Introduction

(a) Computation of analytic (Hodge theoretic) invariant of algebraic cycles — co-dimension 2 on families of Calabi-Yau threefolds.

(b) Interpretation via mirror symmetry, extrapolating from proven cases.

(c) NEW: recent calcuations begin to show arithmetic flavor.

(d) Why: Because we can.

Prototypical example:

$$\mathcal{Y} \to M \ni z\,, \qquad\qquad \mathcal{C}_\pm \subset \mathcal{Y}$$

$$\mathcal{T}_B = \int_{C_-}^{C_+} \Omega$$

$$\mathcal{P}\mathcal{T}_B = \big(\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\big)\mathcal{T}_B = \frac{15}{16\pi^2}\sqrt{z}$$

Solutions:

$$\varpi(z;H) = \sum_{n=1}^{\infty} \frac{\Gamma(5(n+H)+1)}{\Gamma(n+H+1)^5} z^{n+H} = \sum_{i=0}^{3} \varpi_i(z) H^i \bmod H^4$$

$$\mathcal{T}_B(z) = \varpi(z; H = \frac{1}{2})$$

$$\mathcal{T}_A(q) = \frac{\mathcal{T}_B}{\varpi_0}(z(q)) = \sum_{d \text{ odd}} \tilde{n}_d q^{d/2} = \sum_{d,k \text{ odd}} \frac{1}{k^2} n_d q^{dk/2}$$

$\tilde{n}_d \in \mathbb{Q}$: open Gromov-Witten invariants for pair (quintic, real quintic)

$n_d \in \mathbb{Z}$: BPS invariants

Define

$$\Xi(H) = H^4 \frac{\Gamma(5H+1)}{\Gamma(H+1)^5}$$

then

$$\Xi(1/2) = \frac{1}{2^4} \frac{\Gamma(5(1/2)+1)}{\Gamma(1/2+1)^5} = \frac{15}{4\pi^2}$$

with

$15 =$ number of real lines on real quintic (Solomon)

One-parameter hypergeometric Picard-Fuchs differential equations: (...)

Geometry is complete intersection Calabi-Yau threefold

$$\mathbb{P}^{n}_{w_1,\ldots w_{n+1}}[d_1,\ldots,d_{n-3}]$$

and its mirror manifold.

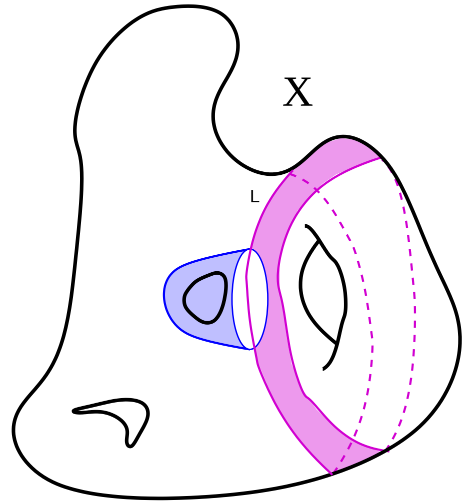
$$\Xi(H) := H^4 \frac{\prod_{j=1}^{n-3}\Gamma(d_j\,H+1)}{\prod_{i=1}^{n+1}\Gamma(w_i\,H+1)}$$

w	d	$\Xi(1/2)$	integr.	cycle	mirror symmetry
$(1,1,1,1,1)$	(5)	$15/4\pi^2$	yes	yes	proven
$(1,1,1,2,5)$	(10)	$32/\pi^2$	yes	yes	conjectured
$(1,1,1,1,1,1,1,1)$	$(2,2,2,2)$	$16/\pi^4$	no	no	??
$(1,1,1,1,1,1)$	$(3,3)$	$9/4\pi^2$	yes	yes	proven
$(1,1,1,1,1,1,1)$	$(2,2,3)$	$6/\pi^3$	no	no	??
$(1,1,1,1,1,1)$	$(2,4)$	$8/\pi^3$	no	no	??
$(1,1,1,1,4)$	(8)	$12/\pi^2$	yes	yes	conjectured
$(1,1,1,1,2)$	(6)	$6/\pi^2$	yes	yes	conjectured
$(1,1,1,1,4,6)$	$(2,12)$	$60/\pi^2$	yes	?	?
$(1,1,1,1,2,2)$	$(4,4)$	$4/\pi^2$	yes	yes	?
$(1,1,1,2,2,3)$	$(4,6)$	$8/\pi^2$	yes	yes	?
$(1,1,1,1,1,2)$	$(3,4)$	$3/\pi^2$	yes	?	?
$(1,1,2,2,3,3)$	$(6,6)$	$16/\pi^2$	yes	yes	?
$(1,1,1,1,1,3)$	$(2,6)$	$16/\pi^3$	no	no	??

Mirror Interpretation

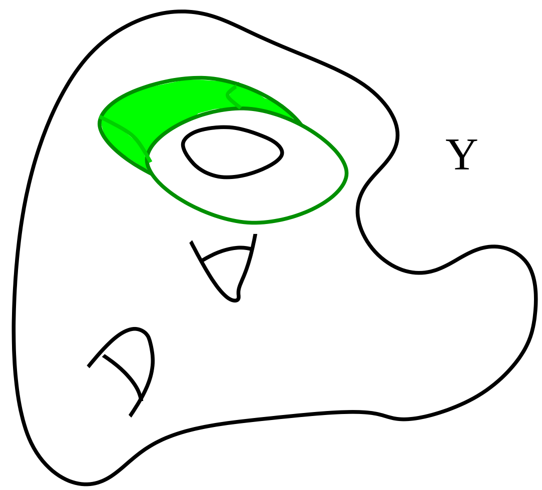
In A-model: four-chain interpolating between, or two-chain ending on, Lagrangian submf.

$$\mathcal{T}_A = \int_{C^{(4)}} \omega \wedge \omega + \int_{C^{(2)}} \omega + \text{hol. disks}$$



In B-model: three-chain interpolating between holomorphic curves

$$\mathcal{T}_B = \int_{C^{(3)}} \Omega$$



Mirror principle:

$$\left.\begin{array}{l}\text{* } \mathcal{T}_B \text{ is holomorphic invariant}\\ \text{* } \mathcal{T}_A \text{ is generating function of enumerative invariants}\end{array}\right\}$$ associated with D-brane configuration

Example: (Krefl-JW) $\mathbb{P}^4_{11114}[8]$

Real locus disconnected: $\mathbb{RP}^3 \cup \mathbb{RP}^3$.

Correspondingly: 4 mirror curves.

Small-volume monodromies mix 4 chains and 2 chains.

Other hypergeometric inhomogeneous Picard-Fuchs: $\mathbb{P}^5_{111144}[6,6]$

$$\Xi(1/3) = \frac{1}{3\pi^2} \qquad \Xi(2/3) = \frac{100}{3\pi^2}$$

Solution of

$$\mathcal{P}\mathcal{T}_B = \Xi(1/3) z^{1/3} + \Xi(2/3) z^{2/3}$$

satisfies integrality

$$\mathcal{T}_A = \sum_{3\nmid d} \tilde{n}_d q^{d/3} = \sum_{3\nmid d,k} \frac{1}{k^2} n_d q^{kd/3}$$

A cycle exists. Mirror unknown. Elementarily,

$$H_1(L;\mathbb{Z}) \cong \mathbb{Z}/3$$

but perhaps only in Floer homology.

Non-hypergeometric inhomogeneity: $\mathbb{P}^5_{111113}[2,6]$

Inhomogeneity

$$\mathcal{PT}_B = \frac{4}{27}\frac{z^{1/3} + 112z^{2/3}}{(1-8z^{1/3})^{5/2}}$$

satisfies integrality.

$$\mathcal{T}_A = \sum_{3\nmid d} \tilde{n}_d q^{d/3} = \sum_{3\nmid d,k} \frac{1}{k^2} n_d q^{kd/3}$$

Cycle computation challenging.

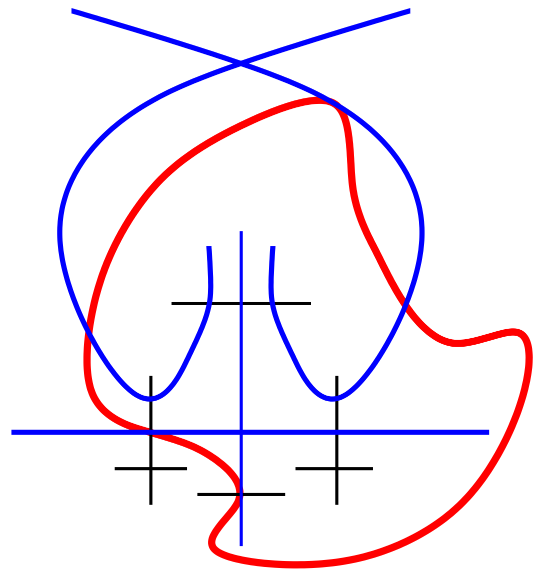
Other work: Jockers-Soroush, Alim-Hecht-Mayr-Mertens, Aganagic-Beem, Klemm-Grimm-Klevers, Li-Liu-Yau,. . .

Multi-parameter models: $\mathbb{P}^4_{11226}[12]$

there is a cycle with inhomogeneous Picard-Fuchs system:

$$\begin{aligned}\big(\theta_y^2(\theta_y - 2\theta_2) - 72y^3 z_2(2\theta_y + 1)(2\theta_y + 3)(2\theta_y + 5)\big)\mathcal{T}_B &= 3\,\frac{4y}{2(1-4y)^{3/2}}\\ \big((\theta_y - 3\theta_2)^2 - 9z_2(2\theta_2 - \theta_y)(2\theta_2 - \theta_y + 1)\big)\mathcal{T}_B &= 3\,\frac{1}{(1-4y)^{1/2}}\end{aligned}$$

with $y = (z_1/z_2)^{1/3}$, $z_1 = \phi\psi^{-6}$,
$z_2 = \phi^{-2}$, $\theta_z = zd/d\ln z$.



Mirror interpretation

The classical closed string mirror map identifies the two Kähler parameters of the A-model geometry as the periods with asymptotic behavior

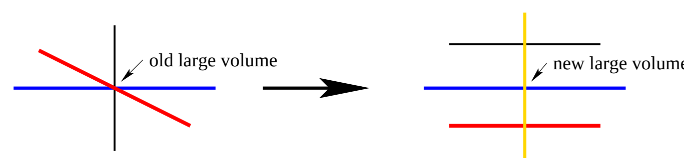
$$t_1 \sim \log z_1 \ , \qquad\qquad t_2 \sim \log z_2$$

The Kähler cone is the subspace of t_1, t_2 real with $t_1 > 0, t_2 > 0$.

Solving inhomogeneous Picard-Fuchs, we deduce that the Lagrangian bounds holomorphic disks of area

$$\mathcal{T}_B^{(1)} = \frac{t_1 - t_2}{3} + \cdots$$

$\longrightarrow$ presence of D-brane introduces additional face in the Kähler cone, where $\frac{t_1 - t_2}{3} = 0$. We require further blowup



$\longrightarrow$ What happens across open string discriminant?

To study behavior accross the open string discriminant, we restrict Picard-Fuchs system to $z_2 = 0$

$$\theta_y^2 \mathcal{T}_B^{(2)} = \frac{1}{\sqrt{1-4y}}$$

with solution around $y \sim \exp\left(-\frac{t_1 - t_2}{3}\right) = 0$,

$$\mathcal{T}_B^{(2)} = \frac{(\log y)^2}{2} - \frac{\pi^2}{6} - \left(\log \frac{1+\sqrt{1-4y}}{2}\right) + 2\mathrm{Li}_2 \frac{1-\sqrt{1-4y}}{2} \sim \frac{(t_1 - t_2)^2}{3} + \cdots$$

Scaling with t_i's implies this must be a domain wall represented by a 4-chain. (integral 4-cycles have tension $\sim 2(t_1^2 + t_1 t_2)$, t_1^2.)

Analytic continuation to $x = 1/y = 0$ yields

$$\mathcal{T}_B^{(2)} = i\pi \log x - \frac{1}{2}\left(\log \frac{\sqrt{x^2 - 4x} + x}{\sqrt{x^2 - 4x} - x}\right)^2 - 2\mathrm{Li}_2\left(\frac{\sqrt{x^2 - 4x} + x}{2}\right) \sim \frac{t_2 - t_1}{3} + \cdots$$

This domain wall must be represented by 2-chain.

$\Rightarrow$ Flop of disk accompagnied by 4-chain $\rightarrow$ 2-chain

For completeness: To extract the instanton corrections to the classical domain wall tension, we go back to the full inhomogeneous Picard-Fuchs system, and expand using Ooguri-Vafa multicover formula

$$\mathcal{W} = \frac{3}{2}\Big(\frac{t_1 - t_2}{3}\Big)^2 + \cdots + \sum_{(m,n)\neq(0,0),k>0} \frac{N_{m,n}}{k^2} p^{km} q^{kn}$$

$$p = e^{2\pi i \frac{t_1 - t_2}{3}}, \qquad q = e^{2\pi i t_2}$$

Mirror Principle: *Expand fully quantum corrected superpotential in terms of classical domain wall tension corrected only by closed string instantons*

$m \setminus n$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	6	6	0	0	0	0	0
2	3	90	3	0	0	0	0
3	6	388	388	6	0	0	0
4	12	-258	2934	-258	12	0	0
5	30	-540	11016	11016	-540	30	0
6	75	-1388	67602	348 774	67602	-1388	75

Pfaffian Calabi-Yaus

Shimizu-Suzuki

$$\#99:\quad \Big[13^2\theta^4 - 13z(4569\theta^4 + 9042\theta^3 + 6679\theta^2 + 2158\theta + 260) + 2^4z^2(6386\theta^4 - 1774\theta^3 - 17898\theta^2 - 11596\theta - 2119) + 2^8z^3(67\theta^4 + 1248\theta^3 + 1091\theta^2 + 312\theta + 26) - 2^{12}z^4(2\theta+1)^4\Big]\mathcal{T}_B = \frac{1183}{16}z^{1/2} + 5278z^{3/2} + 4272z^{5/2}$$

$$\#302:\quad \Big[\theta^4 + 2^2z(500\theta^4 + 976\theta^3 + 677\theta^2 + 189\theta + 19) + 2^4z^2(39688\theta^4 + 39688\theta^3 - 1336\theta^2 - 1164\theta - 177) + 2^{10}z^3(500\theta^4 + 24\theta^3 - 37\theta^2 + 6\theta + 3) + 2^{12}z^4(2\theta+1)^4\Big]\mathcal{T}_B = \frac{93}{4}z^{1/2} + 7440z^{3/2} + 49600z^{5/2}$$

$$\#109:\quad \Big[7^2\theta^4 - 42z(1272\theta^4 + 2508\theta^3 + 1779\theta^2 + 525\theta + 56) + 2^2\cdot 3z^2(43704\theta^4 + 38088\theta^3 - 25757\theta^2 - 20608\theta - 3360) - 2^4\cdot 3^3z^3(2736\theta^4 - 1512\theta^3 - 1672\theta^2 - 357\theta - 14) - 2^6\cdot 3^5z^4(3\theta+1)(2\theta+1)^2(3\theta+2)\Big] = \frac{245}{8}z^{1/2} + 4326z^{3/2} + 10638z^{5/2}$$

$$\#263:\quad \Big[5^2\theta^4 + 2^2\cdot 5z(688\theta^4 + 1352\theta^3 + 981\theta^2 + 305\theta + 35) + 2^4z^2(5856\theta^4 + 7008\theta^3 + 96\theta^2 - 1260\theta - 265) + 2^{10}z^3(176\theta^4 + 120\theta^3 + 69\theta^2 + 30\theta + 5) + 2^{12}z^4(2\theta+1)^4\Big] = \frac{25}{2}z^{1/2} + 1240z^{3/2} + 2048z^{5/2}$$

Lines on the mirror quintic

Albano-Katz, van Geemen, Mustata

Generic member of the Dwork family of quintics has
* 375 isolated coordinate lines
* 2 families of lines parameterized by smooth genus 626 curve

$$375 + 2 \times (2 \times 626 - 2) = 2875$$

5000 special members of family: "van Geemen lines"

Inhomogeneity: JW

$$\mathcal{PT}_B = \frac{96}{\pi^2} \cdot \frac{-\frac{512}{\psi^5} + \frac{1824}{\psi^{10}} + \frac{63}{\psi^{15}}}{\left(3 - \frac{128}{\psi^5}\right)^{5/2}}$$

Integrality of van Geemen lines:

$$\mathcal{T}_A = \sum_{d>0} \tilde{n}_d q^d = \sum_{d,k>0} \frac{\chi(k)}{k^2}\frac{n_d}{3^d} q^{dk}$$

$$\chi(k) = \begin{cases} 1 & k = 1 \bmod 3 \\ -1 & k = 2 \bmod 3 \\ 0 & k = 0 \bmod 3 \end{cases}$$

with $n_d \in \mathbb{Z}$.

Definition: D-log

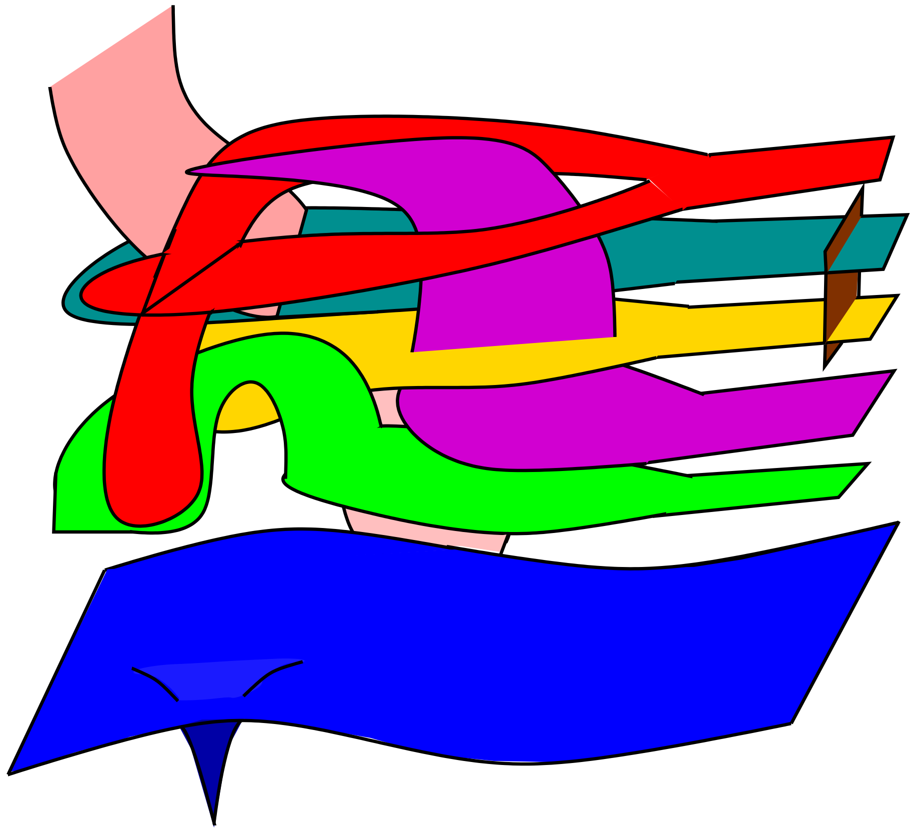
$$Li_2^{\chi}(x) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^2} x^k$$

D is for D-brane

Conics on the mirror quintic

The generic quintic contains 609250 conics (Katz).

The mirror quintic contains several families (Mustata), and many isolated conics. Full structure of Hilbert scheme not available.



One component of Hilbert scheme:

$$
\begin{aligned}
&-140544+1312896u-6157536u^{2}+20560128u^{3}-55739073u^{4}+126082635u^{5}-240562314u^{6}+389983296u^{7}-517794816u^{8}\\
&+526312752u^{9}-386382096u^{10}+195989568u^{11}-64755264u^{12}+12180480u^{13}-890112u^{14}+3526016v-35327360uv\\
&+164085512u^{2}v-490389848u^{3}v+1119877362u^{4}v-2054126078u^{5}v+2822178044u^{6}v-2674914608u^{7}v+1703155648u^{8}v\\
&-783769296u^{9}v+331207776u^{10}v-160872256u^{11}v+74273920u^{12}v-20820224u^{13}v+1915392u^{14}v-12887824v^{2}\\
&+156888240uv^{2}-794842896u^{2}v^{2}+1924669488u^{3}v^{2}-1861954446u^{4}v^{2}-560979783u^{5}v^{2}+2742716878u^{6}v^{2}\\
&-2532259552u^{7}v^{2}+1357646032u^{8}v^{2}-522831968u^{9}v^{2}+146600816u^{10}v^{2}-62851072u^{11}v^{2}+15982784u^{12}v^{2}\\
&+5083904u^{13}v^{2}-966912u^{14}v^{2}-110365024v^{3}+1349538976uv^{3}-5573477584u^{2}v^{3}+9890213496u^{3}v^{3}-8559117395u^{4}v^{3}\\
&+4959540898u^{5}v^{3}-3410214400u^{6}v^{3}+1529015152u^{7}v^{3}+33207472u^{8}v^{3}-335159488u^{9}v^{3}+398796352u^{10}v^{3}\\
&-199492096u^{11}v^{3}+27038976u^{12}v^{3}+206336u^{13}v^{3}-181248u^{14}v^{3}-1058031072v^{4}+7328123536uv^{4}-16155350056u^{2}v^{4}\\
&+17717423024u^{3}v^{4}-18250232092u^{4}v^{4}+21436831296u^{5}v^{4}-16619578848u^{6}v^{4}+8429844448u^{7}v^{4}-3758257792u^{8}v^{4}\\
&+1108256896u^{9}v^{4}-254268672u^{10}v^{4}+129069056u^{11}v^{4}-52310016u^{12}v^{4}+4696064u^{13}v^{4}+135168u^{14}v^{4}-1271515824v^{5}\\
&-3515100512uv^{5}+23558245664u^{2}v^{5}-33532680832u^{3}v^{5}+15994006832u^{4}v^{5}+1748284832u^{5}v^{5}-6786182656u^{6}v^{5}\\
&+5719888128u^{7}v^{5}-2496033024u^{8}v^{5}+544198656u^{9}v^{5}-76355584u^{10}v^{5}+61018112u^{11}v^{5}+3694592u^{12}v^{5}\\
&-1777664u^{13}v^{5}+8485369664v^{6}-39975494784uv^{6}+76393384256u^{2}v^{6}-82428927744u^{3}v^{6}+64625199040u^{4}v^{6}\\
&-46320419072u^{5}v^{6}+28977470976u^{6}v^{6}-13398732800u^{7}v^{6}+5270946816u^{8}v^{6}-1532405760u^{9}v^{6}+165781504u^{10}v^{6}\\
&-21012480u^{11}v^{6}+2473984u^{12}v^{6}+3324777728v^{7}-4295229696uv^{7}-21543773440u^{2}v^{7}+57614347264u^{3}v^{7}\\
&-56924167424u^{4}v^{7}+37388443136u^{5}v^{7}-22467149824u^{6}v^{7}+7877509120u^{7}v^{7}-2088861696u^{8}v^{7}+451772416u^{9}v^{7}\\
&-78036992u^{10}v^{7}+6029312u^{11}v^{7}-8267872256v^{8}+34670743552uv^{8}-42630860800u^{2}v^{8}+7651102720u^{3}v^{8}\\
&+12061375488u^{4}v^{8}-7382695936u^{5}v^{8}+6715727872u^{6}v^{8}-2138890240u^{7}v^{8}+730972160u^{8}v^{8}-79429632u^{9}v^{8}\\
&+8498114560v^{9}-35617423360uv^{9}+49886576640u^{2}v^{9}-22319595520u^{3}v^{9}-1981624320u^{4}v^{9}+2571264000u^{5}v^{9}\\
&-2668953600u^{6}v^{9}+344719360u^{7}v^{9}-2792865792v^{10}+10876387328uv^{10}-12904677376u^{2}v^{10}+2578120704u^{3}v^{10}\\
&+3163045888u^{4}v^{10}-473956352u^{5}v^{10}+320798720v^{11}-891617280uv^{11}+732364800u^{2}v^{11}-87818240u^{3}v^{11}
\end{aligned}
$$

$$= 0$$

In expansion around $z = 0$:

$$\mathcal{PT}_B = \sum_{n=0}^{\infty} a_n z^{\alpha+n}$$

$$\alpha \in \mathbb{Q}_{>0} \qquad a_n \in \mathbb{Q}(\eta)$$

$$\implies \mathcal{T}_A = \sum_{d=0}^{\infty} \tilde{n}_d q^{\alpha+d}$$

$$\tilde{n}_d \in \mathbb{Q}(\eta)\text{unavoidable}$$

It appears that via

$$\sum \tilde{n}_d q^{\alpha+d} \sim \sum_{d,k} \frac{\chi(k)}{k^2} n_d q^{(\alpha+d)k}$$

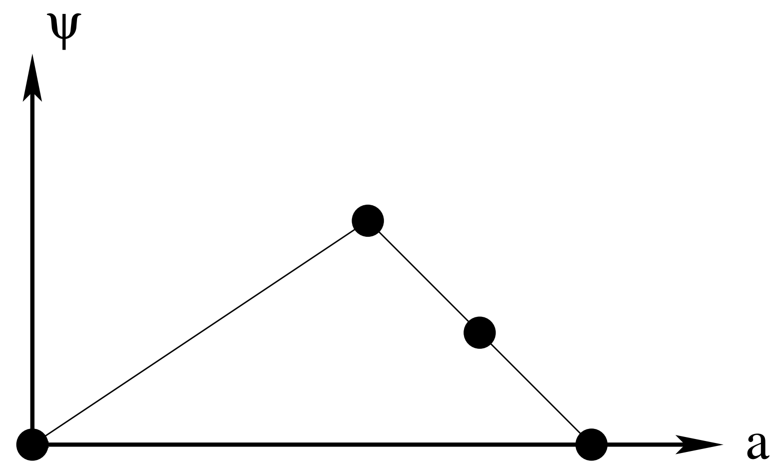
n_d are algebraic integers!

Newton-Puiseux expansion

Example:

$$64 + 5\psi^2 a^3 - 40\psi a^4 + 12a^5 = 0$$

Consider Newton polygon:



As $\psi \to \infty$

$$a = \psi\,\eta_1 + \mathcal{O}(\psi^{-4}) \qquad 5 - 40\eta_1 + 12(\eta_1)^2 = 0$$

$$a = \psi^{-2/3}\,\eta_{2/3} + \mathcal{O}(\psi^{-7/3}) \qquad 64 + 5(\eta_{2/3})^3 = 0$$

Other examples with:

$$\alpha = \frac{1}{7}\,; \qquad 5 + 5\eta^2 + \eta^4$$

$$\begin{aligned}\alpha = 1\,; \qquad & 1 + 243\eta + 27675\eta^2 + 1529140\eta^3 + 49599473\eta^4 \\ & - 221079468\eta^5 + 49599473\eta^6 + 1529140\eta^7 \\ & + 27675\eta^8 + 243\eta^9 + \eta^{10}\end{aligned}$$

To be continued . . . . . .