Deconfined Phases and Phase Transitions of Gauge Theories, & Potential Uses of the Sign Problem

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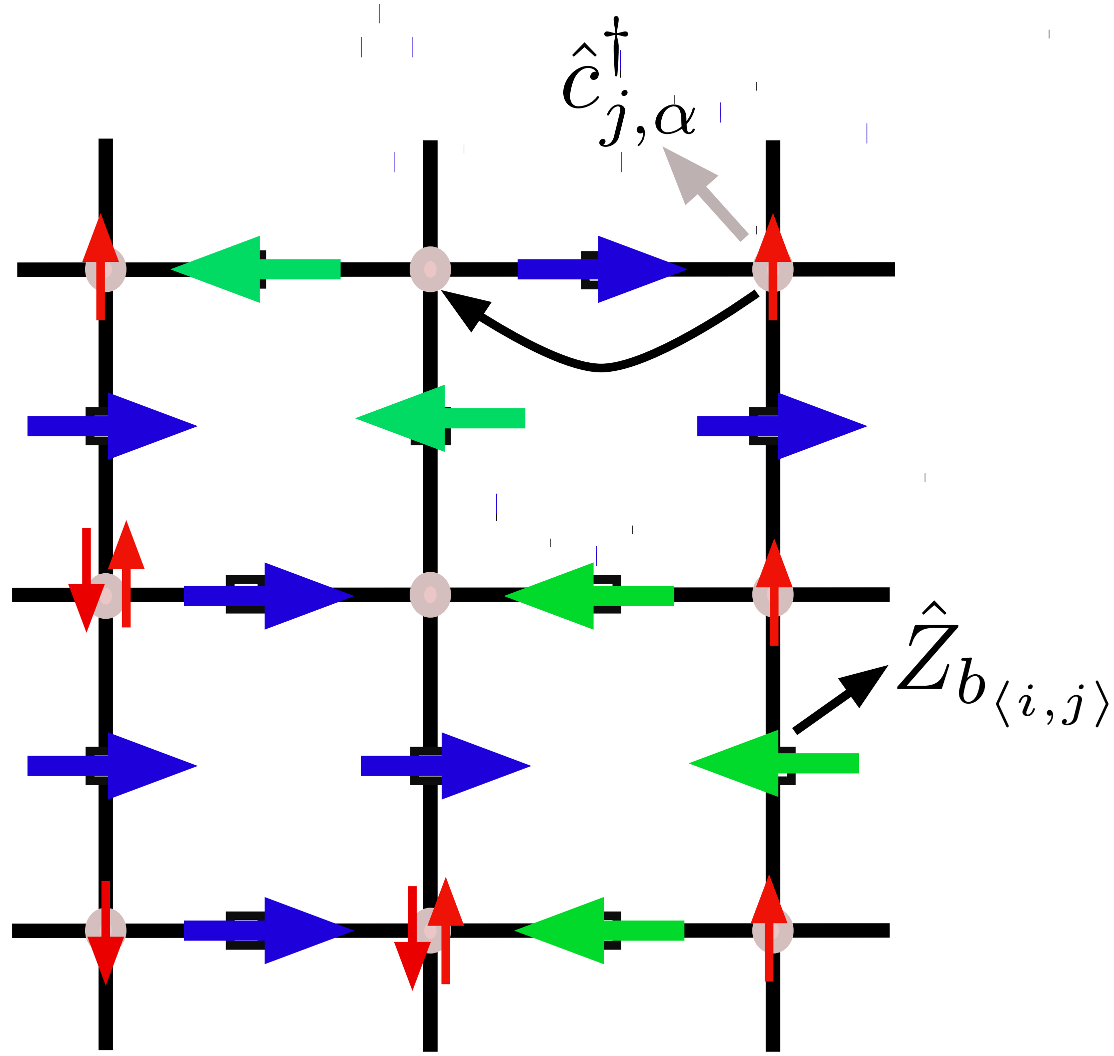
arXiv:1607.03912/PRX 2016

Related work: Gazit, Randeria, Vishwanath, arXiv:1607.03892/nature physics 2017.

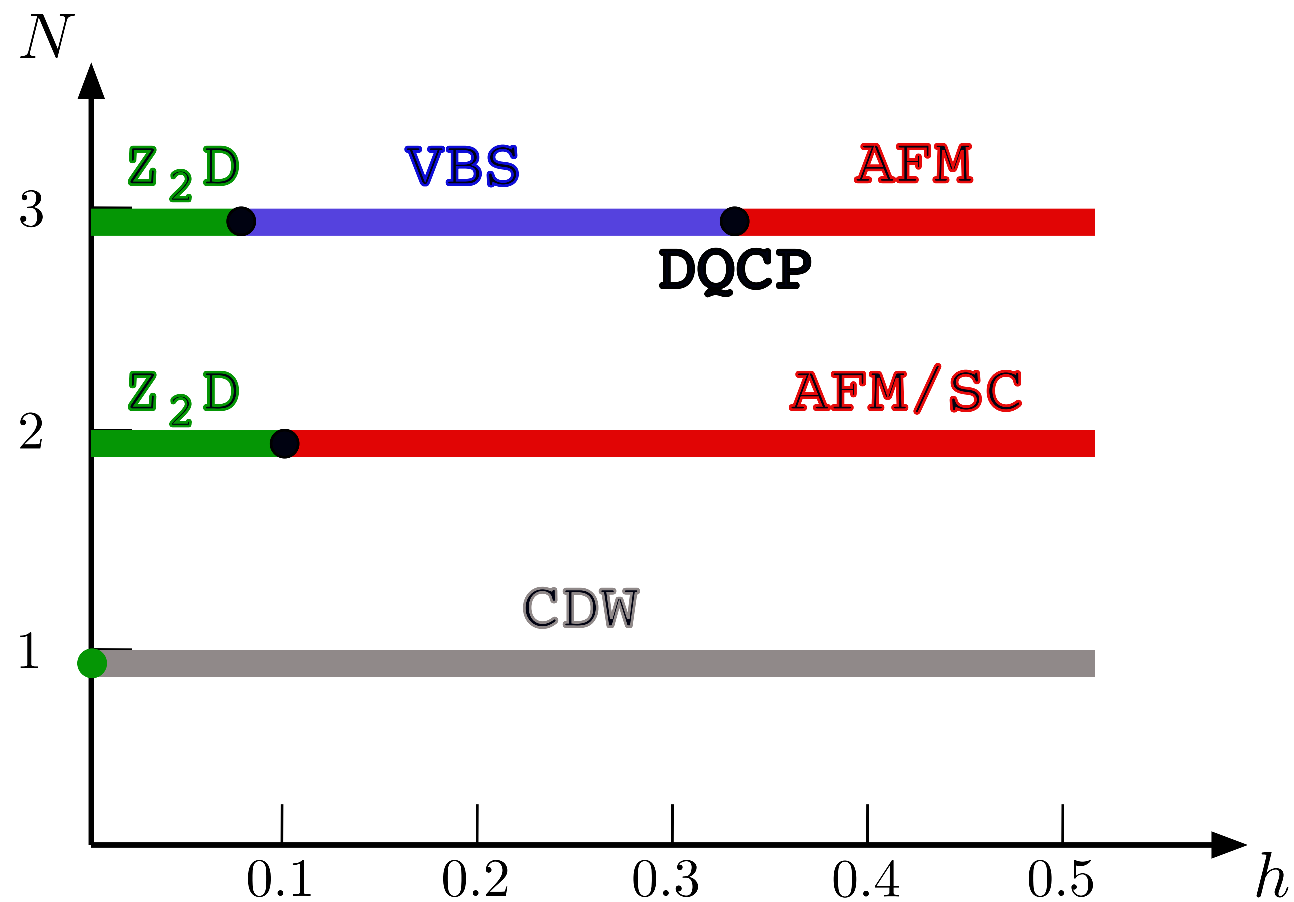
The Model



$$\hat{H} = \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{Z}_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \left(\sum_{\alpha=1}^{N} \hat{c}^{\dagger}_{\boldsymbol{i},\alpha} \hat{c}_{\boldsymbol{j},\alpha} + \mathrm{H.c.} \right) - Nh \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{X}_{\langle \boldsymbol{i},\boldsymbol{j} \rangle}$$

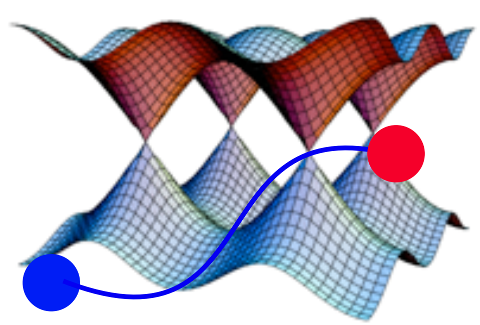
With “normal” hopping: Y. Schattner, S. Lederer, S. A. Kivelson, and E. Berg 2015

The Phase Diagram



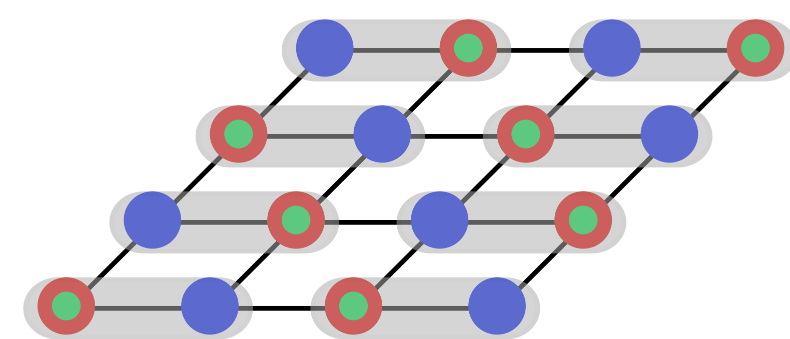
Z_2D

(N=2)



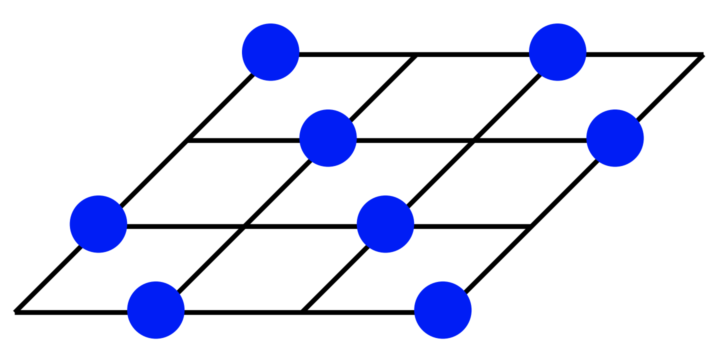
VBS

(N=3)



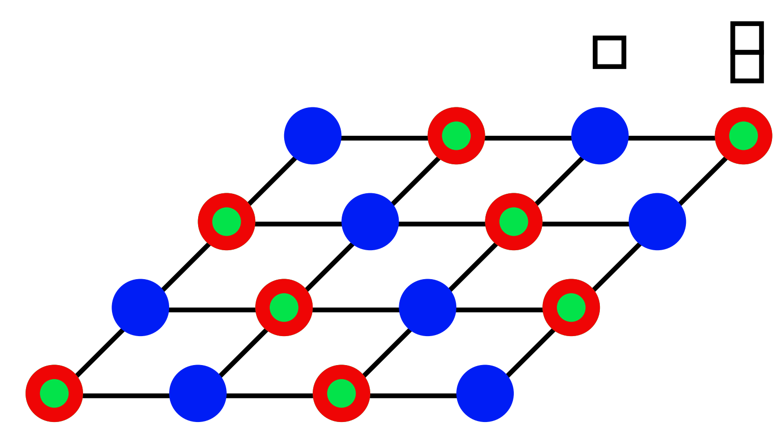
CDW

(N=1)



AFM

(N=3)



- “Simple” model: only one continuous tuning parameter.

- Hosts a gauge theory without being a bona fide gauge theory. Conceptually interesting.

- No fermion sign problem! Results obtained using auxiliary field Monte Carlo.

- Hosts a deconfined quantum critical point and possibly other exotic criticality.

- Related: An application of the sign problem to deduce a result on many-body Hamiltonians.

Symmetries

- Particle-hole symmetries:

$$\hat{P}_\alpha^{-1} \hat{c}^\dagger_{\boldsymbol{i},\beta} \hat{P}_\alpha = \delta_{\alpha,\beta} e^{i\boldsymbol{K}\cdot\boldsymbol{i}} \hat{c}_{\boldsymbol{i},\beta} + (1 - \delta_{\alpha,\beta})$$

Explicitly: $\hat{P}_\alpha = \prod_i (c_{\boldsymbol{i},\alpha} e^{i\boldsymbol{K}\cdot\boldsymbol{i}/2} + \hat{c}^\dagger_{\boldsymbol{i},\alpha} e^{-i\boldsymbol{K}\cdot\boldsymbol{i}/2})$

- O(2N) symmetry:

$$\gamma_i \rightarrow O_{ij}\gamma_j$$

$$c_{\boldsymbol{j},\alpha} = \left(\gamma_{\boldsymbol{j},\alpha,1} + i\gamma_{\boldsymbol{j},\alpha,2}\right)/2$$

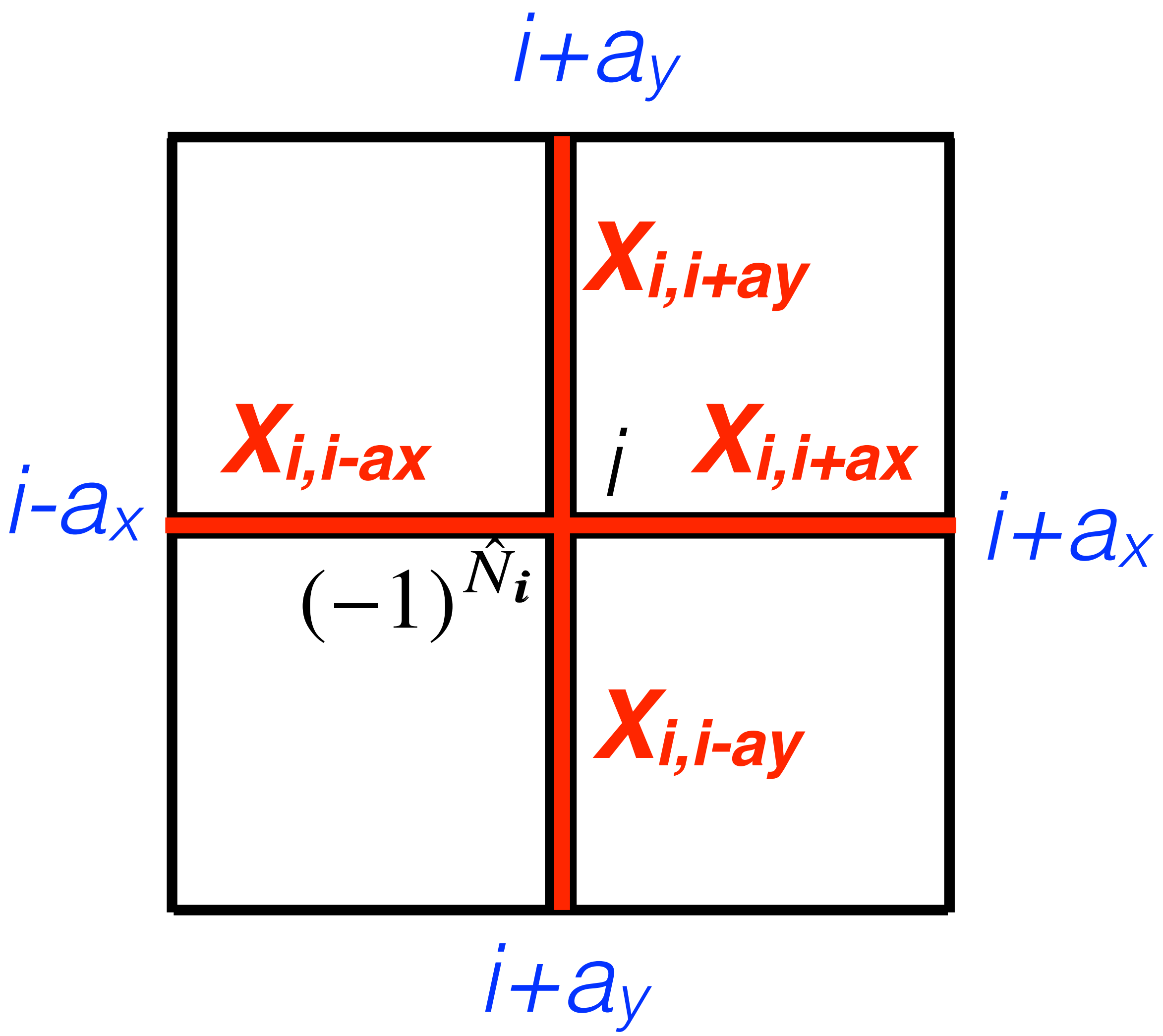
A more interesting symmetry

$$\left[\hat{Q}_{\boldsymbol{i}}, \hat{H}\right] = 0$$

$$\hat{Q}_{\boldsymbol{i}} = \hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y}\hat{p}_{\boldsymbol{i}}$$

$$\hat{p}_{\boldsymbol{i}} = \prod_{\alpha=1}^{N}(1 - 2\hat{n}_{\boldsymbol{i},\alpha}) = (-1)^{\hat{N}_{\boldsymbol{i}}}$$

= Fermion parity



Infinite # of conservation laws but not integrable

Note: $\hat{P}_\alpha^{-1}\hat{Q}_{\boldsymbol{i}}\hat{P}_\alpha = -\hat{Q}_{\boldsymbol{i}}$

Resemblance with Z_2 Gauge-Matter theory

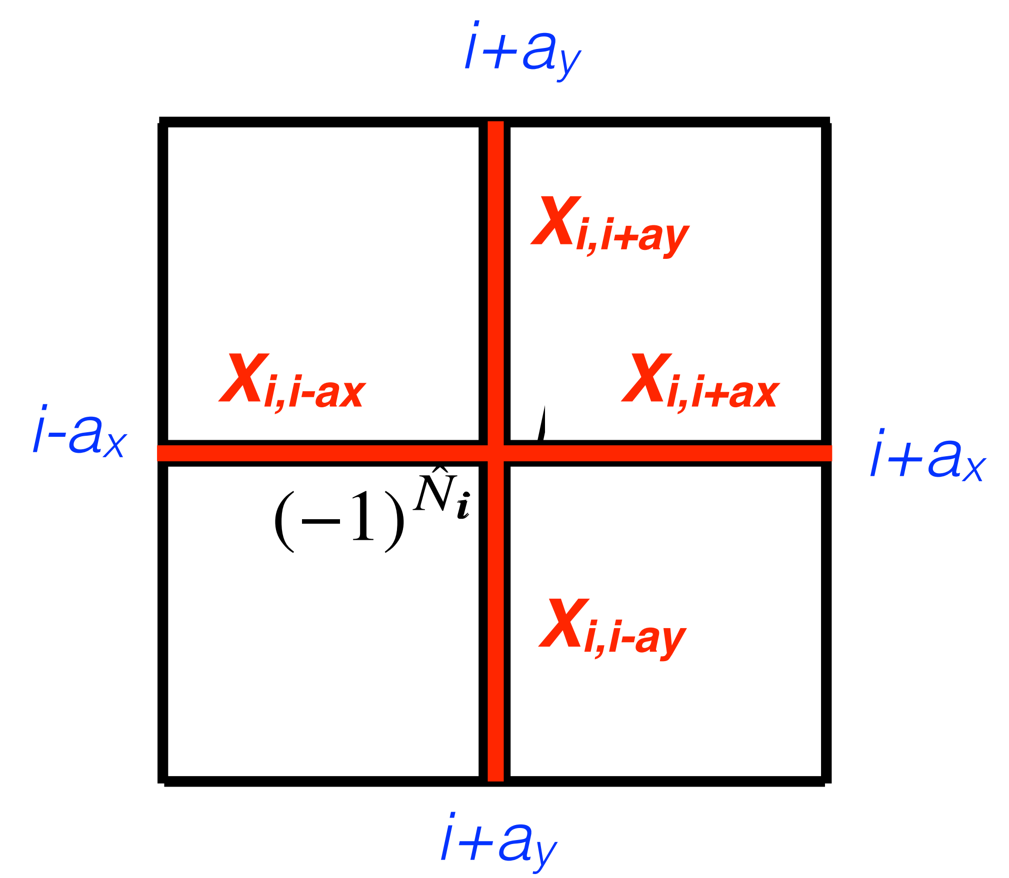
Consider imposing the constraint:

$$\hat{Q}_{\boldsymbol{i}} = \hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y}\hat{p}_{\boldsymbol{i}} = 1$$

Equivalently:

$$\hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y}\ (\equiv \boldsymbol{\nabla}.\boldsymbol{E}) = \hat{p}_{\boldsymbol{i}}\ (\equiv \rho)$$

"Gauss's law"



$$H' = \left(\prod_i \frac{(1+Q_i)}{2}\right) H \left(\prod_i \frac{(1+Q_i)}{2}\right)$$

Gazit, Randeria, Vishwanath, arXiv:1607.03892, N=2 model with Gauss Law.

Our model $\neq$ Gauge Theory

Unlike our model, in a gauge theory, there are no local symmetries.

The would-be local symmetries in a gauge theory are eliminated by imposing Gauss's law $\boldsymbol{\nabla}.\boldsymbol{E} = \rho$

We ***do not*** enforce Gauss's law.

Emergence of Gauss's law at T = 0

$$\hat{Q}_{\boldsymbol{i}} = \hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y}\hat{p}_{\boldsymbol{i}}$$ is conserved

Which set of $\boldsymbol{Q_i}$ will minimize ground state energy?

Emergence of Gauss's law at T = 0

Effective model of Q_i's (classical variables):

$$\hat{H}_Q = \sum_{\boldsymbol{i},\boldsymbol{j}} K_{\boldsymbol{i},\boldsymbol{j}} \hat{Q}_{\boldsymbol{i}} \hat{Q}_{\boldsymbol{j}}$$

Terms linear in Q_i not allowed due to particle-hole symmetry.

Perturbatively in h:

$$K_{i,i+1} \propto (-1)^{N+1}$$

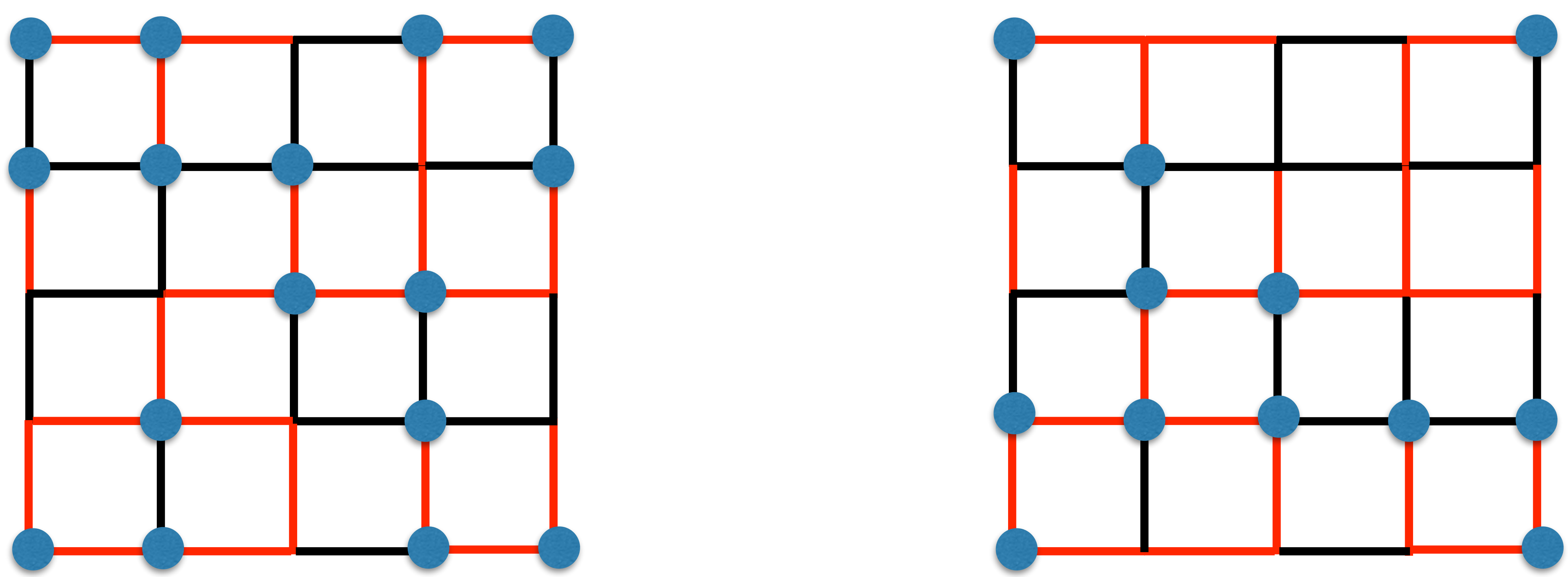
More convincingly, numerics show $\boldsymbol{Q_i}$ order ferromagnetically for even *N* and anti-ferromagnetically for odd *N*.

Emergence of Gauss's law at T = 0

$$\hat{Q}_{\boldsymbol{i}} = \hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x}\hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y}\hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y}\hat{p}_{\boldsymbol{i}}$$

N = even $\quad \boldsymbol{Q_i} = 1$

N = odd $\quad \boldsymbol{Q_i} = (-1)^{x+y}$



● = odd number of fermions

$X_{ij} = -1$ (red)

$X_{ij} = +1$ (black)

Particle-hole Symmetry Breaking due to Ordering of Q_i

At T = 0

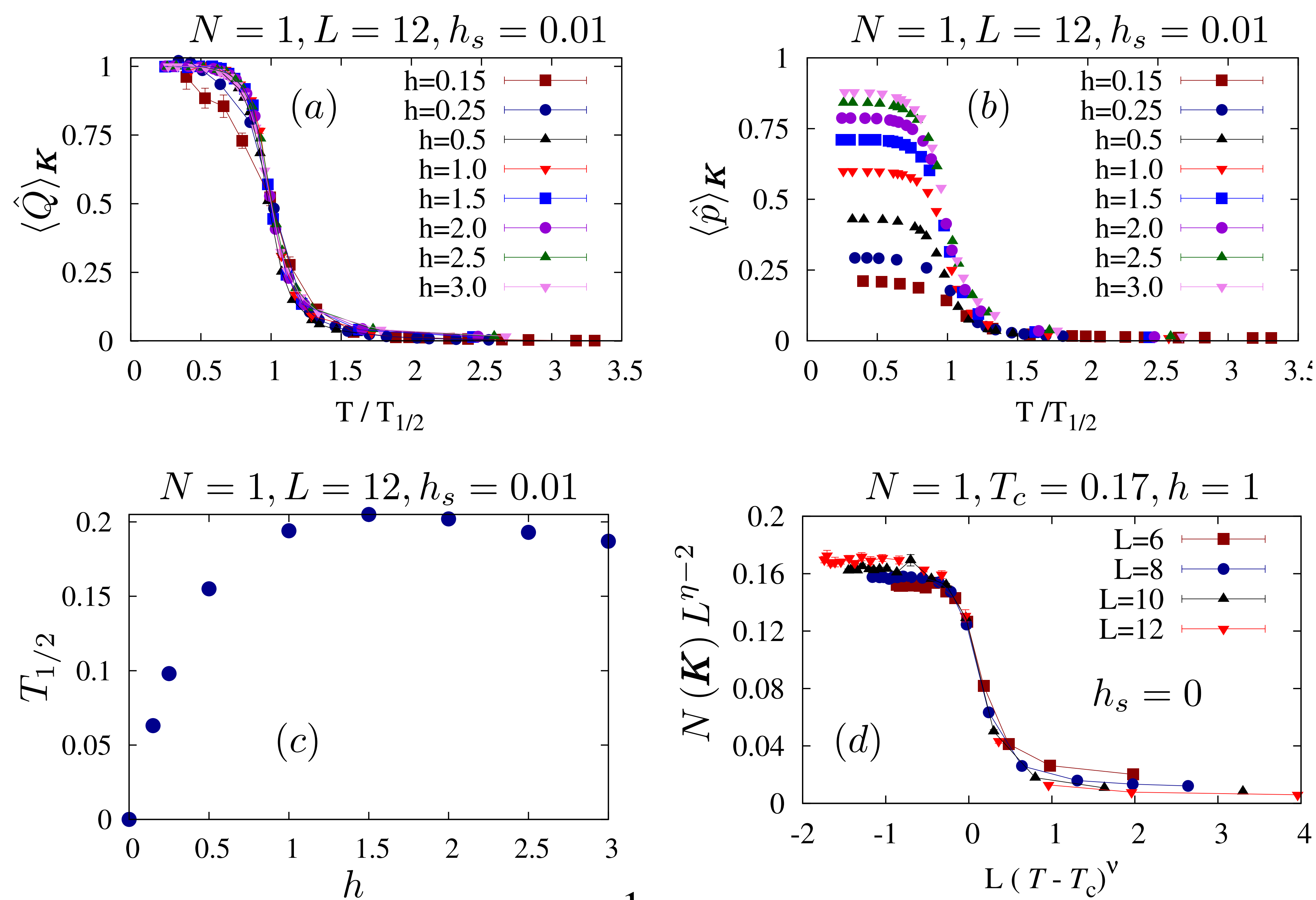
$$\hat{Q}_{\boldsymbol{i}}|0\rangle = e^{iN\boldsymbol{K}\cdot\boldsymbol{i}}|0\rangle$$

This generates a term $\sum_i e^{iN\boldsymbol{K}\cdot\boldsymbol{i}}\hat{p}_{\boldsymbol{i}} = \sum_i e^{iN\boldsymbol{K}\cdot\boldsymbol{i}}\prod_{\alpha=1}^{N}(1-2\hat{n}_{\boldsymbol{i},\alpha})$

in the effective Hamiltonian, thus breaking the particle-hole symmetry.

This symmetry is restored above $T = T_C > 0$

High temperature restoration of Particle-hole symmetry (Ising Transition)

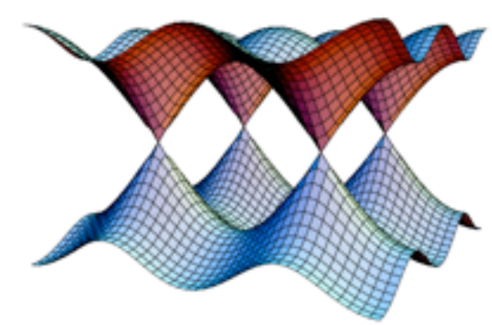


$$N(\boldsymbol{q}) = \frac{1}{4}\sum_{\boldsymbol{r}} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\langle \hat{p}_{\boldsymbol{r}}\hat{p}_{\boldsymbol{0}}\rangle$$

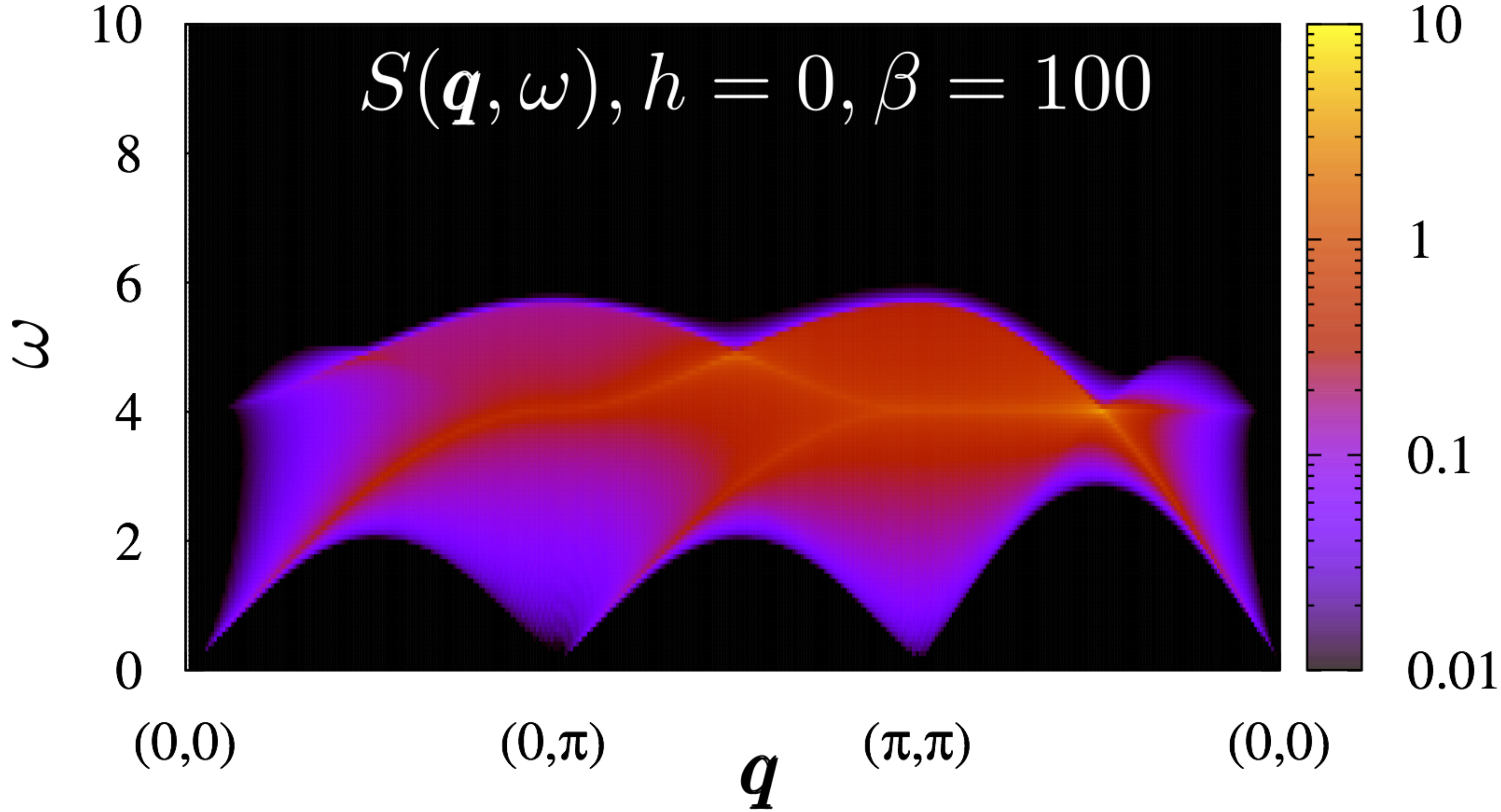
Soluble Limits #1: h = 0

Ground state energy minimized when Z_2 flux through each plaquette is π (Lieb 1994).

Low energy theory: 2N Dirac fermions coupled to static Z_2 gauge field.



Are Dirac fermions stable to dynamic gauge field? (i.e. when $h \neq 0$).



Dynamical spin structure factor

Small h

Particle-hole symmetry breaking dictates small-h phase diagram

N = 1 $$H_{eff} = H_{Dirac} + h^{\alpha} \sum_i e^{i\mathbf{K}\cdot\mathbf{r}} \left(1 - 2\hat{n}_{\mathbf{r}}\right)$$

⇒ CDW gap for *infinitesimal* h.

N = 2 $$H_{eff} = H_{Dirac} + h^{\alpha} \sum_i \left(1 - 2\hat{n}_{\mathbf{i},1}\right)\left(1 - 2\hat{n}_{\mathbf{i},2}\right)$$

⇒ Z_2 Dirac phase stable. Same conclusion for N=3.

Soluble Limits #2: h = ∞

$$\hat{H}_{\infty} = -\frac{1}{2Nh} \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \hat{k}^2_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \qquad \hat{k}_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} = \sum_{\alpha=1}^{N} \left(\hat{c}^{\dagger}_{\boldsymbol{i},\alpha} \hat{c}_{\boldsymbol{j},\alpha} + \text{H.c.} \right)$$

N = 1 $$H = \frac{1}{2Nh} \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \left(n_{\boldsymbol{i}} - \frac{1}{2} \right) \left(n_{\boldsymbol{j}} - \frac{1}{2} \right)$$

⇒ CDW ground state.

N = 2 $$H = \frac{1}{h} \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \left(\hat{\boldsymbol{S}}_{\boldsymbol{i}} \hat{\boldsymbol{S}}_{\boldsymbol{j}} + \hat{\boldsymbol{\eta}}_{\boldsymbol{i}} \hat{\boldsymbol{\eta}}_{\boldsymbol{j}} \right)$$

⇒ AFM/SC+CDW ground state.

N = ∞ Valence bond solid (Affleck, Marston, 1988)

Exploiting
Hardness of Sign Problem

Consider the following equality on *arbitrary* graphs

$$\hat{H}_\infty = -\sum_{\boldsymbol{i},\boldsymbol{j}} J_{\boldsymbol{i},\boldsymbol{j}} \left[\sum_{\sigma=1}^{2} \left(\hat{c}^\dagger_{\boldsymbol{i},\sigma}\hat{c}_{\boldsymbol{j},\sigma} + \text{H.c.}\right)\right]^2 \qquad \left[\hat{\eta}_{\boldsymbol{j},\alpha}, \hat{S}_{\boldsymbol{i},\beta}\right] = 0$$

$$= \sum_{\boldsymbol{i},\boldsymbol{j}} J_{\boldsymbol{i},\boldsymbol{j}} \left[\hat{\boldsymbol{S}}_{\boldsymbol{i}} . \hat{\boldsymbol{S}}_{\boldsymbol{j}} + \left(\hat{\eta}^z_{\boldsymbol{i}}\hat{\eta}^z_{\boldsymbol{j}} - \frac{1}{2}\left(\hat{\eta}^+_{\boldsymbol{i}}\hat{\eta}^-_{\boldsymbol{j}} + \hat{\eta}^-_{\boldsymbol{i}}\hat{\eta}^+_{\boldsymbol{j}}\right)\right)\right]$$

$$\hat{\boldsymbol{S}}_{\boldsymbol{i}} = \frac{1}{2}\sum_{\sigma,\sigma'} \hat{c}^\dagger_{\boldsymbol{i},\sigma}\boldsymbol{\sigma}_{\sigma,\sigma'}\hat{c}_{\boldsymbol{i},\sigma'} \qquad \hat{\boldsymbol{\eta}}_{\boldsymbol{i}} = \hat{P}_1^{-1}\hat{\boldsymbol{S}}_{\boldsymbol{i}}\hat{P}_1$$

"Anderson's Pseudospin"

$$\left[\hat{\eta}_{i,\alpha}, \hat{S}_{j,\beta}\right] = 0$$

$$\hat{P}_\alpha^{-1}\hat{c}^\dagger_{\boldsymbol{i},\beta}\hat{P}_\alpha = \delta_{\alpha,\beta}e^{i\boldsymbol{K}\cdot\boldsymbol{i}}\hat{c}_{\boldsymbol{i},\beta} + \left(1-\delta_{\alpha,\beta}\right)\hat{c}^\dagger_{\boldsymbol{i},\beta}$$

(particle-hole transformation)

$$\hat{H}_\infty = -\sum_{i,j} J_{i,j} \left[\sum_{\sigma=1}^{2} \left(\hat{c}^\dagger_{i,\sigma}\hat{c}_{j,\sigma} + \text{H.c.}\right)\right]^2$$

No sign problem when $J_{ij} > 0$

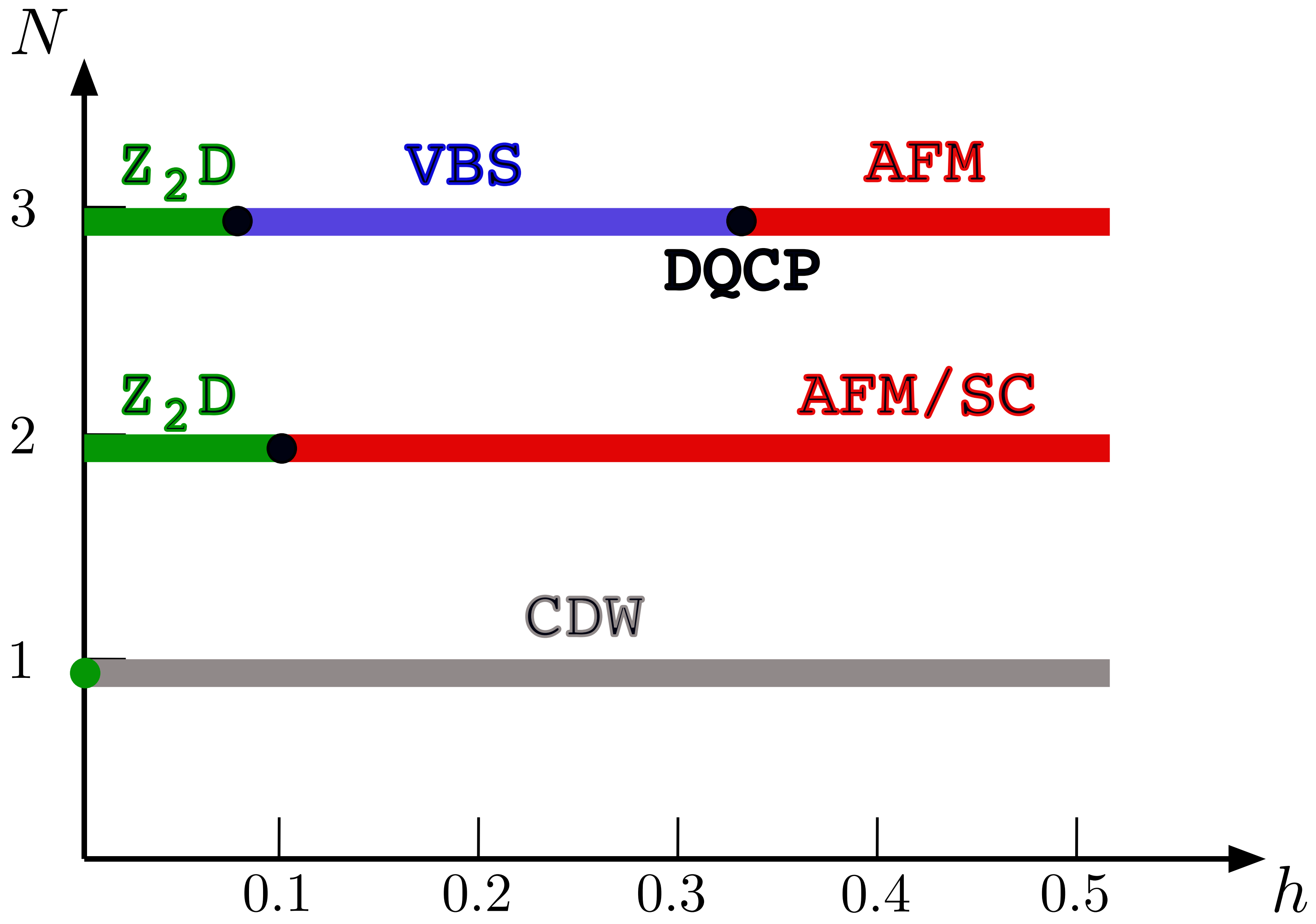
$$= \sum_{i,j} J_{i,j} \left[\hat{\boldsymbol{S}}_i . \hat{\boldsymbol{S}}_j + \left(\hat{\eta}^z_i \hat{\eta}^z_j - \frac{1}{2}\left(\hat{\eta}^+_i \hat{\eta}^-_j + \hat{\eta}^-_i \hat{\eta}^+_j\right)\right)\right]$$

Sign problem for S, but not η when $J_{ij} > 0$

Implications Assuming Sign Problem Can't be Solved on a general Graph

1. The ground state of H_∞ lies in the η sector i.e. only zero or double occupancy of electrons.

2. The ground state energy of $\hat{H}_{\boldsymbol{\eta}} = \sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} J_{\boldsymbol{i},\boldsymbol{j}} \left(\hat{\eta}^z_{\boldsymbol{i}}\hat{\eta}^z_{\boldsymbol{j}} - \frac{1}{2}\left(\hat{\eta}^+_{\boldsymbol{i}}\hat{\eta}^-_{\boldsymbol{j}} + \hat{\eta}^-_{\boldsymbol{i}}\hat{\eta}^+_{\boldsymbol{j}}\right)\right)$ is always lower than that of $\hat{H}_{\boldsymbol{s}} = \sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} J_{\boldsymbol{i},\boldsymbol{j}} \left(\hat{s}^z_{\boldsymbol{i}}\hat{s}^z_{\boldsymbol{j}} + \frac{1}{2}\left(\hat{s}^+_{\boldsymbol{i}}\hat{s}^-_{\boldsymbol{j}} + \hat{s}^-_{\boldsymbol{i}}\hat{s}^+_{\boldsymbol{j}}\right)\right)$. when $J_{\boldsymbol{i},\boldsymbol{j}} \geq 0$, (already proved by Nie, Katsura, Oshikawa 2013).



N
3
2
1
Z_2D
VBS
AFM
DQCP
Z_2D
AFM/SC
CDW
0.1
0.2
0.3
0.4
0.5
h

Supersymmetry for N=2

If ground state has $\boldsymbol{Q_i}$ =1 ⇒ superconducting order.

If ground state has $\boldsymbol{Q_i}$ =-1 ⇒ AFM order.

The AFM and SC/CDW ground states are supersymmetric partners when the number of lattice sites is odd (see Hsieh, Halasz, Grover 2016).

This is because the particle-hole transformation operator is *fermionic*.

$$\hat{P}_\alpha = \prod_i (c_{\boldsymbol{i},\alpha} e^{i\boldsymbol{K}\cdot\boldsymbol{i}/2} + \hat{c}^\dagger_{\boldsymbol{i},\alpha} e^{-i\boldsymbol{K}\cdot\boldsymbol{i}/2})$$

Quick Recap of Supersymmetry

$$[O_1, H] = 0 \qquad [O_2, H] = 0 \qquad \{O_1, O_2\} = 0$$

$H|\psi_1\rangle = E|\psi_1\rangle$ with $O_1|\psi_1\rangle = \pm|\psi_1\rangle$

$\Rightarrow$ $|\psi_2\rangle = O_2|\psi_1\rangle$ is a different eigenstate with the same energy.

If O_1 is bosonic and O_2 fermionic, then one has supersymmetry.

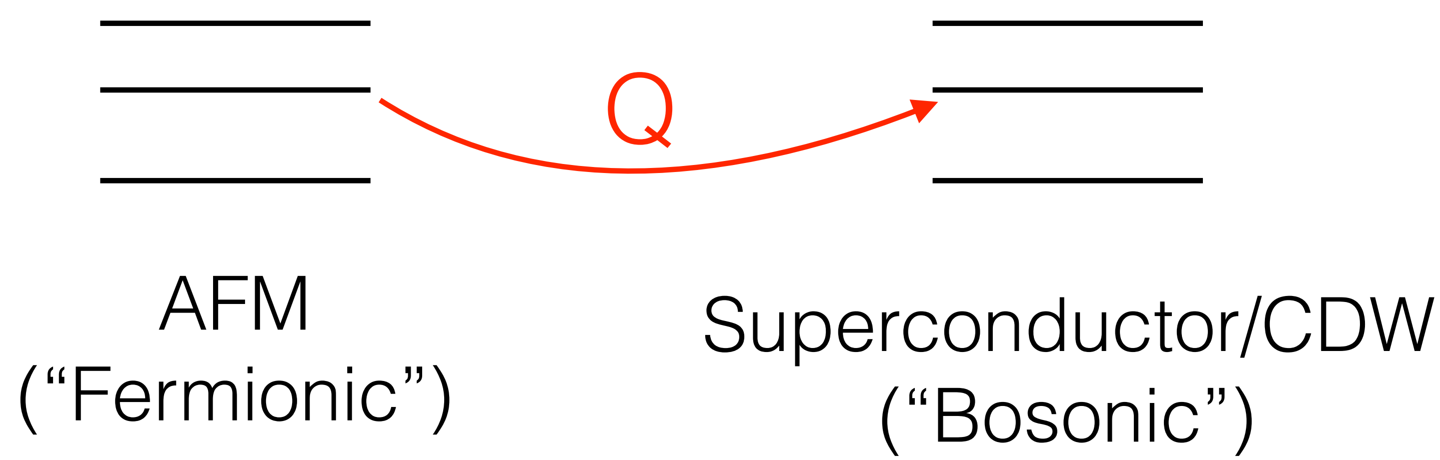
$$\hat{Q} = \sqrt{\frac{H}{2}}\, O_2\,(1 + O_1) \qquad [H, \hat{Q}] = 0 \qquad \{\hat{Q}, \hat{Q}^\dagger\} = 2H$$

In our case,

O_1 = Fermion parity.

O_2 = Particle-hole on up-electrons.

$\hat{P}_\alpha = \prod_i (c_{\boldsymbol{i},\alpha} e^{i\boldsymbol{K}\cdot\boldsymbol{i}/2} + \hat{c}^\dagger_{\boldsymbol{i},\alpha} e^{-i\boldsymbol{K}\cdot\boldsymbol{i}/2})$



Phase transition from Z_2D to AFM

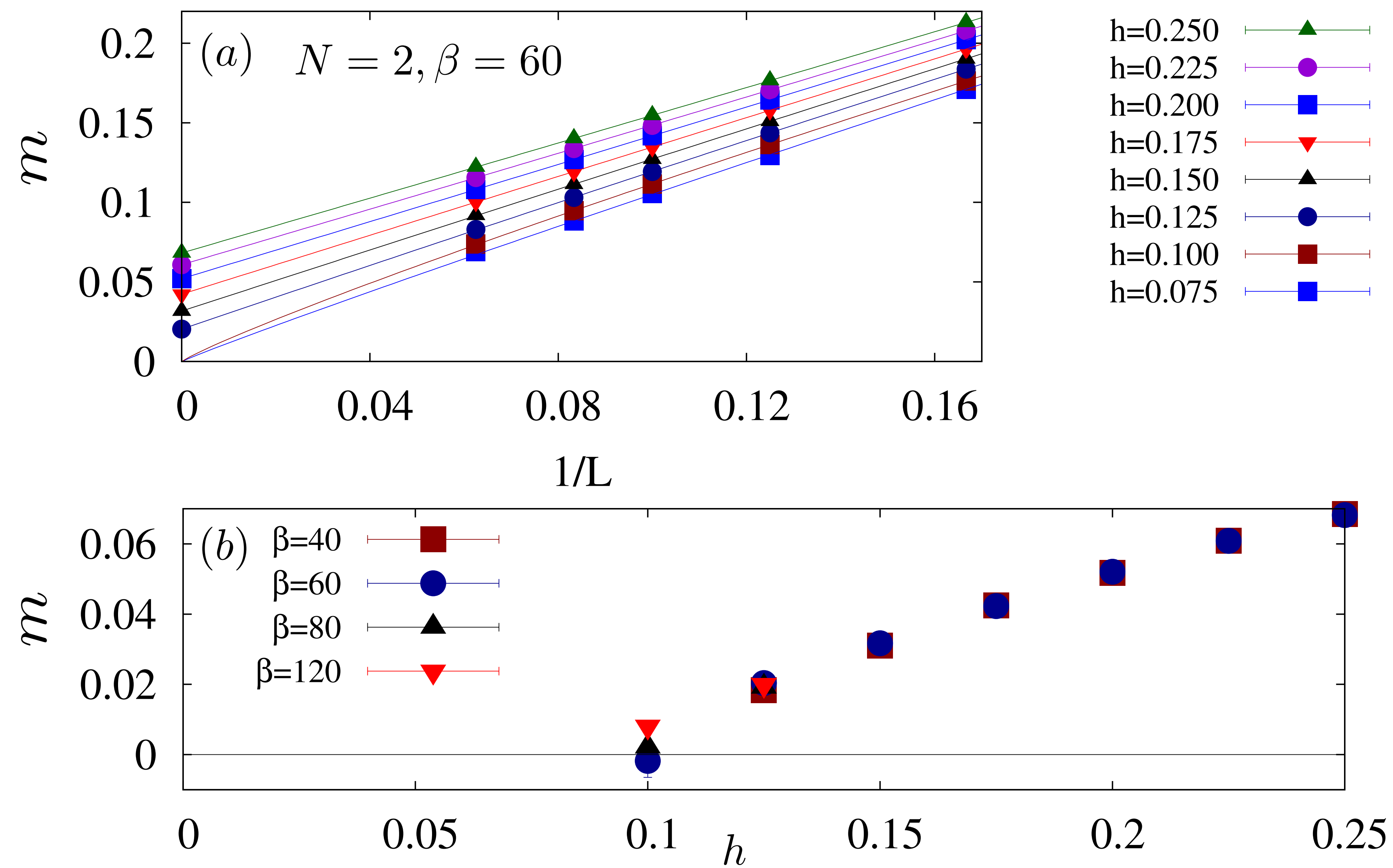
If second order, then no understanding of this transition as yet!

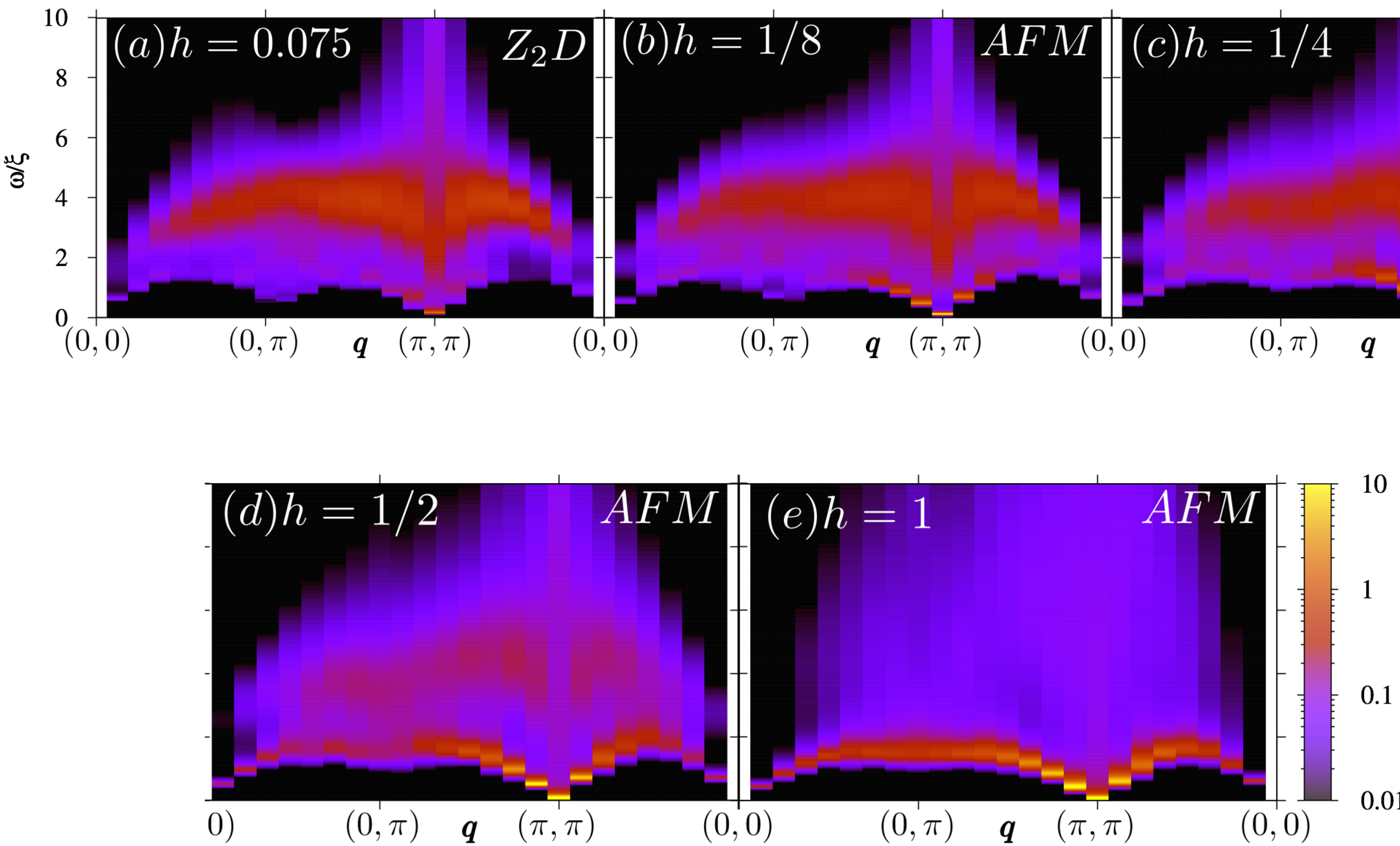
Two things happen at the same point:

Confinement (loss of Z2 topological order) &

AFM ordering.

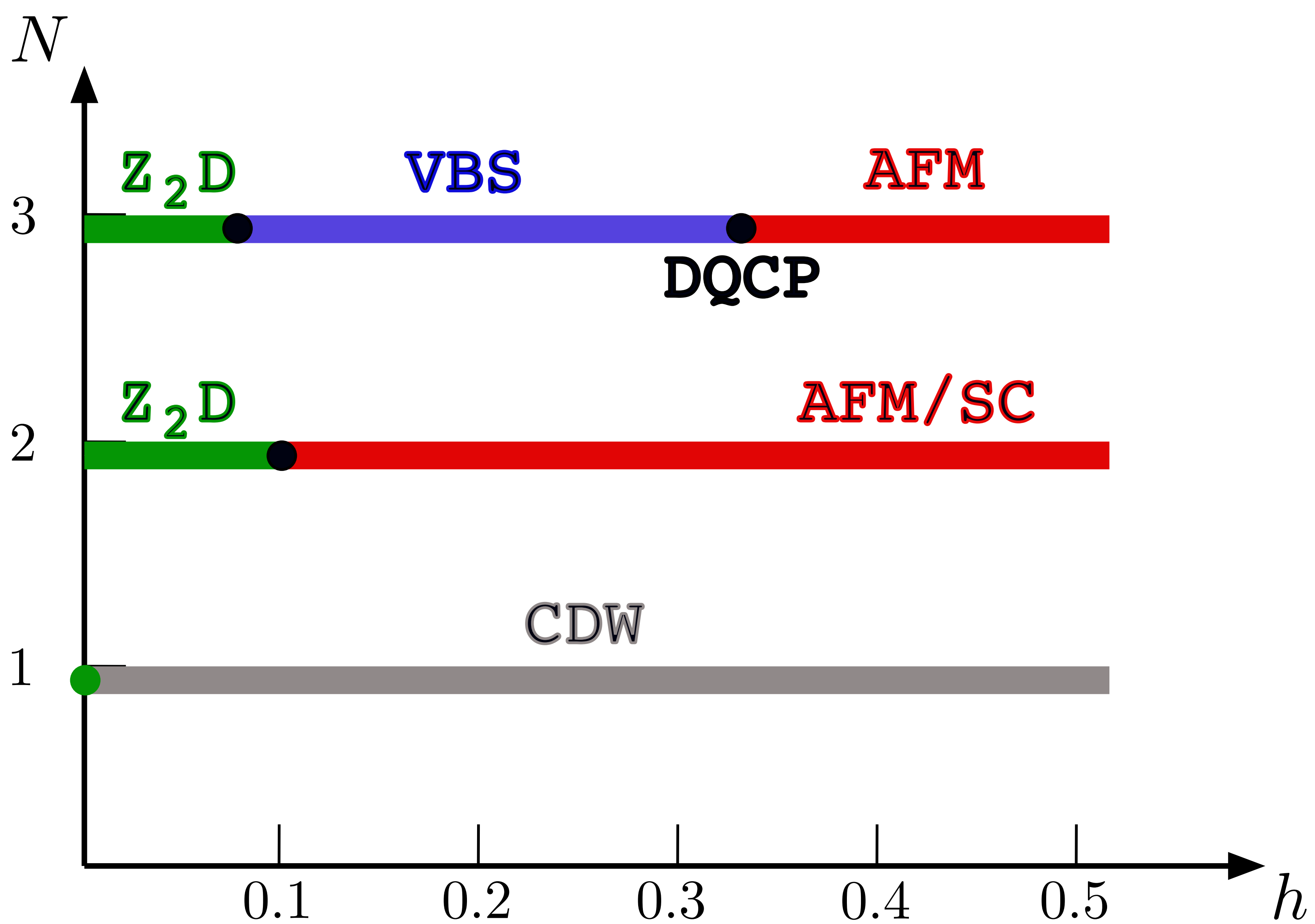
Phase transition from Z_2D to AFM





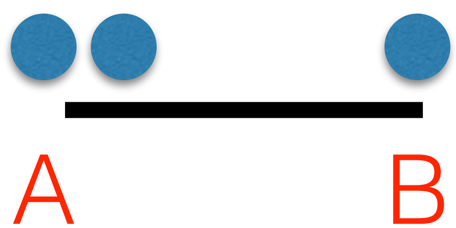
Dynamical spin structure factor for the $N = 2$ model as a function of h

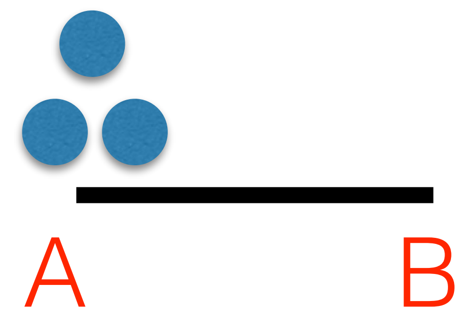
Features of N = 3 Phase Diagram



Pattern of Symmetry Breaking

Two distinct possibilities:





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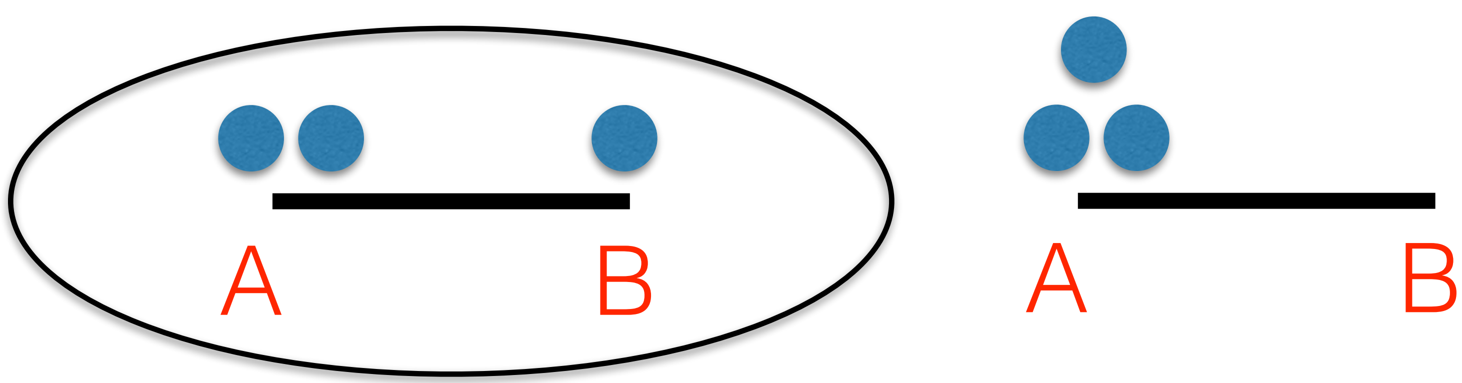
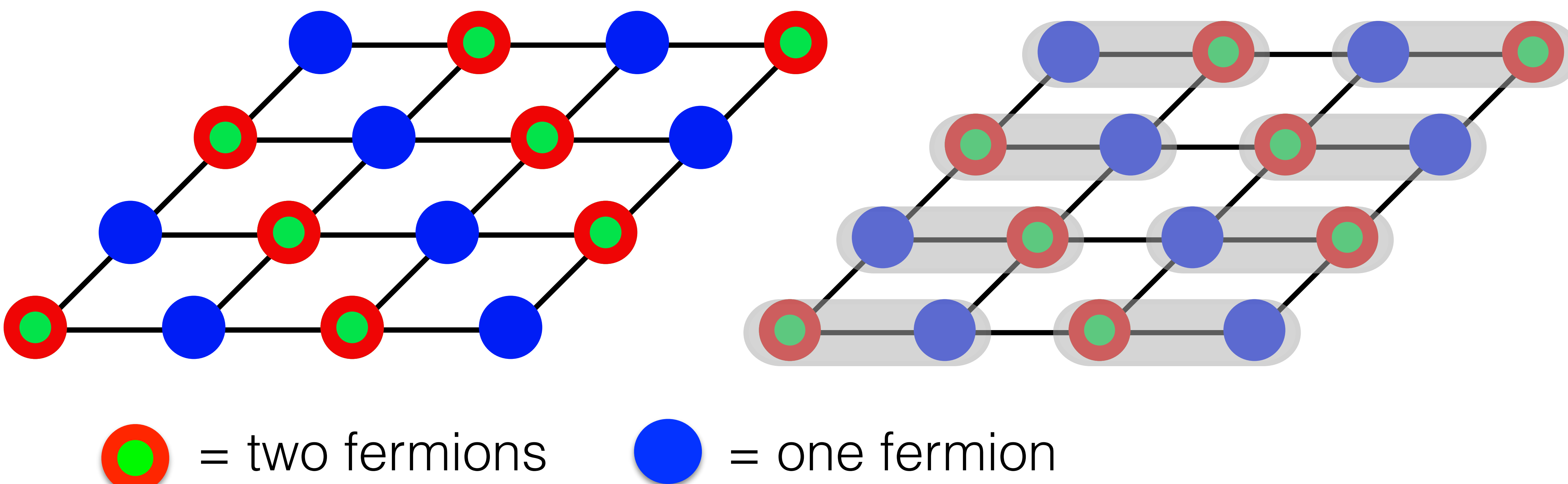


Pattern of Symmetry Breaking

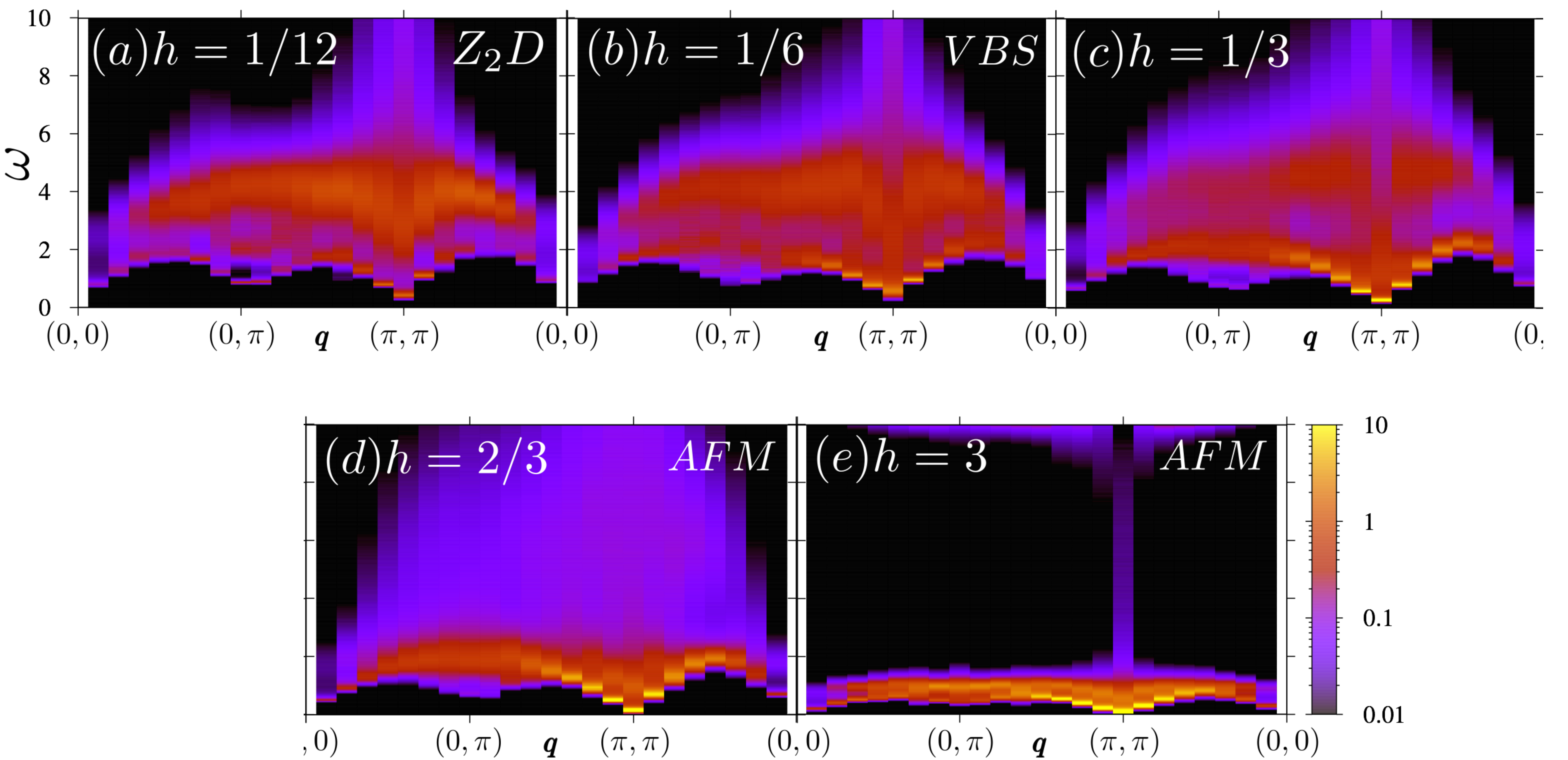
$$\Delta^\dagger_{\alpha,\boldsymbol{i}} = \frac{1}{2}\epsilon_{\alpha,\beta,\gamma}\hat{c}^\dagger_{\beta,\boldsymbol{i}}\hat{c}^\dagger_{\gamma,\boldsymbol{i}}$$

$$|\text{Néel}\rangle = \prod_{\boldsymbol{i}\in \mathrm{A}}\Delta^\dagger_{\alpha,\boldsymbol{i}}\prod_{\boldsymbol{j}\in \mathrm{B}}\hat{c}^\dagger_{\boldsymbol{j},\alpha}|0\rangle$$

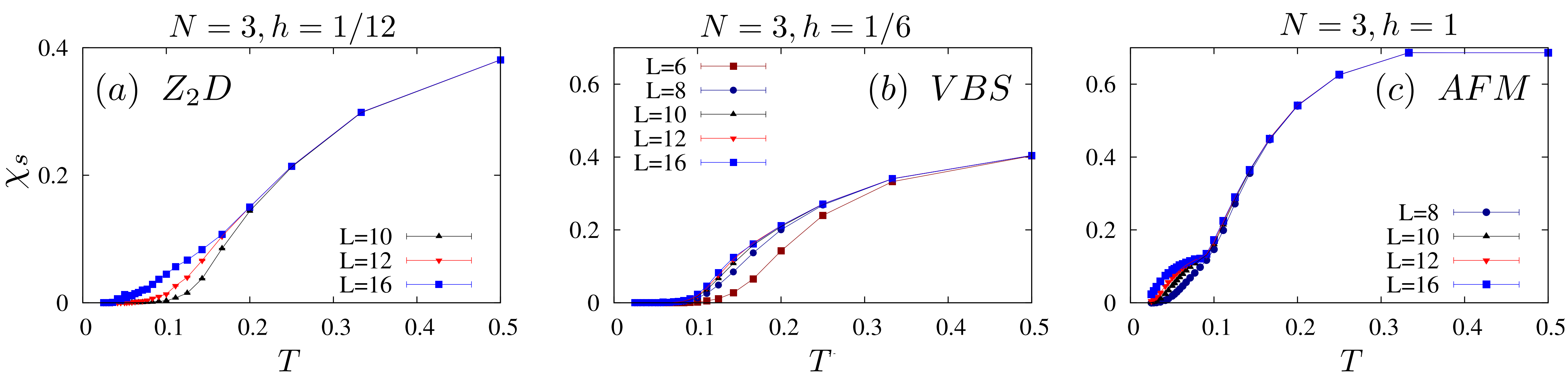
$$|\text{VBS}\rangle = \prod_{\boldsymbol{i},\, e^{i\boldsymbol{q}\cdot\boldsymbol{i}}=1}\left(\sum_{\alpha=1}^{3}\Delta^\dagger_{\alpha,\boldsymbol{i}}c^\dagger_{\alpha,\boldsymbol{i}+\boldsymbol{a}_x}\right)|0\rangle.$$



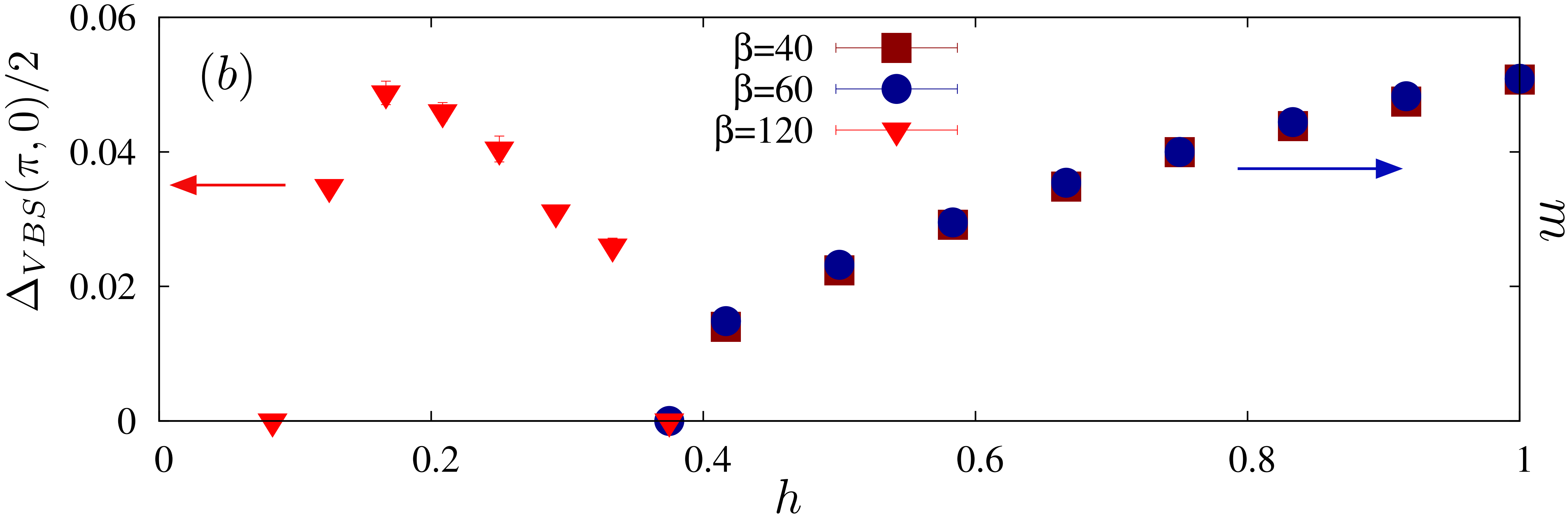
N = 3 Evolution of Dynamical Spin Structure Factor



N = 3 Evolution of Dynamical Spin Structure Factor



Nature of Transition between Neel and VBS?



Seemingly continuous!

Deconfined Criticality Scenario

Senthil, Vishwanath, Balents, Sachdev, Fisher 2004

- Continuous transition between two phases where the broken symmetry in one phase ***is not*** a subgroup of another.

- Defects in one phase carry the quantum numbers of the phase on the other side of the transition.

⇒Proliferating these defects kill the phase

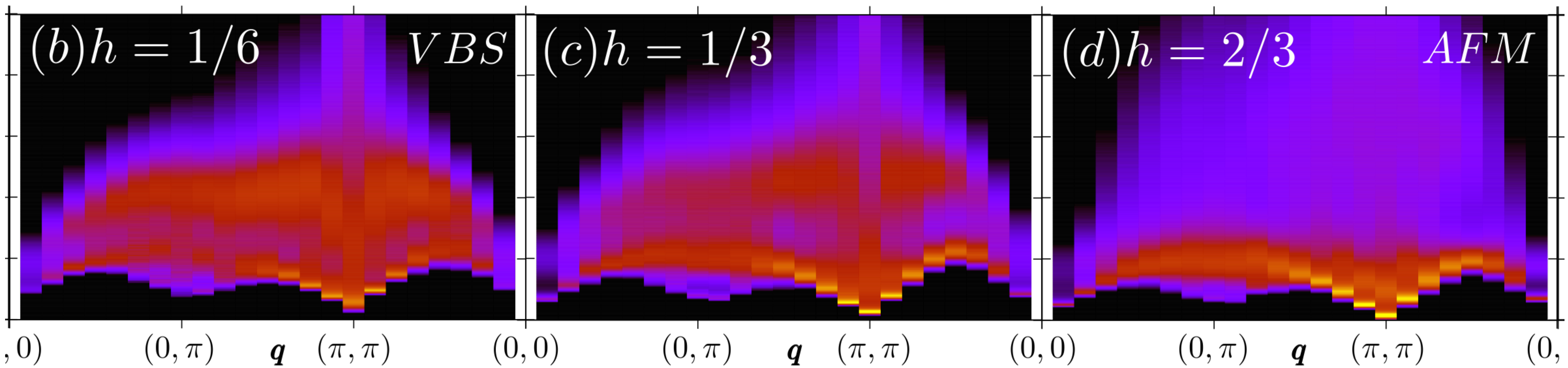
in lieu of generating the other.

Deconfined Criticality Scenario

Deconfined criticality in SU(N) magnets has been previously reported by several authors starting with the work by Sandvik on “J-Q models”. (Sandvik 2007; Melko, Kaul 2008; Kaul, Sandvik 2012; Harada et al 2013; Nahum et al 2015)

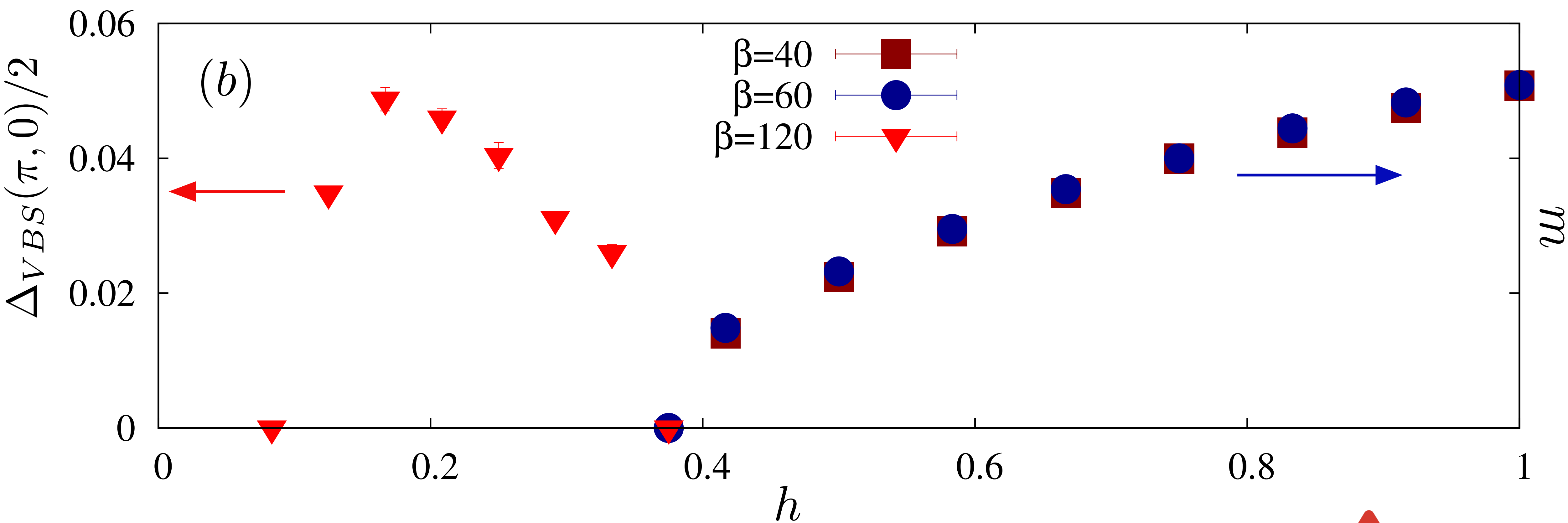
If Neel-VBS transition continuous in our case, then it would provide another example, now in a fermionic model.

Smoking Gun Signals of Deconfined Criticality



Very incoherent excitation spectra close to the transition, unlike Wilson-Fisher transition.

Smoking Gun Signals of Deconfined Criticality



Our Model: $\eta_{VBS} \approx \mathbf{0.6}$ $\eta_{Neel} \approx \mathbf{0.64}$

(L ~ 20, “Small”)

J-Q SU(3) Model: $\eta_{VBS} \approx \mathbf{0.5}$ $\eta_{Neel} \approx \mathbf{0.4}$

(Block, Melko, Kaul 2013)

Summary and Open Questions

- Gauge theory without being a gauge theory. Possible applications to continuous gauge groups?
- Rich phase diagram: Z_2 Dirac Phase, VBS, Neel, CDW. Can tune N as well.
- For N=3, possible deconfined critical point between Neel and VBS.
- Nature of transition between Z_2 Dirac and Neel/VBS?
- Other examples of implications of sign-problem for spectra?
- A model where particle-hole symmetry is not broken at low temperatures?

$$\hat{H} = \sum_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \left[\hat{Z}_{\langle \boldsymbol{i},\boldsymbol{j} \rangle} \left(\hat{c}^{\dagger}_{\boldsymbol{i}} \hat{c}_{\boldsymbol{j}} + \text{H.c.} \right) + u\, \hat{Q}'_{\boldsymbol{i}} \hat{Q}'_{\boldsymbol{j}} \right] + v \sum_{\boldsymbol{i}} \hat{Q}_{\boldsymbol{i}} \qquad \hat{Q}'_{\boldsymbol{i}} = \hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_x} \hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_x} \hat{X}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}_y} \hat{X}_{\boldsymbol{i},\boldsymbol{i}-\boldsymbol{a}_y}$$