Kinematic Space 2:

Seeing into the bulk with Integral Geometry

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How do we probe the bulk?

- From a dual field theory localized on the boundary, we **see into** the bulk.
- Some CFT quantities involve probes **extending inward** from the boundary:
  - Entanglement entropy, Wilson loops
- Local bulk operators however, are represented as **smeared operators** on the boundary
  - How does a smeared operator see **into** the bulk?

Bulk Reconstruction

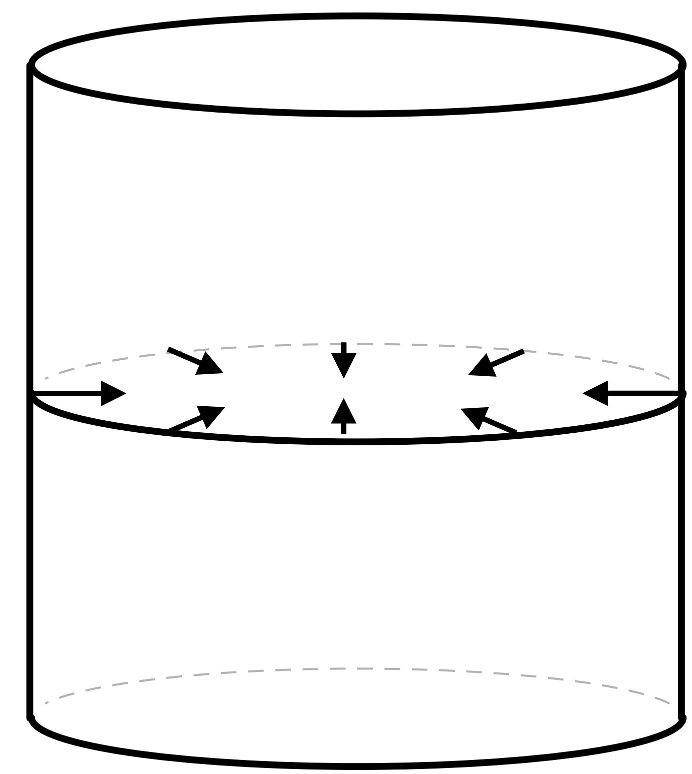
- Let ϕ be a free field in AdS

$$\Box\phi = m^2\phi$$

- The AdS/CFT dictionary relates ϕ to a boundary operator O

$$\phi(z,x) \sim z^{\Delta}\mathcal{O}(x)$$

- How do we solve this boundary value problem?



A **non-standard** Cauchy problem!

Bulk Reconstruction

- Standard construction: bulk local operator is a *smeared* boundary operator
  [Bena 99; Hamilton, Kabat, Lifschytz, Lowe 05]

$$\phi\left(z, x\right) = \int dx' K\left(x' \,|\, z, x\right) \mathcal{O}\left(x'\right)$$

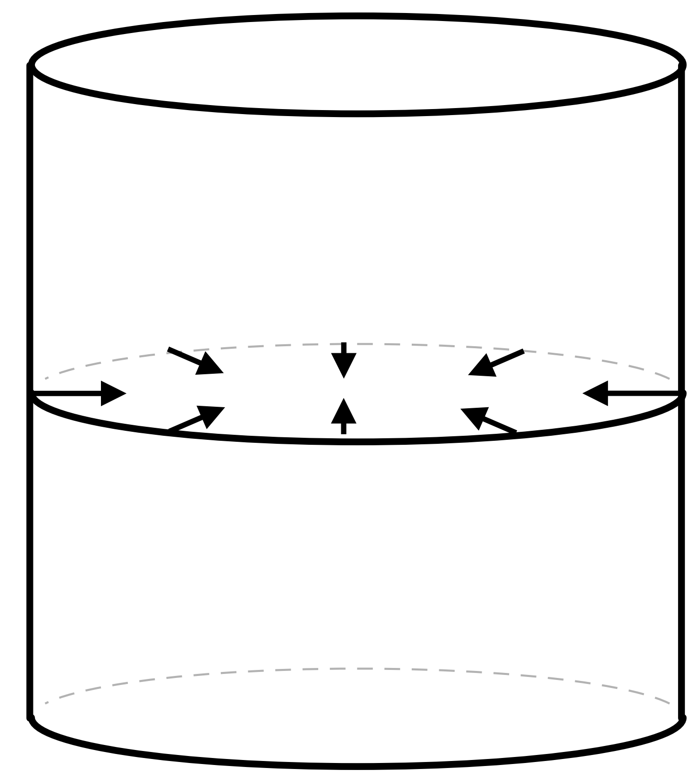
- The smearing function *K* is determined by "brute force" from the bulk mode expansion

- Some remaining questions:

  - Extension to other geometries? (Causal vs Entanglement wedge?)

  - How to write as an operator at one time?

  - Why does this see into the bulk?



A non-standard Cauchy problem!

$$\Box\phi = m^2\phi$$

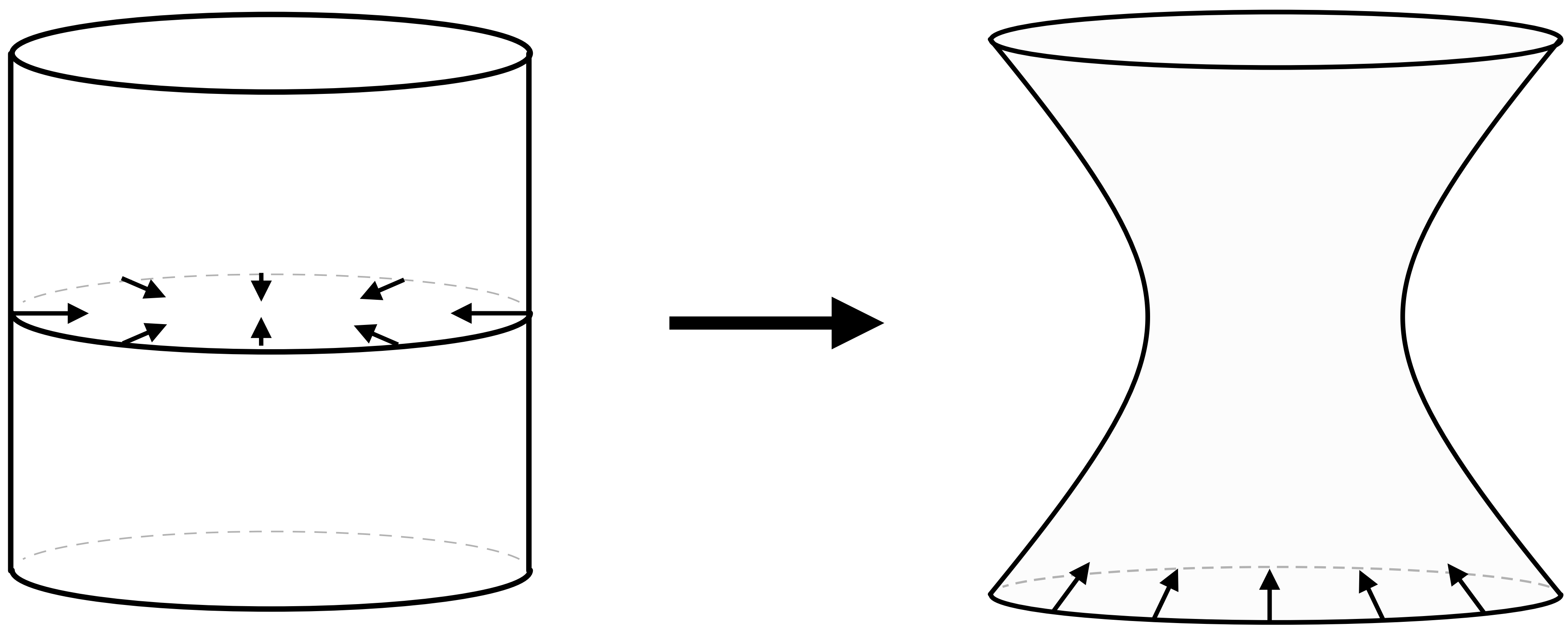
$$\phi(z, x) \sim z^{\Delta}\mathcal{O}(x)$$

Bulk Reconstruction

- A detour through *kinematic space* will provide insight!

Real Space

Kinematic Space



Non-standard
Cauchy problem

Standard
Cauchy problem

Plan

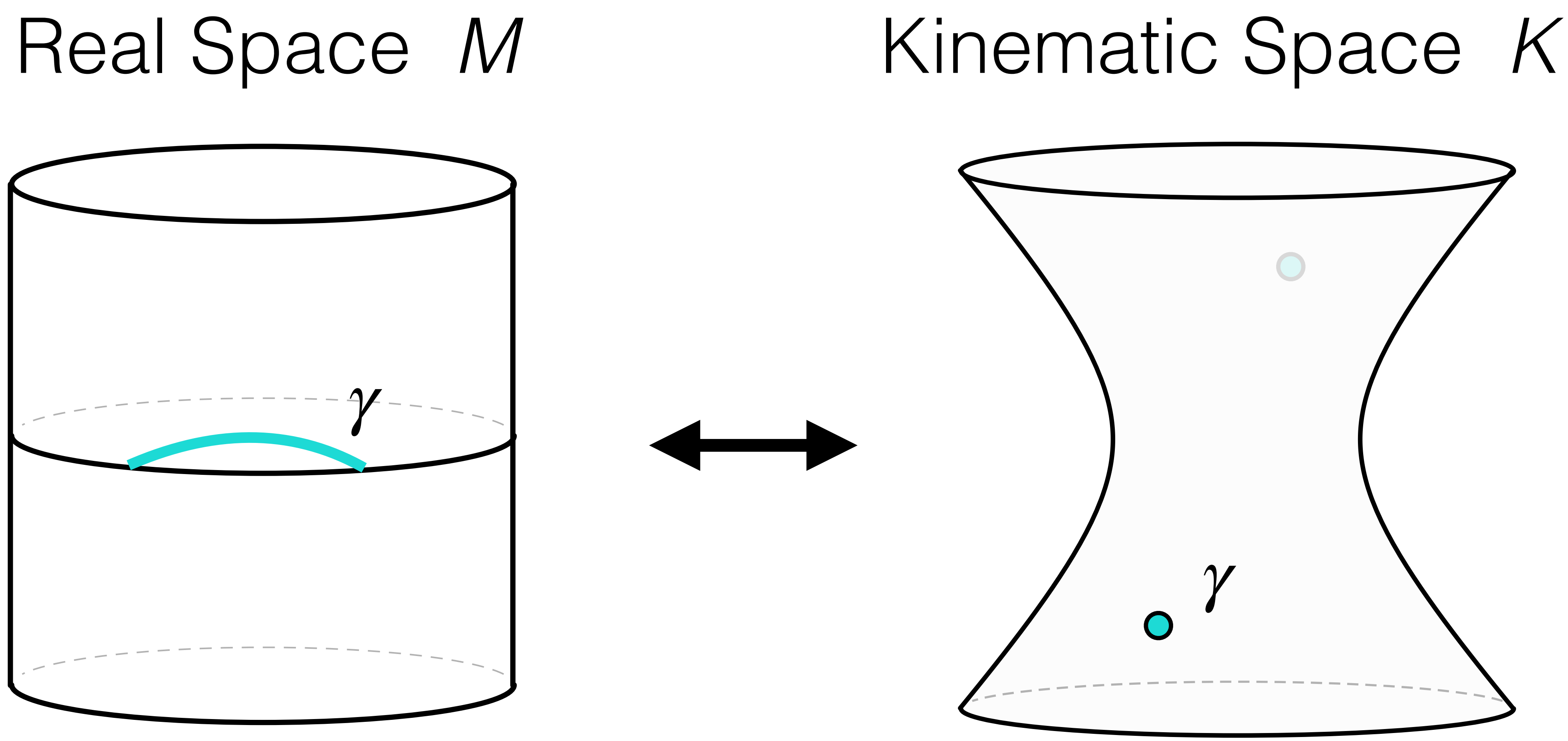
- Bulk field reconstruction and integral geometry
  - The X-Ray transform
  - Structure of kinematic space
  - Intertwining operators and kinematic fields
  - Geodesic operators and Local bulk operators
  - The space of CFT bilocals

- Generalizations and applications (work in progress):
  - Bulk tensor operators and the modular Hamiltonian
  - Higher dimensions
  - MERA
  - Beyond the vacuum
  - Bulk interactions (1/N corrections)

The X-Ray Transform

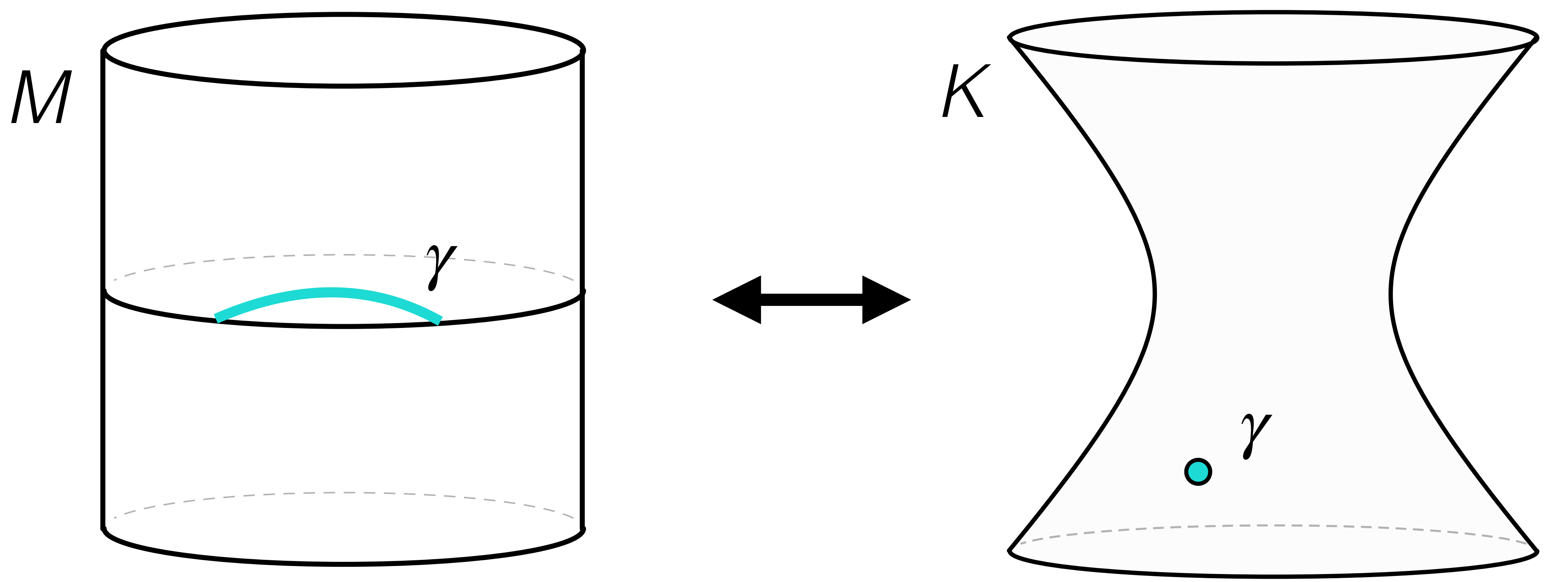
- **Kinematic space** *K*: the space of oriented spacelike geodesics in a manifold *M*

- A point in *K* corresponds to a geodesic in *M*



The X-Ray Transform

- The **X-Ray transform**: maps a function on real space to a function on kinematic space by **integrating over geodesics**
- Inversion formulas are known for some symmetric cases (hyperbolic space, flat space)

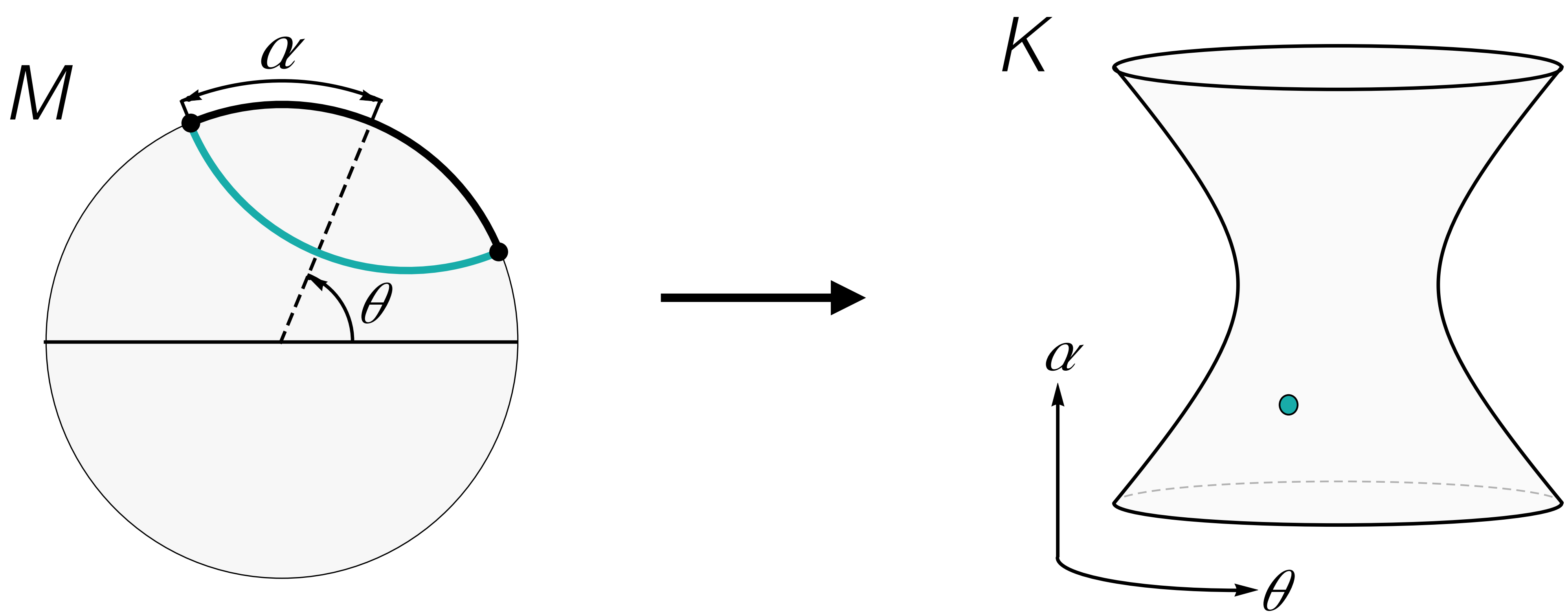


$$f(x) \longrightarrow Rf(\gamma) = \int_\gamma ds\, f(x)$$

The X-Ray Transform

- Example: let *M* be $\mathbb{H}_2$
- Parameterize geodesics by midpoint $\boldsymbol{\theta}$ and opening angle $\boldsymbol{\alpha}$

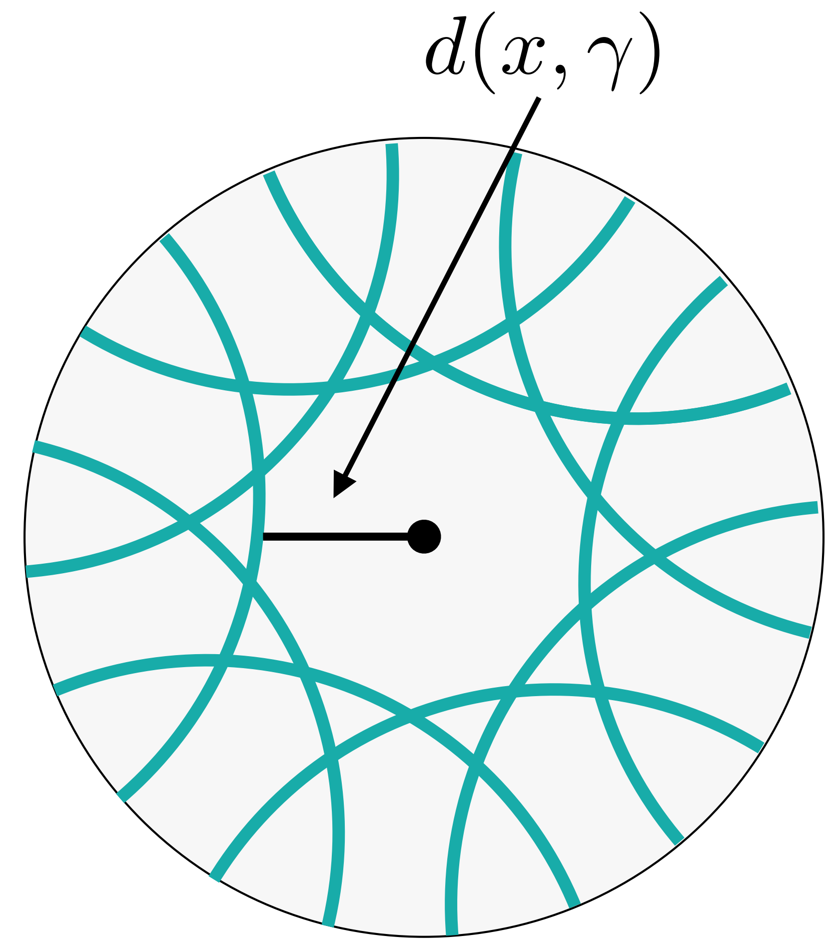
$$f(x) \longrightarrow Rf(\alpha, \theta) = \int_{\gamma_{\alpha,\theta}} ds f(x)$$



The X-Ray Transform

- Definition of X-Ray Transform:

$$f(x) \longrightarrow Rf(\alpha,\theta) = \int_{\gamma_{\alpha,\theta}} ds\, f(x)$$



- **Inversion formula:** [Helgason]

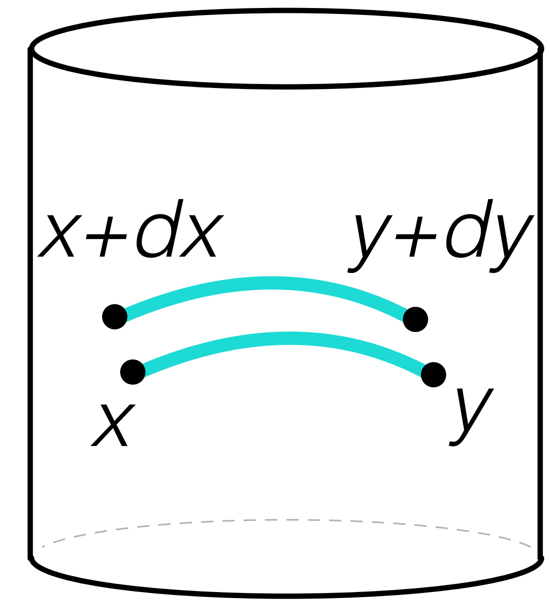
$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{d}{dp}\left(\underset{d(x,\gamma)=p}{\text{average}}\, Rf(\gamma)\right)\frac{dp}{\sinh p}$$

Structure of Kinematic Space

- We want to find an equation of motion for $R\phi(\gamma)$
- First, we must fix a metric on kinematic space - a **distance function** on the **space of geodesics**
- Kinematic Space for AdS_n or H_n is a highly symmetric space
    - No distinguished geodesics: all spacelike geodesics are related by symmetry
- We can fix a unique metric on *K* using this symmetry

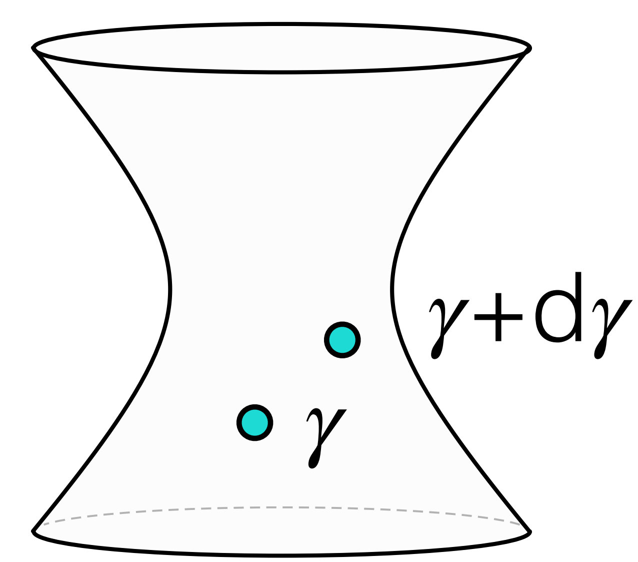
Structure of Kinematic Space

- Fix a **unique metric** on *K* using invariance under **conformal symmetry**



- Parameterize a geodesic by its endpoints
- The metric must be of the form $ds^2 = f_{\mu\nu} dx^\mu dy^\nu$
- Scaling and translation fix $f_{\mu\nu} = \frac{1}{(x-y)^2}\left(a\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} + b\eta_{\mu\nu}\right)$
- Inversion $\left(x^\mu \to \frac{x^\mu}{x^2}\right)$ fixes *a* = –2 *b*

$$ds^2 = \frac{1}{(x-y)^2}\left(\eta_{\mu\nu} - 2\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2}\right) dx^\mu dy^\nu$$

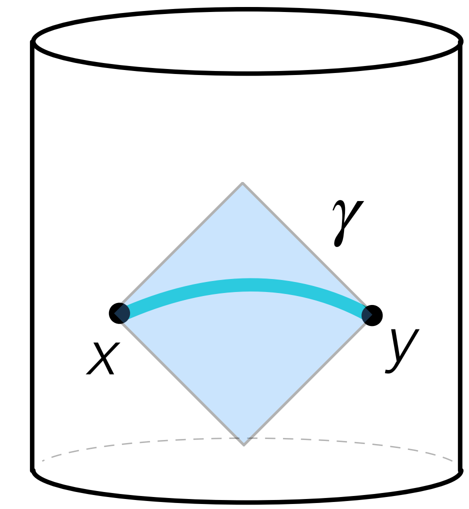


- *Note to experts: These same requirements fix the CFT two-point function of vector fields* $\langle O_\mu(x)\, O_\nu(y)\rangle$

Structure of Kinematic Space

- What is this metric?

$$ds^2 = \frac{1}{(x-y)^2}\left(\eta_{\mu\nu} - 2\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2}\right) dx^\mu dy^\nu$$



$$(u_L, v_L) = \left(\frac{x^1 - x^0}{2}, \frac{x^1 - x^0}{2}\right)$$

- It is (2*d*)-dimensional - *d* space, *d* time

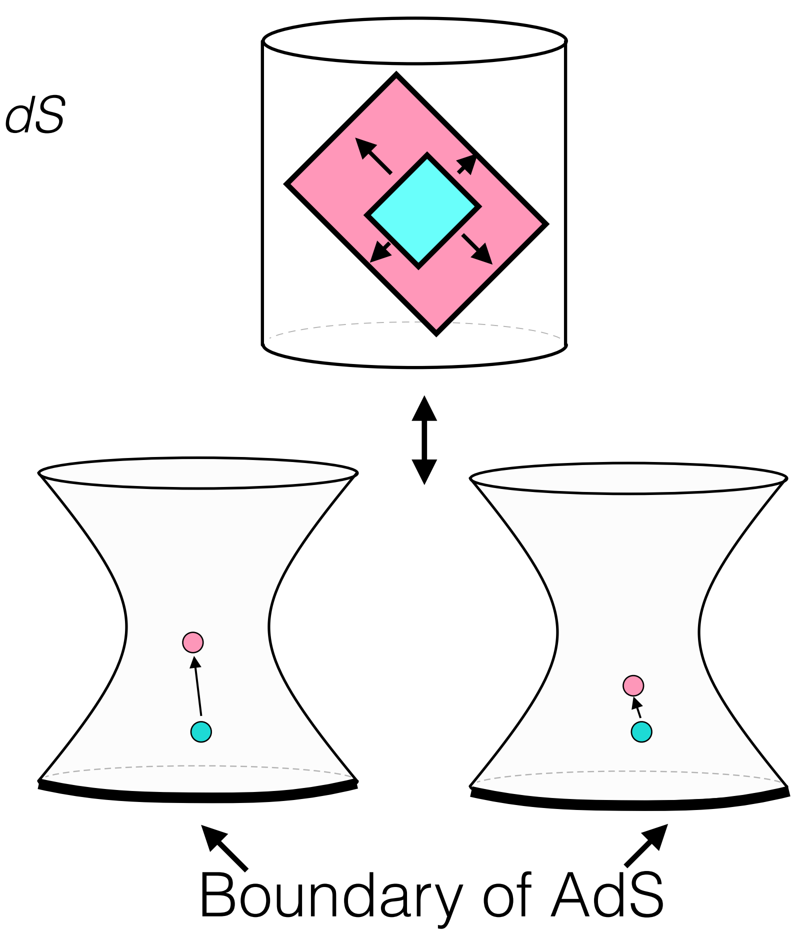
- For **AdS$_3$**, it is ***dS$_2$* x *dS$_2$***: $ds^2 = \frac{1}{2}\frac{du_L du_R}{\left(\frac{u_R - u_L}{2}\right)^2} + \frac{1}{2}\frac{dv_L dv_R}{\left(\frac{v_R - v_L}{2}\right)^2}$

  Left-moving *dS* ↑   Right-moving *dS* ↑

- For Hyperbolic 2-space, it is the diagonal *dS*$_2$

- The **causal structure** is determined by **containment** of boundary causal diamonds

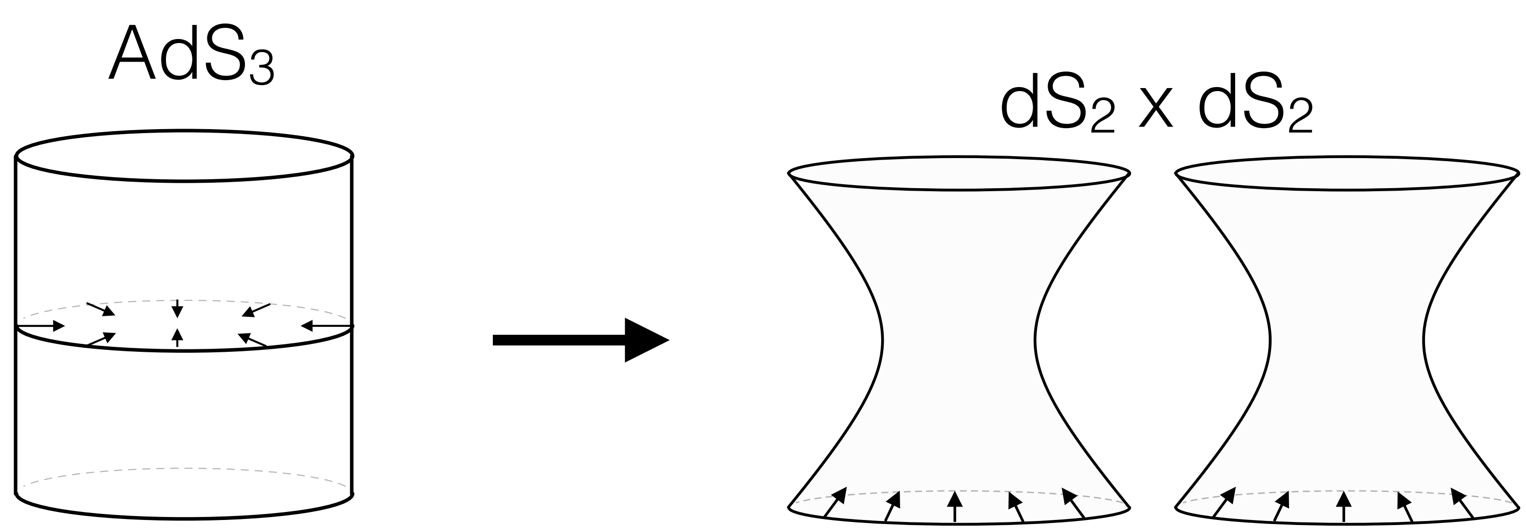
- The **asymptotic past** of *K* is the **boundary** of AdS



Intertwining Operators

- We want to find **two** equations of motion for $R\phi(\gamma)$
- We will find: $R\Box_{AdS_3} = -\Box_{dS\times dS} R$
- Also, $(\Box_{dS_L} - \Box_{dS_R}) \, Rf = 0$

$$\Box_{AdS_3}\phi = m^2\phi \longrightarrow \begin{array}{l} \Box_{dS\times dS} \; R\phi = -m^2 \; R\phi \\ (\Box_{dS_L} - \Box_{dS_R}) \, R\phi = 0 \end{array}$$



Non-standard Cauchy problem

Standard Cauchy problem

Intertwining Operators

- First step: write $Rf(\gamma)$ as an integral over all of space

$$Rf\left(\gamma\right) = \int_\gamma ds\, f\left(x\right) = \int d^d x\, f\left(x\right)\; F\left(d\left(x,\gamma\right)\right)$$

  - For Euclidean space: $F\left(x\right) = \dfrac{\delta\left(x\right)}{x^{d-2} S_{d-2}}$
  - For Lorentzian space: $F\left(x\right) = \dfrac{\delta\left(x\right)}{x^{d-2}\log x\, S_{d-3}}$

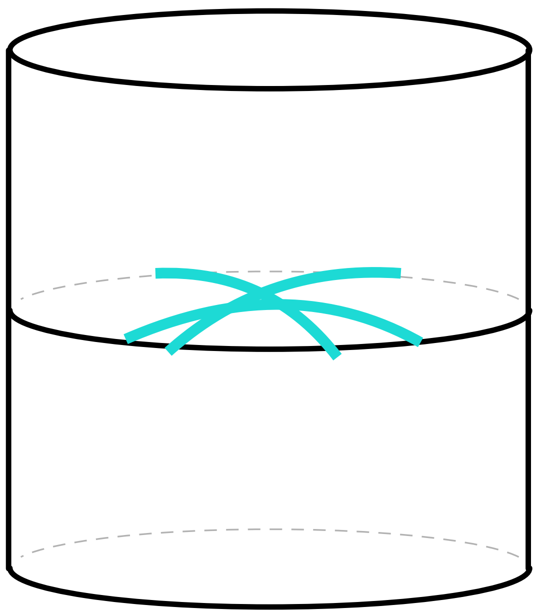
- Can show: $\left(\Box_{AdS_3} + \Box_{dS\times dS}\right) F\left(d\left(x,\gamma\right)\right) = 0$
  - Comes from relationship to conformal Casimir operator
  - Integrating by parts proves the intertwinement relation:

$$R\Box_{AdS_3}\phi = -\Box_{dS\times dS} R\phi$$

Intertwining Operators

$$Rf\left(\gamma\right)=\int_{\gamma}ds\,f\left(x\right)=\int d^{d}x\,f\left(x\right)\,F\left(d\left(x,\gamma\right)\right)$$

- Can also show: $\left(\Box_{dS_L}-\Box_{dS_R}\right)F\left(d\left(x,\gamma\right)\right)=0$
  - Directly implies $\left(\Box_{dS_L}-\Box_{dS_R}\right)Rf=0$
- Comes from a **redundancy** in the Radon transform
- f is a function of 3 variables, but Rf is a function of 4 variables
- The **constraint** reduces this dimensionality by 1
- E.g., can determine boosted geodesics in terms of unboosted geodesics
- Similar result in flat space: “John’s equation”

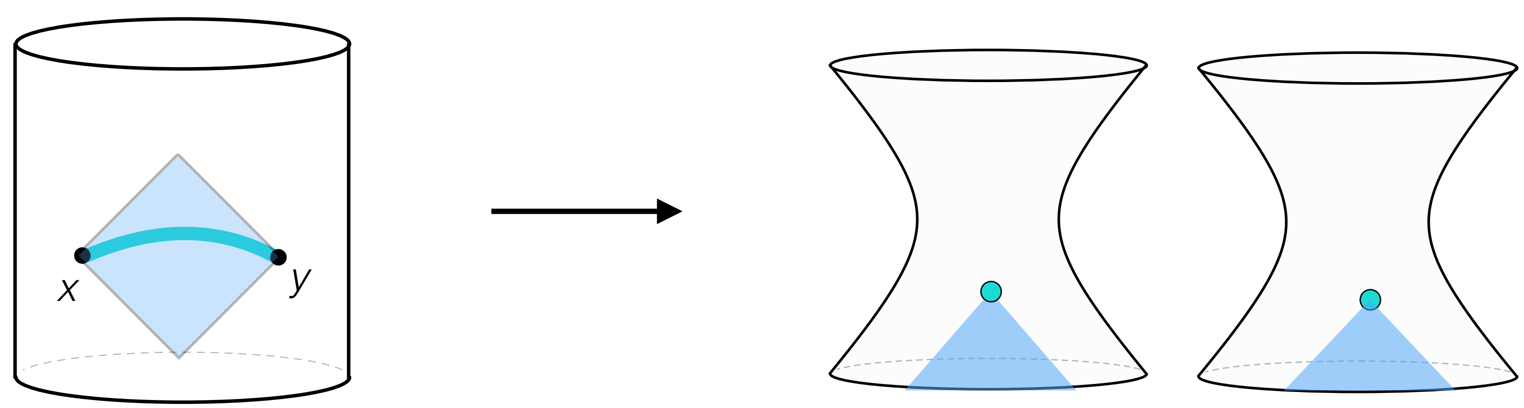


Geodesic Operators

- Now, solve the Cauchy problem to determine the geodesic operators

Equations of Motion: $\Box_{AdS_3}\phi = m^2\phi \longrightarrow \begin{array}{l} \left(\Box_{dS_L} + \Box_{dS_R}\right) R\phi = -m^2 R\phi \\ \left(\Box_{dS_L} - \Box_{dS_R}\right) R\phi = 0 \end{array}$

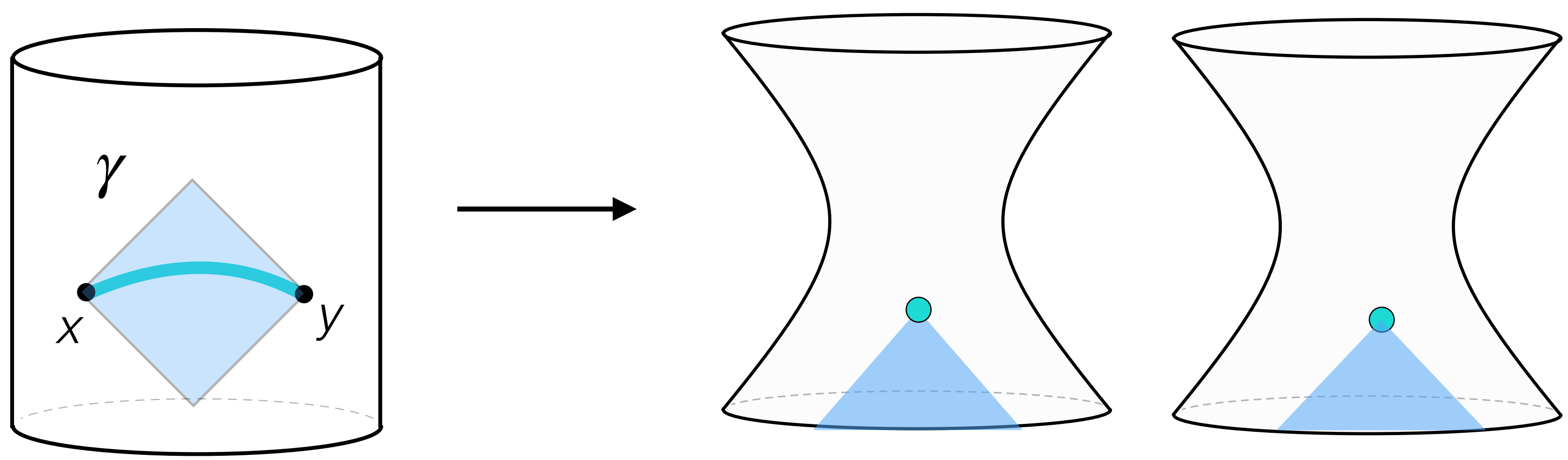
Boundary Conditions: $\phi(z,x) \sim z^{\Delta}\mathcal{O}(x) \longrightarrow R\phi\,(x,y) \sim c_{\Delta}\,(y-x)^{\Delta}\,\mathcal{O}\left(\frac{x+y}{2}\right)$



- The geodesic operator depends **only on** the boundary values in the **causal diamond** it **subtends**!

Geodesic Operators

- Now, solve the Cauchy problem to determine the geodesic operators



- The result: a **smeared operator** on the causal diamond (see previous talk for details)

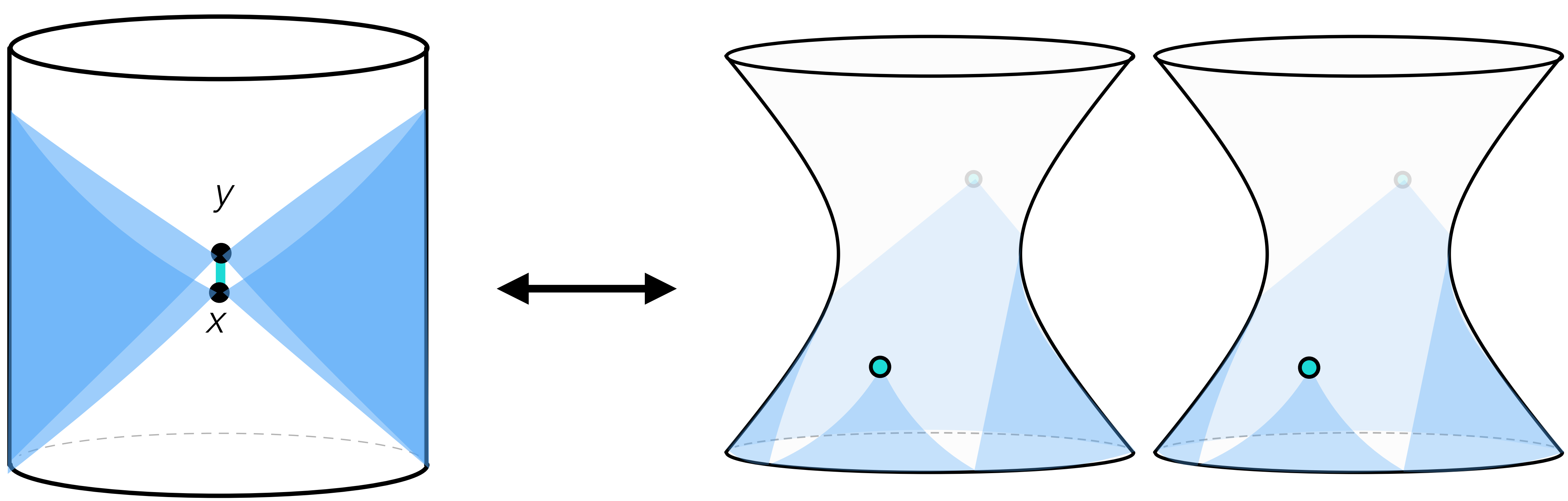
$$R\phi\left(\gamma\right) = \int_{\gamma} ds\,\phi\left(x\right) = \int_{\diamond} d^2x' \left(\frac{|y-x'|\,|x'-x|}{|y-x|}\right)^{\Delta-2} \mathcal{O}\left(x'\right)$$

Geodesic Operators

- The result: a **smeared operator** on the causal diamond
(see previous talk for details)

$$R\phi\left(\gamma\right) = \int_{\gamma} ds\, \phi\left(x\right) = \int_{\diamond} d^2x' \left(\frac{|y-x'|\,|x'-x|}{|y-x|}\right)^{\Delta-2} \mathcal{O}\left(x'\right)$$

- Note: We have **two choices** for the causal diamond – two representations of the geodesic operator



Local Bulk Operators

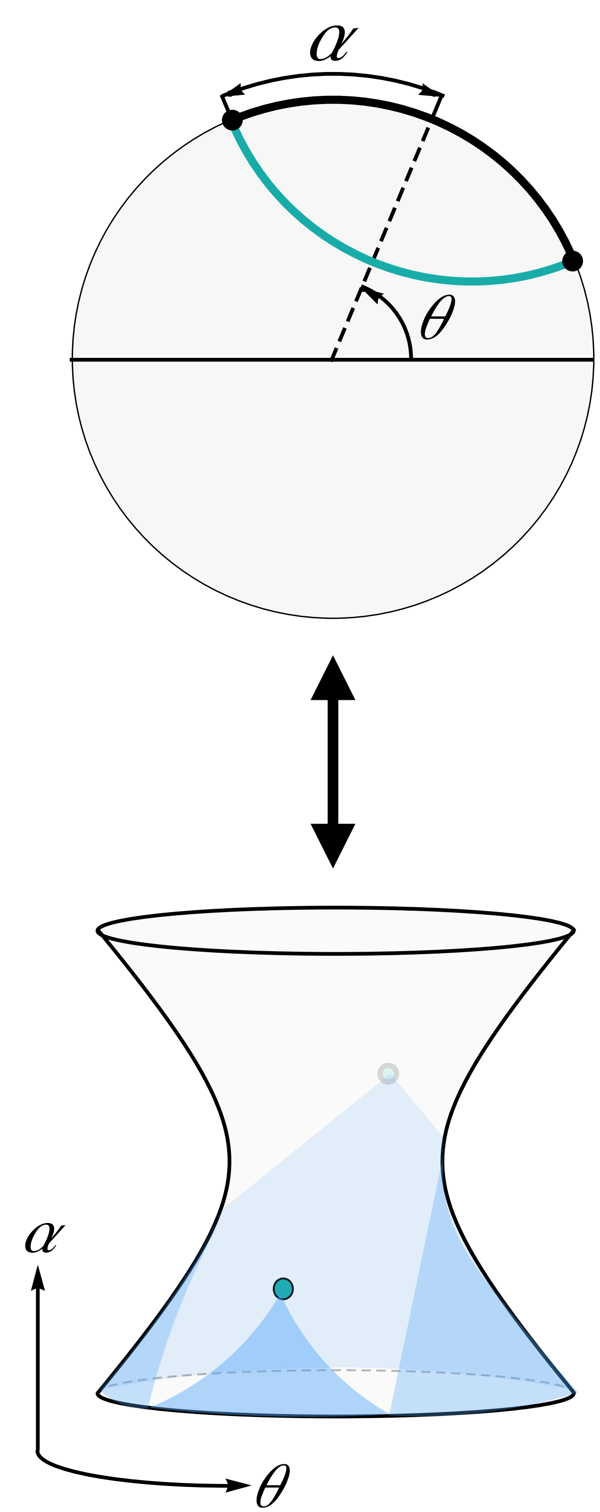
- How can we obtain local bulk operators? **Invert** the **X-ray transform** on a time-slice.

$$f(x) = -\frac{1}{\pi}\int_0^\infty \frac{d}{dp}\left(\underset{d(x,\gamma)=p}{\text{average}}\, Rf(\gamma)\right)\frac{dp}{\sinh p}$$

- We integrate over all geodesics on a slice

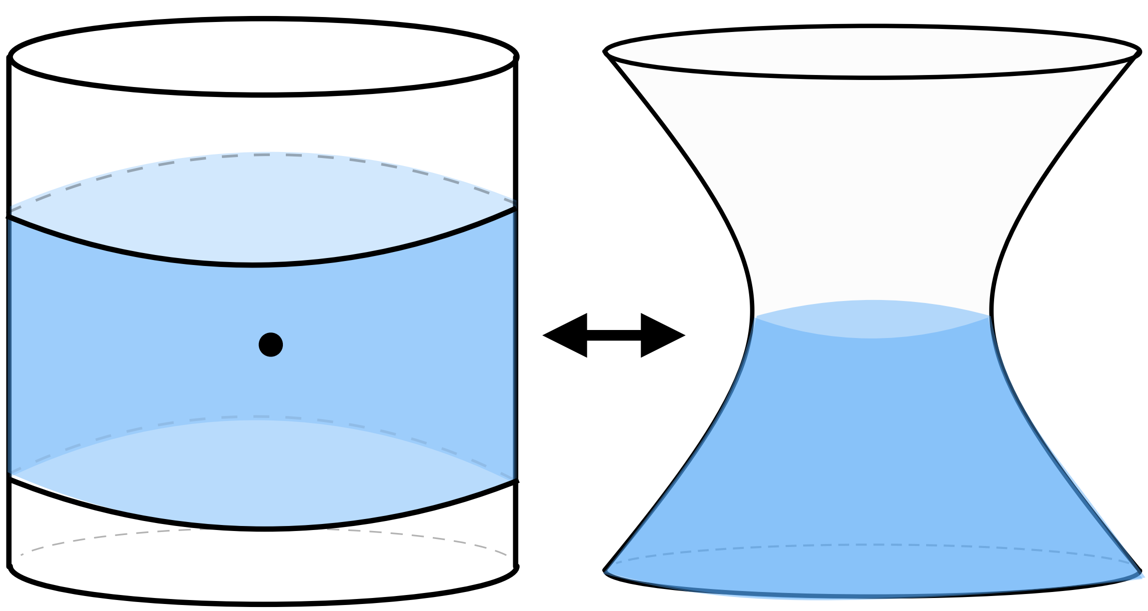
$$\phi\,(\text{center}) = -\frac{1}{2\pi}\iint d\alpha d\theta\, \tan\alpha\, \frac{d}{d\alpha} R\phi\,(\alpha,\theta)$$

- For each geodesic, choose a diamond
- We only need to **integrate** over ***half* of kinematic space** for the time-slice

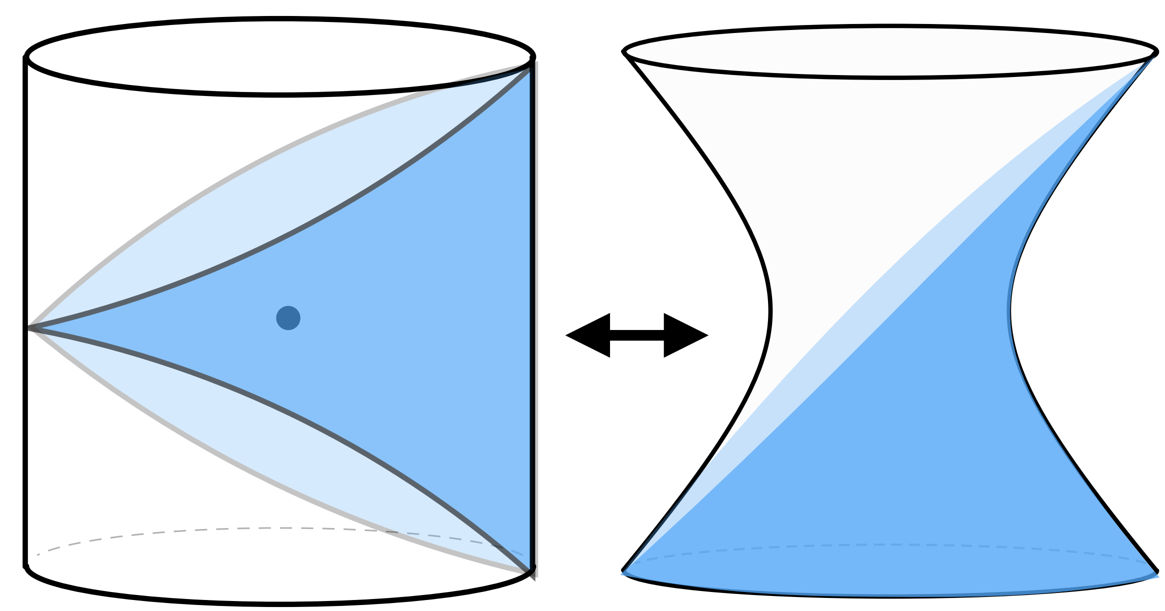


Local Bulk Operators

- Choosing **which half** of kinematic space determines **which smearing representation** of the bulk operator we obtain



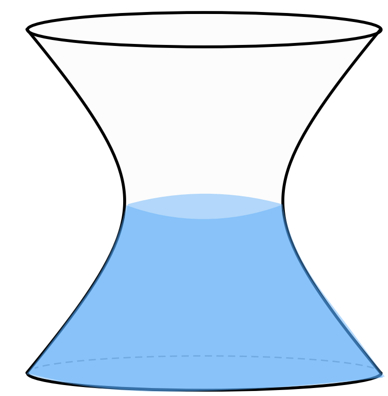
Global smearing



Poincaré smearing

Local Bulk Operators

- Let's obtain the **global smearing function**

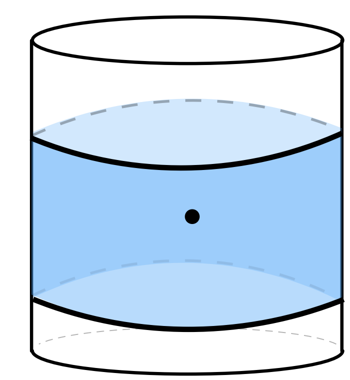


$$\phi\left(\text{center}\right) = -\frac{1}{\pi}\iint_{\text{half } K} d\alpha d\theta\ \tan\alpha\ \frac{d}{d\alpha} R\phi\left(\alpha,\theta\right)$$

- In global coordinates, $R\phi$ becomes

$$R\phi\left(\alpha,\theta\right) = \int_{\diamond} d\phi d\tau \left[2\frac{\left(\cos\tau - \cos\left(\phi+\alpha\right)\right)\left(\cos\tau - \cos\left(\phi-\alpha\right)\right)}{1-\cos\left(2\alpha\right)}\right]^{\Delta/2-1}\mathcal{O}\left(\tau,\phi+\theta\right)$$

- The result **matches the HKLL result**:



$$\phi\left(\text{center}\right) = \int_{\text{strip}} d^2x\ \left[-\left(\cos\tau\right)^{\Delta-2}\log\cos\tau + \left(\cos\tau\right)^{\Delta-2}\log\epsilon\right]\mathcal{O}\left(x\right)$$

HKLL Result — Divergent piece

- The divergent piece **vanishes** – its Fourier modes have no overlap with the bulk mode expansion (see HKLL 2006)

The Operator Product Expansion

- Consider a CFT in *d* dimensions.
- A **product** of local primary operators can be written as a **sum** over the primary operators in the theory:

$$O_1(x)\,O_2(y) = \sum_k C_{12k} \underbrace{\left(1 + \#\partial + \#\partial^2 + \ldots\right)}_{\text{Fixed by conformal invariance}} O_k(x)$$

$$\equiv \sum_k \left[O_1(x)\,O_2(y)\right]_k$$

- The coefficients look a lot like a Taylor expansion. Can we “undo” it?

$$\left[O_1(x)\,O_2(y)\right]_k = N_{12k} \int d^d z \left\langle O_1(x)\,O_2(y)\,\tilde{O}_k^{\mu\nu\ldots}(z)\right\rangle O_{k\,\mu\nu\ldots}(z)$$

- A “shadow operator” with $\tilde{\Delta}_k = d - \Delta_k$ gives the correct conformal transformation properties, but also includes unwanted pieces

  [Simmons-Duffin 2014]

- What integration region? How to normalize? Is this a useful OPE?

The Operator Product Expansion

- The fix: integrate over a **causal diamond**

$$[O_1(x)\,O_2(y)]_k = N_{12k}\int_\diamond d^2z\,\left\langle O_1(x)\,O_2(y)\,\tilde{O}_k^{\mu\nu\ldots}(z)\right\rangle O_{k\,\mu\nu\ldots}(z)$$

- Is this “projected OPE” related to the AdS$_3$ geodesic operator?

- This object obeys a **conformal Casimir equation**

$$(L_1+L_2)^2\,[O_1(x)\,O_2(y)]_k = C^2_{\Delta_k}\,[O_1(x)\,O_2(y)]_k$$

- This Casimir equation is just a kinematic space **wave equation** if $O_1 = O_2$

$$\left(-\Box_{dS\times dS} - m^2\right)[O_1(x)\,O_1(y)]_k = 0$$

$m^2 = C^2_{\Delta_k}$

- It also obeys the **constraint** equation:

$$(\Box_{dS_L} - \Box_{dS_R})\,[O_1(x)\,O_1(y)]_k = \left(h-\bar{h}\right)[O_1(x)\,O_2(y)]_k$$

- Is has **boundary conditions** for small separation of the operators:

$$[O_1(x)\,O_2(y)]_k \sim (y-x)^{\Delta_k-\Delta_1-\Delta_2}\,O_k\left(\frac{x+y}{2}\right)$$

The Operator Product Expansion

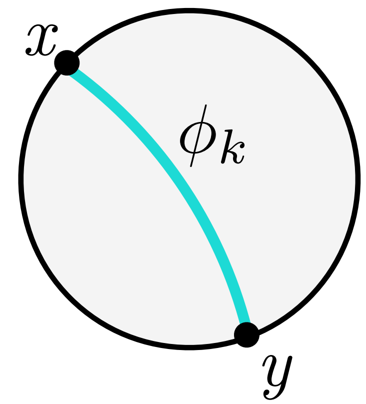
- The projected OPE obeys the **same equations of motion** as the geodesic operator for AdS3, with the **same boundary conditions**

- They must be equal!

$$[O_1(x)\,O_1(y)]_k \propto \frac{1}{(x-y)^{2\Delta_1}} \int_{\gamma_{x\to y}} ds\,\phi_k$$

- A **product of boundary operators** is localized on a **geodesic**

$$O_1(x)O_1(y) = \frac{1}{(x-y)^{2\Delta_1}} \sum_k C_{11k}$$



+ tensors, local descendants

- A bulk local operator can be written as a projected **smeared bilocal**

$$\phi_k\,(\text{center}) = -\frac{1}{\pi} \iint_{\text{half } K} d\alpha d\theta\,\tan\alpha\,\frac{d}{d\alpha}\left(\left(1-\cos(\phi_2-\phi_1)\right)^{2\Delta_1} [O_1(\theta-\alpha)\,O_1(\theta+\alpha)]_k\right)$$

Work in Progress

Bulk Tensor Fields

- What do we do with tensor operators in the bulk?

$$[O_1(x)\,O_2(y)]_k = N_{12k}\int_\diamond d^2z\,\left\langle O_1(x)\,O_2(y)\,\tilde{O}_k^{\mu\nu\ldots}(z)\right\rangle O_{k\,\mu\nu\ldots}(z)$$

- A hint from the **modular Hamiltonian**:

$$\int_\diamond d^2x\,\left\langle O_1(x_0-R)\,O_1(x_0+R)\,\tilde{T}^{\mu\nu}(x)\right\rangle T_{\mu\nu}(x) = \quad 2\pi\int dx\,\frac{(x-x_0)^2-R^2}{2R}T_{00}(x)$$

- The modular Hamiltonian is a kinematic operator!

$$H_{\text{mod}} = \delta A = \int_\gamma ds\,\delta g_{\mu\nu}\hat{v}^\mu\hat{v}^\nu$$

[Nozaki, Numasawa, Prudenziati, Takayanagi]
[de Boer, Myers, Heller, Neiman]
[Lashkari, McDermott, van Raamsdonk]
[Swingle, van Raamsdonk]

Entanglement equation of motion ⟷ **Einstein's equations**

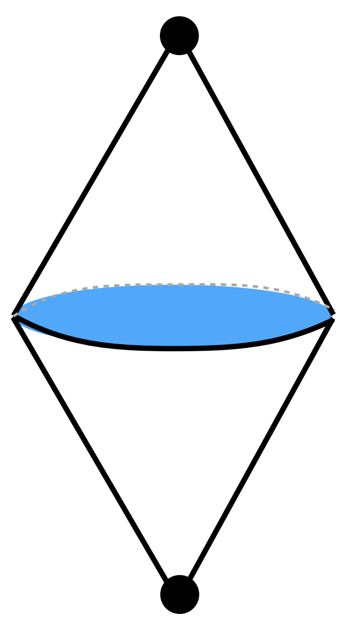
- Take the OPE of twist operators to **derive** the modular Hamiltonian

$$\sigma_n^\dagger(x)\,\sigma_n(y) = \frac{1}{(y-x)^{\frac{c}{6}\left(n-\frac{1}{n}\right)}}\left(1+(1-n)\,H_{\text{mod}} + \underbrace{\text{other operators}}\right)$$

Suppressed by $(1-n)$ or supported on multiple orbifold sheets

Higher Dimensions

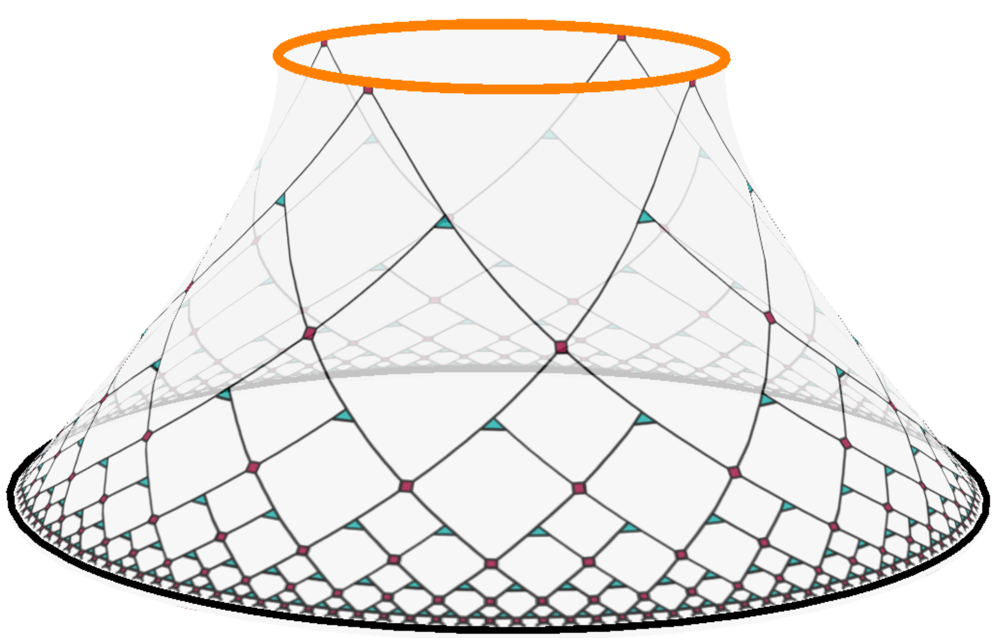
- What do we do in higher dimensions?
- **Spacelike separated** points correspond to **geodesics**
- **Timelike separated** points correspond to **minimal surfaces**
- Define the **Radon transform** of a function - its integral over the minimal surface
- The previous discussion holds - a diamond-smeared boundary operator is a **bulk surface operator**
- The (d-1) boosts provide (d-1) constraints
- The modular Hamiltonian is again a kinematic operator
- Timelike kinematic space for $\mathbb{H}_d$ is dS_d



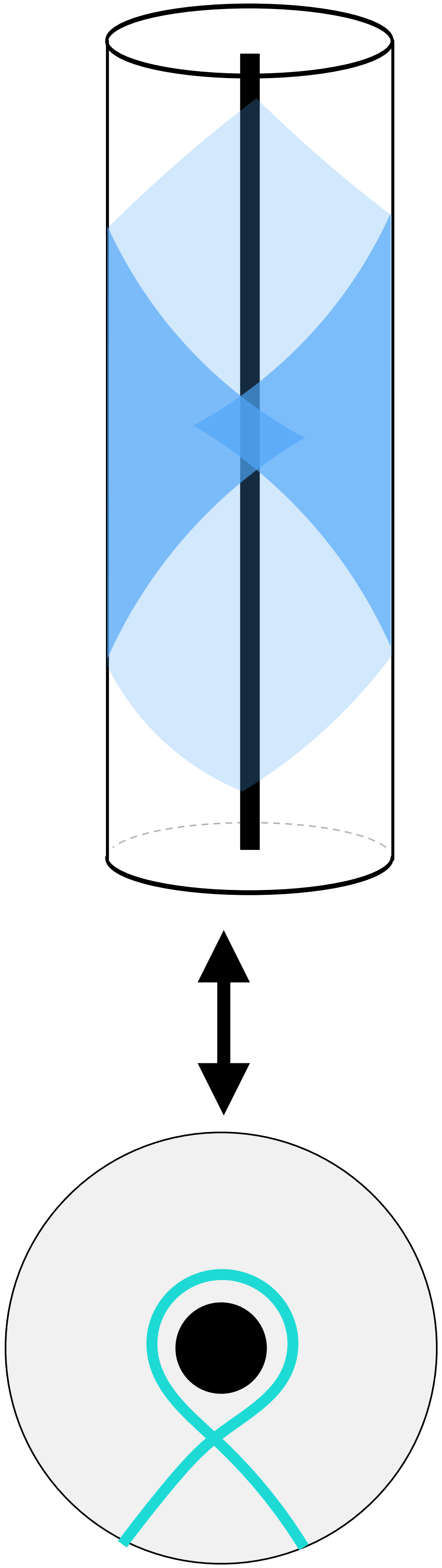
[deBoer, Heller, Myers, Neiman 2015]

Tensor Networks

- The **MERA** tensor network for a 2D CFT ground state naturally **lives on dS_2** – more generally, **on 2D kinematic space**
  [Beny 2011; Czech, Lamprou, McCandlish, Sully 2015]
  - Each tensor is associated with a **boundary ball** in its asymptotic past
- Does kinematic space tell us how to **generalize** MERA to higher dimensions, including time?



Beyond the Vacuum



- The previous discussion holds if we consider a quotient of AdS
- **Kinematic space** is also a **quotient** of the **AdS kinematic space**
- Non-minimal “entwinement” geodesics come from **winding diamond operators**

Interactions

- How do 1/N corrections appear in this description?

- The X-Ray (or Radon) transform transforms the free equation of motion:

$$(\tilde{\phi} = R\phi)$$

$$\left(\Box_X - m^2\right)\phi = 0 \quad \Longrightarrow \quad \left(-\Box_K - m^2\right)\tilde{\phi} = 0$$

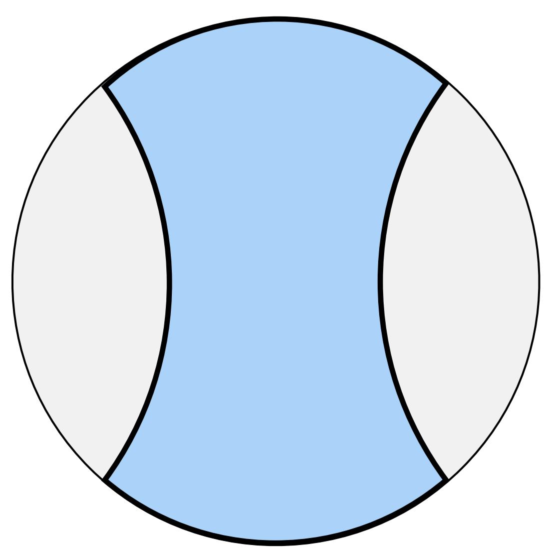
- If we add local bulk interactions, we get a **nonlocal** interaction in kinematic space (similar to momentum space)

$$\left(\Box_X - m^2\right)\phi = \phi^3 \quad \Longrightarrow \quad \left(-\Box_K - m^2\right)\tilde{\phi} = R\left(R^{-1}\tilde{\phi}\right)^3$$

- Can we include Virasoro descendants in 2D?

Entanglement Wedge

- Geodesic operators are Rindler-wedge operators
- How can we get **local** Rindler-wedge operators?
- How can we invert the X-ray transform with **limited data?**
- Is the Ryu-Takayanagi transition a phase transition in the limited data inversion formula for the X-ray transform?



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