Perturbative Proof of the Covariant Entropy Bound

Work in progress, with H. Casini, Z. Fisher, and J. Maldacena

Raphael Bousso
Berkeley Center for Theoretical Physics
University of California, Berkeley

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Area and Information

- We understand the holographic connection between geometry and information best in AdS.
- The Covariant Entropy Bound is a general relation conjectured to hold in arbitrary spacetimes, including cosmology:
- The entropy on a light-sheet is bounded by the difference between its initial and final area.
- In this talk I will present a proof of this relation in a nontrivial limiting regime.

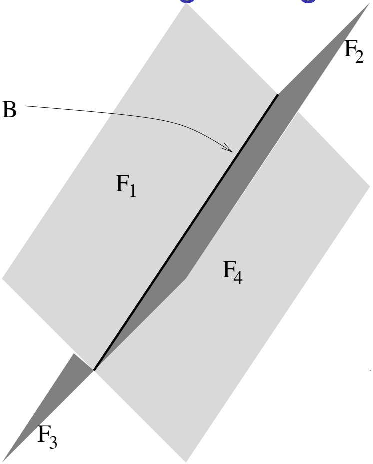
Covariant Entropy Bound

What is the Entropy?

Perturbative Proofs of the GCEB

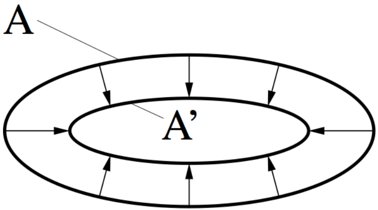
Outlook

Surface-orthogonal light-rays



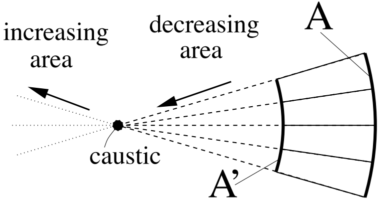
- Any 2D spatial surface B bounds four (2+1D) null hypersurfaces
- Each is generated by a congruence of null geodesics ("light-rays") $\perp B$

Light-sheets



An orthogonal null hypersurface is called light-sheet if the generating light-rays are nonexpanding away from the initial surface.

The Nonexpansion Condition

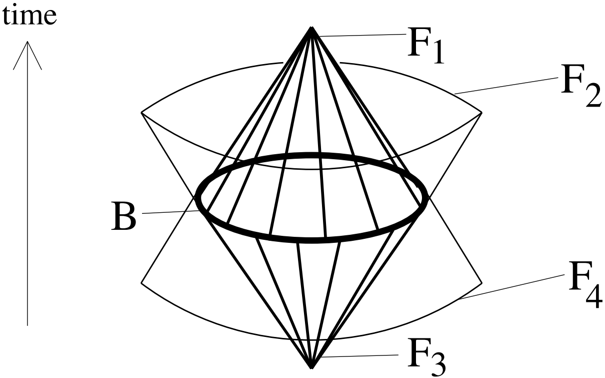


- $\theta \equiv \widehat{\nabla_a k^a}$, where k^a is the affine tangent vector field
- In terms of an infinitesimal area element $\mathcal{A}$ spanned by nearby light-rays,

$$\theta = \frac{d\mathcal{A}/d\lambda}{\mathcal{A}}$$

Demand
$\theta \leq 0 \quad \leftrightarrow$ nonexpansion
everywhere on the
light-sheet.

Light-sheets



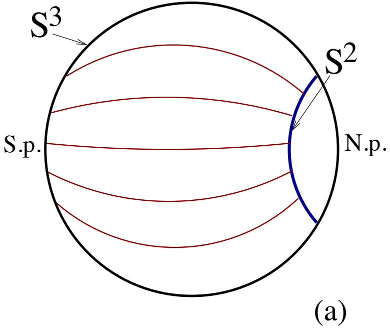
Out of the 4 orthogonal directions, usually at least 2 will initially be nonexpanding.

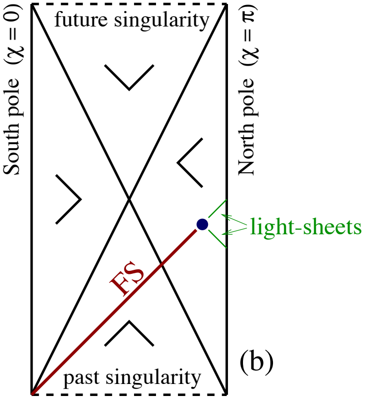
Covariant Entropy Bound

In an arbitrary spacetime, choose an arbitrary two-dimensional surface B of area A. Pick any light-sheet of B. Then $S \leq A/4G\hbar$, where S is the entropy on the light-sheet.

RB 1999

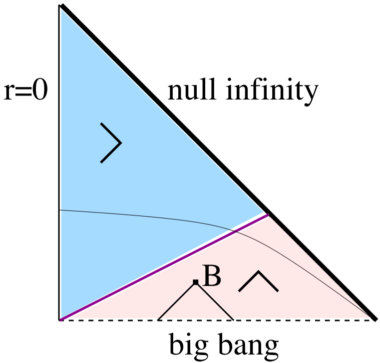
(1) Closed universe





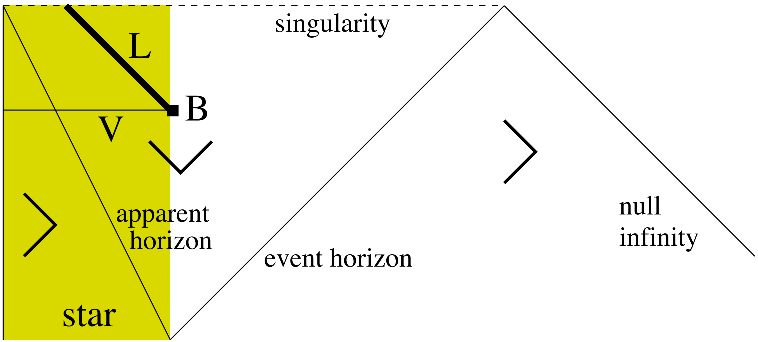
- Consider a small 2-sphere on a closed spatial manifold.
- The light-sheets are directed towards the “small” interior, avoiding an obvious contradiction.

(2) Flat FRW universe



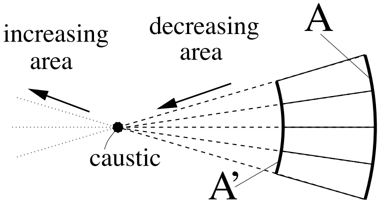
- Naively, $S \sim R^3$, $A \sim R^2$, so S/A diverges
- Sufficiently large spheres at fixed time t are anti-trapped
- Light-sheets are truncated by the singularity
- The entropy on these light-sheets grows only like R^2

(3) Collapsing star



- At late times the surface of the star is trapped
- Only future-directed light-sheets exist
- They do not contain all of the star

Generalized Covariant Entropy Bound

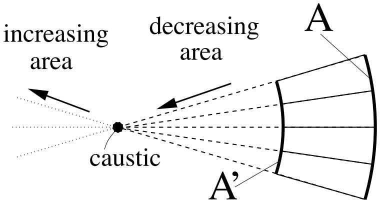


If the light-sheet is terminated at finite cross-sectional area A', then the covariant bound can be strengthened:

$$S \leq \frac{A - A'}{4G\hbar}$$

Flanagan, Marolf & Wald, 1999

Generalized Covariant Entropy Bound



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- Quantum systems are not sharply localized. Under what conditions can we consider a matter system to "fit" on L?
- The vacuum, restricted to L, contributes a (divergent) entropy. What is the justification for ignoring this piece?

How is the entropy defined?

- In cosmology and for large well-separated systems, these subtleties do not present a serious obstruction.
- However, the GCEB is nontrivial even in the perturbative regime, where matter has small backreaction on the geometry. For example, a single wavepacket has $S \sim O(1)$, $\Delta A \sim O(1)$.

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- In cosmology and for large well-separated systems, these subtleties do not present a serious obstruction.
- However, the GCEB is nontrivial even in the perturbative regime, where matter has small backreaction on the geometry. For example, a single wavepacket has $S \sim O(1)$, $\Delta A \sim O(1)$.
- Fortunately, in the $G\hbar \to 0$ limit, a sharp definition of S is possible.
- In the context of spatial regions, this definition was introduced by Casini (2008), building on work of Marolf, Minic, and Ross (2003).

Casini Entropy

Consider two field theory states in Minkowski space: the vacuum, $|0\rangle$, and some excited state ρ_{global}. In the absence of gravity, $G = 0$, the geometry is identical in all matter states, and one can restrict both states to a subregion V:

$$
\begin{aligned}
\rho &\equiv \mathrm{tr}_{-V}\, \rho_{\text{global}} \\
\rho_0 &\equiv \mathrm{tr}_{-V}\, |0\rangle\langle 0|
\end{aligned}
$$

Due to vacuum entanglement entropy, the van Neumann entropy of each density operator diverges like A/ϵ^2, where A is the boundary area of V, and ϵ is a cutoff. However, the difference is finite as $\epsilon \to 0$:

$$\Delta S \equiv S(\rho) - S(\rho_0)\ .$$

Marolf, Minic & Ross 2003, Casini 2008

Properties of the Casini Entropy

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  $\rightarrow$ No Species Problem
- The observer-dependence is physically appropriate: an observer with access only to V is unable to discriminate an arbitrary number of species due to thermal effects.

Assumptions

I will present a perturbative proof of the GCEB for free fields.

RB, Casini, Fisher, Maldacena 2011

The proof builds on work of H. Casini (2008) and of A. Wall (2010, 2011)

Despite the limited regime, the proof is interesting for the following reasons:

- It does not require any explicit assumptions of a relation between entropy and energy, nor classical energy conditions on matter such as the NEC.
- It relies sensitively on the nonexpansion condition, which must be enforced even if the NEC is violated.

Relative Entropy

Given any two states, the (asymmetric!) *relative entropy*

$$S(\rho|\rho_0) = -\mathrm{tr}\,\rho \log \rho_0 - S(\rho)$$

satisfies positivity and monotonicity. That is, under further restrictions of ρ and ρ_0 to a subalgebra (e.g., a subset of V), the relative entropy is nonincreasing.

Lindblad 1975

Modular Hamiltonian

Definition: Let ρ_0 be the vacuum state, restricted to some region V. Then the *modular Hamiltonian*, K, is defined up to a constant by

$$\rho_0 \equiv \frac{e^{-K}}{\operatorname{tr} e^{-K}} .$$

The modular energy is defined as

$$\Delta K \equiv \operatorname{tr} K\rho - \operatorname{tr} K\rho_0 \tag{1}$$

A Central Result

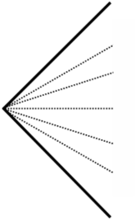
Positivity of the relative entropy implies immediately that

$$\Delta S \leq \Delta K \; .$$

This is useful because ΔK can be related to

- the area increase of a causal horizon — Wall 2011
- the perturbative area difference of an "optimized" light-sheet — RB, Casini, Fisher, Maldacena 2011

GSL for Rindler Space



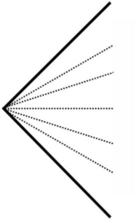
The modular Hamiltonian on the Rindler horizon ($x^+ = x + t > 0$) is given by

$$K = \frac{2\pi}{\hbar} \int d^2x^\perp \int_0^\infty dx^+ \, x^+ \, T_{++} \, ,$$

where $T_{++} = T_{ab}k^a k^b$ and k_a is the affine tangent vector to the horizon.

Bisognano, Wichmann 1975

GSL for Rindler Space



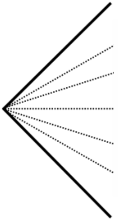
By integrating the Raychaudhuri equation

$$-\frac{d\theta}{dx^+} = \frac{1}{2}\theta^2 + \sigma_{ab}\sigma^{ab} + 8\pi G T_{ab} k^a k^b$$

once, at leading order in G one finds the expansion along the Rindler horizon:

$$\theta(x^+) = 8\pi G \int_{x^+}^{\infty} T_{++} d\hat{x}^+$$

GSL for Rindler Space



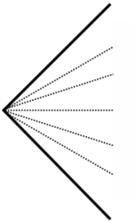
Integrating a second time and using

$$A(x^+) = A(\infty) \exp \int_{x^+}^{\infty} d\hat{x}^+ \theta(\hat{x}^+)$$

one finds that the Rindler horizon grows in area from $x^+ = 0$ to $x^+ = \infty$ by

$$\Delta A = 8\pi G \int d^2x^\perp \int_0^\infty dx^+ \; x^+ \; T_{++} \; .$$

GSL for Rindler Space



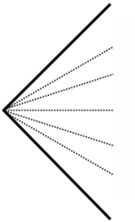
Hence one finds

$$\Delta S \leq \Delta K = \frac{2\pi}{\hbar}\frac{\Delta A}{8\pi G} = \frac{\Delta A}{4G\hbar}$$

and thus, the Generalized Second Law of Thermodynamics for the Rindler horizon.

Wall 2010

GSL for Rindler Space



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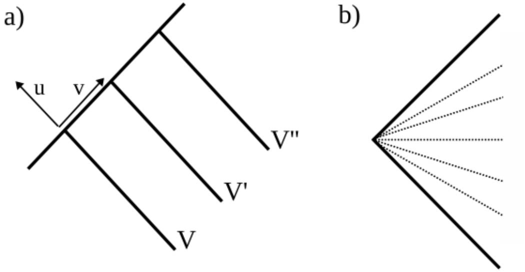
This and all subsequent results obtain at leading order in G (weak backreaction).

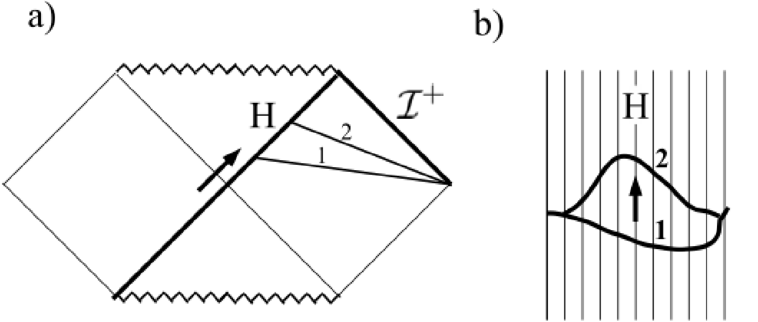
GSL for Rindler Space



- This result applies to the process where all of the entropy in the wedge passes through the Rindler horizon.
- By exploiting monotonicity, it can be generalized to the GSL for a sequence of horizon slices (nested Rindler wedges).

Wall 2010

GSL for Causal Horizons



- By quantizing directly on the light-front, one can further generalize this to arbitrary horizon slices, of arbitrary causal horizons (black hole, de Sitter) Wall 2011
- This exploits the "Ultralocality" of the operator algebra on the null hypersurface: $\mathcal{A}(H) = \prod_i \mathcal{A}(H_i)$.
- Justified so far only for free fields. (Assume for now.)

Finite regions

- In finite volumes, the modular Hamiltonian K is generally nonlocal.
- However, again one finds that null hypersurfaces have special properties: K simplifies dramatically.
- In addition to ultralocality, a special conformal symmetry along each generator was noted (though not needed) by Wall (2011).
- We may obtain the modular Hamiltonian for a finite light-sheet by application of an inversion, $x^+ \to 1/x^+$, to the Rindler Hamiltonian on $x^+ \in (1, \infty)$.

Finite regions

We obtain

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But it is easy to find quantum states for which $T_{++} < 0$. In fact, explicit examples can be found for which $\Delta S > \Delta A/4G\hbar$, if the initial expansion vanishes ($\theta_0 = 0$).

Initial expansion and negative energy

- If the null energy condition holds, initially vanishing expansion is the “toughest” choice for testing the GCEB.
- However, if the NEC is violated, then $\theta_0 = 0$ does not guarantee that the nonexpansion condition holds everywhere.
- To have a valid light-sheet, we must require that

$$0 \geq \theta(x^+) = \theta_0 + 8\pi G \int_{x^+}^{1} d\hat{x}^+ \, T_{++}(\hat{x}^+) \ ,$$

  holds for all $x^+ \in [0, 1]$.
- This can be accomplished in any state.
- But the light-sheet may have to contract initially:

$$\theta_0 \sim O(G\hbar) < 0 \ .$$

Nonzero Initial Expansion Enhances Area Loss

The area loss from $x^+ = 0$ to $x^+ = 1$ is now given by

$$\frac{\Delta A}{A} = -\int_0^1 dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_0^1 dx^+ (1 - x^+) T_{++} \ .$$

at leading order in G.

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One can eliminate θ_0 using the nonexpansion condition: let $f(x^+)$ be any positive function, such that $F(1) = 1$, where $F(x^+) = \int_0^{x^+} d\hat{x}^+ f(\hat{x}^+)$. By nonexpansion, we have $0 \geq \int_0^1 f\theta dx^+$, and thus

$$\theta_0 \leq 8\pi G \int dx^+ [1 - F(x^+)] T_{++} \ .$$

Proof of the GCEB

With the specific choice $f(x^+) = 2 - 2x^+$ we find that the area difference is bounded from below by the modular Hamiltonian:

$$\frac{\Delta A}{4G\hbar} \geq A \times \frac{2\pi}{\hbar} \int_0^1 dx^+ \, x^+(1 - x^+) \, T_{++} = \Delta K \ .$$

The positivity of the relative entropy implies $\Delta S \leq \Delta K$, so the generalized covariant bound follows.

Comments

- Demanding nonexpansion on entire light-sheet is crucial. (As opposed to, e.g., demanding only initial nonexpansion plus some averaged version of the NEC.)
- No classical energy conditions or assumptions restricting entropy in terms of energy density were needed.
- Existence of vacuum on each null generator (which goes into the definition of the modular Hamiltonian) apparently captures all the necessary restrictions.

Generalizations

- Monotonicity can be shown: As the size of the null interval is increased, $\Delta S - \Delta A/4G\hbar$ is nondecreasing.
- The same result follows from any local and concave modular Hamiltonian with the correct Rindler limit.

Interacting Theories

- At linear order in the departure from the vacuum, one has $\Delta S = \Delta K$.
- This fixes the modular Hamiltonian if one can compute ΔS.
- For theories with a bulk dual, one can compute the modular Hamiltonian by vacuum tomography

  Blanco, Casini, Hung, Myers 2013
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- However we also seem to find that $\Delta S = \Delta K$ for *all* states.
- Stay tuned.