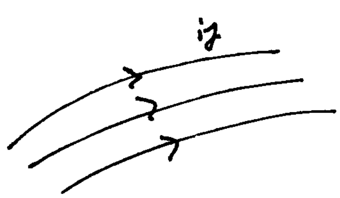
Assume (φ_i) generic $\Rightarrow$ $\begin{array}{c}\Sigma\\ \downarrow \\ C\end{array}$ only has simple branch points

Call the roots (sheets) $\lambda_1, \ldots, \lambda_k$ (locally on C)
Holomorphic 1-forms

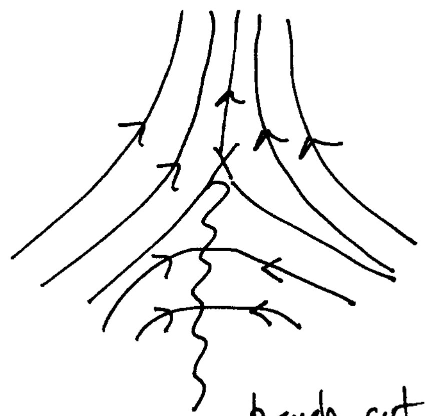
An i-j trajectory is an oriented path on C

along which $e^{i\vartheta}(\lambda_i - \lambda_j)$ is real and positive.

For fixed i,j these give a (local) foliation of C

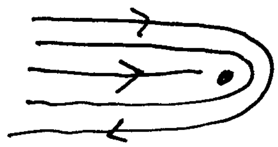


It is singular where $\lambda_i = \lambda_j$ (ij branch point)

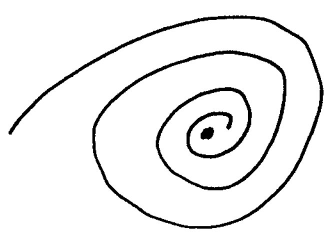


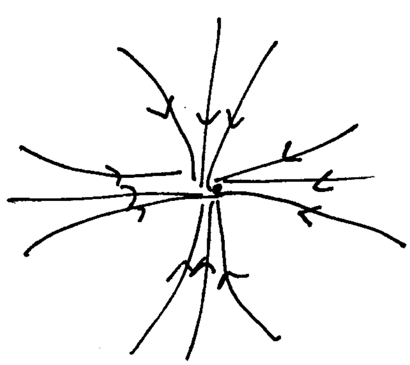
branch cut
trivialize

and singular at the poles of $\lambda_i - \lambda_j$



Higgs field
w/ nilpotent residue



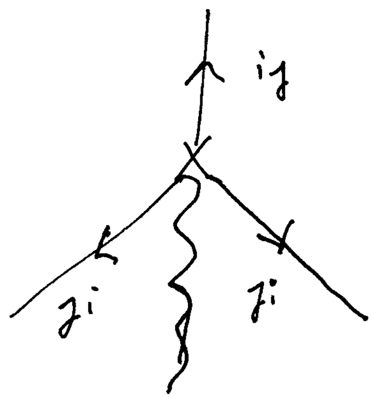


i-j trajectories
ij string supersymmetric

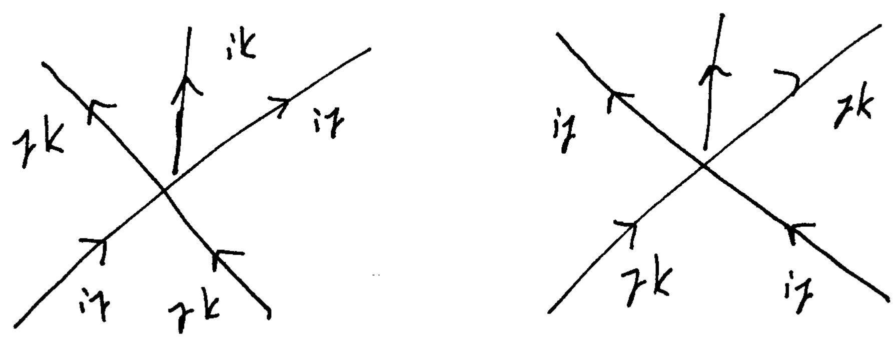
Spectral network

Network of spectral trajectories on C,
built up as follows

Begin at the branch points



Evolve trajectory for ∞ time
If they cross they can spawn a new one



<u>Why</u> (for physics)?

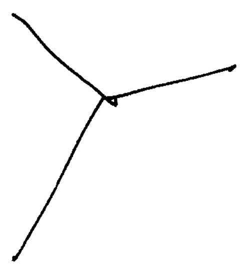
In $S[C, \mathfrak{g}]$ have a <u>surface defect</u>

have <u>surface defect</u> S_z corresponding to point $z \in C$

interfaces between S_z $S_{z'}$ $\leftrightarrow$ paths P from z to z'

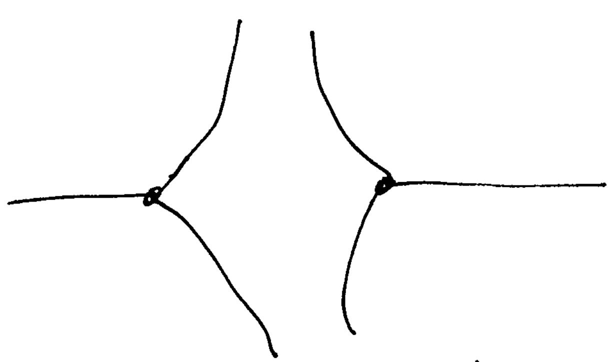
Walls of marginal stability for framed 2d-4d BPS states

<u>Examples</u> $\mathfrak{g} = A_1$



phase rotates picture

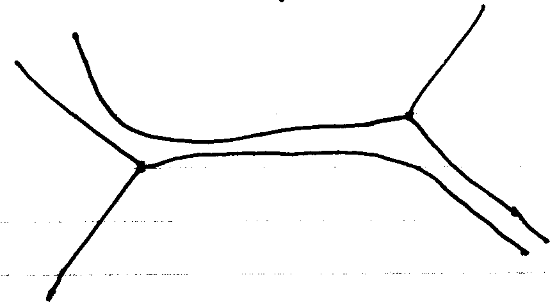
2 branch points



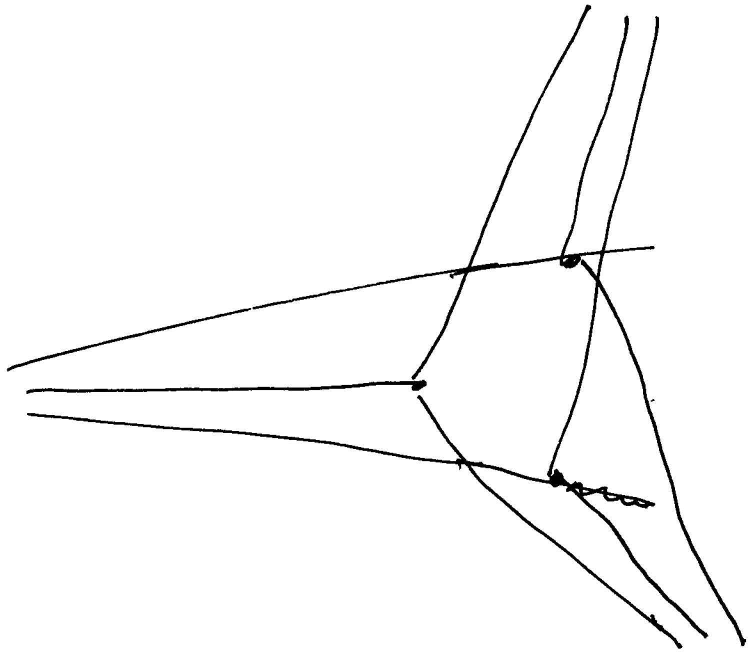
vary phase

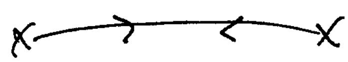
1 BPS state

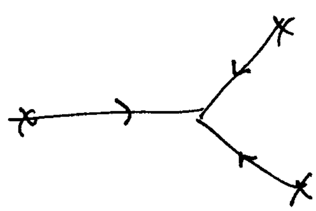
1-disc jump



$k=3$





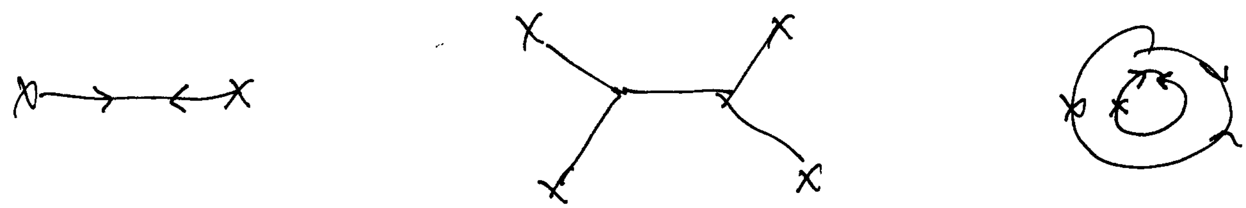


$$\left.\begin{array}{l}\text{Klemm}\\ \text{Lerche}\\ \text{Mayr}\\ \text{Vafa}\\ \text{Warner}\end{array}\right\}\ \text{for } A_1 .$$

BPS degeneracies / DT invariants

At some special values of θ, special network *jumps* discontinuously

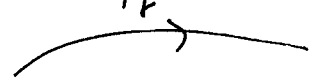
At these θ, $\exists$ collisions - subnetworks where trajectories meet head on.



Whenever we have a subnetwork, it can be canonically *lifted* to a class $\gamma \in H_1(\Sigma, \mathbb{Z})$

IR charge lattice

Lift



lifts to a trajectory on sheet i
and a traj on sheet j
(oppositely oriented)

For any $\gamma \in H_1(\Sigma, \mathbb{Z})$
define $\Omega(\gamma) \in \mathbb{Z}$ to be the # of such

collisions w/ lift γ counted with sign $(-1)^{\# \text{loops}}$

Conj

(1) $\Omega(\gamma)$ are BPS class in $S[\mathfrak{g}, C]$

(2) $\Omega(\gamma)$ depend on $(\varphi_2, \ldots, \varphi_k)$ in a way governed by WCF of Kontsevich-Soibelman

(3) $\Omega(\gamma)$ are generalized DT invariants attached to Fukaya category of local CY

(1), (2) thm for A_1 KLMVW
Bridgeland-Smith in progress

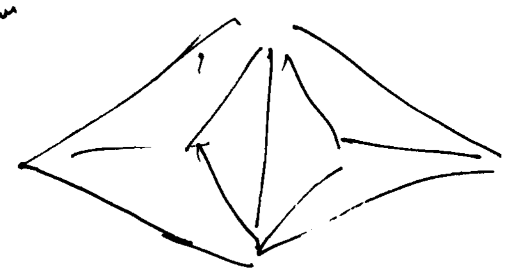
Walls

$$Z_\gamma = \oint_\gamma \lambda$$

walls are where $\frac{Z_\gamma}{Z_{\gamma'}} \in \mathbb{R}_+$ $\quad \Omega(\gamma) \neq 0$, $\Omega(\gamma') \neq 0$

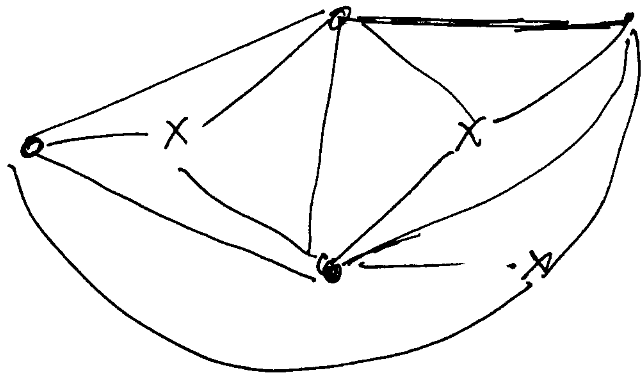
real codim 1 in Hitchin base

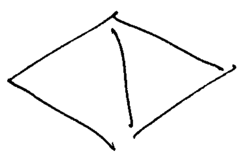
Shear coords
Triangulation



A_1 Edges $\leftrightarrow$ Nodes of quiver

A_1 2nd order poles





rotations $\leftrightarrow$ mutations

disc pump $\Leftrightarrow$ cluster trans

Lines related to WKB
$A_1 \to$ Shrödinger
Stokes wall