O(N) Models, RG and AdS/CFT

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Talk at

From the Renormalization Group to Quantum Gravity: Celebrating the Science of Joe Polchinski

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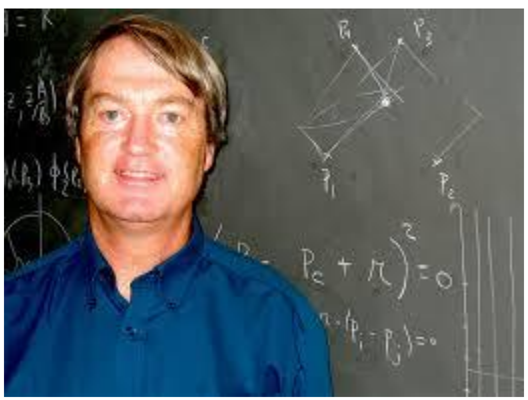
Based mainly on

- S. Giombi, IK, arXiv:1308.2337
- S. Giombi, IK, B. Safdi, arXiv:1401.0825
- S. Giombi, IK, A. Tseytlin, arXiv:1402.5396
- L. Fei, S. Giombi, IK, to appear

Epiphany

- String theorists of my generation remember where they were on the day in early October 1995 when Joe's D-brane paper appeared.
- Finally, we had the "branes" we could think with!



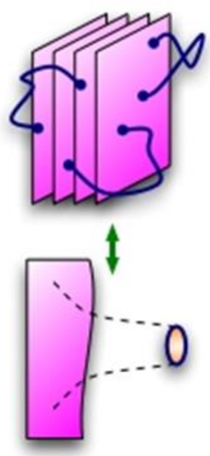
This result for the D-brane charge is new evidence both for string duality and for the conjecture that D-branes are the RR-charged objects required by string duality.

presumably one should think of them as an alternate representation of the black p-branes.

From D-Branes to AdS/CFT

- Stacking D-branes and comparing the gauge theory living on them with the curved background they create led to the gauge/gravity duality.
- In case of N D3-branes, have SU(N) gauge theory with $\mathcal{N}$=4 SUSY.

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-(dx^0)^2 + (dx^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right)$$



- Absorption of closed strings, near-extremal entropy, the "3/4 problem" IK, Gubser, Peet, Tseytlin, …

$$s = \frac{\pi^2}{2} N^2 T^3$$

- Zoom in on the throat: $AdS_5 \times \mathbf{S}^5$ with $L^4 = g_{\rm YM}^2 N \alpha'^2$

Maldacena

Preoccupation

- This has preoccupied us ever since.
- Many insights from weakly curved metrics about strongly coupled gauge theory.
- Some non-BPS tests using integrability and SUSY localization to get at the strongly coupled gauge theory.
- Yet, a proof the duality remains elusive.
- Maybe this is because both $\mathcal{N}$=4 SYM and Type IIB string are really complicated theories?

Simplify!

- Look for AdS duals of CFT's where dynamical fields are in the fundamental of O(N) or U(N) rather than in the adjoint. IK, Polyakov
- Wilson-Fisher O(N) critical points in d=3:

$$S = \int d^3x \left[\frac{1}{2} (\partial_\mu \phi^a)^2 + \frac{\lambda}{2N} (\phi^a \phi^a)^2 \right]$$

- Even simpler: the O(N) singlet sector of the free theory.
- Conserved currents of even spin

$$J_{(\mu_1 \cdots \mu_s)} = \phi^a \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^a + \ldots$$

All Spins All the Time

- Similarly, can consider the U(N) singlet sector in the d-dimensional free theory of N complex scalars. There are conserved currents of all integer spin.
- The dual AdS_{d+1} description must consist of massless gauge fields of all integer spin, coupled together.

$$\text{Spectrum}: \qquad s = 1, 2, 3, \ldots, \infty \qquad \text{gauge fields}$$
$$s = 0, \quad m^2 = -2(d-2) \quad \text{scalar}$$

- Vasiliev and others have constructed the classical EOM for some such interacting theories.
- Complicated. No known action principle.

Evidence for the Duality

- Matching of the spectra. Neatly summarized by the thermal partition functions in singlet sector CFT Shenker, Yin; Giombi, IK, Tseytlin

$$\mathcal{Z}_{U(N)}(\beta) = \left[\mathcal{Z}_0(\beta)\right]^2 = \frac{q^{d-2}(1+q)^2}{(1-q)^{2(d-1)}}$$

- The free scalar partition function on $S^1 \times S^{d-1}$

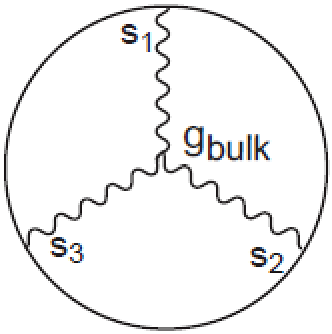
$$\mathcal{Z}_0(\beta) = \sum_{n=0}^{\infty} \mathrm{d}_n \, e^{-\beta[n+\frac{1}{2}(d-2)]} = \frac{q^{\frac{d-2}{2}}(1+q)}{(1-q)^{d-1}}$$

- In the Vasiliev theory $\mathcal{Z}_s(\beta) = \frac{q^{s+d-2}}{(1-q)^d}(\mathrm{d}_s - \mathrm{d}_{s-1}q)$

$$\mathcal{Z}(\beta) = \mathcal{Z}_0^{(d-2)} + \sum_{s=1}^{\infty} \mathcal{Z}_s(\beta) = \frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}}$$

Matching of 3-pt functions

- n-point functions of the currents do not vanish in the free CFT. This requires the bulk theory to be interacting.
- At leading order in N, the classical EOM may be used to calculate the 3-pt functions Giombi, Yin



$$g_{\text{bulk}} \sim \frac{1}{\sqrt{N}}$$

- The inverse Newton constant is quantized
Maldacena, Zhiboedov

$$G_N^{-1} \propto N$$

Sphere Free Energy

- Compare the free energy on the d-sphere at the boundary of Euclidean AdS with the bulk calculation $Z_{\rm bulk} = e^{-\frac{1}{G_N}F^{(0)} - F^{(1)} - G_N F^{(2)} + \dots}$
- Cannot determine the leading classical piece (no known action), but focus on the one-loop correction.
- In the free CFT, $F = -\log Z_{S^d} = N F_{\text{free scalar}}$
- For example, in d=3 U(N) singlet CFT

$$N\left(\frac{\log 2}{4} - \frac{3\zeta(3)}{8\pi^2}\right)$$

- Check cancellation of the $\mathcal{O}(N^0)$ term in F.

$$Z_{1-\text{loop}} = \frac{1}{[\det(-\nabla^2 - 2)]^{\frac{1}{2}}} \prod_{s=1}^{\infty} \frac{\left[\det_{s-1}^{STT}(-\nabla^2 + s^2 - 1))\right]^{\frac{1}{2}}}{\left[\det_{s}^{STT}(-\nabla^2 + s(s-2) - 2)\right]^{\frac{1}{2}}}$$

$$F^{(1)}_{(\Delta,s)} = -\frac{1}{2}\zeta'_{(\Delta,s)}(0) - \frac{1}{2}\zeta_{(\Delta,s)}(0)\log\left(\ell^2\Lambda^2\right)$$

$$\zeta_{(\Delta,s)}(0) = \frac{1}{24}(2s+1)\left[\nu^4 - \left(s+\frac{1}{2}\right)^2\left(2\nu^2+\frac{1}{6}\right) - \frac{7}{240}\right], \qquad \nu \equiv \Delta - \frac{3}{2}$$

- Using zeta-function regularization for summing over spins, the log term cancels. This is evidence for one-loop finiteness of the Vasiliev theory.

- The finite part for each spin Camporesi, Higuchi

$$\zeta'_{(\Delta,s)}(0) = \frac{1}{3}(2s+1)\left[\frac{\nu^4}{8} + \frac{\nu^2}{48} + c_1 + \left(s+\frac{1}{2}\right)^2 c_2 + \int_0^\nu dx \left[\left(s+\frac{1}{2}\right)^2 x - x^3\right]\psi(x+\frac{1}{2})\right]$$

- Sum over spins vanishes using the Hurwitz-Lerch function to regularize:

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty dt \frac{t^{s-1}e^{-vt}}{1-ze^{-t}} = \sum_{n=0}^\infty (n+v)^{-s} z^n$$

- Perfect agreement with the CFT where the $\mathcal{O}(N^0)$ term vanishes!
- In the minimal Vasiliev theory with only even spins we encounter a surprise. The log divergence cancels, but the finite part does NOT vanish.

Free O(N) Model

- The sum over even spins in AdS gives

$$F^{(1)}_{\rm min} = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}$$

- This is exactly the F-value of a real massless scalar in 3 dimensions! Giombi, IK
- This suggests a shift in the identification of the quantized Vasiliev coupling: $N \to N-1$
- We conjecture that the classical term is

$$\frac{1}{G_N} F^{(0)}_{\rm min} = (N-1)\left(\frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}\right)$$

- Then the sum of the classical and one loop terms in the minimal (even spin) Vasiliev theory would agree with the CFT of N real scalars.

Even Boundary Dimensions

- In even d, the CFT sphere free energy is UV logarithmically divergent, the coefficient being related to the Weyl *a-anomaly.*
- In the bulk, this logarithmic divergence is reflected in the IR divergence of the AdS_{d+1} volume for odd d+1

$$\int \mathrm{vol}_{\mathrm{AdS}_{d+1}} = \frac{2(-\pi)^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)} \log R$$

- The coefficient of log R in the bulk free energy is dual to the a-anomaly coefficient on the CFT side.
- There is no UV divergence in the bulk in this case (in odd dimensional spacetime, $\zeta(0)=0$ identically).

Anomaly Matching

- If the CFT is free, the a-anomaly should be Na_{scalar} without 1/N corrections.
- The results we find are consistent with the general picture: Giombi, IK, Safdi

$$F^{(1)} = 0$$

$$F^{(1)}_{\text{min HS}} = F^{\text{conf. scalar}}_{S^d} = a_{\text{scalar}} \log R$$

where a_{scalar} is the a-anomaly coefficient of one real conformal scalar in d-dimensions (e.g. a_{scalar}=1/90, -1/756, 23/113400… in d=4,6,8…).

Example: d=4

- For the Vasiliev theory in AdS_5 with all integer spins, the one-loop bulk free energy

$$F^{(1)} = -\frac{\log R}{360}\sum_{s=1}^{\infty} s^2(1+s)^2\left(3+14s(1+s)\right)$$
$$= -\left(\frac{1}{18}\zeta(-3)+\frac{7}{60}\zeta(-5)\right)\log R = 0$$

- For the minimal theory with even spins only

$$F^{(1)}_{\rm min\ HS} = -\frac{\log R}{360}\sum_{s=2,4,\ldots}^{\infty} s^2(1+s)^2\left(3+14s(1+s)\right)$$
$$= -\left(\frac{4}{9}\zeta(-3)+\frac{56}{15}\zeta(-5)\right)\log R = +\frac{1}{90}\log R$$

- The +1/90 is the a-anomaly coefficient of a real scalar in d=4.
- For the even spin theory in any d, $G_N \sim \frac{1}{N-1}$

Casimir Energies on S^{d-1}

- Must vanish for any CFT in odd d. Regularized sum over spins in global AdS_{d+1} indeed vanishes.
- In even d, one-loop corrections work similarly to the a-anomalies. Giombi, IK, Tseytlin
- Regularized sum over all spins vanishes.
- Sum over even spins equals the Casimir energy of one real scalar field: $\sum_{n=0}^{\infty} \frac{(n+d-3)!}{(d-2)!n!} \left[n + \frac{1}{2}(d-2)\right]^2$
- Further evidence for the one-loop shift $G_N \sim \frac{1}{N-1}$

Interacting CFT's

- A scalar operator $\mathcal{O}(x^\mu)$ in d-dimensional CFT is dual to a field $\Phi(z, x^\mu)$ in AdS_{d+1} which behaves near the boundary as z^Δ
- There are two choices $$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2}$$
- If we insist on unitarity, then Δ_- is allowed only in the Breitenlohner-Freedman range

IK, Witten

$$-(d/2)^2 < m^2 < -(d/2)^2 + 1$$

- Flow from a large N CFT where $\mathcal{O}(x^\mu)$ has dimension Δ_- to another CFT with dimension Δ_+ by adding a double-trace operator. Witten; Gubser, IK
- Can flow from the free d=3 U(N) scalar model in the UV to the Wilson-Fisher interacting one in the IR. The dimension of scalar bilinear changes from 1 to 2 +O(1/N). The dual of the interacting theory is the Vasiliev theory with Δ=2 boundary conditions on the bulk scalar.
- Or can flow from the interacting Gross-Neveu model in the UV to the free U(N) Dirac fermion model in the IR.

$$F_{\rm UV} - F_{\rm IR} = \frac{\zeta(3)}{8\pi^2} + O(1/N)$$

Gross-Neveu CFT

- Multiple Dirac fermions with action

$$\mathcal{S}(\bar{\psi},\psi) = -\int \mathrm{d}^d x \left[\bar{\psi}\cdot\partial\!\!\!/\psi + \tfrac{1}{2N} G\left(\bar{\psi}\cdot\psi\right)^2\right]$$

- In 2 < d < 4 there is a UV fixed point, at least for large N.
- In $d = 4 - \varepsilon$ can also be described as an IR fixed point of the Gross-Neveu-Yukawa model

Zinn-Justin, Moshe; Hasenfratz et al

$$\mathcal{S}(\bar{\psi},\psi,\sigma) = \int \mathrm{d}^d x \left[-\bar{\psi}\cdot\left(\partial\!\!\!/ + g\Lambda^{\varepsilon/2}\sigma\right)\psi + \tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 + \tfrac{1}{2}m^2\sigma^2 + \frac{\lambda}{4!}\Lambda^{\varepsilon}\sigma^4\right]$$

- The beta functions are

$$\beta_\lambda = -\varepsilon\lambda + \frac{1}{8\pi^2}\left(\frac{3}{2}\lambda^2 + N\lambda g^2 - 6Ng^4\right)$$
$$\beta_{g^2} = -\varepsilon g^2 + \frac{N+6}{16\pi^2}g^4,$$

- IR stable fixed point Moshe, Zinn-Justin

$$g_*^2 = \frac{16\pi^2\varepsilon}{N+6}, \quad \lambda_* = 16\pi^2 R\varepsilon \qquad R = \frac{24N}{(N+6)\left[(N-6)+\sqrt{N^2+132N+36}\right]}$$

- In d=3 the U(N) singlet sector of the large N model is conjectured dual to type B Vasiliev theory in AdS_4 with the alternate boundary conditions. Leigh, Petkou; Sezgin, Sundell

Towards Interacting 5-d O(N) Model

- Scalar large N model with $\frac{\lambda}{4}(\phi^i\phi^i)^2$ interaction has a UV fixed point for 4<d<6.
- In $4+\epsilon$ $\beta_\lambda = \epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \dots$
- So, the UV fixed point is at a negative coupling

$$\lambda_* = -\frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2)$$

- At large N, conjectured to be dual to Vasiliev theory in AdS_6 with Δ_- boundary condition on the bulk scalar. Giombi, IK, Safdi
- Check of 5-dimensional F-theorem $-F = \log Z_{S^5}$

$$F_{\rm UV}^{(1)} - F_{\rm IR}^{(1)} = -\frac{3\zeta(5) + \pi^2\zeta(3)}{96\pi^4} \approx -0.0016$$

Perturbative IR Fixed Points

- Work in $d = 6 - \epsilon$ with O(N) symmetric cubic scalar theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma(\phi^i \phi^i) + \frac{g_2}{6}\sigma^3$
- The beta functions Fei, Giombi, IK

$$\beta_1 = -\frac{\epsilon g_1}{2} + \frac{(N-8)g_1^3 - 12 g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3}$$

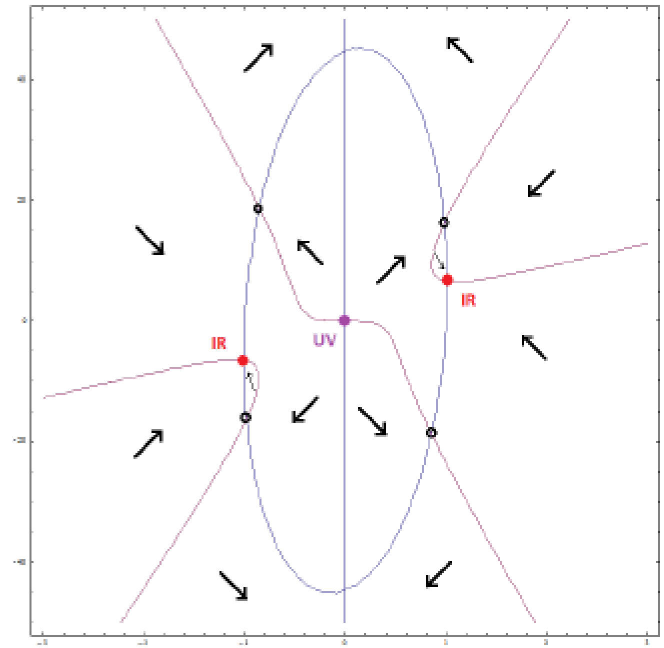
$$\beta_2 = -\frac{\epsilon g_2}{2} + \frac{-4N g_1^3 + N g_1^2 g_2 - 3 g_2^3}{4(4\pi)^3}$$

- For large N, the IR stable fixed point is at real couplings

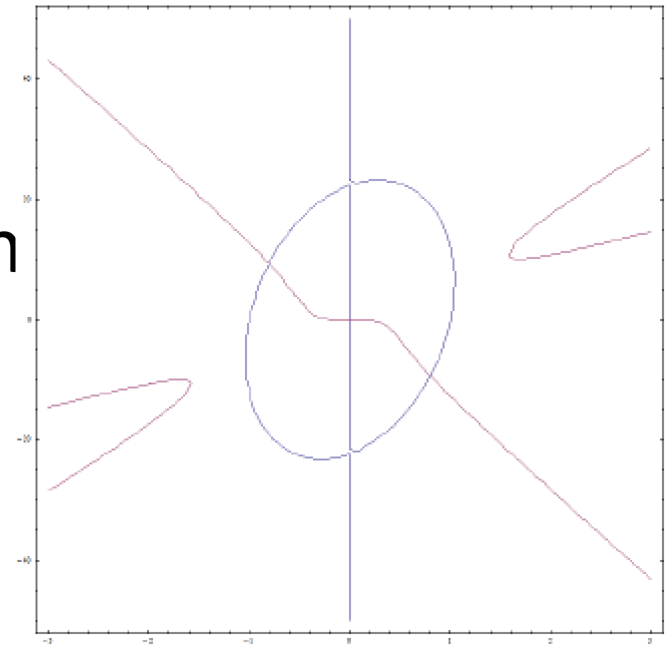
$$g_{1*} = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \qquad g_{2*} = 6 g_{1*}$$

RG Flows

- Here is the flow pattern for N=2000



- The IR stable fixed points go off to complex couplings for N < 1039. Large N expansion breaks down very early!



- The dimension of sigma is $\Delta_\sigma = 2 - \frac{\epsilon}{2} + \frac{Ng_1^2 + g_2^2}{12(4\pi)^3}$
- At the IR fixed point this is $2 + 40\frac{\epsilon}{N}$
- Agrees with the large N “bootstrap” for the O(N) model in d dimensions:

Petkou (1995)

$$2 + \frac{4}{N}\frac{\Gamma(d)}{\Gamma(d/2-1)\Gamma(1-d/2)\Gamma(d/2)\Gamma(d/2+1)}$$

- For N=0, the fixed point at imaginary coupling may lead to a description of the Lee-Yang edge singularity in the Ising model. Michael Fisher (1978)
- For N=0, Δ_σ is below the unitarity bound $2 - \frac{\epsilon}{2}$
- For N>1039, the fixed point at real couplings is consistent with unitarity.

Conclusions

- Higher-Spin AdS/CFT is about Quantum Gravity.
- Renormalization Group methods are important, as in many other fields.
- Yet, there is much room for new tools to explore this duality.
- Happy 60-th Birthday, Joe!!!