(More on) the String Theory of the Ω Deformation

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Outline

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Motivation

- The Gauge–Bethe correspondence – in its simplest manifestation – is the equivalence of the ground states of a supersymmetric gauge theory and the spectrum in a sector of a spin chain.
- There are years of experience in the study of both sides of the correspondence.
- Main interest: We can translate problems from one side to the other
- Fresh perspective on existing problems
- New questions
- Valuable tool for both sides: new insights.

The message

- The gauge–Bethe correspondence relates supersymmetric gauge theories to sectors of spin chains
- Spin chains have symmetries that relate different sectors
- Geometric representation theory describes these symmetries as acting on the ground states of the gauge theories
- String theory provides a framework in which these different gauge theories can be treated in a unified way and the spin chain symmetry understood as symmetry enhancement for coincident D-branes.

Today's talk

- Today I will try to understand this correspondence in the context of String Theory.
- First, I will rephrase the correspondence in terms of geometric representation theory
- Then, I will introduce a String Theory (D-brane) construction of the correspondence, including all the mass parameters.
- Main result: we understand the symmetry relating gauge theories in terms of strings
- Main result: we propose a construction for the twisted masses (Ω background) in terms of D branes in a non-trivial bulk.

Outline

The Gauge-Bethe Correspondence

- The gauge–Bethe correspondence relates the Coulomb branch of certain supersymmetric gauge theories to the Bethe Ansatz equations of integrable systems
- More precisely it identifies the twisted effective superpotential with the Yang–Yang function
- The simplest example is the XXX spin chain, whose Bethe Ansatz equations read:

$$\left(\frac{\sigma_i - i/2}{\sigma_i + i/2}\right)^L = \prod_{j\neq i}^{N} \frac{\sigma_i - \sigma_j - i}{\sigma_i - \sigma_j + i}, \qquad i = 1, \dots, N$$

- These are the same equations that one obtains describing the low energy effective action for a two dimensional $\mathcal{N} = (2,2)$ theory with gauge group $U(N)$, L fundamentals, L antifundamentals and an adjoint field with twisted masses $m_Q = m_{\tilde{Q}} = -i/2$, and $m_\Phi = i$.

Representations

- A representation of the algebra su(2) consists in a vector space V and an action of three operators e, f, k satisfying the relations

$$[e, f] = k\,, \qquad [k, e] = 2e\,, \qquad [k, f] = -2f\,.$$

- Given the tensor product of L copies of the fundamental representation V, there is a natural inclusion of the $(L+1)$–dimensional irreducible representation, $V(L) \hookrightarrow V^{\otimes L}$.

Spin chain

- I will consider the simplest example: the XXX spin chain, with periodic boundary conditions.
- System of L spins on a circle. Each spin can be ↑ or ↓. These are generators of the fundamental representation V of $su(2)$.
- The Hamiltonian is invariant under the action of $su(2)$ defined as

$$E = \sum_{m=1}^{L} \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{m-1} \otimes \mathrm{e} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \ .$$

- The spectrum of the chain is organized into representation of $su(2)$:

$$\mathscr{H} = V^{\otimes L} = \bigoplus_{N=0}^{L} V_{L-2N} \, .$$

V_{L-2N}: magnon states. This is **not** the decomposition into irreps.

The Poor Man's Introduction to Geometric Representation

- Each of the terms $H_*[T^*\mathrm{Gr}(N,L),\mathbb{C}]$ is identified with the $(L-2N)$ weight space, which has dimension $\binom{L}{N}$:

$$H_*[T^*\mathrm{Gr}(L)] \simeq V^{\otimes L} = \bigoplus_{N=0}^{L} V_{L-2N} \simeq \bigoplus_{N=0}^{L} H_*[T^*\mathrm{Gr}(N,L)] \,.$$

- The key point of the construction is the definition of the operators e and f which act between the homologies,

$$\mathrm{e}, f : H_*[T^*\mathrm{Gr}(L)] \to H_*[T^*\mathrm{Gr}(L)] \,.$$

In particular, we need f to act between two components of $\mathrm{Gr}(L)$, raising N by 1:

$$f : H_*[T^*\mathrm{Gr}(N,L)] \to H_*[T^*\mathrm{Gr}(N+1,L)] \,.$$

The Poor Man's Introduction to Geometric Representation

- Introduce a correspondence (in the mathematical sense)

$$Z \subset T^*\mathrm{Gr}(N,L) \times T^*\mathrm{Gr}(N+1,L)$$

$$T^*\mathrm{Gr}(N,L) \xleftarrow{\ \pi_1\ } Z \xrightarrow{\ \pi_2\ } T^*\mathrm{Gr}(N+1,L)$$

  where Z is the diagonal part of the cotangent bundle of the product of two Grassmannians:

$$Z = \left\{ (X, U_N, U'_{N+1}) \mid U_i \in \mathrm{Gr}(N_i, L), X \in \mathrm{End}(\mathbb{C}^L), U \subset U', X(\mathbb{C}^L) \subset U, X(U') = 0 \right\} .$$

- Define the Hecke operator f by first acting with the pullback $\pi_1{}^*$, then intersecting with the fundamental class $[Z]$ and finally acting with the pushforward π_{2*}:

$$f \colon H_*[T^*\mathrm{Gr}(N,L)] \to H_*[T^*\mathrm{Gr}(N+1,L)]$$

$$x \mapsto f(x) = \pi_{2*}([Z] \cap \pi_1{}^*(x)) .$$

The Poor Man's Introduction to Geometric Representation

- The spaces $H_*[T^*\mathrm{Gr}(N,L)]$ are eigenspaces for k:

$$k = \bigoplus_{N=0}^{L} (L-2N)\, \mathbb{1}_{H_*[T^*\mathrm{Gr}(N,L)]} \,,$$

- as we wanted, the $L-2N$ weight space is precisely the homology of $T^*\mathrm{Gr}(N,L)$:

$$H_*[T^*\mathrm{Gr}(N,L)] = \{\, x \in H_*[T^*\mathrm{Gr}(L)] \simeq V^{\otimes L} \mid kx = (L-2N)\,x \,\} \simeq V_{L-2N} \,.$$

- summing over all these spaces:

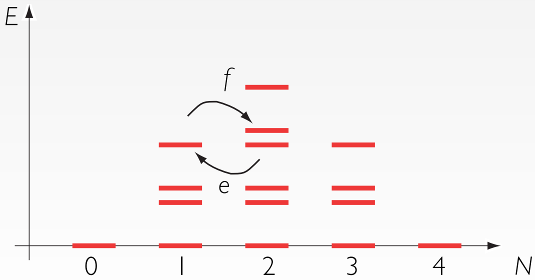
$$H_*[T^*\mathrm{Gr}(L)] \simeq V^{\otimes L} = \bigoplus_{N=0}^{L} V_{L-2N} \simeq \bigoplus_{N=0}^{L} H_*[T^*\mathrm{Gr}(N,L)] \,.$$

The Gauge–Bethe correspondence revisited

- The direct sum of the homologies of the cotangent bundles over the Grassmannians of $\mathbb{C}^L$ corresponds to the ground states of the non-linear sigma models on all the $T^*\mathrm{Gr}(N, L)$ for $N = 0, 1, \dots, L$.
- Via geometric representation theory, this space can be given the structure of the $V^{\otimes L}$ representation of $\mathrm{su}(2)$
- The Hilbert space of the $\mathrm{xxx}_{1/2}$ spin chain has the same structure.
- The homology of the Grassmannian $H_*[T^*\mathrm{Gr}(N, L)]$ is the $(L - 2N)$ weight space V_{L-2N}, which is spanned by the spectrum of the $\mathrm{xxx}_{1/2}$ chain in the N magnon sector
- There is a one-to-one correspondence between the minima of the twisted superpotential of the nlsm and the solutions to the Bethe Ansatz (gauge/Bethe correspondence as an isomorphism of modules).

Another dictionary

physics	mathematics
spectrum of $\mathrm{xxx}_{1/2}$ spin chain	su(2) representation $V^{\otimes L} \simeq H_*[T^*\mathrm{Gr}(L)]$
ground states of the nlsm on $T^*\mathrm{Gr}(N, L)$	cohomology $H^*[T^*\mathrm{Gr}(N, L)]$
spectrum for the N magnon sector	weight space $V_{L-2N} \simeq H_*[T^*\mathrm{Gr}(N, L)]$
ground states of $\mathrm{xxx}_{1/2}$	hw representation $V(L) \simeq H_{top}[T^*\mathrm{Gr}(L)]$
gauge/Bethe correspondence	geometric representation of su_2



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physics	mathematics
spectrum of $\mathrm{xxx}_{1/2}$ spin chain	$su(2)$ representation $V^{\otimes L} \simeq H_*[T^*\mathrm{Gr}(L)]$
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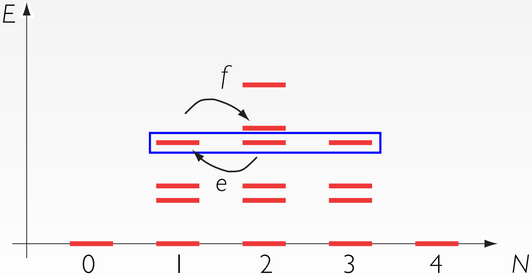

Another dictionary

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spectrum of $xxx_{1/2}$ spin chain	$su(2)$ representation $V^{\otimes L} \simeq H_*[T^*Gr(L)]$
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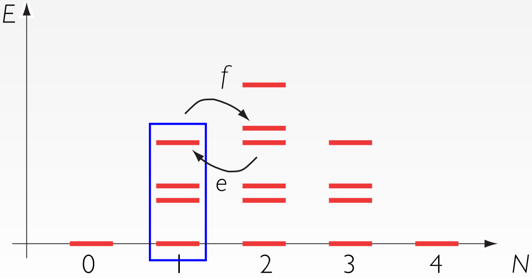
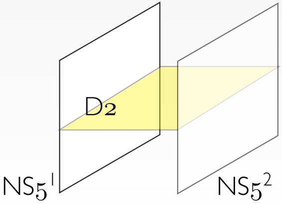


Gauge theory

- $H^*[T^*Gr(N, L)]$: the ground states of a two dimensional $U(N)$ theory with L fundamentals, L antifundamentals and one adjoint
- String Theory: N D2 branes stretching between two NS5 branes with L perpendicular D4 branes

	0	1	2	3	4	5	6	7	8	9
$NS5^{1,2}$	×	×					×	×	×	×
D2	×	×	×							
D4	×	×		×	×	×				



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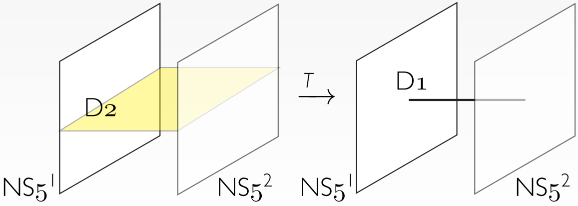
$NS5^{1,2}$	×	×					×	×	×	×
D2	×	×	×							
D4	×	×		×	×	×				



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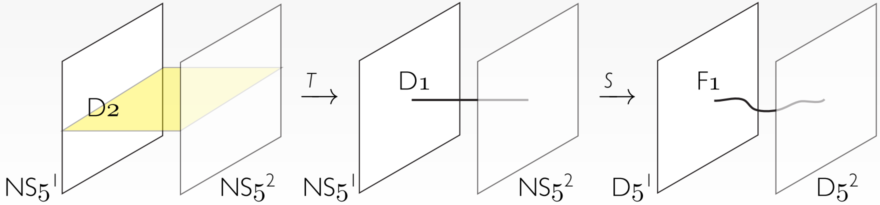


Outline

Twisted masses

- Can we go home? Not yet: the gauge theory has an exotic ingredient: twisted masses for the adjoint and the hypers

$$\int d^4\theta\, X^\dagger e^{\theta^- \bar{\theta}^+ \widetilde{m}_X + \text{h.c.}} X$$

- neither superpotential nor twisted superpotential terms.
- They cannot be viewed as perturbations of the action symmetric under a fixed superalgebra
- They are associated with a deformation of the superalgebra itself.
- A real twisted mass in 2D can be understood by lifting to an $\mathcal{N} = 2$ theory in 3 dimensions.
- The lifted deformation (real mass) gives a central extension of the undeformed theory:

$$\{Q_\alpha, \bar{Q}_\beta\} = \Gamma^{\mu}_{\alpha\beta} P_\mu + \delta_{\alpha\beta} Z\,,$$

where $Z \equiv m^l q_l$ is a linear combination of non-R global symmetries q_l.

Real masses

- This leads to a simple and general construction of the twisted mass deformation in the general case:
- Couple a fictitious (nondynamical) vector multiplet $\{ A_{0,1,2}, \sigma \}$ to the theory, for each Abelian global symmetry.
- Give the real (appropriately normalized) fields σ^I fixed values m^I.
- This preserves $\mathcal{N} = 2$ SUSY in 3 dimensions, but adds the central extension Z.
- This construction gives a simple sufficient condition for real mass deformations of theories with superpotentials which must remain invariant under the $U(1)$ symmetry Z.
- Simplest non-trivial example:

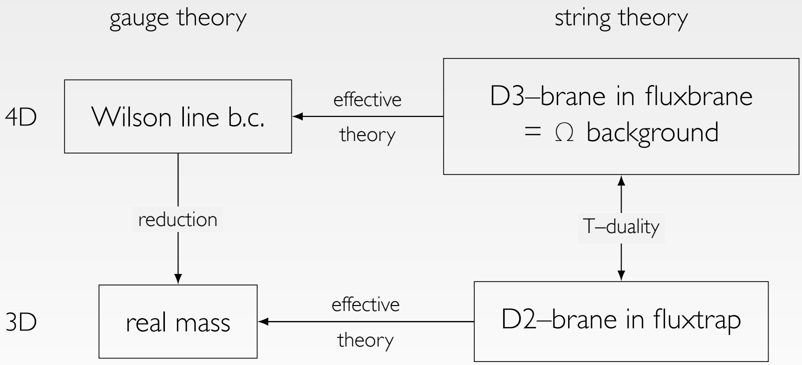
$$W = \tilde{Q}\Phi Q \,,$$

implies $m_Q + m_\Phi + m_{\tilde{Q}} = 0$.

Real mass terms from four dimensions

- For certain 3D theories there's an even simpler description: those that lift to 4D.
- This assumes the global symmetries are exact in 4D as well.
- Then the σ field lifts to $A_{\tilde{8}}$ where $\tilde{8}$ is the fourth dimension we're lifting to and A is again the nondynamical Abelian gauge field.
- The vev of σ in three dimensions corresponds to a compactification with monodromy $\tilde{R} m_I q^I$ - this is the integral of the gauge connection around the fourth dimension.
- Then the fields have generalized momenta $Z = m_I q^I$ and thus Kaluza-Klein masses $Z = \left| m_I q^I \right|$.

Real masses, monodrofolds, and String Theory



Outline

Fluxbrane background

- The fluxbrane background is obtained starting from flat space in ten dimensions and imposing identifications.
- The D2 brane comes from a D3 brane, extended in the directions 0128
- we give a complex structure to the remaining six

$$w_1 = y_1 + i y_2 = \rho_1 e^{i\theta_1}, \quad w_2 = y_3 + i y_4 = \rho_2 e^{i\theta_2}, \quad w_3 = y_5 + i y_6,$$

- we impose the identification

$$\widetilde{x}_8 \simeq \widetilde{x}_8 + 2\pi\widetilde{R}, \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \simeq \begin{pmatrix} e^{2\pi i m\widetilde{R}} & 0 \\ 0 & e^{-2\pi i m\widetilde{R}} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

- In terms of coordinates:

$$\begin{cases} \widetilde{x}_8 \simeq \widetilde{x}_8 + 2\pi\widetilde{R} k_1, \\ \theta_1 \simeq \theta_1 + 2\pi m\widetilde{R} k_1, \\ \theta_2 \simeq \theta_2 - 2\pi m\widetilde{R} k_1, \end{cases} \qquad k_1 \in \mathbb{Z},$$

The Ω background

- Introduce $\varphi_1 = \theta_1 - m\tilde{x}_8$, $\varphi_2 = \theta_2 + m\tilde{x}_8$. The flat space metric:

$$\widetilde{\mathrm{d}s^2} = \mathrm{d}\vec{x}^2_{0\ldots 3} + \mathrm{d}\rho_1^2 + \rho_1^2 \mathrm{d}\varphi_1^2 + \mathrm{d}\rho_2^2 + \rho_2^2 \mathrm{d}\varphi_2^2 + 2m\left(\rho_1^2 \mathrm{d}\varphi_1 - \rho_2^2 \mathrm{d}\varphi_2\right)\mathrm{d}\tilde{x}_8 + \left(1 + m^2\left(\rho_1^2 + \rho_2^2\right)\right)\mathrm{d}\tilde{x}_8^2 + \mathrm{d}x_9^2 ,$$

- or, in rectilinear coordinates

$$x_4 + i x_5 \equiv \rho_1 e^{i\varphi_1}, \qquad x_6 + i x_7 \equiv \rho_2 e^{i\varphi_2},$$

the metric becomes the Ω–deformation of flat space with $\varepsilon_1 = -\varepsilon_2 = m$:

$$\mathrm{d}\vec{x}^2_{0\ldots 3} + \sum_{i=4}^{7}\left(\mathrm{d}x_i + mV^i \mathrm{d}\tilde{x}_8\right)^2 + \mathrm{d}\tilde{x}_8^2 + \mathrm{d}x_9^2 ,$$

where $V^i\partial_i$ is the Killing vector

$$V^i\partial_i = -x^5\partial_{x_4} + x^4\partial_{x_5} + x^7\partial_{x_6} - x^6\partial_{x_7} = \partial_{\varphi_1} - \partial_{\varphi_2}$$

The fluxtrap background

- To get the real mass we need to T–dualize in x_8, in order to get a D2 brane
- The resulting background is a fluxtrap:

$$
\begin{aligned}
ds^2 &= d\vec{x}^2_{0\dots 3} + d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\varphi_1^2 + \rho_2^2 d\varphi_2^2 \\
&\quad + \frac{-m^2 \left(\rho_1^2 d\varphi_1 - \rho_2^2 d\varphi_2\right)^2 + dx_8^2}{1 + m^2\left(\rho_1^2 + \rho_2^2\right)} + dx_9^2 , \\
B &= m \frac{\rho_1^2 d\varphi_1 - \rho_2^2 d\varphi_2}{1 + m^2\left(\rho_1^2 + \rho_2^2\right)} \wedge dx_8 , \\
e^{-\Phi} &= \frac{\sqrt{1 + m^2\left(\rho_1^2 + \rho_2^2\right)}}{g_3^2 \sqrt{\alpha'}} .
\end{aligned}
$$

- The effective theory for a D2 brane in this background acquires a real mass term m.

The NS5 fluxtrap

Even if we can't use linearity, it is still possible to add an NS5 brane in the bulk, impose the identifications and T-dualize:

$$
\begin{aligned}
\mathrm{d}s^2 &= U\left[\mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}\rho_1^2\right] + \mathrm{d}\rho_2^2 + \frac{U\rho_1^2\left[\mathrm{d}\varphi_1^2 + m^2\rho_2^2\left(\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2\right)^2\right] + \rho_2^2\,\mathrm{d}\varphi_2^2}{\Delta^2} \\
&+ \frac{\Lambda^2}{\left(x_3^2+\rho_1^2\right)\Delta^2}\left[\left(x_3^2+\rho_1^2\right)\mathrm{d}x_2 - x_2x_3\,\mathrm{d}x_3 - x_2\rho_1\,\mathrm{d}\rho_1 + \frac{\sqrt{x_3^2+\rho_1^2}\,\mathrm{d}x_8}{\Lambda}\right]^2 , \\
B &= \frac{\Lambda\left[-\left(x_3^2+\rho_1^2\right)\mathrm{d}x_2 + x_2x_3\,\mathrm{d}x_3 + x_2\rho_1\,\mathrm{d}\rho_1\right]\wedge\left[\left(1+m^2\rho_2^2\right)\mathrm{d}\varphi_1 + m^2\rho_2^2\,\mathrm{d}\varphi_2\right]}{m\left(x_3^2-\rho_1^2\right)\Delta^2} \\
&- m\frac{U\rho_1^2\,\mathrm{d}\varphi_1 - \rho_2^2\,\mathrm{d}\varphi_2}{\Delta^2}\wedge\mathrm{d}x_8 , \\
e^{-\Phi} &= \frac{1}{g_3^2\sqrt{\alpha'}}\sqrt{\frac{1+m^2\left(U\rho_1^2+\rho_2^2\right)}{U}} ,
\end{aligned}
$$

where

$$
U = 1 + \frac{N5\,\alpha'}{x_2^2+x_3^2+\rho_1^2} , \qquad \Delta^2 = 1 + m^2\left(U\rho_1^2+\rho_2^2\right) , \qquad \Lambda = \frac{2mx_3\left(x_3^2+\rho_1^2\right)}{\left(x_2^2+x_3^3+\rho_1^2\right)^2\Delta} .
$$

Supersymmetry

- One can understand the supersymmetries of this string theory solution starting from the type IIB picture (D3 brane)
- In the initial flat background (prior to identifications), there are 32 Killing spinors

$$K^{IIB} = \exp[\tfrac{1}{2}\theta_1 \Gamma_{45} + \tfrac{1}{2}\theta_2 \Gamma_{67}]\, \varepsilon_0 ,$$

  where ε_0 is a complex Weyl spinor.
- Introducing φ_1 and φ_2, this becomes

$$K^{IIB} = \exp[\tfrac{1}{2}\varphi_1 \Gamma_{45} + \tfrac{1}{2}\varphi_2 \Gamma_{67}] \exp[\frac{m\widetilde{R}\widetilde{u}}{2}(\Gamma_{45} - \Gamma_{67})]\, \varepsilon_0 .$$

- In the generic case the second exponential must vanish to preserve periodicity. Introduce the orthogonal projectors

$$\Pi^{\text{flux}}_{\pm} = \tfrac{1}{2}\left(\mathbb{1} \pm \Gamma_{4567}\right) ,$$

Supersymmetry

- The fluxtrap background preserves 16 supersymmetries. In the type IIA picture the Killing spinors are:

$$\begin{cases} \varepsilon_L = (\mathbb{1} + \Gamma_{11})\, \Pi_-^{\text{flux}} \exp[\tfrac{1}{2}\varphi_1 \Gamma_{45} + \tfrac{1}{2}\varphi_2 \Gamma_{67}]\, \varepsilon_0 \,, \\ \varepsilon_R = (\mathbb{1} - \Gamma_{11})\, \Gamma_u \Pi_-^{\text{flux}} \exp[\tfrac{1}{2}\varphi_1 \Gamma_{45} + \tfrac{1}{2}\varphi_2 \Gamma_{67}]\, \varepsilon_1 \,, \end{cases}$$

- The NS5-fluxtrap background preserves 8 supersymmetries. In the type IIA picture the Killing spinors are:

$$\begin{cases} \varepsilon_L = (\mathbb{1} + \Gamma_{11})\, \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}} \exp[\tfrac{1}{2}\varphi_1 \Gamma_{45} + \tfrac{1}{2}\varphi_2 \Gamma_{67}]\, \varepsilon_0 \\ \varepsilon_R = (\mathbb{1} - \Gamma_{11})\, \Gamma_u \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \exp[\tfrac{1}{2}\varphi_1 \Gamma_{45} + \tfrac{1}{2}\varphi_2 \Gamma_{67}]\, \varepsilon_1 \end{cases}$$

where ε_0 and ε_1 are constant Majorana spinors,

$$\Pi_\pm^{\text{NS5}} = \frac{1}{2}\left(\mathbb{1} \pm \Gamma_{2345}\right) \,,$$

and

$$\Gamma_u = \frac{m \rho_1 \sqrt{U}}{\Delta} \Gamma_5 - \frac{m \rho_2}{\Delta} \Gamma_7 + \frac{1}{\Delta} \Gamma_8 \,.$$

A tale of many epsilons

- For our problem we only need to get one real (imaginary) twisted mass. Nevertheless, the construction can be easily generalized:
- Wilson loops in another direction (say x_9) will give a complex ε
- Identifications involving other complex planes will give more ε's. In order to preserve supersymmetry:

$$\sum_i \varepsilon_i = 0$$

- Example: 3 real independent ε's:

$$\widetilde{x}_8 \simeq \widetilde{x}_8 + 2\pi \widetilde{R}$$

$$\begin{pmatrix} x_1 + i x_9 \\ x_2 + i x_3 \\ x_4 + i x_5 \\ x_6 + i x_7 \end{pmatrix} \simeq \begin{pmatrix} e^{2\pi i \varepsilon_1} & & & \\ & e^{2\pi i \varepsilon_2} & & \\ & & e^{2\pi i \varepsilon_3} & \\ & & & e^{-2\pi i(\varepsilon_1+\varepsilon_2+\varepsilon_3)} \end{pmatrix} \begin{pmatrix} x_1 + i x_9 \\ x_2 + i x_3 \\ x_4 + i x_5 \\ x_6 + i x_7 \end{pmatrix}$$

Outline

D2 branes in the fluxtrap

- Now we have all the ingredients to write our gauge theory.
- The D2 branes are embedded in the NS-fluxtrap background

direction	0	1	2	3	4	5	6	7	8	9
NS5	×	×					×	×	×	×
fluxtrap	×	×	×	×						×
D2	×	×	×							
D4	×	×		×	×	×				

- From the point of view of the gauge theory on the D2 branes
  - $x_8 + i x_9 = \sigma$ (twisted chiral)
  - $x_6 + i x_7 = \Phi$ (chiral adjoint)
  - the separation of the NS5 in x_3 is the Fayet-Iliopoulos term
  - the separation of the NS5 in x_2 is $1/g^2$

The D2 brane as a BPS object

Consider the static embedding

$$F_{\alpha\beta} = 0, \quad x_0 = \zeta^0, \quad x_1 = \zeta^1, \quad x_2 = \zeta^2, \quad \varphi_1 = \omega \zeta^0, \quad \varphi_2 = -\omega \zeta^0.$$

The bosonic part of the DBI action reads

$$S = -\mu_2 \int d^3 \zeta \sqrt{1 - \frac{1 - \Delta^2 (1 + \Lambda^2 / U)}{m^2} (m^2 - \omega^2)}.$$

The equations of motion can be satisfied in two ways:

- if $\rho_1 = \rho_2 = x_3 = 0$. This is a static D2 brane sitting in the trap.
- if $\omega = m$. This is a rotating D2 brane. A nice feature is that we are not in the linearized approximation but the frequency is fixed only by the twisted mass m and is independent of the amplitude.

Static Embedding

The supersymmetries preserved by the static embedding are those such that

$$\varepsilon_L = \Gamma_{D2}\, \varepsilon_R\,.$$

- Without the NS5 there are 8 preserved supercharges ($\mathcal{N} = 2$ theory in 3d)

$$\begin{cases} \varepsilon_L = \ \Gamma_{1208}\left(\mathbb{1} + \Gamma_{11}\right)\sqcap_-^{\text{flux}} \exp[\tfrac{1}{2}\left(\varphi_1 + \varphi_2\right)\Gamma_{67}]\, \varepsilon_1\,, \\ \varepsilon_R = \left(\mathbb{1} - \Gamma_{11}\right)\Gamma_u \sqcap_-^{\text{flux}} \exp[\tfrac{1}{2}\left(\varphi_1 + \varphi_2\right)\Gamma_{67}]\, \varepsilon_1\,. \end{cases}$$

- With the NS5 there are 4 preserved supercharges ($\mathcal{N} = (2,2)$ theory in 2d)

$$\begin{cases} \varepsilon_L = \left(\mathbb{1} + \Gamma_{11}\right)\sqcap_-^{\text{NS5}} \sqcap_-^{\text{flux}} \Gamma_{1208} \exp[\tfrac{1}{2}\left(\varphi_1 + \varphi_2\right)\Gamma_{67}]\, \varepsilon_2\,, \\ \varepsilon_R = \left(\mathbb{1} - \Gamma_{11}\right)\Gamma_u \sqcap_+^{\text{NS5}} \sqcap_-^{\text{flux}} \exp[\tfrac{1}{2}\left(\varphi_1 + \varphi_2\right)\Gamma_{67}]\, \varepsilon_2\,. \end{cases}$$

Rotating Embedding

The rotating embedding ($\omega = m$) works in the same way and preserves

- 4 real supercharges without the NS5

$$\begin{cases} \varepsilon_L = (\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{flux}} \Gamma_{1208} \exp[\frac{1}{2}(\varphi_1 + \varphi_2) \Gamma_{67}] (\mathbb{1} \mp \Gamma_{08}) \varepsilon_2 , \\ \varepsilon_R = (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_-^{\text{flux}} \exp[\frac{1}{2}(\varphi_1 + \varphi_2) \Gamma_{67}] (\mathbb{1} \mp \Gamma_{08}) \varepsilon_2 . \end{cases}$$

- 2 real supercharges in presence of the NS5

$$\begin{cases} \varepsilon_L = (\mathbb{1} + \Gamma_{11}) \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}} \Gamma_{1208} \exp[\frac{1}{2}(\varphi_1 + \varphi_2) \Gamma_{67}] (\mathbb{1} \mp \Gamma_{08}) \varepsilon_2 , \\ \varepsilon_R = (\mathbb{1} - \Gamma_{11}) \Gamma_u \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \exp[\frac{1}{2}(\varphi_1 + \varphi_2) \Gamma_{67}] (\mathbb{1} \mp \Gamma_{08}) \varepsilon_2 . \end{cases}$$

DBI action for the D2 brane

The last thing that remains to do is to show how the D2 branes in the fluxtrap background acquire a twisted mass.
We simply need to write the Dirac–Born–Infeld action at quadratic order:

$$S = -\mu_2 \int d^3\zeta \;\; e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta})} \left[1 - \frac{1}{2} \bar{\psi} \left((g+B)^{\alpha\beta} \Gamma_\beta D_\alpha + \Delta^{(1)} \right) \psi \right],$$

where

$$D_\alpha = \partial_\alpha X^\mu \left(\nabla_\mu + \frac{1}{8} H_{\mu mn} \Gamma^{mn} \right), \quad \Delta^{(1)} = \frac{1}{2} \Gamma^m \partial_m \Phi - \frac{1}{24} H_{mnp} \Gamma^{mnp}.$$

Then we expand all the terms at their respective leading order in the fields:

$$g_{\mu\nu} \, dX^\mu \, dX^\nu = d\vec{x}^2_{0\dots9} + \mathcal{O}(X^4),$$

$$H_{\mu\nu\rho} \, dX^\mu \wedge dX^\nu \wedge dX^\rho = 2m \left(\rho_1 \, d\rho_1 \wedge d\varphi_1 - \rho_2 \, d\rho_2 \wedge d\varphi_2 \right) \wedge dx_8 + \mathcal{O}(X^5),$$

$$e^{-\Phi} = \frac{1}{g_3^2 \sqrt{\alpha'}} \left(1 + \frac{m^2}{2} \left(\rho_1^2 + \rho_2^2 \right) \right) + \mathcal{O}(X^4).$$

DBI action for the D2 brane

- From the dilaton we get a term

$$m^2 \left(\rho_1^2 + \rho_2^2 \right)$$

- From the B-field we get the mass for the fermions:

$$\frac{m}{2} \bar{\psi} \left(\Gamma_{45} - \Gamma_{67} \right) \Gamma_8 \psi$$

Putting everything together we reproduce the expected form for the real mass term in three dimensions:

$$S = \frac{1}{8 \pi^2 g_3^2 (\alpha')^2} \int \mathrm{d}^3 \zeta \left[\dot{z}_1 \dot{\bar{z}}_1 + \dot{z}_2 \dot{\bar{z}}_2 - m^2 \left(z_1 \bar{z}_1 + z_2 \bar{z}_2 \right) \right. \\ \left. - \bar{\psi} \Gamma_0 \dot{\psi} - i m \bar{\psi} \left(\Pi_-^{z_1} - \Pi_-^{z_2} \right) \Gamma_8 \psi \right],$$

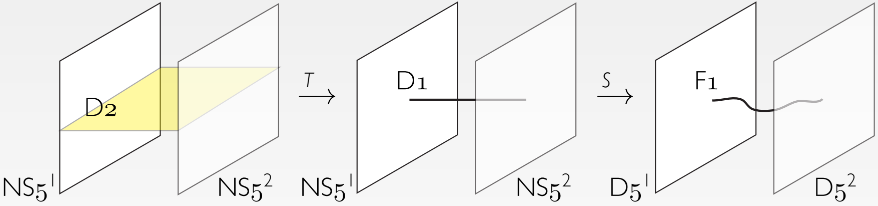
Outline

What have we seen?

- The gauge–Bethe correspondence is special because it relates a single spin chain to different gauge theories
- This suggests the existence of symmetries relating different gauge theories. We want to understand them in terms of String Theory.
- We propose a new approach in which the twisted masses result from a non-trivial bulk theory
- The bulk provides a natural embedding of the Ω background in string theory (as an exact CFT)
- It is easy to study the possible deformations (more than one ε, reality conditions, …)
- It is easy to understand the supersymmetry properties, in terms of BPS objects in the bulk

To do

- Now we have a complete setup in which to study the gauge–Bethe correspondence, choosing the most appropriate duality frame



- We can generalize the action of the $su(2)$ beyond the vacua. At strong coupling the full spectrum is organized under $su(2)$ and we know for sure that there are BPS states that are protected and do not decouple.

To do

- Beyond the correspondence: one can study other gauge theories describing different kinds of branes in the same background
- Natural example: start with a Euclidean brane extended in $4, 5, 6, 7, \tilde{8}$ and reduce (T-dualize) on $\tilde{8}$

direction	0	1	2	3	4	5	6	7	8	9
fluxtrap	×	×	×	×						×
E4					×	×	×	×		

- The E_4 extended in $4, 5, 6, 7$ embedded in the fluxbrane background describe the Ω deformation of $\mathcal{N} = 4$ super-Yang–Mills
- The same bulk that produced the twisted mass, now generates a position-dependent gauge coupling

$$g_{4\Omega}^2 \propto \frac{1}{\sqrt{1 + \varepsilon^2 \left(\rho_1^2 + \rho_2^2\right)}}$$

and the instantons are localized in the trap set by the dilaton in the bulk.

Thank you

for your attention