M-Theory on $\mathrm{K3} \times \mathrm{K3}$.

…an analysis of fluxes in a simple example.

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M-Theory on $\mathrm{K3} \times \mathrm{K3}$
is dual to
Heterotic string on $T^3 \times \mathrm{K3}$
is dual to
IIA string on $X \times S^1$
is dual to
M-Theory on $X \times T^2$.

for some Calabi–Yau threefold, X.

But there are at least tens of thousands of families of Calabi–Yau threefolds.

Many of these are connected in the moduli space by extremal (e.g., conifold) transitions.

The K3 $\times$ K3 compactification of M-theory must also therefore have thousands of components to its moduli space. But there is only one K3 $\times$ K3!

The solution, of course, is that 8-dimensional compactifications require a choice of flux or M2-branes such that

$$n_{M2} + \tfrac{1}{2}G^2 = \frac{\chi}{24}.$$

For $X \times T^2$, $\chi = 0$, so we can't add any fluxes (without breaking supersymmetry).

But for K3 $\times$ K3, we require fluxes and/or M2-branes. It is the choice of these that must provide the many possibilities.

So the rules for putting fluxes in K3 $\times$ K3 must be very interesting.

One approach is to pick a Calabi–Yau threefold, X, on which the type IIA string is compactified.

Then use F-theory rules to determine the dual heterotic string on K3. This allows the duality above to be completed.

Knowledge of the resulting moduli space can usually be used to deduce the required flux on $\mathrm{K3} \times \mathrm{K3}$. E.g., G often freezes moduli, M2-branes yield more moduli.

A Quick Review of IIA Strings on a K3

$$\begin{aligned} H^2(S, \mathbb{Z}) &= \Gamma_{3,19} \\ H^*(S, \mathbb{Z}) &= \Gamma_{4,20} \end{aligned} \tag{1}$$

Moduli Space of Ricci-flat metrics on S of volume 1 is

$$\mathrm{Gr}(\Gamma_{3,19}) = \mathrm{O}(\Gamma_{3,19}) \backslash \mathrm{O}(3, 19)/(\mathrm{O}(3) \times \mathrm{O}(19)).$$

The Grassmannian of space-like 3-planes Σ. Σ is spanned by the real and imaginary parts of Ω; and J.

The type IIA string on S has moduli corresponding to metric on S and the B-field.

Using

$$\frac{\mathrm{O}(4,20)}{\mathrm{O}(4)\times\mathrm{O}(20)} \cong \frac{\mathrm{O}(3,19)}{\mathrm{O}(3)\times\mathrm{O}(19)} \ltimes \mathbb{R}^{22} \rtimes \mathbb{R}_+, \qquad (2)$$

one may show that the moduli space of type IIA strings on S is $\mathrm{Gr}(\Gamma_{4,20})$.

We consider points in $\mathrm{Gr}(\Gamma_{4,20})$ to be given by space-like 4-planes Π.

Now try to build X, such that M-theory on $S_1 \times S_2$ with $G = 0$ is dual to M-theory on $X \times T^2$.

M-theory on S_1 is dual to the heterotic string on T^3.

Assuming S_1 is smooth, we have a gauge group of $\mathrm{U}(1)^{22}$ in seven dimensions.

So the heterotic string compactification on S_2 may involve no nontrivial bundle (see later for $\mathrm{U}(1)$-bundles).

Since $c_2(S_2) = 24$, we must therefore have 24 point-like instantons on S_2.

F-theory methods tell us exactly which manifold we use for $X_{G=0}$ such that IIA on $X_{G=0}$ is dual to the heterotic string on $S_2 \times T^2$ with 24 point-like E_8-instantons. Morrison, Vafa

Begin with the singular hypersurface

$$x_0^2 + x_1^3 + x_2^7 x_3^{10} + x_2^7 x_4^{10} + x_2^5 x_3^{14} + x_2^5 x_4^{14} = 0,$$

in $\mathbb{P}^4_{12,8,2,1,1}$, and resolve the various singularities.

We may also represent $X_{G=0}$ as a hypersurface in a toric variety. This makes various computations (such as Hodge numbers) very easy.

$$h^{1,1}(X_{G=0}) = h^{2,1}(X_{G=0}) = 43.$$

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The Moduli Space

$N = 4$ theories in four dimensions have a moduli space locally of the form $\mathscr{M}_1 \times \mathscr{M}_2$, where $\mathscr{M}_1$ and $\mathscr{M}_2$ are quaternionic Kähler manifolds.

For M-theory on $S_1 \times S_2$, the $\mathscr{M}_j$ factor is clearly associated with S_j.

For M-theory on $X \times T^2$, $\mathscr{M}_1$ is associated with vector multiplets and $\mathscr{M}_2$ is associated with hypermultiplets.

$$\dim \mathscr{M}_1 = h^{1,1}(X) + 1, \qquad \dim \mathscr{M}_2 = h^{2,1}(X) + 1.$$

Exchanging S_1 and S_2 corresponds to *Mirror Symmetry* for X.

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For the heterotic string on S_2 with 24 point-like instantons, the moduli space without instanton corrections is

$$\mathscr{M}_2 = \mathrm{Gr}(4,20) \ltimes \mathrm{Sym}^{24}(S_2).$$

But S_1 and S_2 appear identically in the compactification, so $\mathscr{M}_1$ must also be of this form.

M-theory on S_1 has a moduli space $\mathrm{Gr}(\Gamma_{3,19}) \times \mathbb{R}_+$. But it also had a gauge group of $\mathrm{U}(1)^{22}$, which, in three dimensions yields 22 more scalars. These scalars act like a B-field.

The 24 point-like instantons produce massless tensor fields which account for 24 more quaternionic degrees of freedom.

The uncorrected moduli space is therefore

$$\Big(\mathrm{Gr}(4,20) \times \mathrm{Gr}(4,20) \ltimes \mathrm{Sym}^{24}(S_1 \times S_2)\Big)/\mathbb{Z}_2.$$

The latter $\mathbb{Z}_2$ corresponds to exchanging S_1 and S_2.

The $\mathrm{Sym}^{24}(S_1 \times S_2)$ factor accounts for the 24 M2-branes.

$X_{G=0}$ must be self-mirror.

Now try $G \neq 0$, but still $N = 4$.

For any supersymmetry, supergravity analysis says G is primitive ($G \wedge J = 0$) and of type (2,2).

For $N = 4$,

$$G = \sum_{\alpha} \omega_1^{(\alpha)} \wedge \omega_2^{(\alpha)},$$

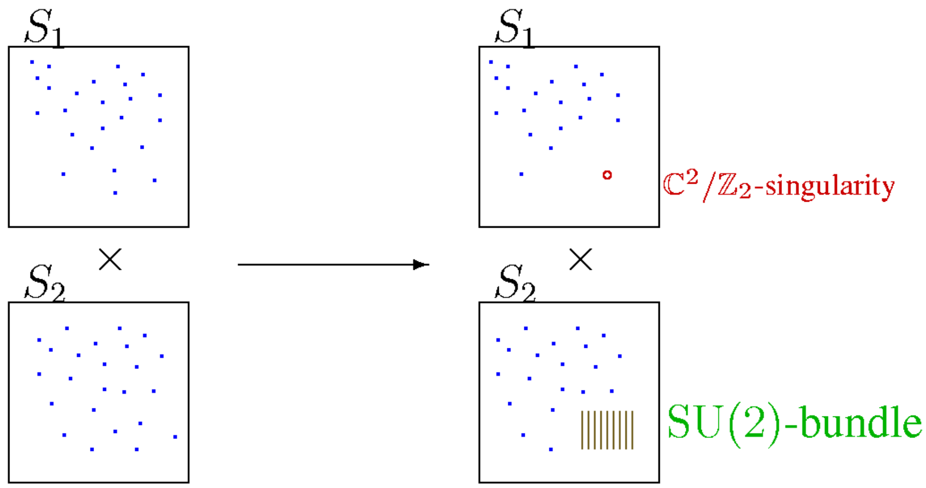
where ω_j^α are primitive (1,1)-forms on S_j.
Becker, Becker; Dasgupta, Rajesh, Sethi

If ω is a primitive (1,1)-form on a K3 surface, then ω^2 is a negative, even integer.

If $\omega^2 = -2$, then the K3 surface must be singular. ω is dual to a vanishing 2-sphere.

So putting $\omega_1^2 = -2$, forces S_1 to be singular, which gives the heterotic string theory a non-abelian gauge group for the bundle $E \to S_2$.

ω_2 now plays the rôle of the curvature of this bundle E. It is Lie algebra valued.



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The condition that ω_2 is primitive, is replaced by $c_1(E)$ is primitive, but this is trivial since $c_1(E) = 0$.

No S_2 moduli are frozen. Indeed, we gain moduli from the nonabelian G-flux (i.e., bundle moduli).

The $\frac{1}{2}G^2$ contribution to the tadpole is replaced by $c_2(E)$.

We may, of course, mix this effect on both sides. S_1 and S_2 may both acquire singularities and bundles with semi-simple structure groups.

"Explains" mirror symmetry structure seen by Perevalov and Rajesh.

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What about flux on a smooth K3 $\times$ K3?

If $\omega_j^2 \leq -4$ then S_j need not be singular.

We have already argued that $c_1(E) = \omega_2$.

So we want to look for heterotic compactifications with $\mathrm{U}(1)$ gauge groups and their Calabi–Yau duals.

At first sight, if the bundle E has a structure group containing $\mathrm{U}(1)$ then this $\mathrm{U}(1)$ is in the centralizer of the structure group and so should appear in the low-energy effective field theory.

As Witten pointed out many years ago, this isn't correct.

The low-energy effective action contains a term H^2, where

$$H = dB + \omega_Y - \omega_L.$$

If E has a first Chern class, then H^2 has a term that looks like $AdB\langle \mathrm{Tr}\, F\rangle$.

Via the Higgs mechanism, this makes the photon massive and eats up a B-field zero mode.

So we have no $\mathrm{U}(1)$!

We may, however, embed a $\mathrm{U}(1)$ identically into both E_8's. The diagonal combination is then broken as before, but the anti-diagonal combination is unbroken.

Consider the Calabi–Yau $X_{\mathrm{U}(1)}$ given by the resolution of

$$x_0^2 + x_1^4 + x_2^5 x_3^6 + x_2^5 x_4^6 + x_2^3 x_3^{10} + x_2^3 x_4^{10},$$

in $\mathbb{P}^4_{\{8,4,2,1,1\}}$.

Following the chain of dualities outlined earlier, one can prove that M-theory on $X_{\mathrm{U}(1)} \times T^2$ is dual to M-theory on a smooth K3 $\times$ K3 with a non-trivial G-flux (and 16 M2-branes).

Aldazabal, Font, Ibanez, Uranga; PSA

In this case,

$$G = \omega_1 \wedge \omega_2,$$

where $\omega_j = \alpha_j - \beta_j$ and $\alpha_j^2 = \beta_j^2 = -2$, $\alpha_j.\beta_j = 0$ on S_j.

Model first written down by Dasgupta, Rajesh, Sethi.

From the heterotic point of view, α_1 and β_1 are roots of $E_8 \times E_8$ denoting how the $\mathrm{U}(1)$ is embedded, and $c_1(E) = \alpha_2 - \beta_2$.

There are many fascinating details of the geometry of this correspondence (involving the Mordell–Weil groups of the various elliptic fibrations) but we do not have time to elucidate this.

$X_{G=0}$ and $X_{\mathrm{U}(1)}$ are both hypersurfaces in toric varieties and therefore connected by extremal (conifold) transitions.

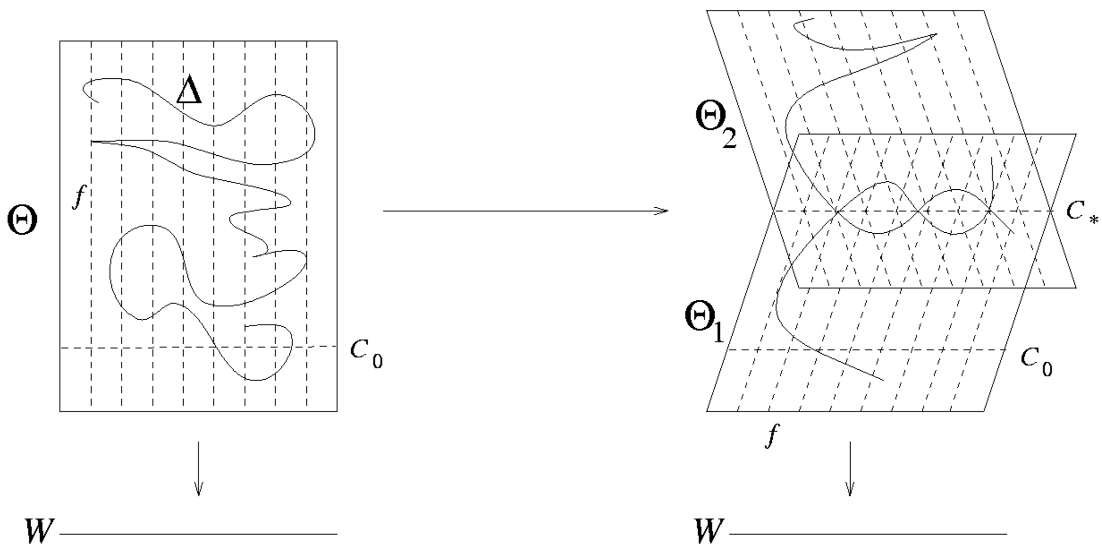
It must therefore be that the M-theory compactifications on $\mathrm{K3} \times \mathrm{K3}$ are connected by varying the moduli.

That is, we may convert M2-branes into G-flux by moving in the moduli space.

We may use our knowledge of F-theory map to follow this transition explicitly.

The assertion that one can do this in general is not new.
Bershadsky, Sadov; Gopakumar, Vafa; etc.

(but a full analysis of the global geometry is).



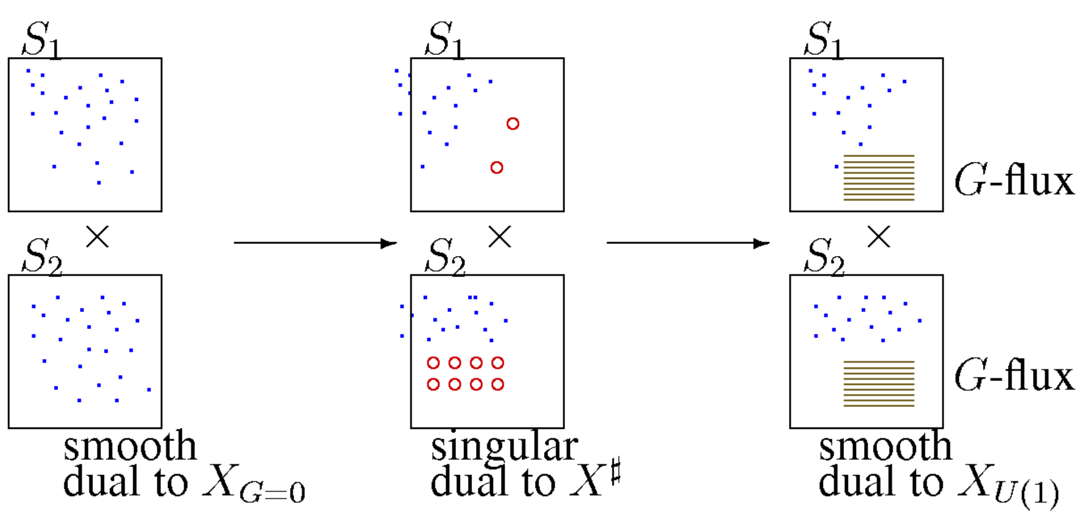
In this transition, S_1 and S_2 necessarily become singular (as they must since G lies in integral cohomology).

S_1 acquires 2 A_1 singularities
S_2 acquires 8 A_1 singularities

Indeed S_2 becomes globally an orbifold $\mathrm{K3}/\mathbb{Z}_2$ with 8 fixed points.

The two singularities on S_1 produce an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ enhanced gauge group.

The holonomy on S_2 exchanges the two $\mathrm{SU}(2)$'s.



We have accounted for some, but by no means all of the possible Calabi–Yau threefolds.

It seems reasonable to assert that all Calabi–Yau threefolds must be dual to some kind of $\mathrm{K3} \times \mathrm{K3}$ compactification whether or not we have a K3 fibration, elliptic fibration, etc.

After all, most Calabi–Yau threefolds are probably connected by extremal transitions.

We don't expect $\mathrm{K3} \times \mathrm{K3}$ to turn into some other space with $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ holonomy!

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So far we have analyzed the duality from F-theory methods which apply when quantum corrections to the moduli space are small. That is, everything is large.

We may follow transitions which necessarily move us into the interior of the moduli space where quantum corrections (M5-brane instantons) dominate.

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Consider the intersection X_1 of two hypersurfaces

$$\begin{aligned}(x_1^4 + x_3^4 - x_4^4)y_0 + (x_0^4 + x_2^4 + x_4^4)y_1 &= 0\\ x_1y_1 + x_2y_2 &= 0,\end{aligned} \quad (3)$$

in $\mathbb{P}^4 \times \mathbb{P}^1$.

This has an extremal transition to the quintic threefold.
Candelas, Green, Hübsch.

In this transition we move from a Calabi–Yau with a K3 fibration to one without.

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This transition forces $\mathrm{Vol}(S_1)^2\,\mathrm{Vol}(S_2)$ (or $\mathrm{Vol}(S_1)\,\mathrm{Vol}(S_2)^2$) to be stuck at the Planck size. If X and its mirror have no K3 fibration then we fix both.

This is morally equivalent to a choice of G-flux given by the volume 4-form of S_2. The Primitivity condition $J \wedge G = 0$ would then force J on S_1 to be zero (or thereabouts).

Thus we may use fluxes to fix the size of our manifold.

This contradicts the supergravity analysis — but we are at small size!!

It would be nice to make this analysis less schematic.

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Finally let us show that there are more choices of flux on K3 $\times$ K3 than there are Calabi–Yau threefolds!

We are considering M-theory on $X \times T^2$, where X is a Calabi–Yau threefold.

We have concerned ourselves purely with X so far. What about T^2?

Could it be that the degrees of freedom associated with the T^2 allow for more extremal transitions?

A naïve answer would be no, since M-theory on T^2 never has enhanced gauge symmetries.

There is a natural chain of theories:

1. 6 dimensions: F-theory on X.
2. 5 dimensions: M-theory on X.
3. 4 dimensions: M-theory on $X \times S^1$ (i.e., type IIA on X).
4. 3 dimensions: M-theory on $X \times T^2$.

The vector multiplet moduli space for these theories is, respectively, (assuming no quantum corrections to moduli space).

1. Nothing.
2. Real. E.g. $\mathrm{O}(1, n-2)/\,\mathrm{O}(n-2)$.
3. Complex. E.g. $\mathrm{O}(2, n-1)/(\mathrm{O}(2) \times \mathrm{O}(n-1))$.
4. Quaternionic. E.g. $\mathrm{O}(4, n)/(\mathrm{O}(4) \times \mathrm{O}(n))$.

The hypermultiplet moduli space is the same in each case!

In each case the moduli space is like a Grassmannian of space-like d-planes in $\mathbb{R}^{d,p}$.

There is also (as always!) a lattice in this space that represents some kind of integral cohomology.
(Algebraic 2-cocycles, 2-cocycles, even cocycles respectively.)

We “expect” some kind of enhanced gauge symmetry every time our d-plane is orthogonal to a lattice vector of length squared -2.

So the vector multiplet moduli space for M-theory on $X \times T^2$ should have more scope for extremal transitions than the type IIA string on X.

That is, M-theory on $X \times T^2$ has more transitions than the Calabi–Yau threefold X has geometric transitions.

We have ignored quantum corrections to the moduli space, but such corrections are not expected to remove transitions.

So, by duality, there are more choices of flux on K3 × K3 than there are Calabi–Yau threefolds!

Conclusions

- M-theory on K3 × K3 with flux is dual to M-theory on $X \times T^2$ without flux. This allows for a very geometric analysis of the properties of flux.
- A transition which changes M2-branes into G-flux may be followed through the moduli space "rigorously".
- A suitable choice of flux appears to fix the volume of a K3 surface — in contrast to supergravity analysis.
- The number of choices of flux on K3 × K3 appears to exceed the number of Calabi–Yau threefolds!