Quantum Mechanics Without State Vectors

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Are there small non-linear corrections to the time-dependent Schrödinger equation?
S. W. 1989

A problem: Instantaneous communication in entangled systems!
Polchinski 1991

Gisin 1990:
For isolated systems I and II, with ρ^I_{mn} fixed, it is always possible to find entangled states such that measurements in system II will put system I in any ensemble of states $|\Psi_\ell>$ with probabilities P_ℓ, provided only that

$$\sum_\ell P_\ell \, |\Psi_\ell >< \Psi_\ell| = \rho^I \; .$$

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This is OK in ordinary quantum mechanics, because

- $\rho \Rightarrow$ all probabilities.
- $d\rho^I/dt = -i\left[H^I, \rho^I\right]$.

Still, if state vectors can be changed instantaneously by distant measurements, can we take them seriously as a representation of reality?

Also, the possibility of instantaneous communication stands in the way of some attempts (e. g. S.W. 2012) to improve on the Copenhagen interpretation, without surrendering realism or accepting many worlds.

A modest proposal: Any statement that a system is in an ensemble of states $|\Psi_\ell >$ with probabilities P_ℓ has no physical significance, except that it implies a density matrix

$$\rho = \sum_\ell P_\ell \, |\Psi_\ell >< \Psi_\ell| \; .$$

The same density matrix will describe many different ensembles, but only the density matrix has physical significance.

Example: Spin 1/2 ‘

Pure State	Probability
$North$	50%
$South$	14%
$East$	35%

or

Pure State	Probability
$Northeast$	75%
$Southwest$	25%

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North	50%
South	14%
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or

Pure State	Probability
Northeast	75%
Southwest	25%

$$\rho = \begin{pmatrix} 0.69 & 0.17 \\ 0.17 & 0.31 \end{pmatrix} .$$

Interpretive postulate: $\overline{A} = \mathrm{Tr}(\rho A)$

- $\rho^\dagger = \rho \Leftrightarrow \overline{A}^* = \overline{A}$ if A Hermitian.
- $\mathrm{Tr}\rho = 1 \Leftrightarrow < \alpha >= \alpha$ if α c-number.
- $\rho \text{ positive} \Leftrightarrow \overline{A} \geq 0$ if A positive.

What's the difference?

Under symmetry transformations, $\Psi \mapsto U\Psi$ (Wigner 1939), so if ρ is defined by $\sum_\ell P_\ell \, |\Psi_\ell >< \Psi_\ell|$ with invariant probabilities P_ℓ, then $\rho \mapsto U\rho U^\dagger$.

Otherwise, $g : \quad \rho \mapsto \rho'$, with

$$\rho'_{M'N'} = \sum_{MN} K_{M'M,N'N}[g] \, \rho_{MN}$$

So far, this has been considered only for time-translation.
General time evolution: (Kossakowski 1972; Lindblad 1976; Sudarshan et al. 1976)
Spontaneous localization: (Ghirardi, Rimini, & Weber, 1986; Pearle, 1989; Bassi & Ghirardi 2003, etc.)
$\Psi(t)$ unnecessary!

An Example $\quad SU(3) \quad [d=3]$

$$\begin{pmatrix} \rho_{12} \\ \rho_{13} \\ \rho_{23} \end{pmatrix} \mapsto \mathcal{U} \begin{pmatrix} \rho_{12} \\ \rho_{13} \\ \rho_{23} \end{pmatrix} \ , \quad \mathcal{U}^\dagger \mathcal{U} = \mathbf{1}$$

and $\rho_{NN} \mapsto \rho_{NN}$.

Here ρ transforms as

$$\mathbf{3} + \overline{\mathbf{3}} + \mathbf{1} + \mathbf{1} + \mathbf{1} \ .$$

This cannot take the form $\rho \mapsto U \rho U^\dagger$ of ordinary quantum mechanics because then ρ would transform as

$$\mathbf{3} \times \overline{\mathbf{3}} = \mathbf{8} + \mathbf{1} \ ,$$

or

$$\mathbf{singlets} \times \mathbf{singlets} = \mathbf{singlets} \ .$$

General Symmetries

$$\rho'_{M'N'} = \sum_{MN} K_{M'M,N'N}[g]\,\rho_{MN}$$

- ρ' Hermitian for all Hermitian ρ

  $\Leftrightarrow K[g]^*_{M'M,N'N} = K_{N'N,M'M}[g]$.

- $\mathrm{Tr}\rho' = \mathrm{Tr}\rho$

  $$\Leftrightarrow \sum_{M'} K_{M'M,M'N}[g] = \delta_{N'N} \ .$$

- ρ' positive for all positive ρ
  $\Leftrightarrow \ ???$.

Transformation of Entangled Systems

m, n, etc. label states of system I
a, b, etc. label states of system II
Entanglement: $\rho_{ma\,nb} \neq \rho^I_{mn}\rho^{II}_{ab}$. But

$$K_{m'a'\,n'b',ma\,nb}[g] = K^I_{m'n',mn}[g]\;K^{II}_{a'b',ab}[g]\;.$$

$$\sum_{m'} K^I_{m'm,m'n}[g] = \delta_{mn}\;,\quad \sum_{a'} K^{II}_{a'a,a'b}[g] = \delta_{ab}\;.$$

Define $\rho^I_{mn} \equiv \sum_a \rho_{ma\,na}$ so that, if
$A_{ma,nb} = A^I_{mn}\delta_{ab}$, then
$\overline{A} \equiv \mathrm{Tr}(A\rho) = \mathrm{Tr}(A^I\rho^I)$.

For a transformation $g : \rho \mapsto \rho'$:

$$\rho'^{(I)}_{m'n'} = \sum_{a'}\sum_{mnab} K^I_{m'm,n'n}[g]\;K^{II}_{a'a,a'b}[g]\;\rho_{ma,nb}$$
$$= \sum_{mn} K^I_{m'm,n'n}[g]\;\rho^I_{mn}\;.$$

Because $K[g]^*_{M'M,N'N} = K_{N'N,M'M}[g]$,

$$K_{M'M,N'N}[g] = \sum_i \eta^{(i)}[g]\; u^{(i)}_{M'M}[g]\; u^{(i)*}_{N'N}[g]\ ,$$

$$\sum_{N'N} K_{M'M,N'N}[g]\, u^{(i)}_{N'N}[g] = \eta^{(i)}[g]\, u^{(i)}_{M'M}[g]\ ,$$

$$\mathrm{Tr}\Big(u^{(i)\dagger}[g]\, u^{(j)}[g]\Big) = \delta_{ij}\ .$$

Trace condition:

$$\sum_i \eta^{(i)}[g]\; u^{(i)\dagger}[g]\; u^{(i)}[g] = \mathbf{1}\ .$$

$$\rho' = \sum_i \eta^{(i)}[g]\; u^{(i)}[g]\; \rho\; u^{(i)\dagger}[g]\ .$$

With several i, this is a generalization of the transformation $\rho \mapsto U\rho U^\dagger$ of ordinary quantum mechanics.

Group Multiplication Law

We require that

$$\sum_{M'N'} K_{M''M',N''N'}[g]\, K_{M'M,N'N}[\overline{g}]$$
$$= K_{M''M,N''N}[g\overline{g}]\ ,$$

and so

$$\sum_{ij} \eta^{(i)}[g]\, \eta^{(j)}[\overline{g}]\, u^{(i)}[g]\, u^{(j)}[\overline{g}] \ \otimes\ u^{(j)\dagger}[\overline{g}]\, u^{(i)\dagger}[g]$$
$$= \sum_{k} \eta^{(k)}[g\overline{g}]\, u^{(k)}[g\overline{g}] \otimes u^{(k)\dagger}[g\overline{g}]\ ,$$

where, for any matrices A and B,

$$[A \otimes B]_{M''M,N''N} \equiv A_{M''M}\ B_{NN''}\ .$$

Continuous Symmetries

$$g[0] = \mathbf{I} \quad K[\mathbf{I}] = \mathbf{1} \otimes \mathbf{1}$$

Eigenvalues and eigenvectors:

$$u^{(1)}[\mathbf{I}] = \mathbf{1}/\sqrt{d} \qquad \eta^{(1)}[\mathbf{I}] = d$$

$$\mathrm{Tr}\; u^{(\alpha)}[\mathbf{I}] = 0 \qquad \eta^{(\alpha)}[\mathbf{I}] = 0$$

For group parameter ϵn with $\epsilon \to 0$,

$$\sqrt{\eta^{(1)}[g(\epsilon n)]}u^{(1)}[g(\epsilon n)] \to \mathbf{1} - i\epsilon \sum_r n^r \tau_r + O(\epsilon^2)$$

$$u^{(\alpha)}[g(\epsilon n)] \to u^{(\alpha)}(n) \;, \quad \eta^{(\alpha)}[g(\epsilon n)] \to \epsilon \Delta^{(\alpha)}(n)$$

$$\sum_{M'MN'N} u^{(\alpha)*}_{M'M}(n) \left[\frac{\partial K_{M'M,N'N}[g(\epsilon n)]}{\partial \epsilon} \right]_{\epsilon=0} u^{(\beta)}_{N'N}(n)$$
$$= \delta_{\alpha\beta}\, \Delta^{(\alpha)}(n)$$

Trace condition:

$$-in{\cdot}\tau+in{\cdot}\tau^\dagger+\sum_\alpha \Delta^{(\alpha)}(n)\, u^{(\alpha)\dagger}(n)\, u^{(\alpha)}(n) = 0\ ,$$

where $T_r^\dagger = T_r$.

$$K[g(\epsilon n)] \to \mathbf{1}\otimes\mathbf{1} + \epsilon\Bigg[-\,i\,n\cdot\tau\otimes\mathbf{1} + \mathbf{1}\otimes\ in\cdot\tau^\dagger$$
$$+\sum_\alpha \Delta^{(\alpha)}(n)\, u^\alpha(n)\otimes u^{\alpha\dagger}(n)\Bigg]$$

Hence

$$\sum_\alpha \Delta^{(\alpha)}(n)\, u^\alpha(n)\otimes u^{\alpha\dagger}(n) \propto n\ ,$$

and in particular

$$\sum_\alpha \Delta^{(\alpha)}(n)\, u^{(\alpha)\dagger}(n)\, u^{(\alpha)}(n) = \sum_r n^r\theta_r\ .$$

$$\text{so}\quad \tau_r = T_r - \frac{i}{2}\theta_r\ ,\qquad T_r^\dagger = T_r$$

(For compact groups, $\sigma = 0$.)

$$
\begin{aligned}
\delta\rho = \epsilon \Bigg[& i \sum_r n^r \, [T_r, \rho] \\
& + \sum_\alpha \Delta^{(\alpha)}(n) \Big[u^{(\alpha)}(n) \, \rho \, u^{(\alpha)(n)\dagger} \\
& - \frac{1}{2} u^{(\alpha)\dagger}(n) \, u^{(\alpha)}(n) \, \rho - \frac{1}{2} \rho \, u^{(\alpha)\dagger}(n) \, u^{(\alpha)}(n) \Big] \Bigg] .
\end{aligned}
$$

General Group Multiplication Rule

$$
g(\epsilon n) g(\epsilon \overline{n}) = g\Big(\epsilon n + \epsilon \overline{n} + \epsilon^2 f(n, \overline{n}) + O(\epsilon^3)\Big)
$$

where

$$
\begin{aligned}
f^r(n, \overline{n}) = & \frac{1}{2} \sum_{st} C^r_{st} n^s \overline{n}^t \\
& + \text{terms symmetric in } n \,\&\, \overline{n}
\end{aligned}
$$

$$
\begin{aligned}
&[n\cdot\tau,\overline{n}\cdot\tau]\otimes\mathbf{1}+\mathbf{1}\otimes[n\cdot\tau,\overline{n}\cdot\tau]^{\dagger}\\
&+i\sum_{\alpha}\Delta^{(\alpha)}(\overline{n})\ [\tau\cdot n,u^{(\alpha)}(\overline{n})]\otimes u^{(\alpha)\dagger}(\overline{n})-n\leftrightarrow\overline{n}\\
&-i\sum_{\alpha}\Delta^{(\alpha)}(\overline{n})\ u^{(\alpha)}(\overline{n})\otimes[\tau\cdot n,u^{(\alpha)}(\overline{n})]^{\dagger}-n\leftrightarrow\overline{n}\\
&-\sum_{\alpha\beta}\Delta^{(\alpha)}(n)\,\Delta^{(\beta)}(\overline{n})u^{(\alpha)}(n)u^{(\beta)}(\overline{n})\\
&\qquad\qquad\otimes u^{(\beta)\dagger}(\overline{n})u^{(\alpha)\dagger}(n)-n\leftrightarrow\overline{n}\\
&=i\sum_{rst}\tau_r\,C^r_{st}n^s\overline{n}^t\otimes\mathbf{1}+i\mathbf{1}\otimes\sum_{rst}\tau_r^{\dagger}\,C^r_{st}n^s\overline{n}^t\\
&-\sum_{rst}\left[\frac{\partial}{\partial(n+\overline{n})^r}\sum_{\alpha}\Delta^{(\alpha)}(n+\overline{n})u^{(\alpha)}(n+\overline{n})\right.\\
&\qquad\qquad\left.\otimes u^{(\alpha)\dagger}(n+\overline{n})\right]_{n+\overline{n}=0}\qquad C^r_{st}n^s\overline{n}^t
\end{aligned}
$$

Ordinary Quantum Mechanics

If all $\Delta^{(\alpha)}(n)$ vanish, then

$$\theta_r = 0 \quad \text{so} \quad \tau_r = T_r \ .$$

$$[T_s, T_t] = i \sum_r C^r_{st} T_r$$

$$\delta\rho = i \sum_r n^r \ [T_r, \rho] \ .$$

A Sample Solution with $\Delta^{(\alpha)}(n) \neq 0$ for Abelian Symmetries

Take $\sum_{rs} n^r \overline{n}^s C^t_{rs} = 0$. Try taking $n \cdot T$ and $\overline{n} \cdot T$ and relevant $u^{(\alpha)}(n)$, $u^{(\alpha)\dagger}(n)$, $u^{(\beta)}(\overline{n})$, $u^{(\beta)\dagger}(\overline{n})$ to all commute with each other.

Then constraints reduce to 0=0.

Adopt a basis with

$$[u^{(\alpha)}(n)]_{MN} = \delta_{MN} u_{\alpha M}(n)\ , [n\cdot T]_{MN} = \delta_{MN}\, n\cdot T_M$$

$$\delta_n \rho_{MN} = \epsilon\, \rho_{MN} \Bigg[in \cdot (T_M - T_N) + \sum_\alpha \Delta^{(\alpha)}(n) \Big[u_{\alpha M}(n) u_{\alpha N}(n)^* - \frac{1}{2}|u_{\alpha M}(n)|^2 - \frac{1}{2}|u_{\alpha N}(n)|^2 \Big] \Bigg] \ ,$$

Positivity

We need ρ' positive for all positive ρ. This is OK if $\eta^{(i)} \geq 0$. But recall

$$\eta^{(\alpha)}[g(\epsilon n)] \to \epsilon \Delta_r^{(\alpha)}(n)$$

This cannot be positive for all ϵ unless $\Delta^{(\alpha)}(n) = 0$. But then ρ transforms as in ordinary quantum mechanics.

Generalization

THEOREM
If g has an inverse g^{-1}, and if all $\eta^{(i)}[g]$ and $\eta^{(j)}[g^{-1}]$ are positive, then $u^{(i)}[g] = c^{(i)}[g]u[g]$, so

$$\rho \to U[g]\rho U^{\dagger}[g] \ ,$$

where $U[g] \equiv \sum_i \eta^{(i)}[g]\, |c^{(i)}[g]|^2|^{1/2} u(g)$.
(The trace condition gives $UU^{\dagger} = \mathbf{1}$.)

PROOF:

For any Hermitian positive ρ,

$$\sum_{ij} \eta^{(i)}[g]\, \eta^{(j)}[g^{-1}]\, u^{(i)}[g]\, u^{(j)}[g^{-1}]$$
$$\times\, \rho\, u^{(j)\dagger}[g^{-1}]\, u^{(i)\dagger}[g] = \rho\ ,$$

Find unitary Ω for which $\rho^{\mathrm{D}} = \Omega\rho\Omega^{-1}$ is diagonal, $[\rho^{\mathrm{D}}]_{MN} = P_M \delta_{MN}$.

$$\sum_{ij} \eta^{(i)}[g]\, \eta^{(j)}[g^{-1}]\, \sum_L \left[u^{(i\mathrm{D})}[g]\, u^{(j\mathrm{D})}[g^{-1}]\right]_{ML} P_L$$
$$\times \left[u^{(i\mathrm{D})}[g]\, u^{(j\mathrm{D})}[g^{-1}]\right]^*_{NL} = P_M \delta_{MN}\ ,$$

where

$$u^{(i\mathrm{D})}[g] = \Omega u^{(i)}[g]\Omega^{-1}\ ,\ u^{(j\mathrm{D})}[g^{-1}] = \Omega u^{(j)}[g^{-1}]\Omega^{-1}$$

This must hold for all real numbers P_N, so it follows that, for all L, M, and N:

$$\sum_{ij} \eta^{(i)}[g]\,\eta^{(j)}[g^{-1}]\,\Big[u^{(i\mathrm{D})}[g]\,u^{(j\mathrm{D})}[g^{-1}]\Big]_{ML}$$
$$\times \Big[u^{(i\mathrm{D})}[g]\,u^{(j\mathrm{D})}[g^{-1}]\Big]^*_{NL} = \delta_{ML}\delta_{NL}\ .$$

In particular, if $M = N \neq L$, then

$$\sum_{ij} \eta^{(i)}[g]\,\eta^{(j)}[g^{-1}]\,\left|\Big[u^{(i\mathrm{D})}[g]\,u^{(j\mathrm{D})}[g^{-1}]\Big]_{ML}\right|^2 = 0\ .$$

If all $\eta^{(i)}[g] > 0$ and $\eta^{(j)}[g^{-1}] > 0$, then for $M \neq L$

$$\Big[u^{(i\mathrm{D})}[g]\,u^{(j\mathrm{D})}[g^{-1}]\Big]_{ML} = 0\ .$$

so

$$\Big[u^{(i\mathrm{D})}[g]\,u^{(j\mathrm{D})}[g^{-1}], \rho^{(\mathrm{D})}\Big] = 0$$

so

$$\Big[u^{(i)}[g]\,u^{(j)}[g^{-1}], \rho\Big] = 0$$

so

$$u^{(i)}[g]\, u^{(j)}[g^{-1}] = c_{ij}[g]\mathbf{1}\ .$$

where $c_{ij}[g] = \mathrm{Det} u^{(i)}[g]\, \mathrm{Det} u^{(j)}[g^{-1}]$. There must be at least one j for which $\mathrm{Det} u^{(j)}[g^{-1}] \neq 0$ and $\Delta^{(j)})[g^{-1}] \neq 0$, since otherwise we would have $u^{(i)}[g]\, u^{(j)}[g^{-1}] = 0$ for all relevant i and j. So for such j,

$$u^{(i)}[g] = \mathrm{Det}\, u^{(i)}[g] \times u[g]\ ,$$

where $u[g] = u^{(j)-1}[g^{-1}] \times \mathrm{Det}\, u^{(j)}[g^{-1}]$. Then

$$\rho \mapsto U[g]\rho U^{\dagger}[g]\ ,$$

$$U[g] = \sum_i \left|\mathrm{Det}\, u^{(j)}[g^{-1}]\right|^2\ u[g]$$

$$\mathbf{1} = \sum_i \eta^{(i)}[g] u^{(i)\dagger}[g] u^{(i)}[g] = U^{\dagger}[g]\, U[g]\ .$$

Time translation: $\qquad n = \mathcal{T}$

$$\eta^{(\alpha)}(\delta t\, \mathcal{T}) \to \delta t\, \Delta^{(\alpha)}(\mathcal{T})\ .$$

$\eta^{(\alpha)} \geq 0$ for $\delta t > 0$ if $\Delta^{(\alpha)}(\mathcal{T}) \geq 0$.
Define $L^{(\alpha)} \equiv \sqrt{\Delta^{(\alpha)}(\mathcal{T})}\, u^{(\alpha)}(\mathcal{T})$ and $H = -T_{\mathcal{T}}$. Then

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_\alpha \Bigg[L^{(\alpha)}\,\rho(t)\,L^{(\alpha)\dagger} - \frac{1}{2} L^{(\alpha)\dagger} L^{(\alpha)}\,\rho(t) - \frac{1}{2}\rho(t)\,L^{(\alpha)\dagger} L^{(\alpha)} \Bigg] ,$$

Lindblad 1976

Do we need all $\eta^{(i)}[g] \geq 0$?

We can have ρ' positive for all positive ρ, even if some $\eta^{(i)}[g] < 0$.
Example: $K_{M'M,N'N} = \delta_{M'N}\delta_{N'M}$ has eigenvalues ± 1. But here $\rho' = \rho^{\mathrm{T}}$, which is positive if ρ is.

Complete Positivity (Stinespring 1955):

Consider a system I, and a linear mapping $K^I : \rho^I \mapsto \rho^{I\prime}$ for which $\rho^{I\prime}$ positive for all positive ρ^I. Introduce an isolated system II and extend this mapping to K, which acts as K^I on I and as the identity on II. If K maps all positive (entangled) ρ into positive ρ' for all finite d_{II}, then K^I is *completely positive*. In this case $\eta^{(i)} \geq 0$. (Choi 1975) (But in the real world, $d_{II} = 1$ or $d_{II} = \infty$.)

$$\dot{\rho} = \mathcal{L}\rho$$

where, for any $d \times d$ matrix f,

$$\mathcal{L}f \equiv -i[H, f] + \sum_{\alpha} \Big[L^{(\alpha)} \, f \, L^{(\alpha)\dagger} - \frac{1}{2} L^{(\alpha)\dagger} \, L^{(\alpha)} \, f - \frac{1}{2} f \, L^{(\alpha)\dagger} \, L^{(\alpha)} \Big] ,$$

Measurement

Take $L^{(\alpha)}$ Hermitian, and ignore H.

$$\mathcal{L}f = -\frac{1}{2} \sum_{\alpha} [L^{(\alpha)}, \, [L^{(\alpha)} \, f]] \, .$$

so $\mathrm{Tr}\Big(g^\dagger \mathcal{L} f\Big) = \mathrm{Tr}\Big((\mathcal{L}g)^\dagger f\Big)$.

The general solution of the Lindblad equation is

$$\rho(t) = \sum_n f_n \, \exp(\lambda_n t) \ ,$$

where

$$\mathcal{L} f_n = \lambda_n f_n \ , \qquad (\lambda_n \text{ real}).$$

Also,

$$(f, \mathcal{L} f) = -\frac{1}{2} \sum_\alpha \mathrm{Tr}([L^{(\alpha)}, f]^\dagger [L^{(\alpha)}, f]) \leq 0 \ ,$$

so $\lambda_n \leq 0$. At late time, $\rho(t)$ dominated by zero modes, $\lambda_n = 0$. (E.g., $f_n \propto \mathbf{1}$.)

Suppose we measure a physical quantity A, for which $A\Lambda^{(\alpha)} = a_\alpha \Lambda^{(\alpha)}$, where $\Lambda_\alpha \Lambda_\beta = \delta_{\alpha\beta}\Lambda_\alpha$ and $\sum_\alpha \Lambda_\alpha = \mathbf{1}$. Take $L^{(\alpha)} = c_\alpha \Lambda_\alpha$, with c_α real. Solution of Lindblad equation:

$$\rho(t) = \sum_\alpha \Lambda_\alpha \rho(0) \Lambda_\alpha + \sum_{\alpha \neq \beta} e^{-(c_\alpha^2 + c_\beta^2)t/2} \Lambda_\alpha \rho(0) \Lambda_\beta$$

$$\to \sum_\alpha P_\alpha \Lambda_\alpha \text{ where } P_\alpha = \mathrm{Tr}(\rho(0)\Lambda_\alpha) \ .$$

NB: If

$$\rho(0) = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix} \ , \quad a^* = a, \quad |b|^2 < a(1{-}a) \ .$$

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

Then

$$\rho(t) = \begin{pmatrix} a & b\exp(\lambda t) \\ b^* \exp(\lambda t) & 1-a \end{pmatrix} \ ,$$

where $\lambda \equiv -(c_1^2 + c_2^2)/2$.

This has a negative eigenvalue for $t < -\ln[(1-a)a/|b|^2]/2|\lambda|$.

Testing Quantum Mechanics

Now assume that $\mathcal{L}_0 \gg \mathcal{L}_1$, where $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$:

$$\mathcal{L}_0 f \equiv -i[H, f]$$

$$\mathcal{L}_1 f \equiv \sum_\alpha \left[L^{(\alpha)}\, f\, L^{(\alpha)\dagger} - \frac{1}{2} L^{(\alpha)\dagger}\, L^{(\alpha)}\, f \right.$$
$$\left. - \frac{1}{2} f\, L^{(\alpha)\dagger}\, L^{(\alpha)} \right] .$$

Let $H|a>= E_a|a>$, $<a|b>= \delta_{ab}$. Then $\mathcal{L}_0$ has eigenmatrices

$$f^{(ab)} = |a><b|$$

& eigenvalues $-i(E_a - E_b)$.

The first-order corrections to non-degenerate eigenvalues $-i(E_a - E_b)$ (with $a \neq b$) are

$$
\begin{aligned}
\delta\lambda_{ab} &= \mathrm{Tr}(f^{(ab)\dagger}\mathcal{L}_1\, f^{(ab)}) \\
&= \sum_\alpha \Bigg[[L^{(\alpha)}]_{aa}[L^{(\alpha)\dagger}]_{bb} - \frac{1}{2}[L^{(\alpha)\dagger}\, L^{(\alpha)}]_{bb} \\
&\qquad - \frac{1}{2}[L^{(\alpha)\dagger}\, L^{(\alpha)}]_{aa} \Bigg] \\
&= \sum_\alpha \Bigg[i\mathrm{Im}([L^{(\alpha)}]_{aa}[L^{(\alpha)}]^*_{bb}) \\
&- \frac{1}{2}\Big|[L^{(\alpha)}]_{aa} - [L^{(\alpha)}]_{bb}\Big|^2 \\
&\qquad - \frac{1}{2}\sum_{c\neq b}\Big|[L^{(\alpha)}]_{cb}\Big|^2 - \frac{1}{2}\sum_{c\neq a}\Big|[L^{(\alpha)}]_{ca}\Big|^2 \Bigg] .
\end{aligned}
$$

The first-order corrections to degenerate zero eigenvalues with unperturbed eigenvectors f_{aa} are the eigenvalues of the matrix

$$M_{a'a} \equiv \ell_{a'a} - \delta_{a'a} \sum_b \ell_{ba} \ ,$$

where $\ell_{ba} \equiv \sum_\alpha |[L^{(\alpha)}]_{ba}|^2$. This always has at least one zero eigenvalue, with eigenvectors

$$v = \begin{bmatrix} \ell_{12} \\ \ell_{21} \end{bmatrix} \ , \ \ v = \begin{bmatrix} \ell_{12}\ell_{13} + \ell_{32}\ell_{13} + \ell_{12}\ell_{23} \\ \ell_{21}\ell_{13} + \ell_{21}\ell_{23} + \ell_{23}\ell_{31} \\ \ell_{31}\ell_{12} + \ell_{31}\ell_{32} + \ell_{21}\ell_{32} \end{bmatrix} \ ,$$

etc. So for $t \to \infty$,

$$\rho \to \frac{\sum_a v_a \, |a><a|}{\sum_a v_a} \ .$$