Quantized moduli spaces of flat connections, Liouville theory, and integrable models

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Motivation

Low energy physics of certain N=2 4d SUSY gauge theories admits a family of interesting regularizations / compactifications:

$\mathbb{R}^4$ `(Seiberg-Witten)`

$\mathbb{R}^3 \times S^1$ `(Moore-Gaiotto-Neitzke)` $\mathbb{R}^2 \times \mathbb{R}^2_\epsilon$ `(Nekrasov-Shatashvili)`

$\mathbb{R}^4_{\epsilon_1\epsilon_2}$ `(Moore-Nekrasov-Shatashvili)`, or
S^4 `(Pestun)`

In a large class of examples $\mathcal{G}_C$ `(Gaiotto)` there exist interesting relations to integrable models, CFT, and (quantized) Hitchin moduli spaces: `(Nekrasov-Shatashvili, Alday-Gaiotto-Tachikawa, Gaiotto-Moore-Neitzke)`

$$\begin{tikzcd} & \text{Hitchin system} \arrow[dl, "\epsilon_2"'] \arrow[dr, "\epsilon_1"] & \\ \text{Flat connections} \arrow[dr, "\epsilon_1"'] & & \text{quantum Hitchin system} \arrow[dl, "\epsilon_2"] \\ & \text{Liouville theory} & \end{tikzcd} \tag{1}$$

To understand **why** this is so, we may need to understand the relations in (1) a little better.

Motivation II

Putting gauge theory in finite volume

$\Rightarrow$ quantization of zero modes like constant parts of scalars Φ.

Emerging picture: — (Drukker-Gomis-Okuda-J.T., Nekrasov-Witten)

- Quantization of zero modes $\Rightarrow$ subspace $\mathcal{H}_0$ of zero energy states.

- Algebra of operators on $\mathcal{H}_0$ generated by SUSY loop operators (Wilson, 't Hooft).

- Expectation values of loop operators reduce (localize) to overlaps in $\mathcal{H}_0$, e.g.

$$\langle\,\mathcal{L}\,\rangle\;=\;\langle\,q\,|\,\mathcal{L}_0\,|\,q\,\rangle_{\mathcal{H}_0}\,.$$

In order to fully understand the relations above, we will in particular need to answer the following questions:

(A) How is Liouville theory related to quantized moduli spaces of flat connections?

(B) How is Liouville theory related to the quantized Hitchin systems?

Moduli spaces of flat connections

Consider flat **complex** connections

$$d + \mathcal{A} \;=\; (\partial_z + \mathcal{A}_z)dz + (\partial_{\bar{z}} + \mathcal{A}_{\bar{z}})d\bar{z}\,,$$

modulo gauge transformations. Flat connections modulo gauge transformations characterized by holonomies. They define representations

$$\rho\,:\,\pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})\,.$$

The moduli space of flat connections is therefore isomorphic to

$$\mathcal{M} \;=\; \mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})\,.$$

There is a holomorphic symplectic form on $\mathcal{M}$,

$$\varpi \;=\; \frac{1}{2}\int_C \mathrm{tr}(\delta\mathcal{A}\wedge\delta\mathcal{A})\,.$$

Nothing depends on a (possible) choice of complex structure so far.

Coordinates for $\mathcal{M}$

One may distinguish

- Coordinates associated to triangulations.

  `Penner, Fock, Fock-Goncharov, Gaiotto-Moore-Neitzke....`

- Coordinates associated to pants decompositions.

  `Fenchel-Nielsen, Nekrasov-Rosly-Shatashvili,...`

We shall be interested in the latter.

Preparation:

Each cutting curve γ specifies either a four-holed sphere or a once-punctured torus embedded in C. Choosing a numbering of the boundary components of each pair of pants one can associate to γ a pair of dual curves $\check{\gamma}$, $\hat{\gamma}$.

Loop coordinates for $\mathcal{M}$

To a point in $\mathcal{M}$ let us associate coordinate functions

$$X_\gamma := -\mathrm{Tr}(M_\gamma)\,, \qquad Y_\gamma := -\mathrm{Tr}(M_{\check\gamma})\,, \qquad Z_\gamma := -\mathrm{Tr}(M_{\hat\gamma})\,,$$

where $M_\gamma := \mathrm{Pexp}(\int_\gamma \mathcal{A})$. There are relations: For given four-punctured sphere $C_\gamma \hookrightarrow C$:

$$\begin{aligned} X_\gamma Y_\gamma Z_\gamma \;=&\; X_\gamma^2 + Y_\gamma^2 + Z_\gamma^2 - 4 \\ &+ X_\gamma(M_1M_2 + M_3M_4) + Y_\gamma(M_2M_3 + M_1M_4) + Z_\gamma(M_1M_3 + M_2M_4) \\ &+ M_1^2 + M_2^2 + M_3^2 + M_4^2 + M_1M_2M_3M_4\,, \end{aligned}$$

where $M_k = -\mathrm{Tr}(M_{\gamma_k})$, γ_k: k-th boundary curve of C_γ.

The symplectic structure on $\mathcal{M}$ induces a simple Poisson structure:

$$\{\, X\,,\, Y\,\} \;=\; 2Z - XY + M_1M_3 + M_2M_4\,.$$

Darboux-coordinates for $\mathcal{M}$, I — (`Nekrasov-Rosly-Shatashvili`)

Introduce (**Darboux coordinates**) $(\lambda_\gamma, \kappa_\gamma)$ with $\{\lambda_\gamma, \kappa_{\gamma'}\} = \delta_{\gamma,\gamma'}$ and reconstruct loop coordinates as:

"Wilson loop": $$X_\gamma \,=\, 2\cosh(\lambda_\gamma)\,,$$

"'t Hooft loop":

$$\begin{aligned} Y_\gamma(X_\gamma^2 - 4) \,=\, & 2(M_2M_3 + M_1M_4) + X_\gamma(M_1M_3 + M_2M_4) \\ & + 2\cosh(\kappa_\gamma)\sqrt{c_{12}(X_\gamma)c_{34}(X_\gamma)}\,, \end{aligned}$$

Furthermore:

$$(-2Z_\gamma + X_\gamma Y_\gamma - M_1M_3 - M_2M_4)2\sinh(\lambda_\gamma) \,=\, 2\sinh(\kappa_\gamma)\sqrt{c_{12}(X_\gamma)c_{34}(X_\gamma)}\,,$$

Here $c_{ij}(X) = X^2 + M_i^2 + M_j^2 + XM_iM_j - 4$.

Darboux-coordinates for $\mathcal{M}$, II

Remarks:

- Restricted to the real slice where $X_\gamma, Y_\gamma, Z_\gamma \in \mathbb{R}_+$ for all γ one recovers the **Fenchel-Nielsen** coordinates

- There is a freedom to to redefine $\kappa_\gamma \to \kappa_\gamma + f(\lambda)$. This freedom is fixed by demanding existence of a **natural origin**: Expressions for loop coordinates are **symmetric** w.r.t. $(\lambda_\gamma, \kappa_\gamma) \to (-\lambda_\gamma, -\kappa_\gamma)$.
  – For FN-coordinates the origin is defined geometrically!

Complex-structure dependent Darboux coordinates, I

An interesting part of $\mathcal{M}$, called $\mathcal{M}_0$, can be represented in terms of connections gauge equivalent to the form

$$\nabla' = \partial_y + \begin{pmatrix} 0 & t(y) \\ 1 & 0 \end{pmatrix}.$$

Under changes of local coordinates $y = y(x)$, $t(y)$ transforms as

$$t(y) \mapsto (y'(x))^2\, t(y(x)) - \frac{1}{2}\{y, x\}\,, \qquad \{y, x\} := \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2.$$

The equation for parallel transport implies the Fuchsian differential equation

$$\mathcal{P}_y \psi(y) = (\partial_y^2 - t(y))\psi(y) = 0\,.$$

Connections of this form are nowadays often called $\mathfrak{sl}_2$-opers.

Complex-structure dependent Darboux coordinates II

The difference between two opers $t(y) - t'(y)$ transforms as a quadratic differential, so

$$t(y) \,=\, t_0(y) + Q(y) \,=\, t_0(y) + \sum_{r=1}^{3g-3+n} H_r \, \vartheta_r(y)\,,$$

Q: quadr. differential, $\vartheta_r(y)(dy)^2$: basis for the $3g-3+n$-dimensional space of quadratic differentials, $t_0(y)$: reference oper.

By standard Teichmüller theory one may associate to any Q a cotangent vector to the Teichmüller space $\mathcal{T}(C)$.

Key fact (`Kawai`):

> The map $T^*\mathcal{T} \to \mathcal{M}_0$ defined by the associating to $\partial_y^2 - t(y)$ its monodromy $\rho : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ is locally biholomorphic and symplectic.

In other words, there exist coordinates $\mathbf{z} = (z_1, \ldots, z_{3g-3+n})$, such that $(z_1, \ldots, z_{3g-3+n}; H_1, \ldots, H_{3g-3+n})$ are complex analytic coordinates for $\mathcal{M}_0$ with Poisson brackets

$$\{\, z_r\,, z_s \,\} \,=\, 0\,, \qquad \{\, H_r\,, z_s \,\} \,=\, \delta_{r,s}\,, \qquad \{\, H_r\,, H_s \,\} \,=\, 0\,.$$

Complex-structure dependent Darboux coordinates III

Example: n-punctured sphere, $C = C_{0,n}$:

Opers: Fuchsian differential operators $\partial_y^2 + t(y)$ with

$$t(y) \, = \, \sum_{r=0}^{n-1} \left(\frac{\delta_r}{(y-z_r)^2} + \frac{H_r}{y-z_r} \right) \, ,$$

where $\delta_r = \left(\frac{\lambda_r}{2\pi}\right)^2 + \frac{1}{4}$. Assume w.l.o.g. $z_n = \infty$, $z_{n-1} = 1$, $z_0 = 0$.

There are three relations among $H_1, \dots, H_n$:

$$\sum_{r=0}^{n-1} z_r^k (z_r H_r + (k+1)\delta_r) \, = \, 0 \, , \qquad k = -1, 0, 1 \, .$$

$\Rightarrow$ May pick $(H_1, \dots, H_{n-3}; z_1, \dots, z_{n-3})$ as independent coordinates for $\mathcal{M}_0$.

Change of coordinates

Question: How to characterize the change of Darboux coordinates $(\lambda, \kappa) \to (z, H)$. Define $H_r(\lambda, z)$ as the accessory parameters which gives the oper P the monodromy $\rho_P : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ such that

$$2 \cosh \lambda_r \; := \; \mathrm{tr}(\rho_P(\gamma_r)) \, ,$$

for curves $\gamma_1, \ldots, \gamma_{3g-3+n}$ that constitute a cut system. We claim that the function $\mathcal{W}(\lambda, z)$ which does the job can be defined by the equations

$$H_r(\lambda, z) \; = \; -\frac{\partial}{\partial z_r} \mathcal{W}(\lambda, z) \, .$$

Variables canonically conjugate to the λ_r are then found as

$$\kappa_r \; = \; 2\pi i \frac{\partial}{\partial \lambda_r} \mathcal{W}(\lambda, m) \, .$$

By carefully choosing the complex-structure independent part of $\mathcal{W}(\lambda, z)$, we recover the variables κ_γ — $\mathcal{W}(\lambda, z)$ **is the Yang's function of** `Nekrasov-Rosly-Shatashvili` !

Quantization of the Darboux coordinates, I

The first steps look fairly easy: Recall e.g. formulae for Wilson and 't Hooft loops

$$\begin{aligned} X_\gamma &= 2\cosh(\lambda_\gamma)\,, \\ Y_\gamma &= \frac{1}{X_\gamma^2-4}\Big[2(M_2M_3+M_1M_4)+X_\gamma(M_1M_3+M_2M_4) \\ &\qquad + 2\cosh(\kappa_\gamma)\sqrt{c_{12}(X_\gamma)c_{34}(X_\gamma)}\;\Big]\,, \end{aligned}$$

where $\{\lambda_\gamma, \kappa_{\gamma'}\} = \delta_{\gamma,\gamma'}$. This may be quantized as $[\lambda_\gamma, \kappa_{\gamma'}] = 2\pi i b^2 \delta_{\gamma,\gamma'}$,

$$\begin{aligned} \mathcal{X}_\gamma &= 2\cosh(\lambda_\gamma)\,, \\ \mathcal{Y}_\gamma &= \frac{1}{(2\sinh(\lambda_\gamma))^2}\Big(2(M_2M_3+M_1M_4)+X_\gamma(M_1M_3+M_2M_4)\Big) \\ &\quad + \frac{1}{\sqrt{2\sinh(\lambda_\gamma)}}e^{+\kappa_\gamma/2}\frac{\sqrt{c_{12}(X_\gamma)c_{34}(X_\gamma)}}{2\sinh(\lambda_\gamma)}e^{+\kappa_\gamma/2}\frac{1}{\sqrt{2\sinh(\lambda_\gamma)}} \\ &\quad + \frac{1}{\sqrt{2\sinh(\lambda_\gamma)}}e^{-\kappa_\gamma/2}\frac{\sqrt{c_{12}(X_\gamma)c_{34}(X_\gamma)}}{2\sinh(\lambda_\gamma)}e^{-\kappa_\gamma/2}\frac{1}{\sqrt{2\sinh(\lambda_\gamma)}}\,, \end{aligned}$$

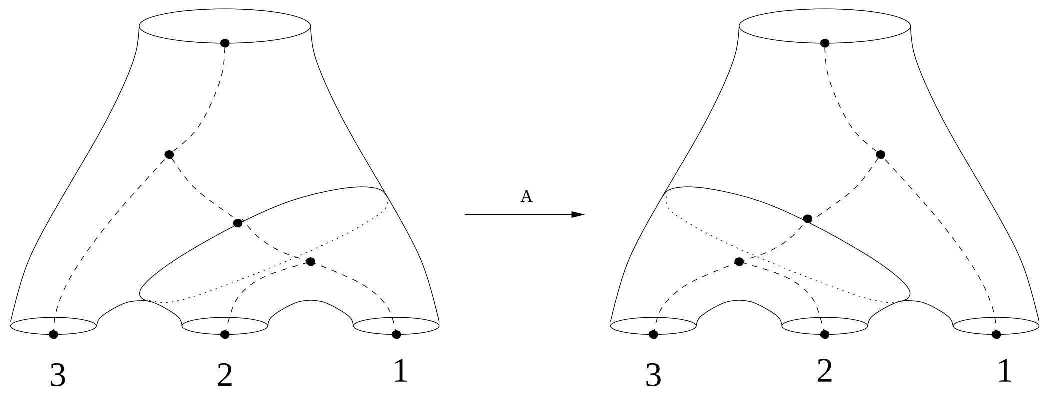
This defines noncommutative deformation $\mathcal{A}_b$ of the algebra of functions on $\mathcal{M}$.

Quantization of the Darboux coordinates, II

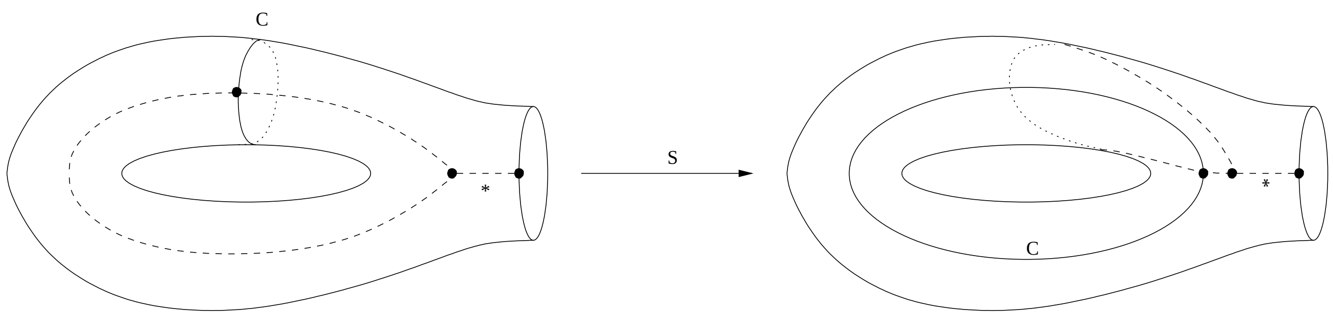
Problem:

The FN/NRS Darboux coordinates depend on a choice of pants decomposition. How to quantize the changes of FN/NRS-coordinates associated to changes of the pants decompositions? This boils down to quantizing elementary F- and S- moves.

F-move:



S-move:



Quantization of the Darboux coordinates, III

The Wilson loop operators associated to non-intersecting curves commute and can be simultaneously diagonalized $\Rightarrow$ **Length representation:**

States represented by wave-functions $\psi(\lambda) = \langle \lambda | \psi \rangle$, $\lambda = (\lambda_1, \ldots, \lambda_{3g-3+n})$. Note that representation depends on choice of pants decomposition $\sigma \Rightarrow \psi(\lambda) \to \psi_\sigma(\lambda)$.

The unitary operators $\mathsf{U}_{\sigma_2\sigma_1}$ relating the length representations associated to two pants decompositions σ_2 and σ_1 can be represented as integral operators of the form

$$\psi_{\sigma_2}(\lambda_2) = (\mathsf{U}_{\sigma_2\sigma_1}\psi_{\sigma_1})(\lambda_2) = \int d\lambda_1\, U_{\sigma_2\sigma_1}(\lambda_2, \lambda_1)\, \psi_{\sigma_1}(\lambda_1)\,,$$

which can be decomposed into the elementary fusion, braiding and S-moves.

The kernels can be **characterized** as solutions to the difference equations

$$(\mathcal{X}_\gamma - \mathcal{Y}'_\gamma)\, U(\lambda, \lambda') = 0 \quad \Leftrightarrow \quad \mathsf{X}_{\gamma,\sigma_2} \cdot \mathsf{U}_{\sigma_2\sigma_1} = \mathsf{U}_{\sigma_2\sigma_1} \cdot \mathsf{X}_{\gamma,\sigma_1}\,.$$

Wilson loops mapped to 't Hooft loops under S-duality !

Fixing Darboux coordinates fixes length representation completely !

Quantization of the Darboux coordinates, IV

Example: F-move:

$$U(\lambda,\lambda') \;=\; \frac{N(\alpha_4,\alpha_3,\beta)N(\beta,\alpha_2,\alpha_1)}{N(\alpha_4,\beta',\alpha_1)N(\beta',\alpha_3,\alpha_2)}\;\frac{S_b(u_1)}{S_b(u_2)}\frac{S_b(w_1)}{S_b(w_2)}\int_{i\mathbb{R}} dt\;\prod_{i=1}^{4}\frac{S_b(t+r_i)}{S_b(t+s_i)},$$

where the special function $S_b(x)$ is closely related to the noncompact quantum dilogarithm and (using $\alpha_i=\frac{Q}{2}+i\frac{\mu_i}{2\pi b}$, $\beta=\frac{Q}{2}+i\frac{\lambda}{2\pi b}$, $\beta'=\frac{Q}{2}+i\frac{\lambda'}{2\pi b}$, $Q=b+b^{-1}$)

$$\begin{aligned}
r_1 &= \alpha_1-\alpha_2, & s_1 &= \alpha_4-\alpha_2+\beta', & u_1 &= \beta+\alpha_2-\alpha_1,\\
r_2 &= Q-\alpha_2-\alpha_1, & s_2 &= Q+\alpha_4-\alpha_2-\beta', & u_2 &= 2Q-\beta-\alpha_3-\alpha_4,\\
r_3 &= \alpha_4+\alpha_3-Q, & s_3 &= \beta, & w_1 &= \beta'+\alpha_2-\alpha_3,\\
r_4 &= \alpha_4-\alpha_3, & s_4 &= Q-\beta, & w_2 &= 2Q-\beta'-\alpha_2-\alpha_3,
\end{aligned}$$

$$\begin{aligned}
&N(\alpha_1,\alpha_2,\alpha_3)=\\
&\quad=\sqrt{\frac{S_b(2\alpha_1)S_b(2\alpha_2)S_b(2\alpha_3)}{S_b(2Q-\alpha_1-\alpha_2-\alpha_3)S_b(Q-\alpha_1-\alpha_2+\alpha_3)S_b(\alpha_2+\alpha_3-\alpha_1)S_b(\alpha_3+\alpha_1-\alpha_2)}}.
\end{aligned}$$

Quantization of the oper-coordinates

States get represented by holomorphic multi-valued wave-functions

$$\Psi(z) \;=\; \langle\, z \,|\, \Psi \,\rangle\,, \qquad\quad z = (z_1, \ldots, z_{3g-3+n})\,,$$

such that the operators z_i, H_i associated to z_i and H_i get represented as

$$\mathsf{z}_i \Psi(z) \;=\; z_i \Psi(z)\,, \qquad\quad \mathsf{H}_i \Psi(z) \;=\; \frac{1}{b^2}\frac{\partial}{\partial z_i}\,\Psi(z)\,.$$

The state $|\, z \,\rangle$: generalization of a coherent state (eigenstate of z_i) in quantum mechanics.

Quantum change of Darboux coordinates: Oper vs. FN/NRS-coordinates, I

Consider the wave-function $\Psi_z(\lambda) \equiv \langle\, \lambda \,|\, z \,\rangle$. It describes the change of representation

$$\tilde{\phi}(z) \,=\, \int d\lambda \,\langle\, z \,|\, \lambda \,\rangle \phi(\lambda)\,.$$

The wave-function $\langle\, \lambda \,|\, z \,\rangle$ is characterized by

- Monodromies
$$\psi_{m.z}(\lambda_2) \,=\, \int d\lambda_2 \, U_{m.\sigma,\sigma}(\lambda_2,\lambda_1)\,\psi_z(\lambda_1)\,,$$
where $\psi_{m.z}(\lambda)$: analytic continuation of $\psi_z(\lambda)$ along closed path m in $\mathcal{M}_{g,n}$.

- Asymptotic behavior fixed by quantizing $z_i H_i \sim \left(\frac{\lambda_i}{4\pi}\right)^2 - \frac{1}{4}$, when $z_i \to 0$.

This defines a Riemann-Hilbert type problem. The solution is essentially unique. It can be constructed in terms of certain vertex operators (`J.T. '01,'03`).

Quantum change of Darboux coordinates: Oper vs. FN/NRS-coordinates, II

- Consider asymptotics of $\langle\, \lambda \,|\, z \,\rangle$ at boundaries of Teichmüller space, e.g. for $C = C_{0,4}$:

$$\langle\, \lambda \,|\, z \,\rangle = \sqrt{C_{43}(\lambda)C_{21}(-\lambda)}\, z^{\Delta_\lambda - \Delta_{\mu_1} - \Delta_{\mu_2}}(1 + \mathcal{O}(z))\,, \tag{2}$$

where $C_{ij}(\lambda) = C\big(\alpha_1, \alpha_j, \frac{Q}{2} + i\frac{\lambda}{2\pi b}\big)$,

$$C(\alpha_1, \alpha_2, \alpha_3) = \Big[\pi\mu\gamma(b^2)b^{2-2b^2}\Big]^{(Q-\sum_{i=1}^3 \alpha_i)/b} \times$$

$$\times\, \frac{\Upsilon_0\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}.$$

$C(\alpha_1, \alpha_2, \alpha_3)$ is the **Liouville three-point function**.

- $\langle\, \lambda \,|\, z \,\rangle$: **Liouville conformal block:** $\langle\, \lambda \,|\, z \,\rangle = \big\langle\, \prod_{r=1}^n e^{2\alpha_r(z_r)} \big\rangle$, also known as co-invariant in tensor products of representations of the Virasoro algebra.

Having fixed the Darboux coordinates fixes normalization (2)!

Intermediate conclusion

Main result:

Liouville conformal blocks are the quantization of the generating function for the change of Darboux coordinates $(\lambda, \kappa) \to (z, H)$.

Remarks:

- Quantization of FN/NRS-coordinates fixes usual ambiguities in holomorphic factorization of Liouville theory.
- Classical limit determines complex-structure independent part of Yang's function – no direct derivation known! (— but see gauge theory approach of NRS.)

We've recovered **all** the main characteristics of Liouville theory from **quantization of moduli space $\mathcal{M}_0$ of flat connections.**

Quantum Hitchin system — Example: $SL(2,\mathbb{C})$-Gaudin model

Consider the tensor product of n principal series representations $\mathcal{P}_j$ of $SL(2,\mathbb{C})$. It corresponds to the tensor product of representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ generated by differential operators $\mathcal{J}_r^a$ acting on functions $\Psi(x_1,\bar{x}_1,\ldots,x_n,\bar{x}_n)$ as

$$\mathcal{J}_r^- = \partial_{x_r}, \quad \mathcal{J}_r^0 = x_r\partial_{x_r} - j_r, \quad \mathcal{J}_r^+ = -x_r^2\partial_{x_r} + 2j_r x_r,$$

and complex conjugate operators $\bar{\mathcal{J}}_r^a$. Casimir parameterized via j_r as $j_r(j_r+1)$. Let

$$\mathsf{H}_r \equiv \sum_{s\neq r} \frac{\mathcal{J}_{rs}}{z_r - z_s}, \qquad \bar{\mathsf{H}}_r \equiv \sum_{s\neq r} \frac{\bar{\mathcal{J}}_{rs}}{\bar{z}_r - \bar{z}_s},$$

where the differential operator $\mathcal{J}_{rs}$ is defined as

$$\mathcal{J}_{rs} := \eta_{aa'}\mathcal{J}_r^a\mathcal{J}_s^{a'} := \mathcal{J}_r^0\mathcal{J}_s^0 + \frac{1}{2}\left(\mathcal{J}_r^+\mathcal{J}_s^- + \mathcal{J}_r^-\mathcal{J}_s^+\right),$$

while $\bar{\mathcal{J}}_{rs}$ is the complex conjugate of $\mathcal{J}_{rs}$. The Gaudin Hamiltonians are mutually commuting,

$$[\,\mathsf{H}_r\,,\,\mathsf{H}_s\,] = 0\,, \qquad [\,\mathsf{H}_r\,,\,\bar{\mathsf{H}}_s\,] = 0\,, \qquad [\,\bar{\mathsf{H}}_r\,,\,\bar{\mathsf{H}}_s\,] = 0\,.$$

Quantum $SL(2,\mathbb{C})$-Gaudin model from Liouville theory

The eigenvalue equations $\mathsf{H}_r\psi = E_r\psi$ emerge in the critical level limit of the KZ-equations

$$(k+2)\partial_z\Psi(x,z) \;=\; \mathsf{H}_r\,\Psi(x,z)\,,$$

which are solved by the WZNW-correlation functions

$$\mathcal{Z}_W(x,z) \;=\; \left\langle\,\prod_{r=1}^{n}\phi^{j_r}(x_r,\bar{x}_r|z_r,\bar{z}_r)\,\right\rangle.$$

The WZNW-correlation functions $\mathcal{Z}_W(x,z)$ can be constructed from Liouville correlation functions

$$\mathcal{Z}_L(y,z) \;=\; \left\langle\,\prod_{r=1}^{n}e^{2\alpha_r\phi(z_r,\bar{z}_r)}\prod_{s=1}^{n-2}e^{-\phi(y_r,\bar{y}_r)/b}\right\rangle$$

by some explicitly known integral transformation `(Sklyanin)`

$$\mathcal{Z}_W(x,z) \;=\; \int dy\;K_z(x,y)\mathcal{Z}_L(y,z).$$

Quantum $SL(2,\mathbb{C})$-Gaudin model from Liouville theory

In the critical level limit $k \to -2$ we may use relation between $\mathcal{Z}_W$ and $\mathcal{Z}_L$ to show

$$\mathcal{Z}_W(x,z) \underset{k\to -2}{\sim} \psi(x)\,\exp\left(\frac{1}{k+2} 2\mathrm{Re}(\mathcal{W}(\lambda_{\mathrm{cl}},z))\right)\,,$$

where $\mathcal{W}(\lambda,z)$ is the Yang's function, and λ_{cl} is defined by solving

$$\frac{\partial}{\partial\lambda_r}\mathrm{Re}(\mathcal{W}(\lambda,z))\Big|_{\lambda=\lambda_{\mathrm{cl}}} \;=\; 0\,, \qquad r = 1,\dots,n-3.$$

This condition characterizes the saddle point of the integral in the factorization expansion

$$\left\langle\, e^{2\alpha_n\phi(z_n,\bar{z}_n)}\cdots e^{2\alpha_1\phi(z_1,\bar{z}_1)}\,\right\rangle_L \;=\; \int d\mu(p)\;|\mathcal{F}^{\sigma}_{\alpha,C_q}(p)|^2\,.$$

In other words:

$$\boxed{\begin{array}{l}\text{Single-valuedness of Liouville correlations functions } \;\Rightarrow \\ \text{Quantization conditions for } E_r \text{ in terms of Yang's function.}\end{array}}$$

Conclusions

We have discussed how the items in

$$\begin{tikzcd}
 & \text{Hitchin system} \arrow[dl, "\epsilon_2"'] \arrow[dr, "\epsilon_1"] & \\
\text{Flat connections} \arrow[dr, "\epsilon_1"'] & & \text{quantum Hitchin system} \arrow[dl, "\epsilon_2"] \\
 & \text{Liouville theory} &
\end{tikzcd}$$

are related.

The resulting relation between **quantum Hitchin system** and **moduli spaces of flat connections** explains why Yang's function $\mathcal{W}(\lambda, z)$ gives quantization conditions.

It is hoped that these results will guide the investigation of the links between gauge theory, integrable models and conformal field theory.