Notes on higher spin symmetries.

AdS_5/CFT_4

$$g_{str} \simeq g_{YM}^2$$

$$\left(\frac{R}{l_{str}}\right)^4 \simeq g_{YM}^2 N$$

$R/l_{str.} \to 0$: complicated in string theory but :

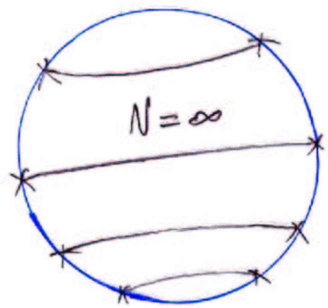
Boundary S-matrix $\simeq$
$\simeq$ correlation functions in the free field theory on the bndry.

Idea: learn about the string theory in the $R/l_{str} \to 0$ limit by looking at the boundary S-matrix.

(P. Haggi-Mani, B. Sundborg, hep-th/0002189.)

1) Spectrum of primaries on the Boundary $\to$
$\to$ field content in the bulk

2) Parameter N : when $N=\infty$, the correlation functions on the boundary factorize into the product of pair correlators



$\Downarrow$

$N=\infty$ corresponds to free theory in the bulk.

Considering $0 < \frac{1}{N} \ll 1$ corresponds to turning on interactions.
The leading connected contribution is given by the classical theory with the coupling constant $\frac{1}{N}$.

For example: $\left\langle \frac{1}{N} \operatorname{tr} \phi^2(x_1) \cdots \frac{1}{N} \operatorname{tr} \phi^2(x_n) \right\rangle$

Locality?

$$= \sum_{\substack{\sigma:\\ \sigma(1)=1}} \frac{1}{N^{n-2}} \frac{1}{\|x_{\sigma(1)} - x_{\sigma(2)}\|^2} \times \cdots \times \frac{1}{\|x_{\sigma(n)} - x_{\sigma(1)}\|^2}$$

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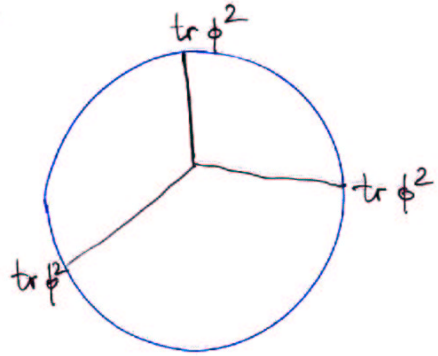
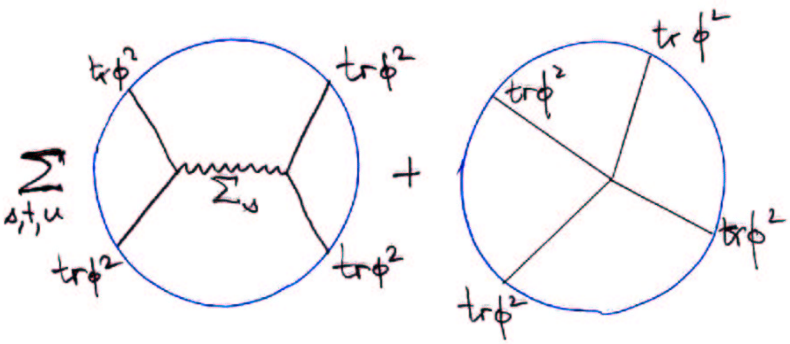
AdS pictures:

n = 3:



n = 4:



$$\stackrel{?}{=} \frac{1}{\|x_1-x_2\|^2}\times\frac{1}{\|x_2-x_3\|^2}\times\frac{1}{\|x_3-x_4\|^2}\times\frac{1}{\|x_4-x_1\|^2}+$$

\+ permutations

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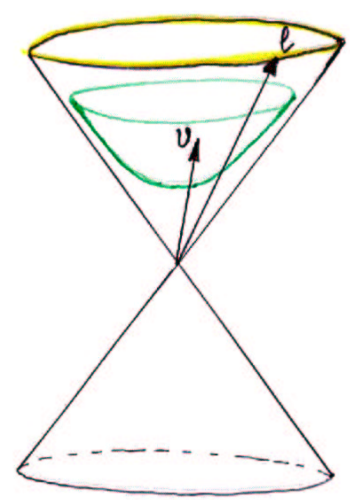
Some notations:

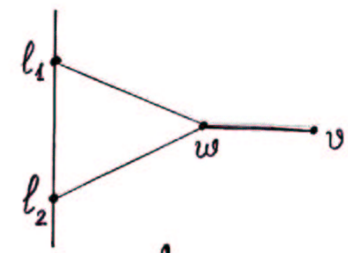
$$\Delta(\Delta-4)=m^2R^2$$

$$D_\ell(v)=\frac{1}{(v\cdot\ell)^\Delta}$$

For $\mathrm{tr}\,\phi^2$, $\Delta=2$,

$$D_2(v)=\frac{1}{(v\cdot\ell)^2}$$



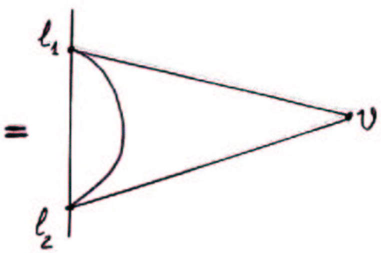


$=$

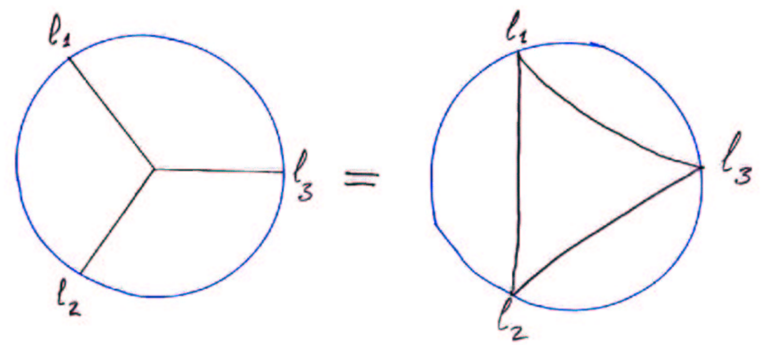
$$=\int d^5w\; D_{\Delta=2}(v;w)\,\frac{1}{(w\cdot\ell_1)^2(w\cdot\ell_2)^2}=$$

$$=\frac{1}{(v\cdot\ell_1)}\;\frac{1}{(v\cdot\ell_2)}\;\frac{1}{(\ell_1\cdot\ell_2)}=$$

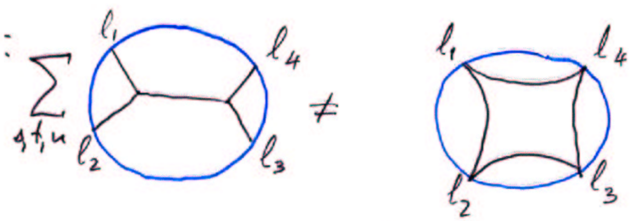
$=$



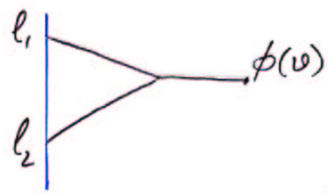
(E. D'Hoker, D.Z. Freedman, L. Rastelli: hep-th/9905049)



But:



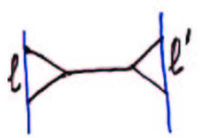
Obvious obstacle: anomalous dimension of $\mathrm{tr}(\phi^2)^2$ $\updownarrow$



$$\phi(v) = \phi^{(0)}(v) + \phi^{(1)}(v)$$

$$(\Box + m^2)\,\phi^{(0)}(v) = 0$$

$$\lim_{l_1 \to l_2} \phi^{(1)}(v) \approx \frac{1}{(v\cdot l)^4}$$



$\int d^5 v \, \frac{1}{(v\cdot l)^4} \frac{1}{(v\cdot l')^4}$ is log-divergent

which means there are terms $\sim \log(l_1 \cdot l_2)$

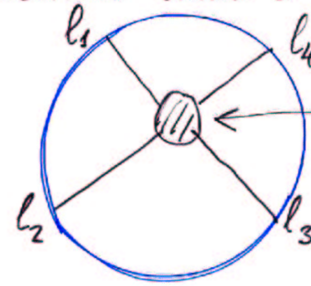
Log-divergence should cancel when we sum over intermediate spins.

Another suggestive formula:

$$\frac{1}{(l_1\cdot l_2)(l_2\cdot l_3)(l_3\cdot l_4)(l_4\cdot l_1)} = \sum_{n=0}^{\infty} \int d^5 v \,\times$$

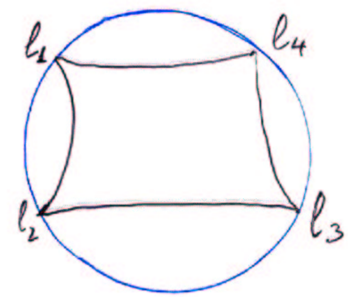
$$\frac{1}{(n+1)!}\left(\left(\frac{\partial}{\partial v^{(1)}} - \frac{\partial}{\partial v^{(2)}}\right)\cdot\left(\frac{\partial}{\partial v^{(3)}} - \frac{\partial}{\partial v^{(4)}}\right)\right)^n \frac{1}{(v\cdot l_1)^2 (v\cdot l_2)^2 (v\cdot l_3)^2 (v\cdot l_4)^2}$$

(index $^{(1)}$ in $\frac{\partial}{\partial v^{(1)}}$ means that one has to differentiate w.r.to v in $(v\cdot l_1)$, etc.)



Nonlocal vertex comes from integrating out infinite series of higher spins

||



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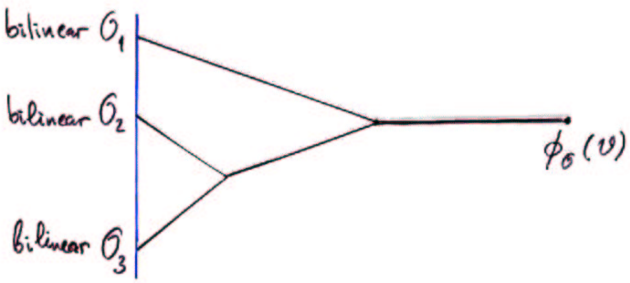
<u>Idea:</u> The free field theory has large symmetry group. $\delta\phi = \partial_\mu \phi$, $\delta\phi = \partial_\mu \partial_\nu \partial_\lambda \phi, \ldots$
Look for the theory in the bulk which has the same group of symmetries, as gauge symmetries.

Consistent truncation: The symmetries of the free field theory are generated by the currents bilinear in free fields.

The subset of operators, which are bilinears in free fields, is closed under the OPE.



$= 0$ unless if O is bilinear in free fields

This suggests that the theory in the bulk has <u>consistent truncation</u> to the set of operators which are bilinears in free fields.

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Interacting classical theories with the required symmetry group were constructed by

<u>E. S. Fradkin and M. A. Vasiliev</u>:

Phys. Lett. <u>B189</u> (1987) 89
Nucl. Phys. <u>B291</u> (1987) 141
hep-th / 9910096, 0104246, 0106200

Formulation of the problem:
<u>C. Fronsdal</u>, Phys. Rev. <u>D18</u> (1978) 3624

Plan:

- Bilinear operators and conserved currents
- Algebraic structure of the h.s. symmetries
- Free higher spin fields in AdS
  AdS/CFT for $N=\infty$
- Global symmetries on the boundary and gauge symmetries in the bulk

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$$S = \int d^D x \; \partial_i \phi^* \partial^i \phi$$

Invariant under special conformal transformations:

$$\delta_v \phi = (v\cdot x)(x\cdot\partial)\phi - \frac{1}{2}\|x\|^2 (v\cdot\partial)\phi + \frac{D-2}{2}(v\cdot x)\phi$$

Primary operators, which are bilinear in scalars:

$$\boxed{O[V] = V^{i_1\cdots i_s}\sum_{k=0}^{s}\frac{(-1)^k}{k!\left(k+\frac{D-4}{2}\right)!\,(s-k)!\left(s-k+\frac{D-4}{2}\right)!}\times \partial_{i_1}\cdots\partial_{i_k}\phi^*\,\partial_{i_{k+1}}\cdots\partial_{i_s}\phi}$$

$V^{i_1\cdots i_s}$ is symmetric, $\underline{g_{ij}V^{ijj_3\cdots j_s}=0}$

The relation between primary operators and conserved currents:

$$j_{i_1\cdots i_s} = \sum_{k=0}^{s}\frac{(-1)^k}{k!\left(k+\frac{D-4}{2}\right)!\,(s-k)!\left(s-k+\frac{D-4}{2}\right)!}\partial_{i_1}\cdots\partial_{i_k}\phi^*\partial_{i_{k+1}}\cdots\partial_{i_s}\phi - \text{traces}\,(i_1\cdots i_s)$$

$j_{i_1\cdots i_s}$ is a primary tensor, its conformal dimension is $s+D-2$.

$$\delta_v j^{i_1\cdots i_s} = (D-2+s)(v\cdot x)j^{i_1\cdots i_s} + \sum_{p=1}^{s}(v^{i_p}x_k - x^{i_p}v_k)j^{i_1\cdots i_{p-1}k\,i_{p+1}\cdots i_s}$$

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On shell, $\partial_i\partial^i\phi = 0$, we have

$$\underline{\partial_i j^{i i_2\cdots i_s} = 0}$$

[For any tensor primary $j^{i_1\cdots i_s}$ of conformal dimension $D-2+s$, it turns out that $\partial_i j^{i i_2\cdots i_s}$ is again a tensor primary. It has conformal dimension equal spin plus D. But all primaries bilinear in free fields have conf. dimension equal spin plus $D-2$.

Therefore, $\partial_i j^{i i_2\cdots i_s} = 0$.]

Higher spin tensor currents are related to higher derivative symmetries.

Conformal Killing tensor $\xi^{i_2\cdots i_s}$:

$$\partial^{(i_1}\xi^{i_2\cdots i_s)} = g^{(i_1 i_2}\chi^{i_3\cdots i_s)}$$

$$g_{i_2 i_3}\xi^{i_2\cdots i_s} = 0$$

Given the Conformal Killing tensor $\xi^{i_2\cdots i_s}$, the contraction $j^i[\xi] = \xi_{i_2\cdots i_s} j^{i i_2\cdots i_s}$ is a conserved current in the usual sense of the word, i.e. it generates symmetry.

One can prove a converse statement:

any higher derivative symmetry of the action is related to the conformal Killing tensor $\xi^{i_2 \dots i_s}$:

$$\delta_\xi \phi = \left(\xi^{i_2 \dots i_s} \partial_{i_2} \dots \partial_{i_s} + \dots \right) \phi$$

Conclusion: free theory has infinitely many symmetries, corresponding to the conformal Killing tensors.

These symmetries form an infinite-dim'l nonabelian algebra.

We want now to describe, in some simple way, the structure of this algebra.

Suppose that $\delta \phi = L(x, \partial_x) . \phi$ is a symmetry of the action. Then, it preserves the Laplace equation: $\Delta \phi = 0 \Rightarrow \Delta (L . \phi) = 0$.

Let us consider a special solution to the Laplace equation,

$$\Phi_{q,\bar q}(x) = e^{q_A \bar q_{\bar A} x^{A\bar A}}$$

$$\left(\Delta \, \Phi_{q,\bar q}(x) = 0 \right)$$

$$L(x, \partial_x) . \, e^{q_A \bar q_{\dot A} x^{A\dot A}} = \mathcal{P}_L (x, q\bar q) \, e^{q_A \bar q_{\dot A} x^{A\dot A}}$$

$$\Delta \left(\mathcal{P}_L (x, q\bar q) . \, e^{q_A \bar q_{\dot A} x^{A\dot A}} \right) = 0$$

$$\Downarrow$$

$$\mathcal{P}_L (x, q\bar q) e^{q_A \bar q_{\dot A} x^{A\dot A}} = \tilde{\mathcal{P}}_L (q, \bar q, \partial_q, \bar\partial_q) . \, e^{q_A \bar q_{\dot A} x^{A\dot A}}$$

Therefore, there is a one to one correspondence between the differential operators L preserving $\Delta \phi = 0$ and the polynomials $\tilde{\mathcal{P}}_L (q, \bar q, \partial_q, \bar\partial_q)$ which are invariant under $U(1)$ generated by $\delta q = iq$.

Consider the operators in the free theory which are linear in the free fields:

$$\mathcal{O}_f = \int d^4x \; f(x) \phi(x)$$

$\mathcal{O}_f = \int d^4x \; f(x) \phi(x)$	$\tilde f(q, \bar q) = \int d^4x \; f(x) \, e^{q\bar q x}$
$\phi(0)$ $\partial_{A\dot A} \phi(0)$ ...	1 $q_A \bar q_{\dot A}$ ...
$L . \mathcal{O}_f$	$\tilde{\mathcal{P}}_L (q, \bar q, \partial_q, \partial_{\bar q}) . \tilde f$

Consider the algebra of oscillators,
4 coordinates and 4 momenta.
Suppose that on the space of oscillators there is a complex structure I, $I^2=-1$, such that:

- commutation relations are invariant under the action of I (ω of type (1,1))
- Kahler metric has signature $(4^+, 4^-)$.

Useful for realization of $su(2,2)$:

$$\underset{\substack{\text{quadratic hamiltonians}\\ \text{invariant under } I\\ \text{(modulo center)}}}{su(2,2)} \subset \underset{\substack{\text{all quadratic}\\ \text{hamiltonians}}}{sp(4,\mathbb{R})}$$

Infinite dimensional extension of $su(2,2)$:

$$\underline{hs(2,2)}: \quad i\alpha^{(0)} + \alpha^{(1)}_{i_1, i_2}\,\theta^{i_1}\theta^{i_2} + i\,\alpha^{(2)}_{i_1\cdots i_4}\,\theta^{i_1}\dots\theta^{i_4} + \dots$$

This is the algebra of symmetries of the free complex scalar field.

Doubleton representation:

$$\left.\begin{aligned} Q^I &\longrightarrow q^I\cdot \\ \bar{P}^I &\longrightarrow \frac{\partial}{\partial q^I} \end{aligned}\right\} \text{ on the space of functions } f(q,\bar{q}). \text{ invariant under } q\mapsto e^{i\theta}q,\ \bar{q}\to e^{-i\theta}\bar{q}$$

Operators linear in ϕ are in $\mathcal{F}$ and operators linear in ϕ^* are in $\overline{\mathcal{F}}$.
This means that the bilinears are in $\mathcal{F}\otimes\overline{\mathcal{F}}$
There is a hermitean scalar product

$$\mathcal{F}\otimes\overline{\mathcal{F}} \to \mathbb{C}$$

Therefore $\overline{\mathcal{F}} \simeq \mathcal{F}^*$;

this means that the <u>n-point function is the element of $(\mathcal{F}\otimes\mathcal{F}^*)^{\otimes n}$, or hs-invariant operator in $\mathcal{F}^{\otimes n}$</u>.

It turns out that an arbitrary operator in $\mathcal{F}$ can be represented as a linear combination (perhaps infinite sum) of generators of $hs(2,2)$.

Therefore, the hs-invariant operator in $\mathcal{F}^{\otimes n}$ should commute with any operator in $\mathcal{F}$ acting on $\mathcal{F}^{\otimes n}$ as symmetries act on tensor product:

$$X.\,(v_1\otimes\dots\otimes v_n) = Xv_1\otimes v_2\otimes\dots\otimes v_n + v_1\otimes Xv_2\otimes\dots\otimes v_n + \dots + v_1\otimes\dots\otimes Xv_n.$$

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This implies that any hs-invariant operator in $\mathcal{F}^{\otimes n}$ is in fact a linear combination of permutations.

Returning to correlation functions,

$$\boxed{\begin{aligned}&\langle \phi^*(x_1)\phi(y_1)\cdots\phi^*(x_n)\phi(y_n)\rangle = \\ &= \sum_{\sigma\in S_n} A_\sigma \frac{1}{\|x_1-y_{\sigma(1)}\|^2}\times\cdots\times\frac{1}{\|x_n-y_{\sigma(n)}\|^2}.\end{aligned}}$$

— the only freedom left by the hs-invariance is in the choice of A_σ.

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Higher spin fields. Fronsdal '78

$h^{\mu_1\cdots\mu_s}$: symmetric in $\mu_1,\ldots,\mu_s$,

$$g_{\mu_1\mu_2}g_{\mu_3\mu_4}h^{\mu_1\cdots\mu_s}=0$$

Gauge transformations:

$$\delta_\Lambda h^{\mu_1\cdots\mu_s}=\nabla^{\mu_1}\Lambda^{\mu_2\cdots\mu_s}, \qquad g_{\mu_2\mu_3}\Lambda^{\mu_2\cdots\mu_s}=0$$

Free eqs. of motion:

$$\nabla_\rho\nabla^\rho h_{\mu_1\cdots\mu_s} - s\nabla_\rho\nabla_{\mu_1}h^\rho{}_{\mu_2\cdots\mu_s} +$$
$$+\frac{1}{2}s(s-1)\nabla_{\mu_1}\nabla_{\mu_2}h^\rho{}_{\rho\mu_3\cdots\mu_s} +$$
$$+2(s-1)(s+d-3)h_{\mu_1\cdots\mu_s}=0$$

de Donder gauge:

$$F^{\mu_2\cdots\mu_s}[h]=\nabla^\rho h_{\rho\mu_2\cdots\mu_s}-\frac{s-1}{2}\nabla_{\mu_2}h^\rho{}_{\rho\mu_3\cdots\mu_s}=0$$

Special gauge for solutions:

1) $\nabla^\rho h_{\rho\mu_2\cdots\mu_s}=0$

2) $g^{\rho\sigma}h_{\rho\sigma\mu_3\cdots\mu_s}=0$

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Boundary to bulk propagator.

Boundary conditions:

consider the traceless tensor field $V^{i_1 \cdots i_s}(x)$ on the boundary. Given $V^{i_1\cdots i_s}(x)$ we can deform the action:

$$\delta S = \int d^D x \; j_{i_1\cdots i_s}(x)\, V^{i_1\cdots i_s}(x)$$

In the bulk, the corresponding solution of the free higher spin equations should have the following boundary behaviour:

$$(\tilde{g}_{\cdot\cdot} = r^2 g_{\cdot\cdot}) \qquad \boxed{\frac{1}{r^2} h[V]\Big|^{i_1\cdots i_s}_{\text{restr.}}(x) = V(x)^{i_1\cdots i_s}}$$

(Explanation: $h[V] = h[V]^{\mu_1 \cdots \mu_s}$ —
— contravariant tensor of rank s in the bulk;
we use natural restriction of vector fields to the boundary)

It is not obvious that the restriction of $\frac{1}{r^2} h[V]$ to the boundary is traceless. If one chooses the gauge $\nabla_\rho h^{\rho\mu_2\cdots\mu_s} = h_\rho{}^{\rho\mu_3\cdots\mu_s} = 0$ then one can show that $h^{zz i_3\cdots i_s} \sim z^4$, $h^{i_1\cdots i_s} \sim z^2$.

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Boundary to bulk propagator:

$$G_0(z_0,\vec{z})^{i_1\cdots i_s}_{\mu_1\cdots\mu_s} = \left[\frac{z_0}{z_0^2+\vec{z}^2}\right]^{d-3} \partial_{\mu_1}\frac{2z^{i_1}}{z_0^2+\vec{z}^2}\cdots\partial_{\mu_s}\frac{2z^{i_s}}{z_0^2+\vec{z}^2} - \text{traces}$$

$$h[V]^{\mu_1\cdots\mu_s} = \frac{1}{\mathcal{N}(s,d)}\int d^{d-1}\vec{x}\; G_{\vec{x}}(z_0,\vec{z})^{\mu_1\cdots\mu_s}_{i_1\cdots i_s} V^{i_1\cdots i_s}(\vec{x})$$

Global transformations on the boundary and gauge transformations in the bulk.

$$\boxed{\frac{\partial}{\partial x^i} G_{\vec{x}}(z_0,\vec{z})^{i\, i_2\cdots i_s}_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\Lambda^{i_2\cdots i_s}_{\mu_2\cdots\mu_s)}}$$

where $\Lambda^{i_2\cdots i_s}_{\mu_2\cdots\mu_s} = \left(\frac{z_0}{z_0^2+\vec{z}^2}\right)^{d-1}\partial_{\mu_2}\frac{2z^{i_2}}{z_0^2+\vec{z}^2}\cdots\partial_{\mu_s}\frac{2z^{i_s}}{z_0^2+\vec{z}^2} - \text{traces}$

Suppose that $\xi_{i_2\cdots i_s}$ is a conformal Killing tensor on the boundary, $\partial_{(i_1}\xi_{i_2\cdots i_s)} G^{i_1\cdots i_s} = 0$

$$\nabla_{(\mu_1}\int d^{d-1}\vec{x}\;\xi_{i_2\cdots i_s}(\vec{x})\,\Lambda^{i_2\cdots i_s}_{\mu_2\cdots\mu_s)}(\vec{x}\,|\,v) = 0$$

therefore

$$\boxed{\Lambda[\xi]_{\mu_2\cdots\mu_s} := \int d^{d-1}\vec{x}\;\xi_{i_2\cdots i_s}(\vec{x})\,\Lambda^{i_2\cdots i_s}_{\mu_2\cdots\mu_s}(\vec{x}\,|\,v)}$$

is a traceless Killing tensor in the bulk.

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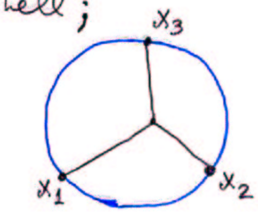
Free higher spin theory is invariant under

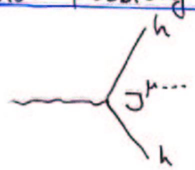
$$\delta_\Lambda h^{\mu_1\cdots\mu_s} = \nabla^{\mu_1}\Lambda^{\mu_2\cdots\mu_s}$$

Suppose that we have the triple interaction invariant under this δ_Λ on shell;

then $\langle \partial_i j^{i\cdots}(x_1)\, j^{\cdots}(x_2)\, j^{\cdots}(x_3)\rangle = 0$:



If all the three point functions of the theory on the boundary are correctly reproduced, then the triple interaction cannot possibly be invariant off shell.



$$\partial_\mu J^{\mu\cdots} \sim \Box h \cdot h$$

To compensate for that, we need to deform δ_Λ:

$$\delta_\Lambda = \delta_\Lambda^{(0)} + \lambda\,\delta_\Lambda^{(1)} + \ldots$$

$\delta_\Lambda^{(0)} \leftarrow \nabla\otimes\Lambda$, $\quad \delta_\Lambda^{(1)} \leftarrow \Lambda\cdot h$

For Λ a Killing tensor, $\delta_\Lambda^{(1)}$ should be a symmetry of the free action, and of the triple interaction.

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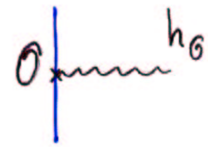


$$\boxed{\delta^{(1)}_{\Lambda[\xi]}\, h_\mathcal{O} = h_{\delta_\xi \mathcal{O}}}$$

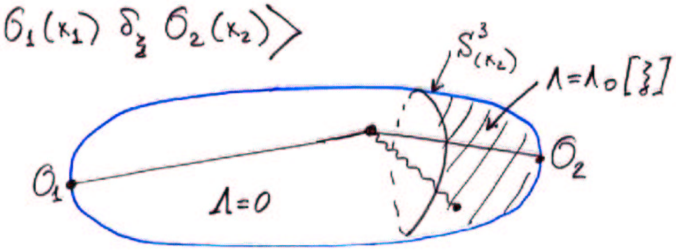
(if the three point functions are correctly reproduced)

Indeed, consider the correlation function:

$$\left\langle \mathcal{O}_1(x_1) \oint_{S^3_{(x_1)}} * \left(\xi_{i_2\cdots i_s}\, j^{i i_2\cdots i_s}\right) \mathcal{O}_2(x_2)\right\rangle =$$

$$= \left\langle \mathcal{O}_1(x_1) \int_{D^4_{(x_2)}} \partial_i j^{i i_2\cdots i_s}\, \xi_{i_2\cdots i_s}\, \mathcal{O}_2(x_2)\right\rangle =$$

$$= \left\langle \mathcal{O}_1(x_1)\, \delta_\xi\, \mathcal{O}_2(x_2)\right\rangle$$



Consider δS_{free}.

$\delta^{(0)} S_{free}[h_{\mathcal{O}_1}, h_{\mathcal{O}_2}] = 0$ because Λ goes either to 0 or to Killing near both x_1 and x_2

$$\delta^{(1)}_\Lambda S_{free}[h_{\mathcal{O}_1}, h_{\mathcal{O}_2}] = S_0[\delta^{(1)}_\Lambda h_{\mathcal{O}_1}, h_{\mathcal{O}_2}] + \underbrace{S_0[h_{\mathcal{O}_1}, \delta^{(1)}_\Lambda h_{\mathcal{O}_2}]}_{=0}$$

$$S_0[\delta^{(1)}_\Lambda h_{\mathcal{O}_1}, h_{\mathcal{O}_2}] = S_0[\delta_{\Lambda[\xi]} h_{\mathcal{O}_1}, h_{\mathcal{O}_2}]$$

Conclusion: if the three point functions are correctly reproduced, <u>and</u> $\delta^{(2)}_{\Lambda} = \delta^{(3)}_{\Lambda} = \ldots = 0$ on shell,

then the boundary S-matrix is the correlation functions of the free field theory.