

“Catastrophic” Holography

By

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(Based on [hep-th/9902170](#), with A. Chamblin, R. Emparan and R. Myers)

15th March, 1999



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Introduction and Motivation

Background

A sharp statement has been made about the long anticipated map between large N gauge theory and string theory.

The answer to: Is there an effective theory of large N ?

...was motivated as follows:

The Lagrangian of pure gauge theory is:

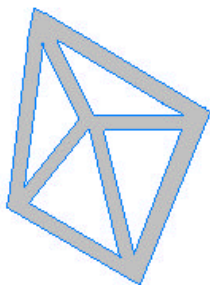
$$L = \frac{N}{g^2} \text{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$

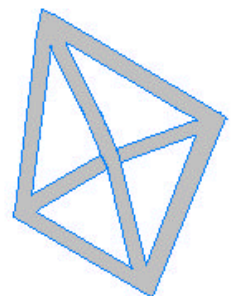
Expanding this, we see that vacuum graphs, for instance, get a factor:

- $\Rightarrow 1/N$ for propagators (E)
- $\Rightarrow N$ for vertices (V)
- $\Rightarrow N$ for loops (F)

$$N^{F+V-E} = N^\chi = N^{2-2h}$$



Perturbation theory in $1/N$ is organized by topology!



This looks like the world-sheet expansion of a string theory, where $N \sim 1/g_s$!



What is this effective string theory?

Using the results about geometrical properties of string theory from numerous calculations using D-brane technology, the sharp statement is (for example):

Large N , $D=4$, $N=4$ Susy
 $SU(N)$ Yang-Mills



Type IIB Supergravity
"compactified" on $AdS_5 \times S^5$

Furthermore:

$1/N$ corrections are generated by type IIB *string theory* on $AdS_5 \times S^5$

The effective string of large N gauge theory is the type IIB string!

Note that, for consistency:

- The radius of the S^5 is set by $\alpha'(4\pi g_s N)^{1/2}$, as is the inverse cosmological constant of the AdS_5 space.keep the radii finite by sending $\alpha' \rightarrow 0$ as $N \rightarrow \infty$
- Keep the radii large (and hence curvatures small), to make sure that the gravity solution is valid. So we must have $g_s N = g^2 N \gg 1$.

So the string coupling is weak, while the 't Hooft coupling is strong.

So the effective theory of large N is a "fundamental" string theory. (compare to expectations from the 80's).

The gauge theory is a conformal field theory in this case, and this is the motivation for "AdS-CFT Correspondence".

The change in dimensionality is behind the name "holography".

There is a version of this AdS/CFT correspondence for AdS₄:

Large N , $D=3$, $N=8$ Susy
CFT _{N} . (IR fixed point of
 $SU(N)$ Yang-Mills)



$D=11$ Supergravity
“compactified” on AdS₄×S⁷

...and similar for other dimensions (AdS₇, AdS₃...)

The content of the correspondence comes when a statement is made about computations in each theory. *Part* of such a statement:

$$Z_{CFT}(M) = \sum_i Z_{Sugra}(B_i)$$

- M is the manifold ($dim=n$) upon which the field theory lives. B_i is a family of manifolds ($dim=n+1$) related to M in a certain way.
- Z_{Sugra} is the supergravity partition function. Away from strictly large N it is either the string or M-theory partition function. We will stay at strict large N for this talk.

In the case where we consider the insertions of point-like operators in the field theory, we can say a bit more about M and B_i :

- ⇒ M is a n dimensional manifold ($n=4$ for most of this talk)
- ⇒ B_i is the family of manifolds that have M as their boundary.

Large N Phase Transitions

In computing the contributions to the right hand side, we have already seen that quantum corrections are small and so:

$$\sum_i Z_{Sugra}(B_i) = \sum_i e^{-I(B_i)}$$

where $I(B_i)$, the classical supergravity action, goes for large N as:
 $I(B_i) = N^\chi F(B_i)$, with $\chi > 0$.

So the field theory free energy $F(M) = -\ln Z_{CFT}(M)$ is dominated by the smallest $F(B_i)$:

$$\frac{F(M)}{N^\chi} \xrightarrow{N \rightarrow \infty} F(B_i)$$

Varying some parameter in M (and hence the B 's) can make two $F(B_i)$'s equal at some point in moduli space.... giving a phase transition, as the theory moves from one B to another. (Witten)

But M is compact...
How can we have a
phase transition?



N also measures the
*number of degrees of
freedom*, so phase
transitions meaningful
for infinite N .

Here, we will study when M is Euclidean, defining the theory at finite temperature.

Witten's Example (Hawking/Page)

Witten studied the case of Euclidean AdS₅, with (e.g.) boundary $M=S^3 \times S^1$. The radius, β , of the S^1 represents compact time and thus the inverse temperature. $T=1/\beta$

In this case, there were two choices for B ,

Euclidean AdS₅ itself (topology $R^4 \times S^1$),
Euclidean AdS₅-Schwarzschild solution (topology $R^2 \times S^3$).

Using this case, one is then able to study possible finite temperature phase transitions for the field theory.

Hawking and Page had already studied the thermodynamics of the bulk system in the context of semi-classical quantum gravity, and it was simply a matter of reinterpreting that result.

As solutions to Einstein's equations $R=4\Lambda$, the metrics are:

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right) d\tau^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2$$

$$ds^2 = \left(1 + \frac{r^2}{l^2} - \frac{m}{r^2}\right) d\tau^2 + \left(1 + \frac{r^2}{l^2} - \frac{m}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2$$

$$d\Omega_3^2 \text{ is round } S^3 \text{ metric; } l^2 = -\frac{6}{\Lambda}; \quad M = \frac{3\omega_3}{16\pi G} m$$

One has to analyze the allowed temperatures for these systems. For AdS₅, the period $\Delta\tau = \beta$, is arbitrary, but the τ -Killing vector has a zero at finite r for AdS₅-Sch, which gives a condition for β .

There's a horizon at the larger root, r_+ , of:

$$V(r_+) \equiv 1 + \frac{r_+^2}{l^2} - \frac{m}{r_+^2} = 0$$

So τ and r want to make an \mathbb{R}^2 , leaving a horizon with topology \mathbb{S}^3 .

- This subspace will have conical singularities for arbitrary period $\Delta\tau$, and we can avoid them by making circles have circumference/radius= 2π at r_+ .

$$\left. \frac{d(V^{\frac{1}{2}}(r) \Delta\tau)}{V^{-\frac{1}{2}}(r) dr} \right|_{r=r_+} = 2\pi$$

⇓

$$\Delta\tau V'(r_+) = 4\pi$$

This gives:

$$\beta = \frac{2\pi l^2 r_+}{2r_+^2 + l^2}$$

Notice that:

- β has a maximum, β_{\max} , at $r_+ = r_0$, so there is a minimum temperature T_{\min} below which black holes cannot exist.
- For $r_+ > r_0$, there are two allowed masses of black hole:

$$M_1 < M_2$$

The action, I , of AdS₅-Sch is computed (details later), using AdS₅ as a background....

$$I = \frac{\omega_3 (l^2 r_+^3 - r_+^5)}{4G(4r_+^2 + 2l^2)}$$

And we get these thermodynamic quantities, which give what we might expect them to....

$$\log Z = -I = F / T$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z = M$$

$$S = \beta \langle E \rangle + \log Z \equiv \beta \frac{\partial I}{\partial \beta} - I = \frac{A}{4G_N}$$

For large $\beta > \beta_{\max}$, we are at low temperature. The physics is dominated by AdS₅ geometry.

□ Above T_{\min} , there are two masses of black hole, M_1, M_2 , allowed.

□ M_1 , the smaller, has $C < 0$ and is unstable. It has +ve free energy.

□ M_2 , the larger, has $C > 0$, so is stable, but has:

C is the specific heat:

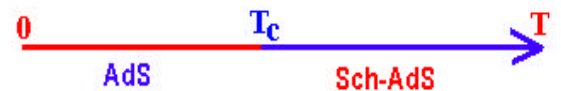
$$C = -\beta \frac{\partial S}{\partial \beta}$$

⇒ +ve free energy if $T < T_c$

⇒ -ve free energy if $T > T_c$

How does this compare to the field theory? To interpret, we must recall that we've taken $N \rightarrow \infty$, so *any free energy difference is made infinite*. (c.f. the action and partition function.)

Therefore, there is a genuine large N phase transition at $T = T_c$, where the physics is dominated by AdS₅-Sch.



In this high temperature phase ($\beta \rightarrow 0$), we note that

$$F \approx V_3 T^4 N^2 \quad \text{and} \quad S \approx V_3 T^3 N^2$$

...which is just what we expect for a field theory in this deconfined regime, as the number of “colour” degrees of freedom is N^2 .

In summary, the finite temperature phase transition of the field theory is controlled by a jump from AdS₅ to Sch-AdS₅.

Observations and Conjecture

Notice:

- ❑ The quantum formation and evaporation of black holes is implied as we change temperature.
- ❑ These, and other such bulk processes are often speculated to be non-unitary. (e.g., Hawking, et. al.)
- ❑ We do not think of phase transitions in field theory as non-unitary. However, there is a holographic map between the two processes.

So this suggests the following picture:

- ❑ transitions between solutions in the old quantum gravity approach (with $\Lambda < 0$ at least) might be given a unitary QFT interpretation, via a holographic relation.
- ❑ This includes even such drastic processes involving *space-time topology change*.
- ❑ Perhaps this is the beginning of directly *proving that all quantum gravity processes are unitary*.

The rest of this seminar will show that the holographic relationship holds for a class of charged black holes.

We will be able to set up the thermodynamics on these space-times and see that the phase structure has a field theory interpretation.

Charging up the Holographic Principle

A natural question:

What about charged black holes?

The Reissner-Nordstrom black hole would be a natural candidate.

Problem: It is NOT a solution of Einstein, but Einstein-Maxwell.

Another natural question:

Is there an Einstein-Maxwell truncation of gauged supergravity?

...and also:

Is this at all relevant to the AdS-CFT correspondence?

Answers:

It is seemingly very messy, but there *is* a very natural way to obtain EM-adS:

Put 10 or 11D sugra on “spinning spheres”: S^5 or S^7

⇒ Arrange the spheres to spin symmetrically in their natural embedding spaces, R^6 and R^8 . (So S^5 has 3 planes, S^7 has 4).

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + l^2 \sum_{i=1}^3 \left[d\mu_i^2 + \mu_i^2 \left(d\phi_i + \frac{2}{\sqrt{3}} A_\mu dx^\mu \right)^2 \right]$$

$$I = -\frac{1}{16\pi G} \int dx^5 \sqrt{-g} \left[R + \frac{12}{l^2} - l^2 F^2 - \frac{l^3}{6\sqrt{3}} \epsilon^{\mu\alpha\beta\gamma\delta} A_\mu F_{\alpha\beta} F_{\gamma\delta} \right]$$

⇒ In this way, you can get EM-adS₄ and EM-adS₅.

⇒ (Also EM-adS₃ but NOT EM-adS₇)

Relevance to Holography?

In “adS” holography, the most important feature is the negative cosmological constant.

EMadS should have some relevance too.

Anything to do with branes?

The “spinning” ansatz corresponds to symmetrically rotating branes!

So the dual field theory is simply the field theory on rotating branes.

e.g., for D=4 field theory, the $SO(6)$ R-symmetry current has the diagonal $U(1)$ switched on.

We are studying the same dual field theories, but in the presence of a background $U(1)$ current. Charge simply descends from rotation.

(Note: We will study RN-adS solutions of Einstein-Maxwell for all dimensions greater than two. Where there is no supergravity, we still propose a holographically dual field theory.)

The Charged Black Hole Solutions

Our (rescaled) action is:

$$I = -\frac{1}{16\pi G} \int d^{n+1}x \sqrt{-g} [R - 2\Lambda - F^2]$$

the solutions are:

$$\Lambda = -n(n-1)/(2l^2)$$

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_{n-1}^2$$

with

$$V(r) = 1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{l^2},$$

and

$$A = \left(-\frac{1}{c} \frac{q}{r^{n-2}} + \Phi \right) dt$$

Also:

$$M = \frac{(n-1)\omega_{n-1}}{16\pi G} m$$

$$Q = \sqrt{2(n-1)(n-2)} \frac{\omega_{n-1}}{8\pi G} q$$

$$c = \sqrt{\frac{2(n-2)}{n-1}}$$

□ Condition for a (non-singular) horizon at $r=r_+$:

$$\left(\frac{n}{n-2} \right) r_+^{2n-2} + l^2 r_+^{2n-4} \geq q^2 l^2$$

□ Saturate for an extremal (but NOT supersymmetric) black hole.

□ Choose the electrostatic potential at infinity by: $\Phi = \frac{1}{c} \frac{q}{r_+^{n-2}}$

(Let's focus on $n=4$, i.e., D=5 RN-adS; relevant to D=4 Yang Mills.)

Euclideanize by sending $t \rightarrow i\tau$, with period β ($=1/T$) and then:

In terms of charge:

$$\beta = \frac{4\pi l^2 r_+^5}{4r_+^6 + 2l^2 r_+^4 - 2q^2 l^2}$$

In terms of potential:

$$\beta = \frac{4\pi l^2 r_+}{2l^2 \left(1 - \frac{4}{3} \Phi^2 \right) + 4r_+^2}$$

Lets plot them and have a look.

Follow links to plots here (β vs r_+ for fixed potential)...

...and here (β vs r_+ for fixed charge fixed charge)

Notice that there are multiple available branches of solutions for a range of temperatures...below a critical value of Φ , or Q .

This will be crucial to the phase structure later....

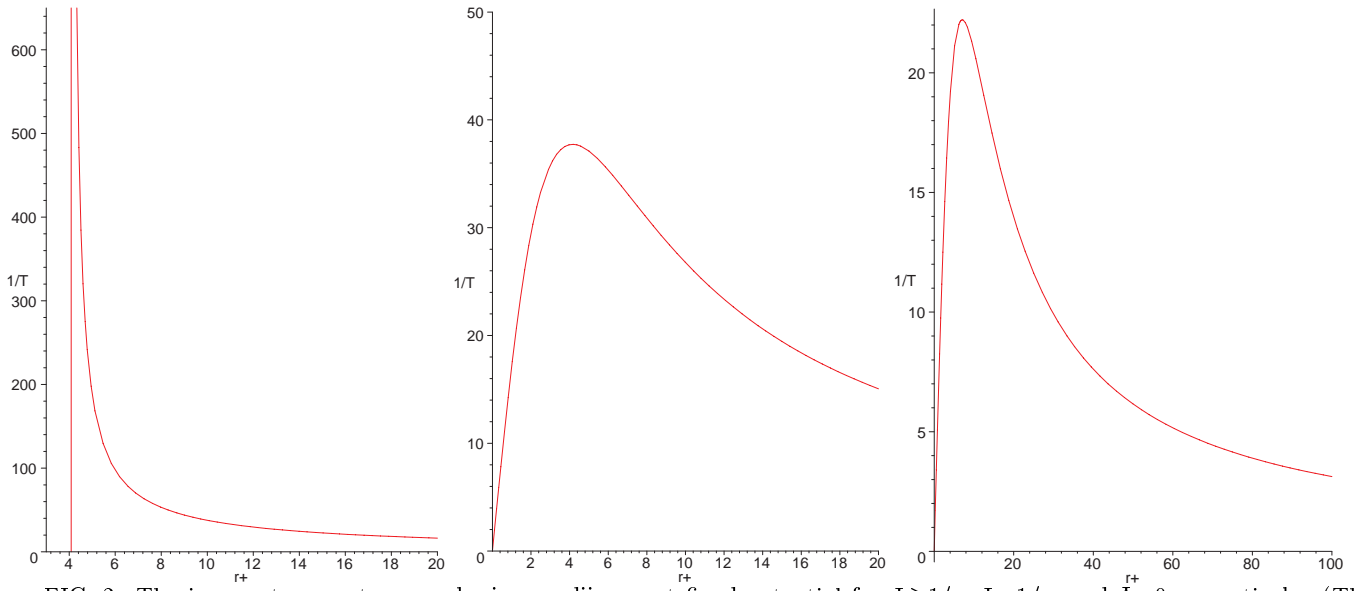


FIG. 2. The inverse temperature *vs.* horizon radii, r_+ , at fixed potential for $\Phi \geq 1/c$, $\Phi < 1/c$, and $\Phi = 0$ respectively. (The values $n=4$, $G=1$, $l=10$ and $\Phi = 1, 0.7, 0$ have been used here.) The divergence in the first graph (here, shown with a vertical line at $r_e=4.08$) is at zero temperature, where the black hole is extremal. This divergence goes away for $\Phi < 1/c$, in general, and the curve is similar to that of the uncharged situation with zero potential, shown last.

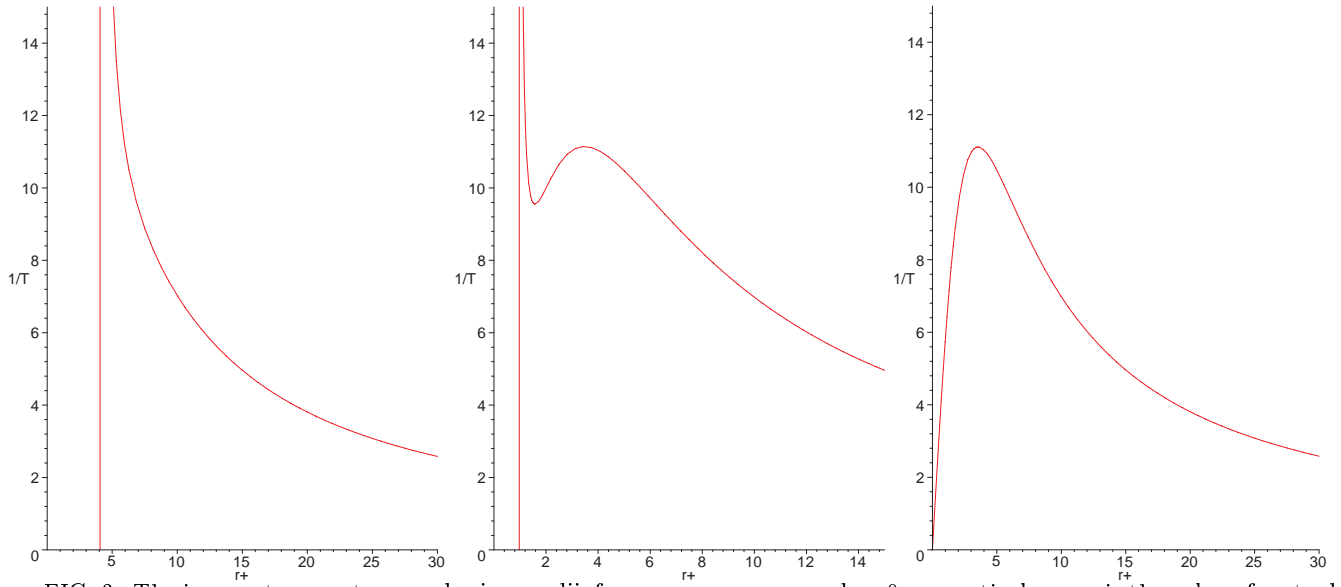


FIG. 3. The inverse temperature *vs.* horizon radii, for $q > q_{\text{crit}}$, $q < q_{\text{crit}}$, and $q = 0$, respectively. q_{crit} is the value of q at which the turning points of $\beta(r_+)$ appear or disappear. (The values $n=4$, $l=5$ and $q = 25, 5, 0$ have been used here.) The divergences (here, shown by the vertical lines at $r_e=0.98$ and 4.05) are at zero temperature, where the black hole is extremal. The final graph, for the uncharged case, may be thought of as a limit of the previous graphs where the divergence disappears, showing that small Schwarzschild black holes have high temperature.

The Action Calculation

The Euclidean action is computed using:

(Hawking, Horowitz, Ross)

$$I = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^5x \sqrt{g} (R - 2\Lambda - F^2) - \int_{\partial\mathcal{M}} d^4x \sqrt{\gamma} \Theta - \frac{1}{4\pi G} \int_{\partial\mathcal{M}} d^4x \sqrt{\gamma} F^{\mu\nu} n_\mu A_\nu$$

- \mathcal{M} is a compact region of spacetime $\partial\mathcal{M}$ is the boundary.
- $\gamma_{\mu\nu}$ is the induced metric on the boundary and Θ is the trace of the extrinsic curvature on it.
- **It's all infinite** as one sends the boundary to infinity. *Regulate by subtracting a contribution from a background.*
- Match the background to the solution of interest on the boundary to sufficient accuracy so that their difference disappears in the limit.

For example, looking at just the non-Maxwell pieces, we would get:
So define action of AdS to be zero, and we get the black hole action:

The volume difference is the $R \rightarrow \infty$ limit of:
$$I = -\frac{\Lambda}{8\pi G} \int_{\mathcal{M}} d^5x \sqrt{g} = \frac{12}{8\pi G l^2} [\text{Vol}_{\text{bh}} - \text{Vol}_{\text{bkgn d}}]$$

$$\text{Vol}_{\text{bkgn d}}(R) = \omega_3 \int_0^{\beta'} d\tau \int_0^R r^2 dr$$

and

$$\text{Vol}_{\text{bh}}(R) = \omega_3 \int_0^\beta d\tau \int_{r_+}^R r^2 dr$$

We match the solutions at radius R by rescaling to give the same period, β , to sufficiently high order.

Ultimately, we get the results:

Fixed potential

$$I = \frac{\omega_3}{16\pi G l^2} \beta \left[l^2 r_+^2 - r_+^4 - \frac{q^2 l^2}{r_+^2} \right] = \frac{\omega_3}{16\pi G l^2} \beta \left[l^2 r_+^2 \left(1 - \frac{4}{3} \Phi^2\right) - r_+^4 \right]$$

Fixed charge

$$I = \frac{\omega_3}{16\pi G l^2} \beta \left[l^2 r_+^2 - r_+^4 + \frac{5q^2 l^2}{r_+^2} - \frac{3}{2} l^2 r_e^2 - \frac{9}{2} \frac{q^2 l^2}{r_e^2} \right]$$

Note:

For fixed potential, we have used as background AdS + charged gas.

For fixed charge, we have used as background, an extremal black hole.
(AdS with fixed charge is NOT a solution of EMadS)

Thermodynamics

- We identify temperature with the inverse of the period $\Delta\tau$, so $T=1/\beta$.
- Once again, we can compute thermodynamic quantities of interest:

Fixed potential:

$$E = -\left(\frac{\partial I}{\partial \beta}\right)_{\Phi} - \frac{F}{\beta} \left(\frac{\partial I}{\partial F}\right)_{\beta} = \frac{3\omega_3}{16\pi G} m = M$$
$$S = \beta \left(\frac{\partial I}{\partial \beta}\right)_{\Phi} - I = \frac{\omega_3 r_+^3}{4G} = \frac{A_H}{4G}$$
$$Q = -\frac{1}{\beta} \left(\frac{\partial I}{\partial \Phi}\right)_{\beta} = 2\sqrt{3} \frac{\omega_3}{8\pi G} q$$

Fixed charge, similarly:

$$E = M - M_e; \quad S = \frac{A_H}{4G}; \quad \Phi - \Phi_e = \sqrt{\frac{4}{3}} \frac{q}{r_+^2}$$

Consistency:

- The energy is indeed the mass difference.
- The entropy is the area of the horizon.
- The charge and potential are what they should be.

Notice that at any given potential (below $\Phi_c=1/c$) the phase structure is just like Hawking–Page, but the temperature at which the transition to black holes happens is increasingly lower for higher potential. See summary of full phase diagram later...

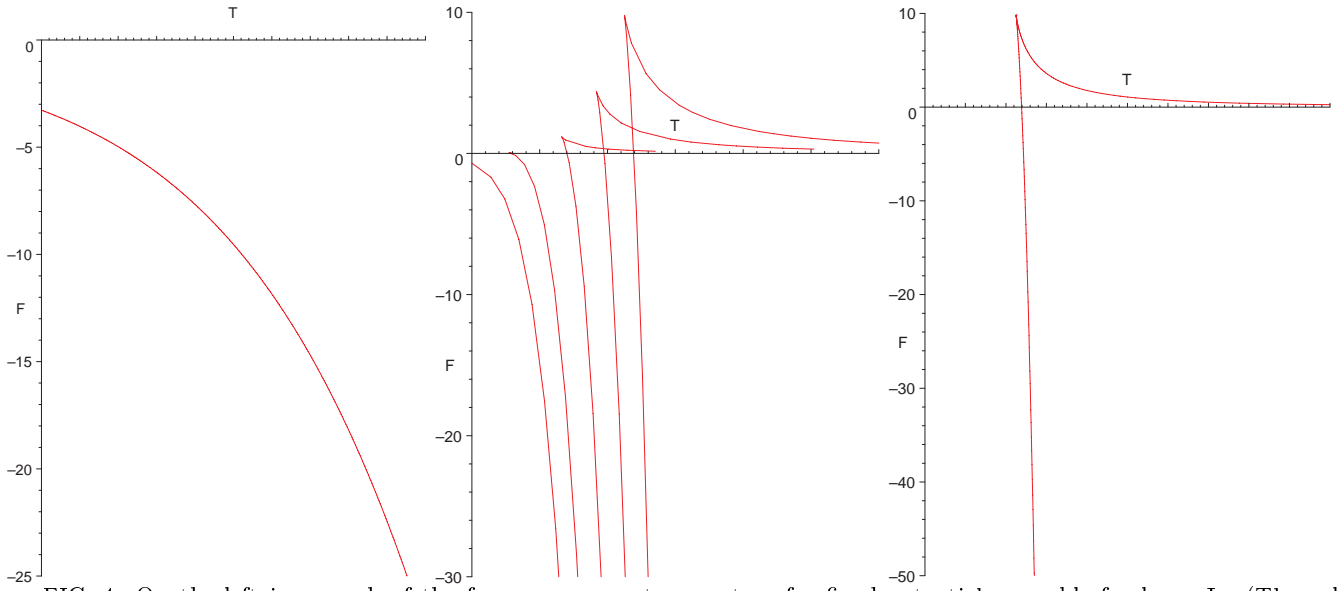


FIG. 4. On the left is a graph of the free energy *vs.* temperature for fixed potential ensemble for large Φ . (The values $n=4$, $G=1$, $l=10$, $\Phi=1$ have been used here.) The center graph depicts a family of free energy curves for different values of Φ . Note the crossover from the cusp ($\Phi < 1/c$) to the single branch ($\Phi > 1/c$) behaviour. On the right is the free energy curve for the uncharged (or $\Phi=0$) ensemble, showing the physics familiar from the Schwarzschild case: visible are the two branches consisting of smaller (unstable) and large (stable) black holes. The entire unstable branch has positive free energy while the stable branch's free energy goes (rapidly, on this scale) negative for all $T > T_c$.

This plot has very new behaviour....one moves along “branch 1” from zero temperature, and then moves onto “branch 3” at some finite temperature..... F is continuous, but its first derivative is discontinuous, so this is a classic first order phase transition. (See blowup of this on next page). As Q increases, the temp. at which this happens comes down. Above some Q_c , the transition goes away. See summary of phases later...

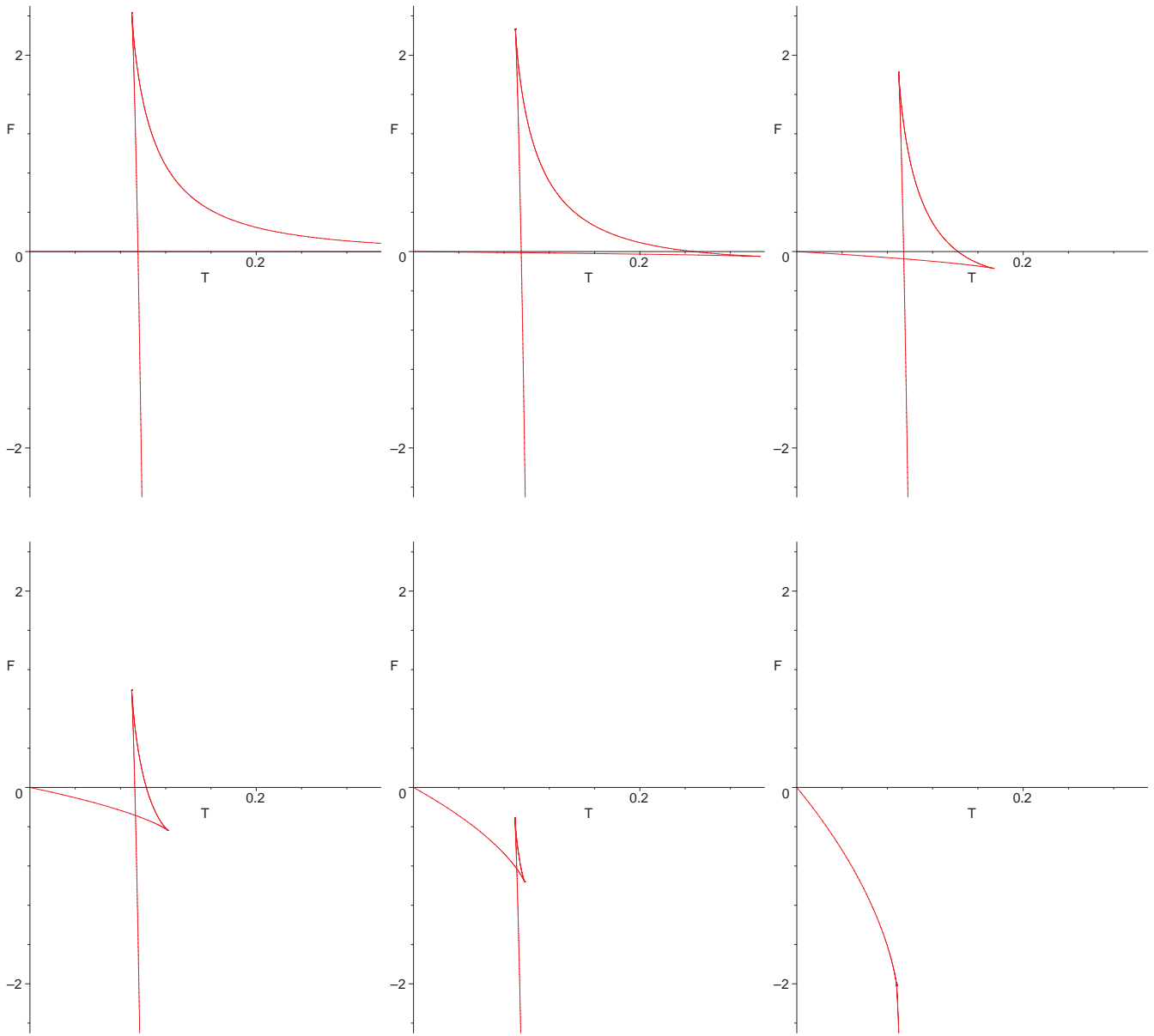


FIG. 6. The free energy *vs.* temperature for the fixed charge ensemble, in a series of snapshots for varying charge, starting from (near) zero charge (top left) and finishing with large charge (bottom left). The values $l=5$, $G=1$, and $n=4$ are used here. This complete evolution describes the two dimensional “swallowtail” catastrophe.

This is a blowup of the transition region....with a snapshot of the existence curve for the black holes alongside...

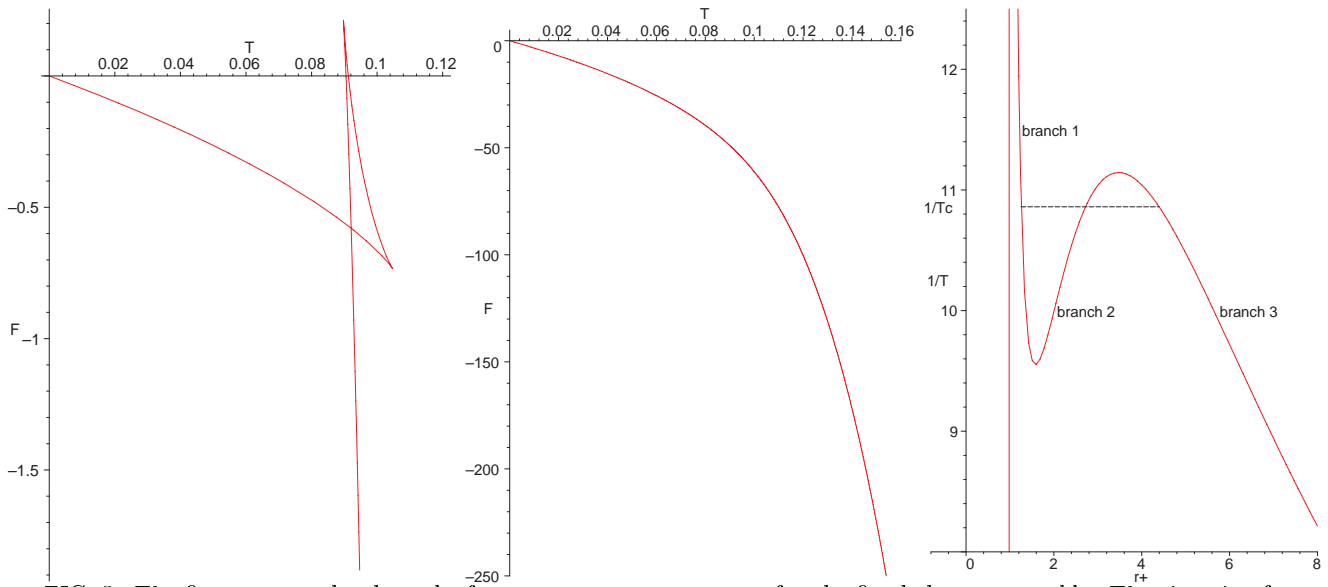
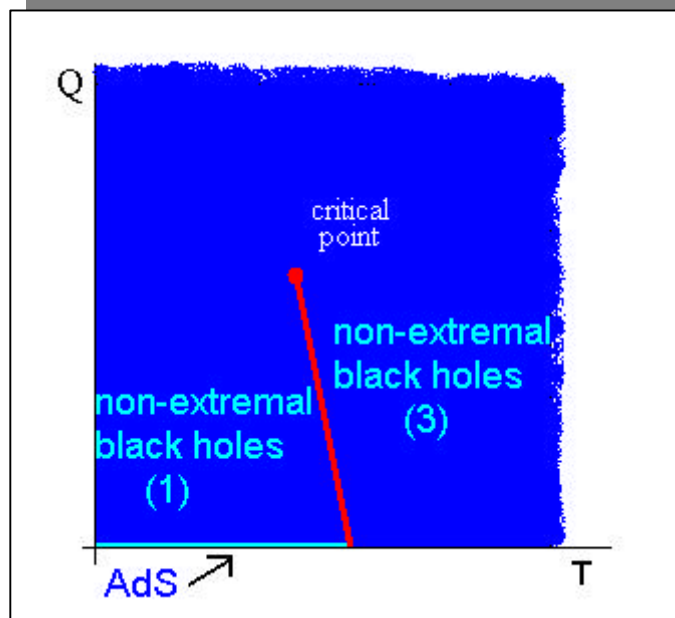
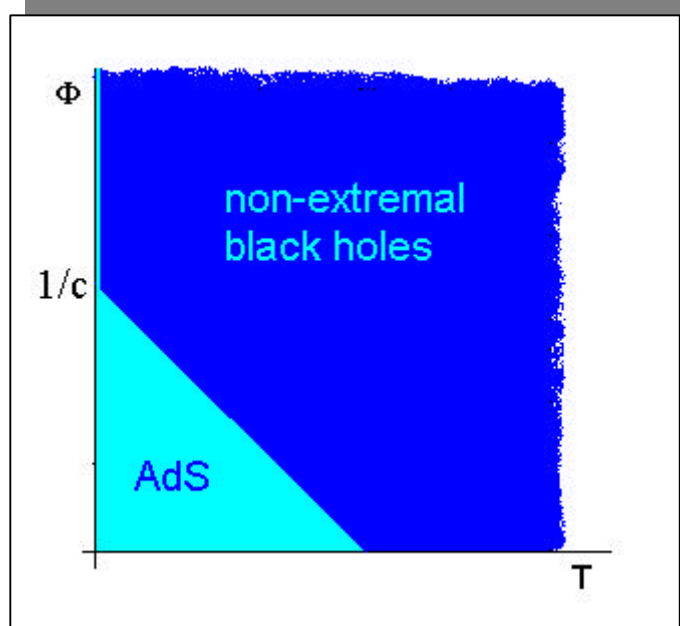
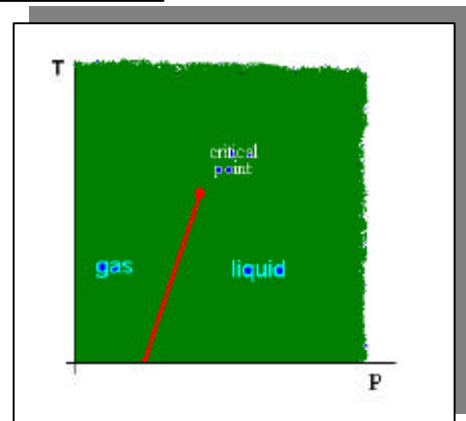


FIG. 5. The first two graphs show the free energy *vs.* temperature for the fixed charge ensemble. The situation for $q < q_{\text{crit}}$ and $q \geq q_{\text{crit}}$, respectively, are plotted. (The values $n=4$, $G=1$, $l=5$ and $q=1, 25$ have been used here.) The first graph is the union of three branches. Branch 1 emanates from the origin, and merges with branch 2 at a cusp. Branch 3 forms a cusp with the other end of branch 2, and continues towards the bottom right. The graph on the right shows how the branches arise from the inverse temperature curves of eqn. (??). (See text for discussion of critical temperature T_c .)

Summary of Phase Structure

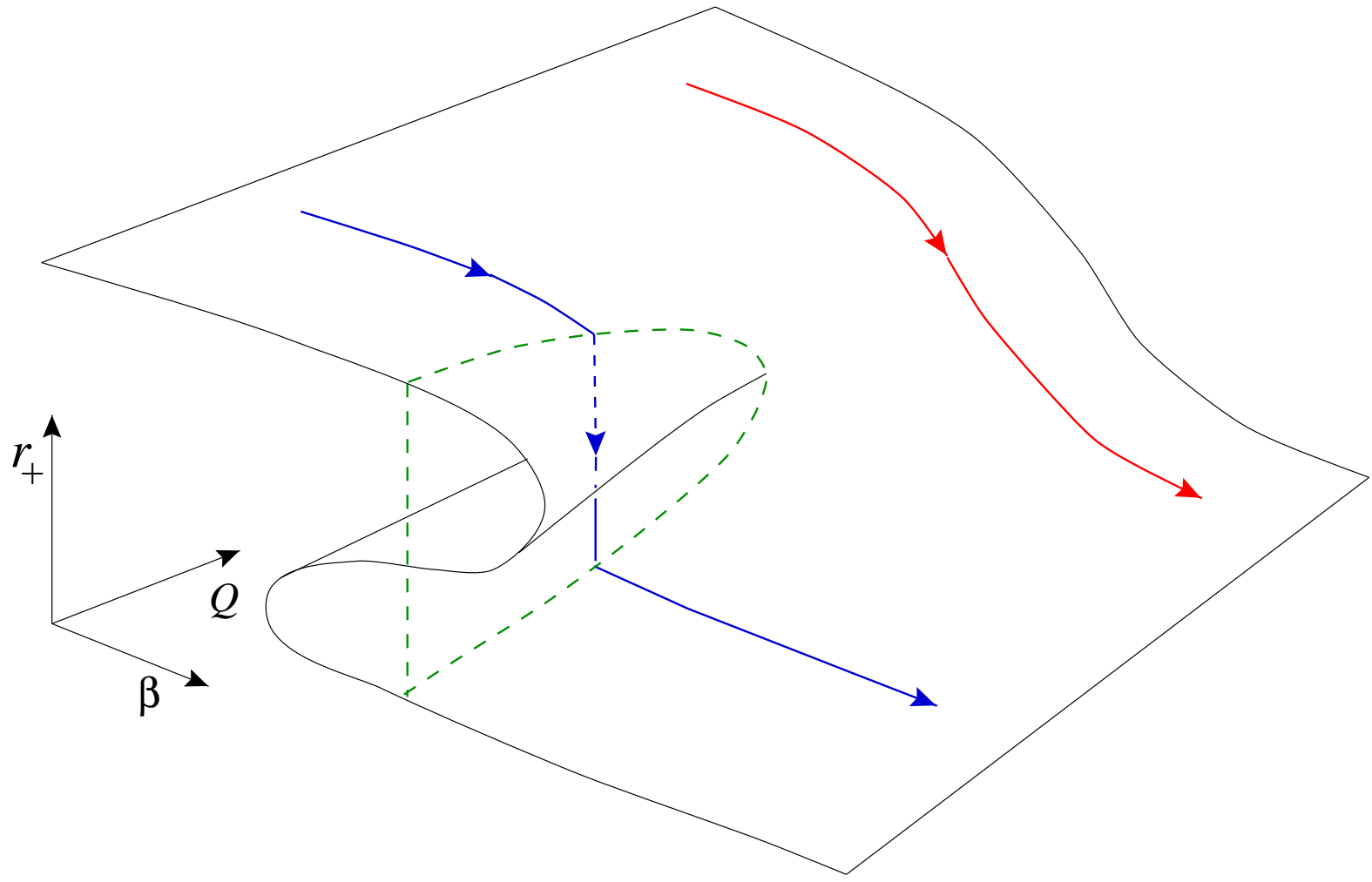


Note similarity to van der Waals !!



The cusp in action: It controls our allowed solutions

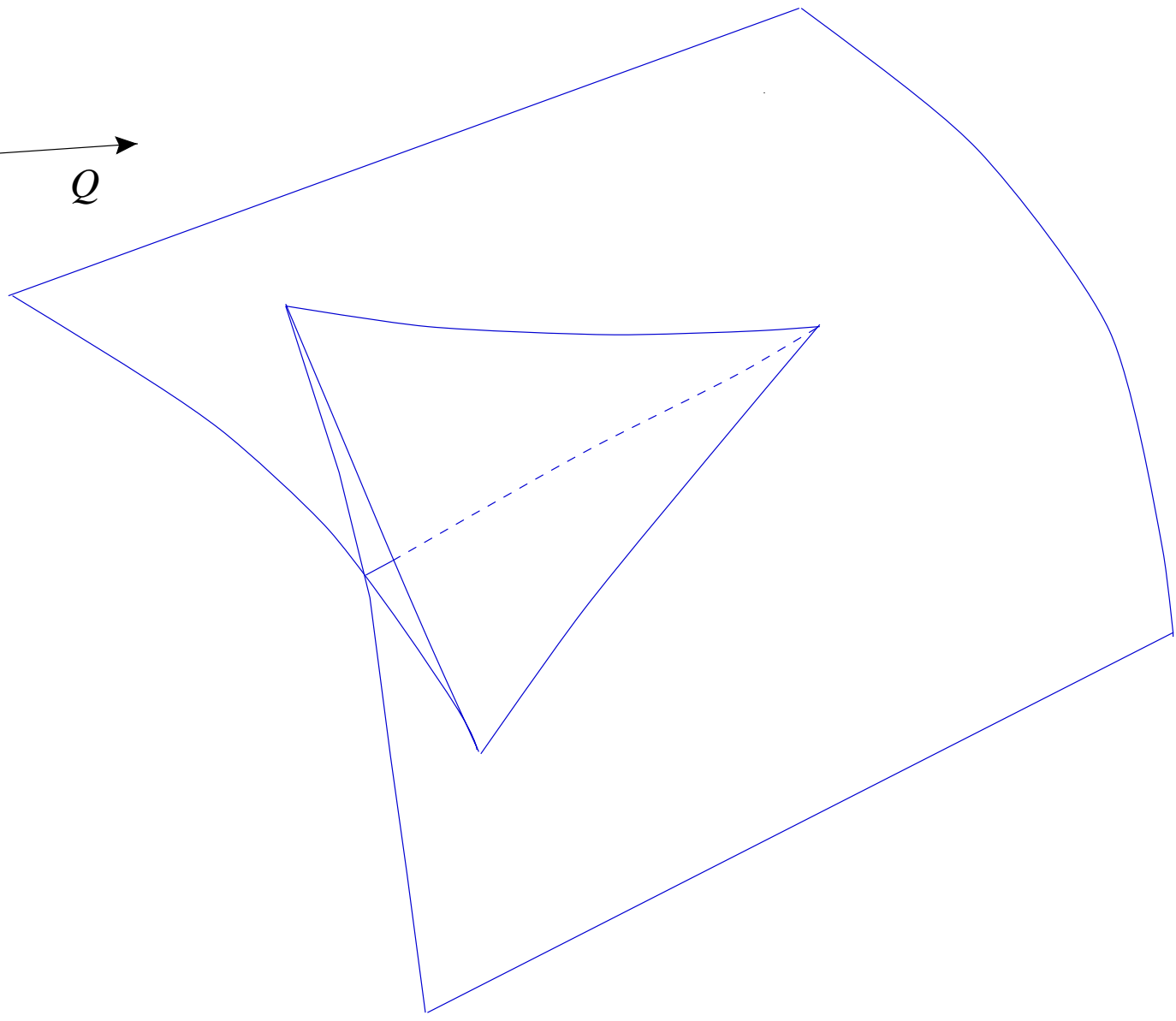
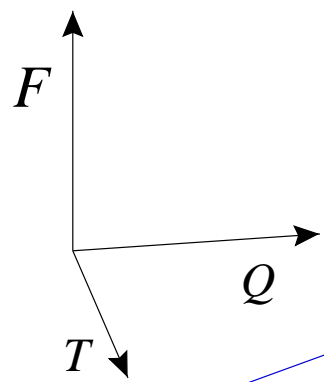
This is the classic universal shape which occurs in many situations in the theory of phase transitions. The “cusp catastrophe” is a member of a well known family of shapes from “catastrophe theory”. (They have an A-D-E classification!)



The swallowtail in flight: It tells us where on the cusp to jump

This is the full three dimensional behaviour of the free energy
as a function of Q and T .





Field Theory Remarks

The phase structure we have discovered is relevant to a dual field theory, at temperature T , with a background global $U(1)$ current.

- The dual 3+1 dimensional field theory lives on the boundary S^3 . (We can also do the infinite volume case.)
- We observe a pattern of large N first order phase transitions in the (Q,T) or (Φ,T) plane.
- Note that at ultra-high temperature, the large non-extremal black holes dominate, and, converting:

$$F \approx V_3 T^4 N^2 \quad \text{and} \quad S \approx V_3 T^3 N^2$$

...which is what one expects for an unconfined phase.

- Intriguingly, at $T=0$, in the fixed charge ensemble, we have:

$$S \approx V_3 \rho$$

...where ρ is the charge density in Kaluza-Klein units.

It would certainly be interesting to derive this entropy formula in the field theory, using ideas similar to the D-brane counting of years ago.

Concluding Remarks

- We have successfully demonstrated that holography extends to charged black holes in AdS.
- The thermodynamics is much richer than the uncharged system, as we have an extra parameter (charge, or potential) to vary.
- The pattern of phase transitions is (surprisingly?) similar to familiar structures from equilibrium thermodynamics of ordinary matter.
- This in itself is further proof of the likelihood of a complete duality, and a further demonstration that the quantum properties of black holes are similar to other systems. (at least in AdS.)
- Appearance of cusp and swallowtail intriguing.....general statements to be made?
- The firm connection we have made with the rich subject of familiar thermodynamic systems with multiple intensive parameters is novel, and worth exploring further.

We will report on these interesting issues in the near future.

-cvj

