Higher rank AGT and non-local operators

Filippo Passerini
Humboldt University, Berlin

KITP

August 24, 2011

Introduction

- Spring 2009: AGT duality, an explicit connection between 4D $\mathcal{N} = 2$ gauge theories and 2D CFT [Alday, Gaiotto, Tachikawa]

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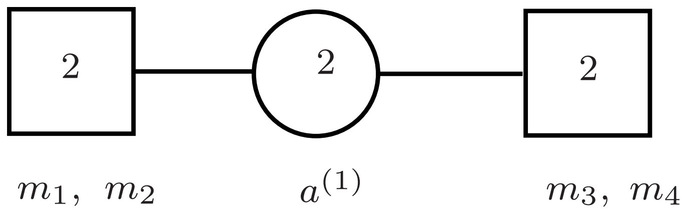

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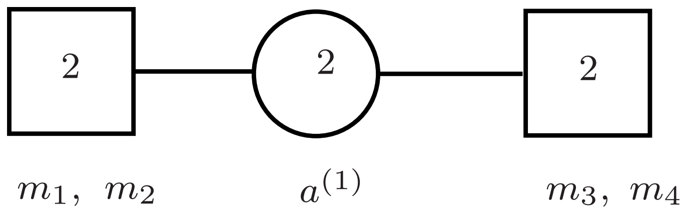

AGT:

$$Z_{S^4} = \langle V_{\beta_1} V_{\beta_2} V_{\beta_3} V_{\beta_4} \rangle_{\text{Liouville}}$$

The partition function is equivalent to a 4-point correlation function in Liouville CFT

Gauge Theory:

The partition function of $\mathcal{N}=2$ SCYM on S^4 results [Pestun]

$$Z_{S^4} = \int da^{(1)}\; Z_{\text{cl}}\; Z_{\text{1-loop}}\; Z_{\text{inst}}$$

the partition function localizes to a matrix integral in $a^{(1)}$, the VEV of the scalar in the vector multiplet that parameterize the Coulomb branch

- where
  - $Z_{\text{cl}} = e^{-\frac{4\pi^2 r^2}{g^2}\text{Tr}\,(a^{(1)})^2}$ is the classical contribution
  - $Z_{\text{1-loop}} = Z_{\text{1-loop}}\,(a, m)$ is the perturbative contribution
  - $Z_{\text{inst}} = |Z_{\text{Nek}}(q, 1/r, 1/r, a, m)|^2$ is the instanton contribution ($q = e^{2\pi i \tau}$)

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- Half-BPS Wilson loop $\langle W\rangle = \int da^{(1)}\ Z_{\text{cl}}\ Z_{\text{1-loop}}\ Z_{\text{inst}}\ \text{Tr}_R e^{2\pi r i a^{(1)}}$

Liouville CFT:

- Primaries: $V_\alpha(z,\bar{z}) = e^{2\alpha\phi(z,\bar{z})}$ characterized by one imaginary parameter
- Conformal Bootstrap Approach: any correlator can be expressed in terms of 3-point functions and conformal blocks

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For our basic example

$$\langle V_{\beta_1}(\infty)V_{\beta_2}(1)V_{\beta_3}(z)V_{\beta_4}(0)\rangle$$
$$= \int d\alpha^{(1)}\langle\beta_1|V_{\beta_2}|\alpha^{(1)}\rangle\langle\alpha^{(1)}|V_{\beta_3}|\beta_4\rangle\mathcal{F}_{\alpha,\beta}(z)\bar{\mathcal{F}}_{\alpha,\beta}(\bar{z})\,|z|^{2(\Delta_\alpha-\Delta_{\beta_3}-\Delta_{\beta_4})}$$

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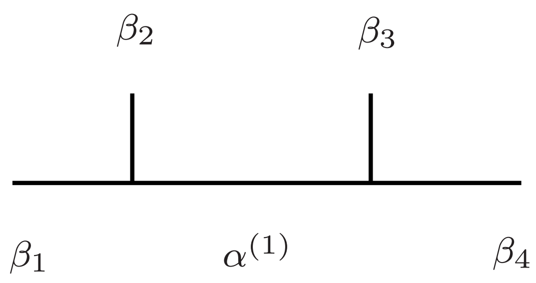
and this specific decomposition is schematized by the diagram



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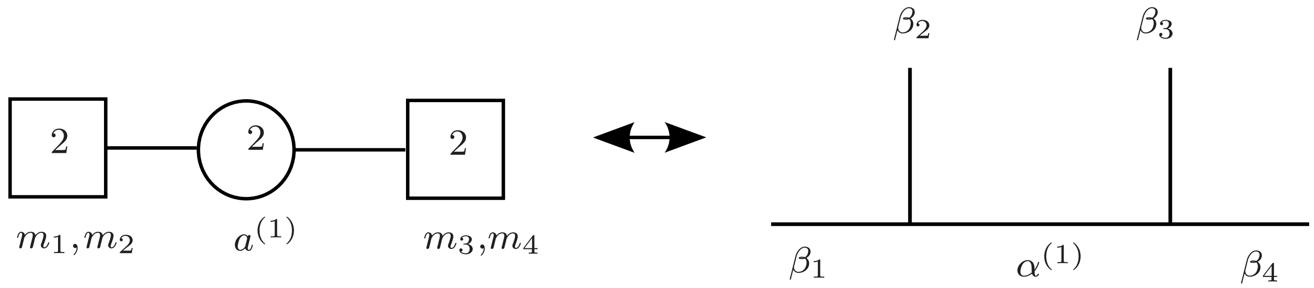

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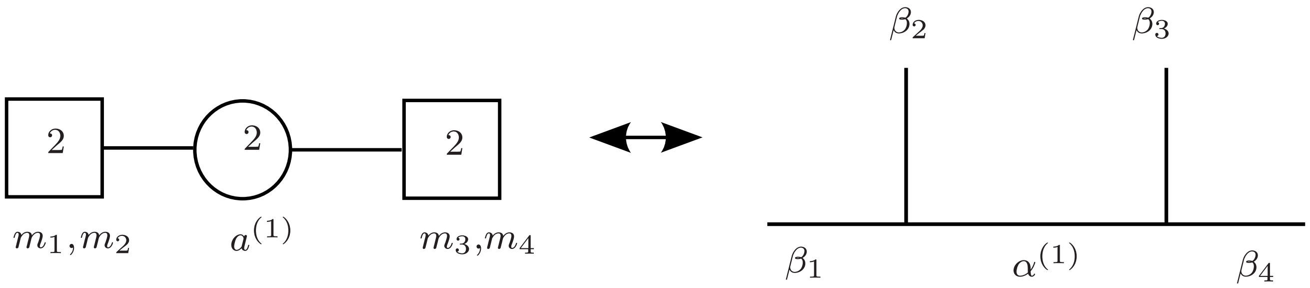

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- Dictionary:
  - Coulomb branch parameters $a^{(1)}$ $\Leftrightarrow$ primaries in the internal channels $\alpha^{(1)}$
  - Hypermultiplets masses m $\Leftrightarrow$ primaries in the external channels β
  - $Z_{\text{1-loop}}$ $\Leftrightarrow$ product of 3-point functions (DOZZ formulas)
  - $Z_{\rm inst}$ $\Leftrightarrow$ conformal block $\mathcal{F}_{\alpha,\beta}$
  - $q = e^{2\pi i \tau}$ $\Leftrightarrow$ z

M-theory picture: 4D $\mathcal{N} = 2$ gauge theories describe the low energy dynamics of a stack of M5-branes compactified on a punctured Riemann surface [Gaiotto][Witten]

- punctures $\Leftrightarrow$ flavor symmetry of the gauge theory
- thin tubes connecting the pair of pants $\Leftrightarrow$ weakly coupled gauge groups

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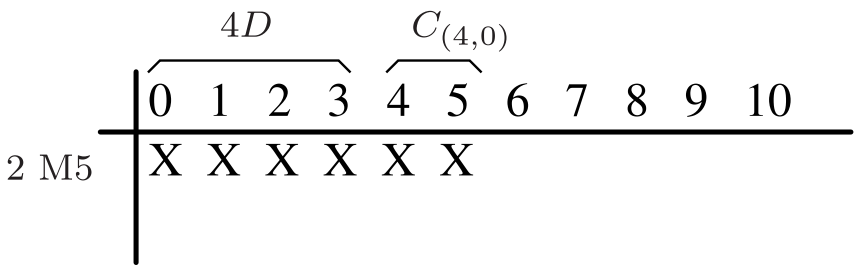

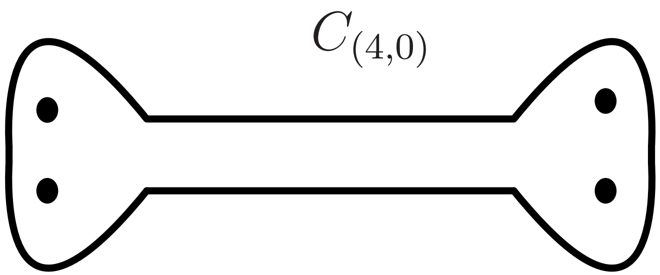

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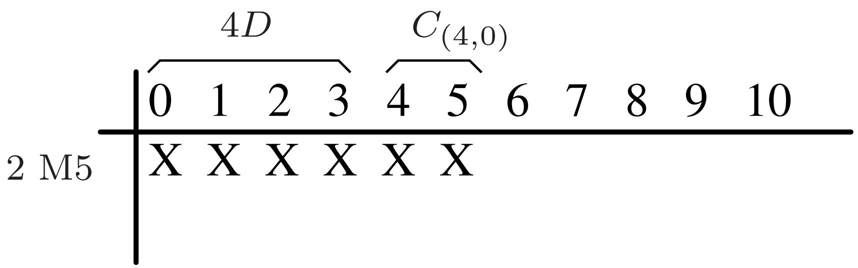

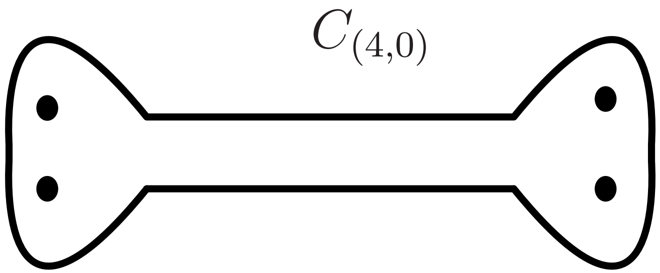

- 4 punctures $\Longleftrightarrow$ $(SU(2))^4$ flavor group (one puncture for any $SU(2)$)
- 1 tube $\Longleftrightarrow$ $SU(2)$ gauge group

Liouville theory is defined on the Gaiotto curve $C_{(4,0)}$. To any puncture is associated an insertion in the correlation function.

Outline

- introduction
  - $SU(2)$ AGT
  - higher rank AGT
- AGT for quiver tails
- adding non-local operators to the AGT proposal
  - surface operators from 2D CFT
- Conclusion

Higher Rank AGT: The partition function of conformal $\mathcal{N} = 2$ theories on S^4 with gauge group $SU(N)^k$ is equivalent to a correlation function in A_{N-1} Toda CFT [Wyllard][Kanno, Matsuo, Shiba]

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A_{N-1} Toda theory:

It is described by a 2D Lagrangian $S_{A_{N-1}} = S_{A_{N-1}}(\phi, b)$

- $\phi = \sum_{k=1}^{N-1} \varphi_k e_k$ where e_k is a simple root of A_{N-1} algebra
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The $\mathcal{W}_N$ primaries are given by $V_\alpha \sim e^{\langle \alpha, \phi \rangle}$

- $\langle \cdot, \cdot \rangle$ is the scalar product in the root space
- α is a vector in the root space of the A_{N-1} algebra, $\alpha = Q\rho + \gamma$
- the conformal dimension is given by $\Delta(\alpha) = \frac{1}{2}\langle \alpha, 2Q\rho - \alpha \rangle$

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V_α depend on $N-1$ parameters γ

Semi-degenerate representations of Toda CFT

- in the orthonormal basis $\alpha = Q(-\frac{N+1}{2}, \dots, \frac{N+1}{2}) + (\gamma_1, \dots, \gamma_N)$
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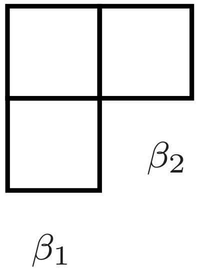
- Semi-degenerate states have descendants that are null vectors

- Physical semi-degenerate states have all the null states at level 1 [Kanno, Matsuo, Shiba, Tachikawa]
  - γ has real components
  - α is invariant under a subgroup of the permutation group S_N
  - subgroups of S_N are described by partition of N, i.e.Young diagram with N boxes

- For instance, the degenerate state of A_2 Toda



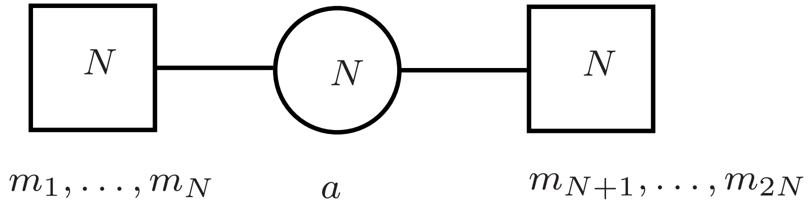
- [2,1] diagram, is invariant under $S_2 \times S_1 \subset S_3$

- The momentum is given by

$$\alpha = Q\rho + (\beta_2,\ \beta_1 + \delta_{2,1},\ \beta_1 + \delta_{2,2})$$
$$= (\beta_2 + Q/2,\ \beta_1 - Q/2,\ \beta_1 - Q/2)$$

- $\delta_{n,j} = (2j - n - 1)Q/2$

For example, $\mathcal{N}=2$ $SU(N)$ SCYM:



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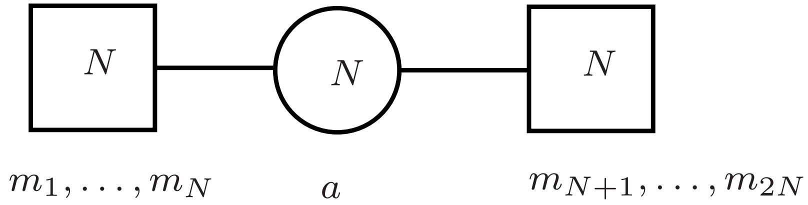

- The flavor group is $(U(1)\times SU(N))^2$. Two types of punctures
  - simple puncture associated to $U(1)$ flavor group (1 mass parameter)
  - full puncture associated to $SU(N)$ flavor group ($N-1$ mass parameters)

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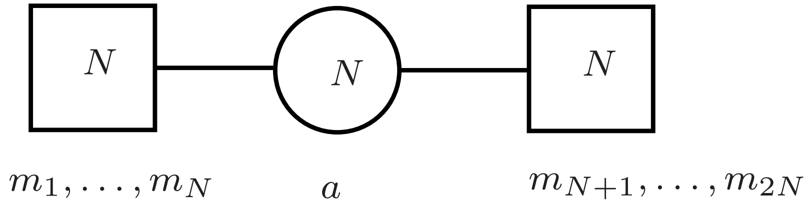

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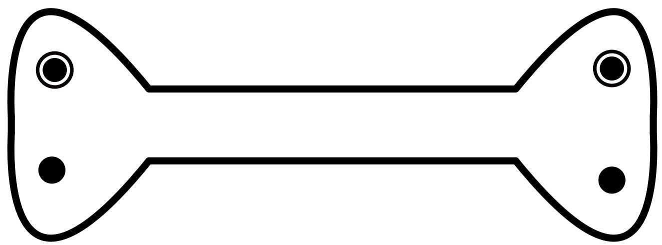
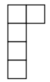
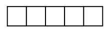


A_{N-1} Toda CFT

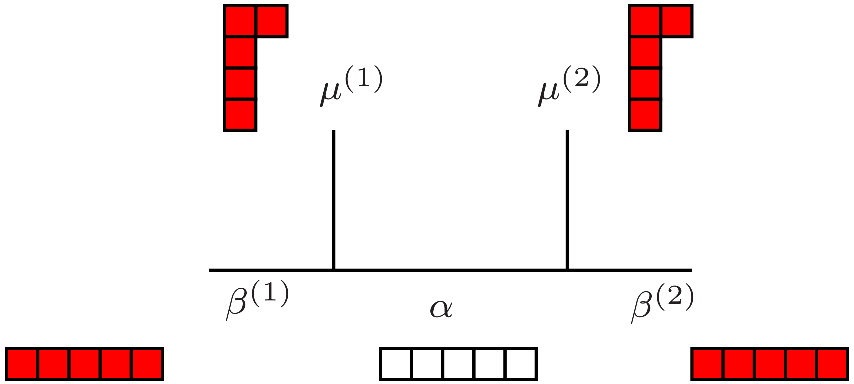
- simple puncture $\Leftrightarrow$ simple degenerate state V_μ, $\mu \sim$  (1 parameter)
- full puncture $\Leftrightarrow$ non-degenerate state V_β, $\beta \sim$  ($N - 1$ parameters)

A_{N-1} Toda CFT

- simple puncture $\Leftrightarrow$ simple degenerate state V_μ, $\mu \sim$ (1 parameter)
- full puncture $\Leftrightarrow$ non-degenerate state V_β, $\beta \sim$ ($N-1$ parameters)

It results:
$$Z_{S^4} = \langle V_{\beta^{(1)}} V_{\mu^{(1)}} V_{\mu^{(2)}} V_{\beta^{(2)}} \rangle_{A_{N-1} \text{ Toda}}$$

where the correlation function is decomposed as

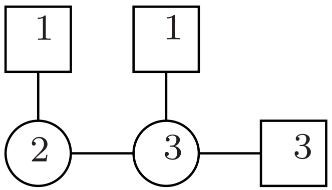


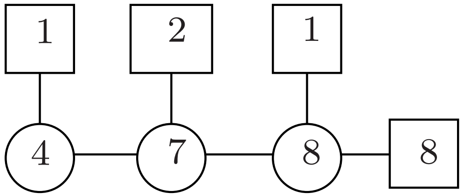
- The primary state in the internal channel α, is non-degenerate and depends on $N-1$ parameters. This is the dimension of the Coulomb branch of $SU(N)$ gauge group.

The problem

- There are also conformal gauge theories where the gauge group is $\prod_i SU(N_i)$ with different values of N_i.

  For instance:





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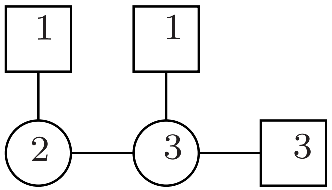
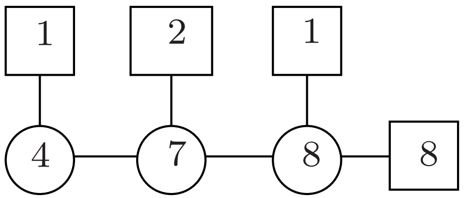




- How the partition function of this class of theories is reproduced in Toda CFT ?

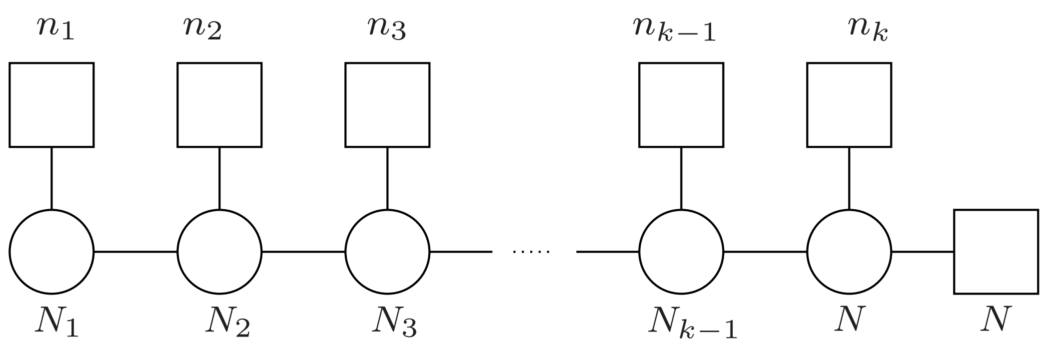
Quiver Tails

- A $\mathcal{N} = 2$ $SU(N)$ gauge theory is conformal when it couples to $N_F = 2N$ matter fields in the fundamental and anti-fundamental representation

Quiver Tails

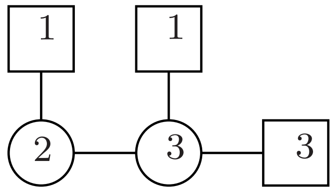
- A $\mathcal{N} = 2$ $SU(N)$ gauge theory is conformal when it couples to $N_F = 2N$ matter fields in the fundamental and anti-fundamental representation

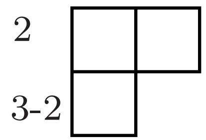
- It is possible to end a linear quiver with a finite series of gauge groups of decreasing rank: a quiver tail



  - the theory is conformal when $n_i = 2N_i - N_{i+1} - N_{i-1} \geq 0$
  - the tail is characterized by the series $N_1 < N_2 < \ldots < N_k = N$.
    This information can be encoded in a Young diagram with N boxes with the r^{th} row of length $N_r - N_{r-1}$

- For instance





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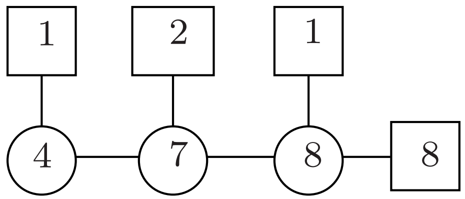

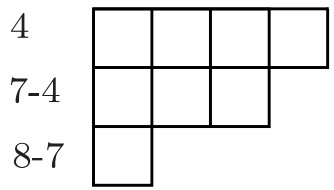

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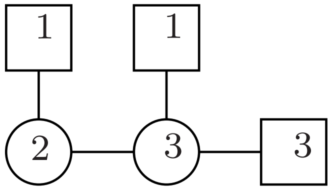

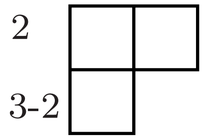

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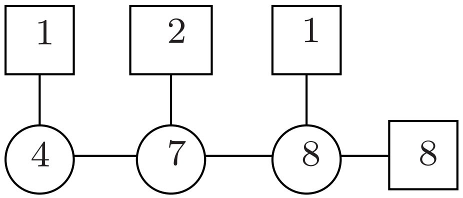
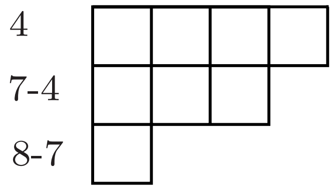
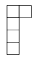




- Physical semi-degenerate states in A_{N-1} Toda CFT are characterized by a Young diagram with N boxes.

AGT for Quiver Tails

- Claim: The partition function of a quiver gauge theory with a tail, can be expressed as a correlation function in Toda CFT [Kanno, Matsuo, Shiba, Tachikawa][Kanno, Matsuo, Shiba][Drukker,FP]
  - $SU(N)$ is maximum rank group $\Leftrightarrow$ A_{N-1} Toda CFT
  - 1 insertion is a semi-degenerate state with the same Young diagram of the tail
  - $k+1$ simple insertions $\mu^{(l)} \sim$ 

AGT for Quiver Tails

- Claim: The partition function of a quiver gauge theory with a tail, can be expressed as a correlation function in Toda CFT
  [Kanno, Matsuo, Shiba, Tachikawa][Kanno, Matsuo, Shiba][Drukker,FP]
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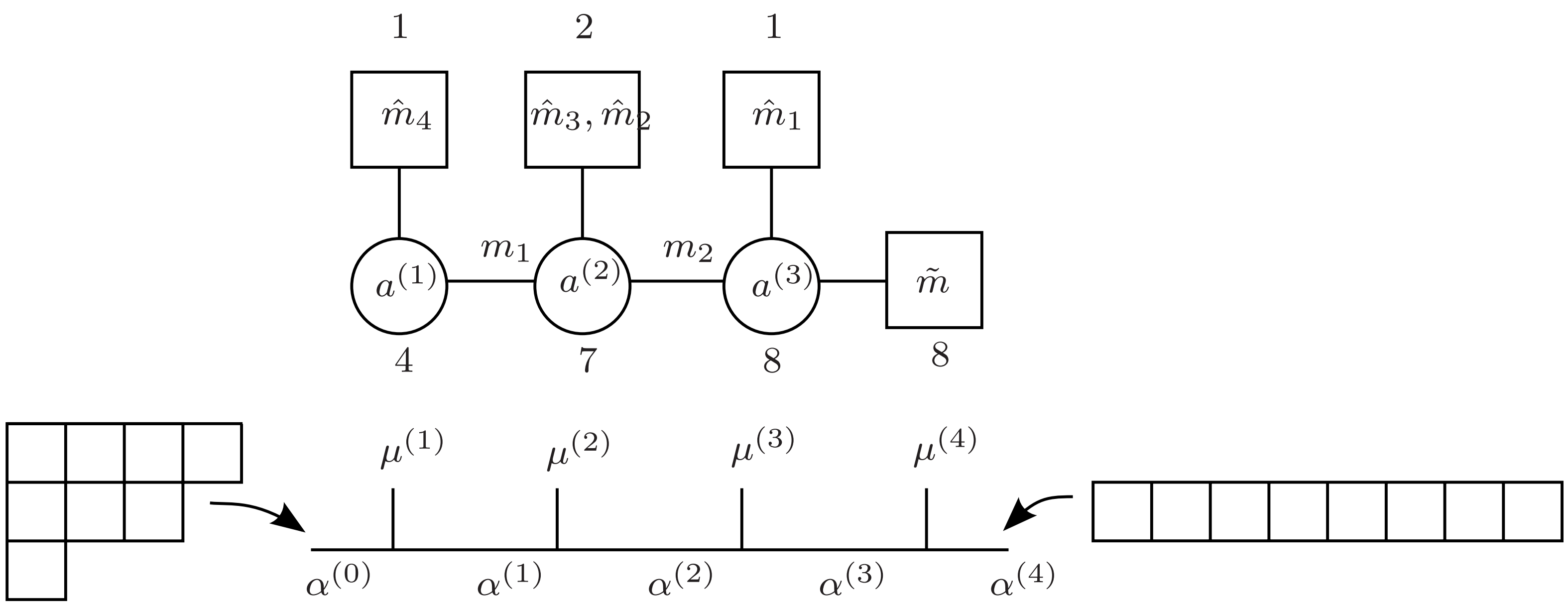
For instance



Questions to address: [Drukker, FP]

- How are the masses m_l, $\hat{m}_l$ encoded in the states $\mu^{(l)}$, $\alpha^{(0)}$, $\alpha^{(4)}$
- What are the allowed states $\alpha^{(l)}$ in the internal channels, and how are they related to the Coulomb branch parameters $a^{(l)}$

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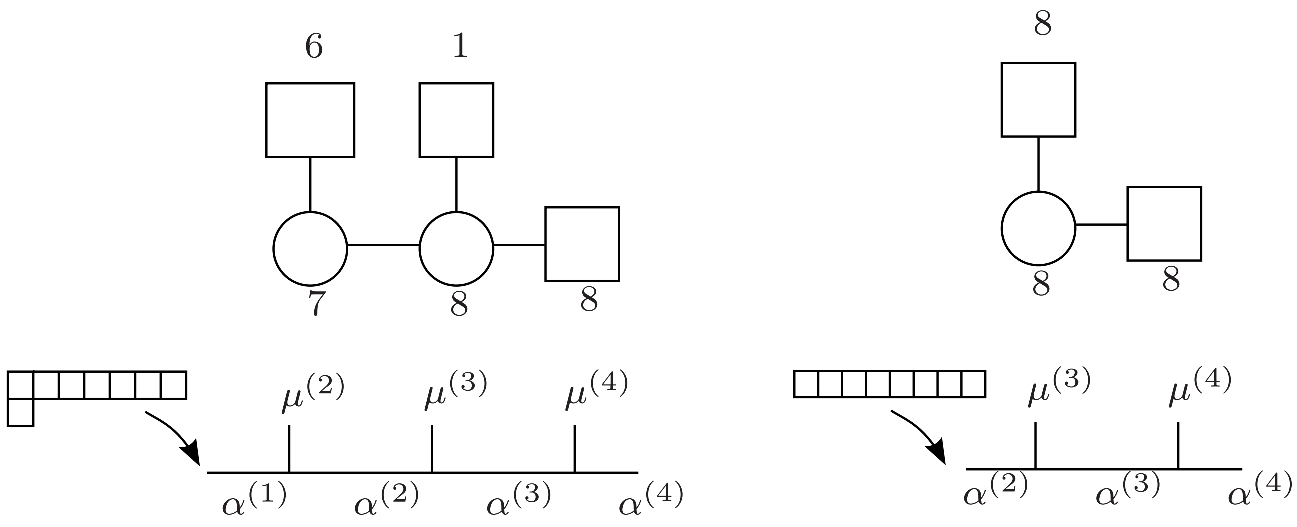

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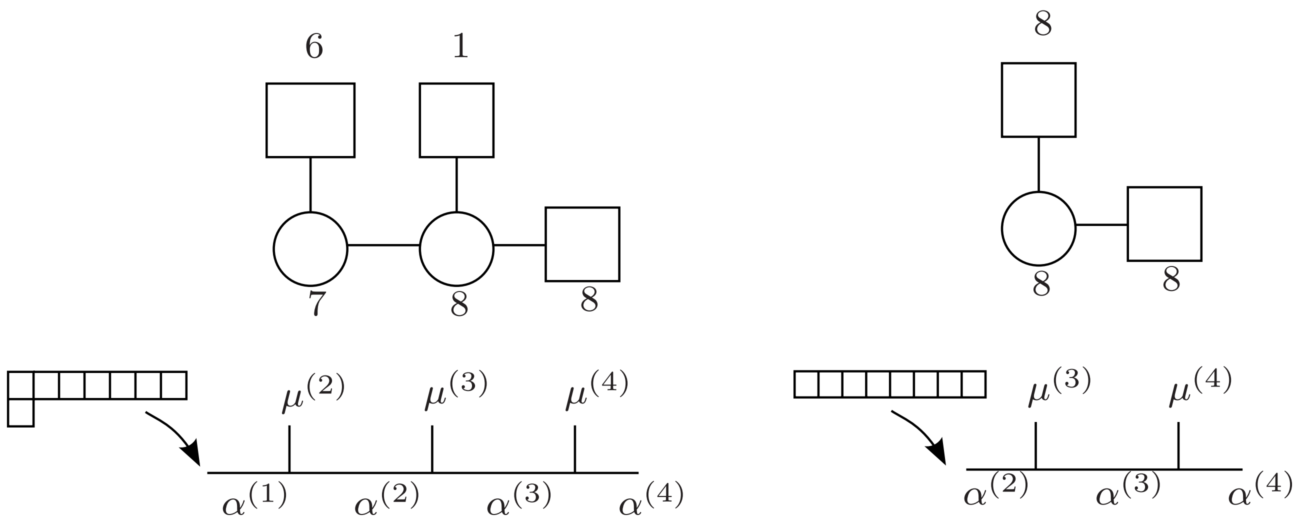
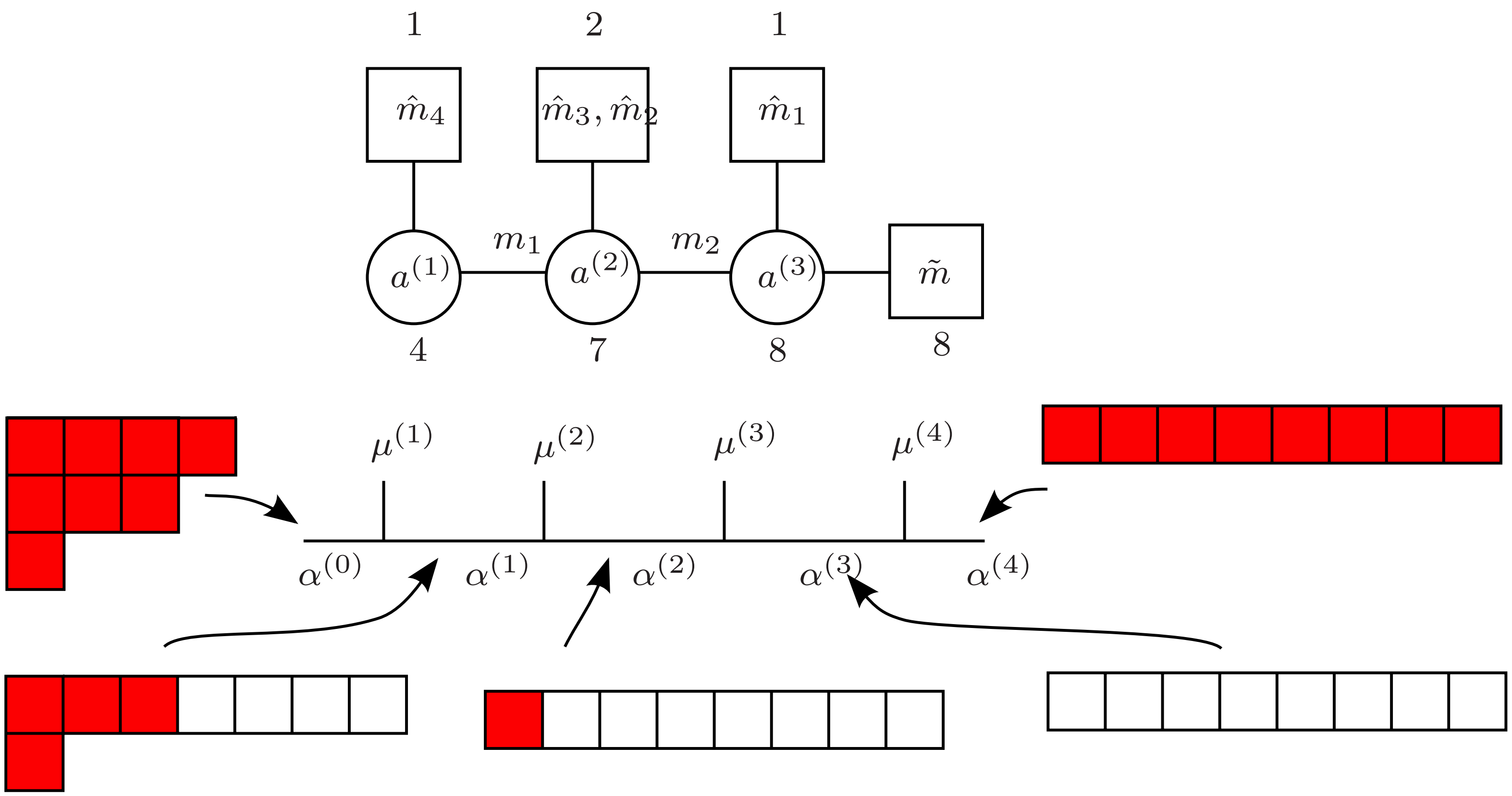


- $\alpha^{(1)}$ has 7 parameters: 3 are related to the masses, 4 to the Coulomb branch
- $\alpha^{(2)}$ has 8 parameters: 1 is related to the masses, 7 to the Coulomb branch

The proposal: [Drukker, FP]



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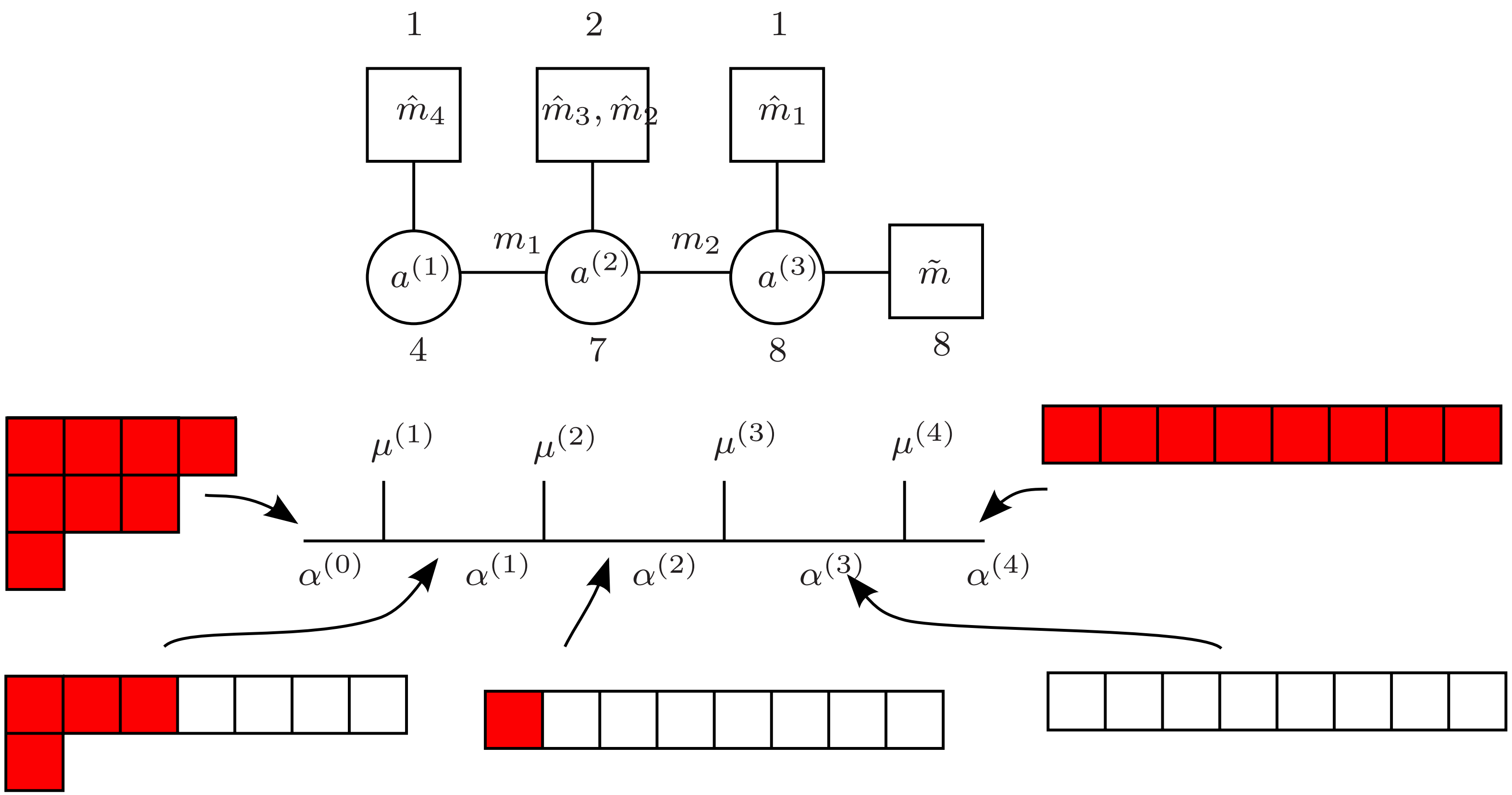

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Proof: study fusion rules for semi-degenerate Toda state.

AGT and non-local operators

Consider supersymmetric M-brane intersections

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For example, M5-M5 intersection:

$$\overbrace{0\ 1\ 2\ 3}^{4D}\ \overbrace{4\ 5}^{C_{(n,g)}}\ 6\ 7\ 8\ 9\ 10$$

	0	1	2	3	4	5	6	7	8	9	10
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The extra M5-brane

- 2D on 4D space $\Leftrightarrow$ surface operator
- wrap the $C_{(n,g)}$ $\Leftrightarrow$ different 2D CFT

[Aldday, Tachikawa]

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The extra M5-brane

- 2D on 4D space ⇔ surface operator
- wrap the $C_{(n,g)}$ ⇔ different 2D CFT [Alday, Tachikawa]

The symmetry of the 2D CFT depends on the type of surface operator.

Surface operators

A surface operator can be defined imposing a singular behavior of the gauge field on a 2D subspace [Gukov, Witten]

- (z_1, z_2) complex coordinates of R^4. The operator is placed at $z_2 = 0$, $z_2 = re^{i\theta}$

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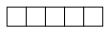
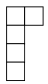
- The structure of the singularity is described by a partition of N, $[n_i]$

$$(\alpha_1, \ldots, \alpha_N) = (\underbrace{\alpha^{(1)}, \ldots, \alpha^{(1)}}_{n_1}, \underbrace{\alpha^{(2)}, \ldots, \alpha^{(2)}}_{n_2}, \ldots, \underbrace{\alpha^{(M)}, \ldots, \alpha^{(M)}}_{n_M})$$

- the extremal cases are
  - full surface operator has $\alpha_1 \neq \ldots \neq \alpha_N$, ($[n_i] \sim$ )
  - simple surface operator has $[n_i] \sim$ 

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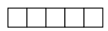
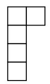
  - full surface operator has $\alpha_1 \neq \ldots \neq \alpha_N$, ($[n_i] \sim$ )
  - simple surface operator has $[n_i] \sim$ 

Same classification of the punctures, since also the punctures can be realized as intersecting M5-branes.

Conjecture: Given a surface operator characterized by a partition $[n_i]$, the symmetry of the 2D CFT is $W(\hat{SL}(N), [n_i])$.

[Braverman, Feigin, Finkelberg, Rybnikov]

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Full surface operators. The expected symmetry is $\hat{SL}(N)$.

- the unbroken gauge group is $U(1)^{N-1}$
- $N-1$ monopole numbers $\ell_i = \frac{1}{2\pi}\int_{z_2=0} F_i$
- the instantons are classical solutions with a singularity

$$A_\mu dx^\mu = \bar{A}_\mu dx^\mu + f(r)\mathrm{diag}(\alpha_1, \ldots, \alpha_N) i d\theta$$

- instanton number k is the instanton number of $\bar{A}_\mu$
- N topological quantities $k, \ell_1, \ldots, \ell_{N-1}$

$$\vec{k} = (k_1, \ldots, k_N) \qquad \text{where} \qquad k_1 = k \qquad k_{i+1} = k_i + \ell_i$$

The instanton partition function is known

[Braverman][Braverman, Etingof][Negut][Alday, Tachikawa][Kanno, Tachikawa]

$$Z_{inst} = \sum_{\lambda} Z_{\vec{k}}(\lambda) \prod_{i} y_i^{k_i}$$

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the N parameters y_i correspond to the $N-1$ parameters of the full surface operator and the usual instanton expansion parameter

- $\lambda = (\lambda_1, \dots, \lambda_N)$ is a vector of Young tableau
- $k_i = \sum_{j \geq 1} \lambda_j^{i-j+1}$
- $Z_{\vec{k}}(\lambda)$ depends on the field content

$\mathcal{N} = 2$ $SU(N)$ gauge theory with $N_f = 2N$

We define [Kozcaz, Pasquetti, FP, Wyllard]

- $Z^{(0),i}$ the sum of all terms with $k_i \neq 0$ and $k_j = 0$ for $i \neq j$
- $Z^{(1),i,j}$ the sum of all terms with $k_i \neq 0$, $k_j = 1$ for and $k_r = 0$ for $r \neq i, j$

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Thus

$$Z_{inst} = \sum_i Z^{(0),i} + \sum_{i,j} Z^{(1),i,j} \dots$$

where

$$Z^{(0),i} = \sum_{n=1}^{\infty} \frac{(\frac{\mu_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + 1)_n (\frac{\tilde{\mu}_i}{\epsilon_1} - \frac{a_i}{\epsilon_1})_n}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + 1)_n \, n!} (-y_i)^n$$

- a_i are the Coulomb branch parameters
- $\mu_i, \tilde{\mu}_i$ are the hypermultiplets masses

Claim: the instanton partition function for $\mathcal{N} = 2$ $SU(2)$ gauge theories with a full surface operator are equivalent to modified affine $SL(2)$ conformal blocks [Alday, Tachikawa]

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Affine $SL(2)$

$$[J_n^0, J_m^0] = \frac{k}{2}\, n\, \delta_{n+m,0}\,, \quad [J_n^0, J_m^\pm] = \pm J_{n+m}^\pm\,, \quad [J_n^+, J_m^-] = 2J_{n+m}^0 + k\, n\, \delta_{n+m,0}$$

- $|j\rangle$ primary state, $J_0^0|j\rangle = j|j\rangle$ and $J_{1+n}^-|j\rangle = J_{1+n}^0|j\rangle = J_n^+|j\rangle = 0$
- $V_j(x,z)$ primary field, x is an isospin variable and z is the worldsheet coordinate

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Differential operators

$$[J^A_n, V_j(x,z)] = z^n D^A V_j(x,z)$$

$$D^+ = 2\, j\, x - x^2\partial_x\,, \qquad D^0 = -x\partial_x + j\,, \qquad D^- = \partial_x$$

$\mathcal{N}=2$ $SU(2)$ SCYM

$$Z_{\text{instanton}} = (1-z)^{2j_2(-j_3+k/2)}\langle j_1|\mathcal{V}_{j_2}(1,1)\mathcal{V}_{j_3}(x,z)|j_4\rangle$$

where

$$\mathcal{V}_j = \mathcal{K}V_j \qquad\qquad \mathcal{K}(x,z) = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(z^{n-1}x\,J^-_{1-n} + \frac{z^n}{x}J^+_{-n}\right)\right]$$

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It can be evaluated pertubatively considering the decomposition

$$\sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1 | \mathcal{V}_{j_2}(1,1) | \mathbf{n}, \mathbf{A}; j \rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j) \langle \mathbf{n}', \mathbf{A}'; j | \mathcal{V}_{j_3}(x,z) | j_4 \rangle$$

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The components $Z^{(0),i}$ of the instanton function are reproduced by terms with the following internal states

- $(J^-_0)^n |j\rangle$ that gives a x^n term
- $(J^+_{-1})^n |j\rangle$ that gives a $\left(\frac{z}{x}\right)^n$ term

The complete dictionary is

- $y_1 = x\,, \qquad y_2 = \frac{z}{x}\,, \qquad j = -\frac{1}{2} + \frac{a_1}{\epsilon_1}\,, \quad k = -2 - \frac{\epsilon_2}{\epsilon_1}$
- $j_1 = -\frac{\epsilon_1+\epsilon_2+\mu_1-\mu_2}{2\epsilon_1}\,, \qquad j_2 = -\frac{2\epsilon_1+\epsilon_2+\mu_1+\mu_2}{2\epsilon_1}$
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What about the $SU(N)$ gauge theories with $N > 2$?

$\mathcal{N} = 2$ $SU(N)$ theories and affine $SL(N)$ algebra [Kozcaz, Pasquetti, FP, Wyllard]

Affine $SL(N)$

$$J^i_n\,, \qquad J^{i+}_n\,, \qquad J^{i-}_n\,, \qquad J^{il}_n \quad (i \neq l)$$

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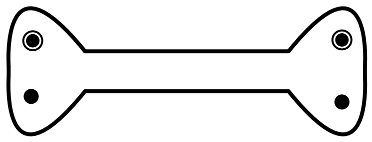
$$J_n^i\,, \qquad J_n^{i+}\,, \qquad J_n^{i-}\,, \qquad J_n^{il} \quad (i \neq l)$$

- $|j\rangle$ primary state, is labeled by $j = \sum_{i=1}^{N-1} j^i \omega_i$ where ω_i are the fundamental weights of $SL(N)$
- $J_0^i|j\rangle = j^i|j\rangle$ and $J_0^{i+}|j\rangle = 0\,, \quad J_0^{il}|j\rangle = 0 \;\; (i > l)\,, \quad J_n^A|j\rangle = 0 \;\; (n > 0)$
- $V_j(x,z)$ primary field, x is a vector of isospin variables and z is the worldsheet coordinate

Let's focus on the conformal $\mathcal{N}$=2 $SU(N)$ gauge theory coupled to $N_f = 2N$ hypermultiplets, i.e.



- simple puncture associated to a state V_χ (1 parameter)
- full puncture associated to a non-degenerate state V_j with generic j (N-1 parameters)

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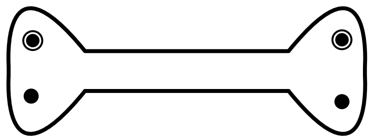

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The primary field with $j = \chi = \kappa\omega_1$ depends only on $N-1$ isospin variables and the action of the generators on these fields is expressed in terms of differential operators

$$[J_n^A, V_j(x,z)] = z^n D^A V_j(x,z)$$

The instanton function is equivalent to

$$\sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1|\mathcal{V}_{\chi_2}(1,1)|\mathbf{n},\mathbf{A};j\rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)\langle \mathbf{n}',\mathbf{A}';j|\mathcal{V}_{\chi_3}(x,z)|j_4\rangle$$

where

$$\mathcal{V}_{\chi_i}(x,z) = V_{\chi_i}(x,z)\mathcal{K}^\dagger(x,z)$$

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- $y_1 = x_1\,, \qquad y_{i+1} = \frac{x_{i+1}}{x_i} \quad (1 \le i \le N-2)\,, \qquad y_N = \frac{z}{x_{N-1}}$
- $j^i = -\frac{1}{2} + \frac{a_i - a_{i+1}}{2\epsilon_1}\,, \qquad k = -N - \frac{\epsilon_2}{\epsilon_1}$
- $\frac{\tilde{\mu}_i}{2\epsilon_1} = -\frac{\kappa_3}{N} + \langle h_i, j_4 + \frac{\rho}{2}\rangle\,, \qquad \frac{\mu_i}{2\epsilon_1} = \frac{\kappa_2}{N} + \langle h_i, j_1 + \frac{\rho}{2}\rangle$

Conclusion

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- AGT works also for quiver tails
- the surface operators modify the 2D CFT
- non-local operators in quiver tails?
- can we reproduce the full partition function considering the correlator of some CFT with affine algebra?
- what is the physical meaning of the $\mathcal{K}$ operator?