Topological Strings and Black Holes

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based on:

A. Strominger, C. Vafa + H.O.	hep-th/0405146
M. Aganagic, N. Saulina, C. Vafa + H.O.	hep-th/0411280
C. Vafa, E. Verlinde + H.O.	hep-th/0502211

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Topological String Theory

(1) Start with a Calabi-Yau 3-fold.

$\mathcal{N}=2$ superconformal sigma-model

$$\Sigma_g \to CY_3$$

(2) Topologically twist the sigma-model.

(3) Couple it to the topological gravity on Σ_g .

$$F_g = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} \left(\underset{\substack{\uparrow \\ \text{Beltrami differential}}}{\eta_i} , \underset{\substack{\uparrow \\ \text{supercurrent}}}{G^-} \right) \right|^2 \right\rangle$$

$$Z^{top}_{pert.} = \exp\left[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g \right]$$

The series is typically not summable.

Topological String Partition Function = Wave Function

Consider the topological B-model

moduli space $\mathcal{M}_B$: z^i complex structure

λ topological string coupling

$$\text{tangent space to } \mathcal{M}_B = H^3(CY_3, \mathbb{R})$$

$$\dim_{\mathbb{C}} H^3 = \underset{\substack{\uparrow \\ \delta z^i}}{h^{2,1}} + \underset{\substack{\uparrow \\ \lambda}}{1}$$

For a given background, we can define the B-model.

F_g is also a holomorphic function of $H^3(CY_3)$.

$$F_g \sim \left\langle \int_{\mathcal{M}_g} (G_i^-)^{3g-3} (G_{\bar{i}}^-)^{3g-3} \, e^{x^I \int_{\Sigma_g} \mathcal{O}_I} \right\rangle$$

$$x^I \in H^3 \, , \quad I = 0, 1, \cdots, h^{2,1}$$

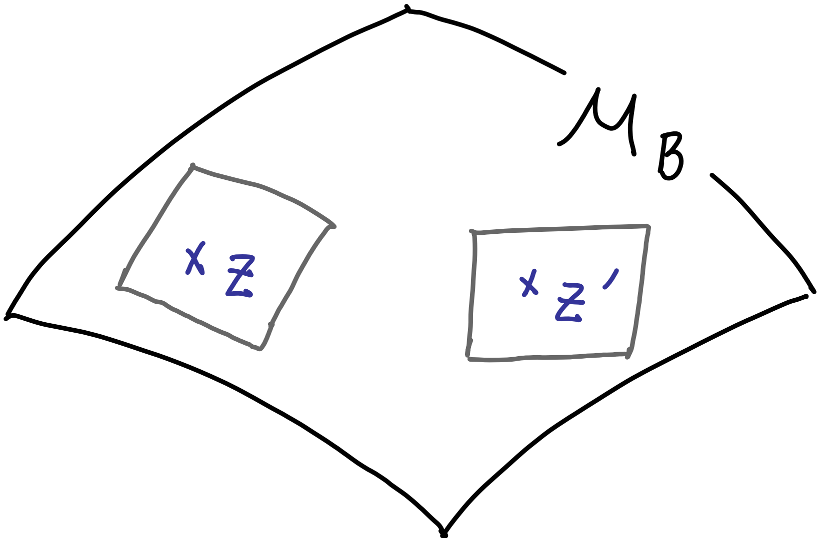
$$\psi_{top}(\underset{\substack{\uparrow \\ \text{including } \lambda}}{x^I} ; z^i, \bar{z}^i) = \exp\left(\sum_g F_g\right)$$

The holomorphic anomaly equations

derived in BCOV,
interpreted by Witten,
refined by DVV

$$\frac{\partial}{\partial z^I}\psi_{top}(x;z,\bar{z}) = \left(\frac{\partial}{\partial x^I}+\cdots\right)\psi_{top}$$

$$\frac{\partial}{\partial \bar{z}^I}\psi_{top}(x;z,\bar{z}) = \left(\bar{C}_I{}^{JK}\frac{\partial^2}{\partial x^J\partial x^K}+\cdots\right)\psi_{top}$$



Interptetation:

(1) On each tangent space, there is a Hilbert space.

(2) The holomorphic anomaly equation is describing the paralell transport between tangent spaces at different points.

More on (1):

- $H^3(CY_3,\mathbb{R})$ has a symplectic structure.
- Topological string uses a holomorphic polarization.

$$\underbrace{H^{3,0}\oplus H^{2,1}}_{x^I}\oplus H^{2,1}\oplus H^{0,3}$$

More on (2):

- The polarization depends on $(z^i,\bar{z}^i)$

$\Rightarrow$ Wave-functions are related by Fourier transformation.

topological string partition function $Z_{top} = \exp\left(\sum_g F_g\right)$

= wave function $\psi_{top}(x; z, \bar{z})$

for quantization of $H^3(CY_3, \mathbb{R})$

(H^3 = tangent space to the moduli space of CY3)

On the other hand,
the topological string partition function gives
superpotential terms for CY3 compactification
of type II superstring. *(BCOV)*

Can we intepret the topological string partition function as a physical wave function in type II superstring?

Black Hole Entropy

Consider type IIB superstring on $CY_3 \times \mathbb{R}^{3,1}$

RR 4-form ⇒ gauge field A_μ^I in 4d.

$I = 0, 1, \ldots, h^{2,1}$

D branes wrapping on 3 cycles in CY_3

q_I times on A_I

p^I times on B^I

$A_I, B^I \in H_3(CY_3)$

$A_I \cap A_J = B^I \cap B^J = 0$

$A_I \cap B^J = \delta_I{}^J$

= BPS black hole in four dimensions

with electric charges q_I , magnetic charges p^I

Conjecture (Strominger, Vafa + H.O.)

$$Z_{BH} \equiv \sum_q \Omega(p,q)\, e^{-q\phi}$$

$$= |\psi_{top}(X)|^2$$

where $X^I = p^I + \frac{i}{\pi}\phi^I$

Black hole partition function:

$$Z_{BH} = \sum_{q} \Omega(p,q)\, e^{-q\cdot\phi}$$

p : magnetic charges of the black hole

ϕ : chemical potentials for electric charges

Perturbative topological string partition function:

$$\psi_{top}^{(pert.)} = \exp\left[\sum_g F_g(X) \right]$$

$\lambda = 4\pi i / X^0$: string coupling constant

X^i / X^0 : complex structure of CY_3

Black Hole Charges <==> Calabi-Yau Moduli.

$$X^I = p^I + \frac{i}{\pi}\phi^I \qquad (I = 0, 1, \cdots, h^{2,1})$$

$$Z_{BH}(p,\phi) = |\psi_{top}(X)|^2$$

Why?

The perturbative topological string amplitudes give low energy effective action terms.

BCOV/1994

It turns out that these are the terms that are relevant in computing perturbative string corrections to the Bekenstein-Hawking entropy formula.

Lopez-Cardoso, de Wit + Mohaupt/1998-99

Define $\mathcal{F} = \log |\psi_{top}^{(pert)}(X)|^2$

$(X^I : \mathcal{N}=2$ chiral superfield $)$

Black Hole Attractor Eguations:

$$X^I = p^I + \frac{i}{\pi}\phi^I, \quad q_I = \frac{\partial}{\partial \phi^I}\mathcal{F}(p,\phi)$$

Black Hole Entropy (all order in string perturbation):

$$S_{BH}(p,\phi) = q_I \phi^I + \mathcal{F}(p,\phi)$$

This is the Legendre transformation:

$$q \longleftrightarrow \phi$$

Entropy and Wave Function

The OSV conjecture can be inverted as:

$$\Omega(p,q) = \int d\phi \, |\psi_{p,q}(\phi)|^2$$

where

$$\psi_{p,q}(\phi) \equiv e^{-\frac{1}{2} q_I \phi^I - \frac{1}{2} p^I \tilde{\phi}_I} \, \psi_{top}(\phi)$$

$$\left(\tilde{\phi}_I = -\frac{2i}{\pi} \frac{\partial}{\partial \phi^I} \right)$$

$e^{-\frac{1}{2} q_I \phi^I - \frac{1}{2} p^I \tilde{\phi}_I}$: flux changing operator

$\psi_{top}(\phi)$: topological string partition function

One can interpret Ω as

(1) $$\Omega(p,q) = \mathrm{Tr}_{H_{pq}} (-1)^F \lambda^2 e^{-\beta H}$$

over BH Hilbert space

AND

(2) type IIB string partition function on AdS2 x S2 x CY3, euclideanized & periodically identified.

Near Horizon

Near Horizon Geometry of the Black Hole

$$AdS_2 \times S^2 \times CY_3$$

RR 4-form ⇒ 4d gauge field $A_\mu{}^I$

on $AdS_2 \times S^2$

electric charge q_I , magnetic charge p^I

⇒ 5-form flux F_5

Flux Compactification on $S^2 \times CY_3$

GVW superpotential

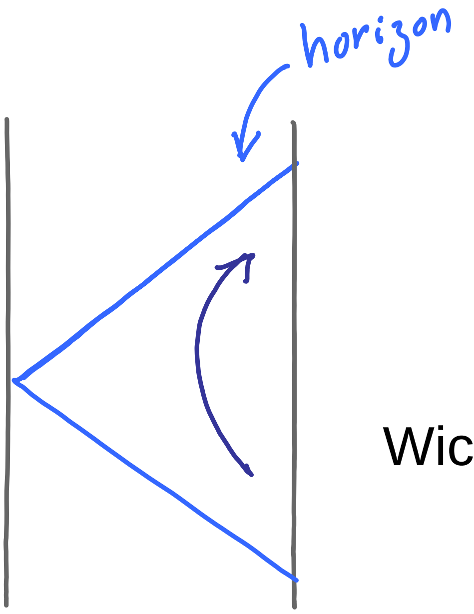
$$W = \int_{S^2 \times CY_3} F_5 \wedge \Omega$$

holomorphic 3 form on CY_3

$$= q_I X^I - p^I \frac{\partial}{\partial X^I} F_0(X)$$

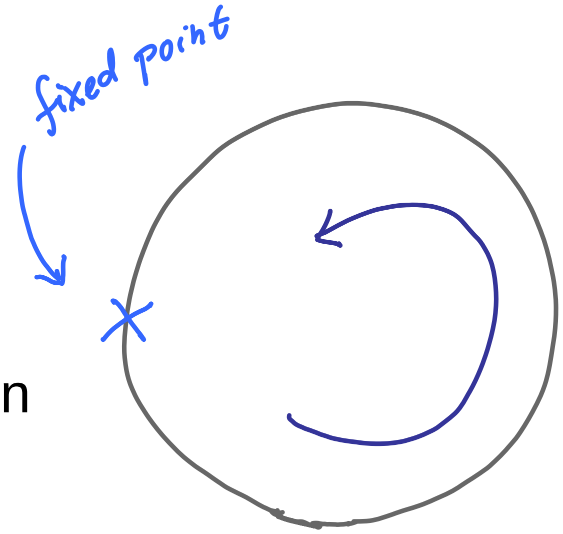
$$dW = 0 \Rightarrow \begin{array}{l} \mathrm{Re}\, X^I = p^I \\ \mathrm{Re}\, \partial_I F_0 = q^I \end{array}$$

Classical Attractor



$\Rightarrow$

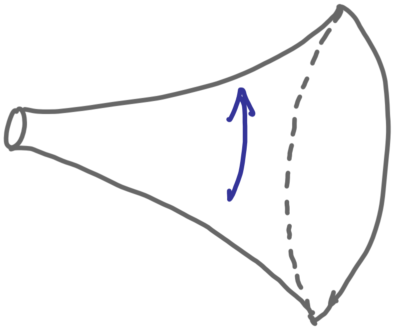
Wick rotation



AdS_2 Euclidean AdS_2

$$\Omega(p,q) = \mathrm{Tr}_{H_{pq}} (-1)^F \lambda^2 e^{-\beta H}$$

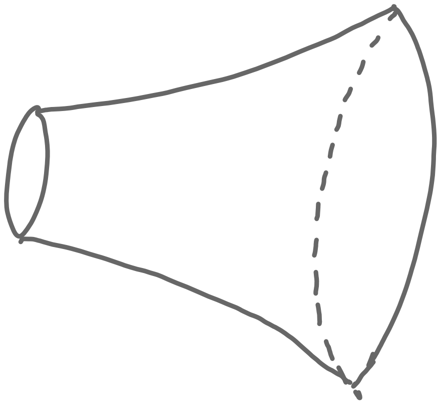
$$=$$



$$= \int d\phi \, | \psi_{pq}(\phi) |^2$$

This suggests that the topologial string partition function is the Hartle-Hawking type wave-function for AdS_2 , Euclideanized and periodically identified.

$$\psi_{p\delta}(x) = x$$



This is similar in sprit to the earlier suggestion by de Boer, H. Verlinde, and V. Verlinde on the relation between the Weeler-de Witt equation and the renormalization group equation in the context of the AdS/CFT correspondence.

One of the new features is that our wave-function takes into account quantum corrections to all orders in the string perturbation theory.

Can we compare vacua with different amount of fluxes?

$$\mathbb{R}^{10} \to CY_3 \times \mathbb{R}^4$$

vector multiplets : X^I, A_μ^I

$$\to CY_3 \times S^1 \times \mathbb{R}^3$$

$$X^I, \quad \underset{\oint_{S^1} A^I}{\overset{\phi^I}{\shortparallel}}, \quad \underset{\oint_{S^1} \tilde{A}_I}{\overset{\tilde{\phi}_I}{\shortparallel}}$$

A natural wave-function is a function of :

$$X^I, \quad \boxed{\phi^I, \ \tilde{\phi}_I}$$

$$\psi_{\boxed{p^I,\ q_I}}(X)$$

conjugate

This means that we have a wave-function over the entire landscape of string vacua.

$$\psi_{p,q}(x) \longleftrightarrow \tilde{\psi}(x, \phi, \tilde{\phi})$$

Vacua with different amount of fluxes are weighted by:

$$\int |\psi_{p,q}|^2 = \Omega(p,q)$$

Entropic Principle

So far, the story has been at the level of
the string perturbation theory.

$\longleftrightarrow$ Large N asymptotic expansion
in the dual gauge theory.

What happens if we go beyong
the perturbative expansion?

in progress.....

In the following, I will show how the OSV conjecture works in an explicit example.

The example is based on the A-model,
so we will take the mirror of the story.

Consider D branes in type IIA string theory wrapping 0, 2, 4, and 6 cycles of a Calabi-Yau 3-fold.

q_0 = # of D0 branes

q_i = # of D2 branes $i = 1, 2, \cdots, h^{1,1}$

p^i = # of D4 branes

p^0 = # of D6 branes

Classical attractor equations:
$$\mathrm{Re}\, X^I = p^I$$
$$\mathrm{Re}\, \partial_I F_0 = q_I$$

$\lambda = \frac{4\pi i}{X^0}$: topological string coupling

$t^i = \frac{X^i}{X^0}$: Kähler moduli

$$CY_3 : \mathcal{O}(-p) \oplus \mathcal{O}(p+2g-2) \rightarrow \Sigma_g$$

Two line bundles of degrees *-p* and *p+2g-2* over a genus *g* Riemann surface

The total space is a Calabi-Yau manifold.

$Z^{top}_{pert.}$

Topological string amplitudes on this CY was recently computed to all order in the perturbative expansion.

Bryan + Pandharipande/2004

Z_{BH}

Consider *N* D4 branes on the total space of the degree *-p* bundle over the Riemann surface.

$$\mathcal{O}(-p) \rightarrow \Sigma_g$$

$$X^0 = \frac{4\pi i}{\lambda} \qquad (\text{no } D_6 \text{ charge})$$

$$X^1 = \underbrace{(p+2g-2)N}_{D_4 \text{ charge}} + \frac{i}{\pi}\phi^1$$

We want to compute the partition function of the N=4 super Yang-Mills on this 4-manifold.

$\mathcal{N}=4$ SYM on $\mathcal{O}(-p) \to \Sigma_g$

$$S = \frac{i}{2\lambda} \int \mathrm{tr} \left(F \wedge F + 2\theta F \wedge k \right)$$

k : Kähler form on Σ_g

$$\Big\Downarrow$$

Equivariant reduction on the fiber
with respect to the $U(1)$ rotation.

2d YM with q-deformation

$$S_{qYM} = \frac{i}{\lambda} \int_{\Sigma_g} \mathrm{tr} \left(\Phi F + \theta \Phi + i \frac{p}{2} \Phi^2 \right)$$

q-deformed since Φ is periodic.

($e^{\Phi} = e^{\oint A}$ is a good variable.)

$$q = e^{-\lambda}$$

2d *q*-YM partition function

$q = e^{-\lambda}$

$$Z_{BH} = S^{2-2g} \sum_{R} (\dim_q R)^{2-2g} \, q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$R = [R_1, \cdots, R_N]$: representation of $U(N)$

$$C_2 = \sum_i R_i (R_i - 2i + 1) + N R_i$$

$$C_1 = \sum_i R_i$$

$$\dim_q R = \prod_{i<j} \frac{[R_i - R_j - i + j]_q}{[i-j]_q}$$

$$\left(\text{where } [m]_q = q^{m/2} - q^{-m/2} \right)$$

$$S = q^{\rho^2} \prod_{i<j} [i-j]_q \,, \quad \vec{\rho} : \text{Weyl vector}$$

The chemical potentials are given by

$$\phi^0 = \frac{4\pi^2}{\lambda} \,, \quad \phi^1 = \frac{2\pi\theta}{\lambda}$$

The above expansion is not in the form expected for Z_{BH}

Z_{BH} can be brought into the form expected from the black hole state counting as well as from the instanton expansion of the N=4 SYM:

$$Z_{BH} = \sum_{q} \Omega(p,q)\, e^{-q\cdot\phi}$$

by the S-duality transformation, $\lambda \to \frac{1}{\lambda}$.

The resulting expression reproduces mathematical facts about cohomologies on the moduli space of U(N) instantons on the 4-manifold: $\mathcal{O}(-p) \to \Sigma_g$

For $\mathcal{O}(-2) \to \mathbb{P}^1$,
the cohomologies of the moduli space of of U(N) instantons make representations of the affine SU(2) Lie algebra of level N. (Nakajima)

Thus, Z_{BH} should be expressed in terms of the SU(2) affine Lie algebra characters and indeed it does.

Z_{BH} also reproduce the blow-up formula in the case of $\mathcal{O}(-1) \to \mathbb{P}^1$. (Yoshioka)

Large N Limit

$$Z_{BH} = S^{2-2g} \sum_{R} (\dim_q R)^{2-2g} \, q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$$\sim \sum_{m=-\infty}^{\infty} \sum_{R_i} Z_{R_1 \cdots R_{|2g-2|}}(\lambda ; t + p\lambda m)$$

$$\times \bar{Z}_{R_1 \cdots R_{|2g-2|}}(\lambda ; \bar{t} - \underbrace{p\lambda m}_{\text{RR flux}})$$

$$Z_{R_1 \cdots R_{2g-2}}(\lambda ; t)$$

g>1 case

$$= C(t) \sum_{R} q^{\frac{1}{2}(p+2g-2)K_R} e^{-t|R|}$$

$$\times \frac{W_{RR_1} \cdots W_{RR_{2g-2}}}{(W_{R0})^{4g-4}}$$

where

$$W_{R_1R_2}(q) = \lim_{N\to\infty} \underset{\substack{\uparrow \\ \text{modular } S\text{-matrix} \\ \text{of the WZW model}}}{S_{R_1R_2}}(q, N)$$

The chiral components:

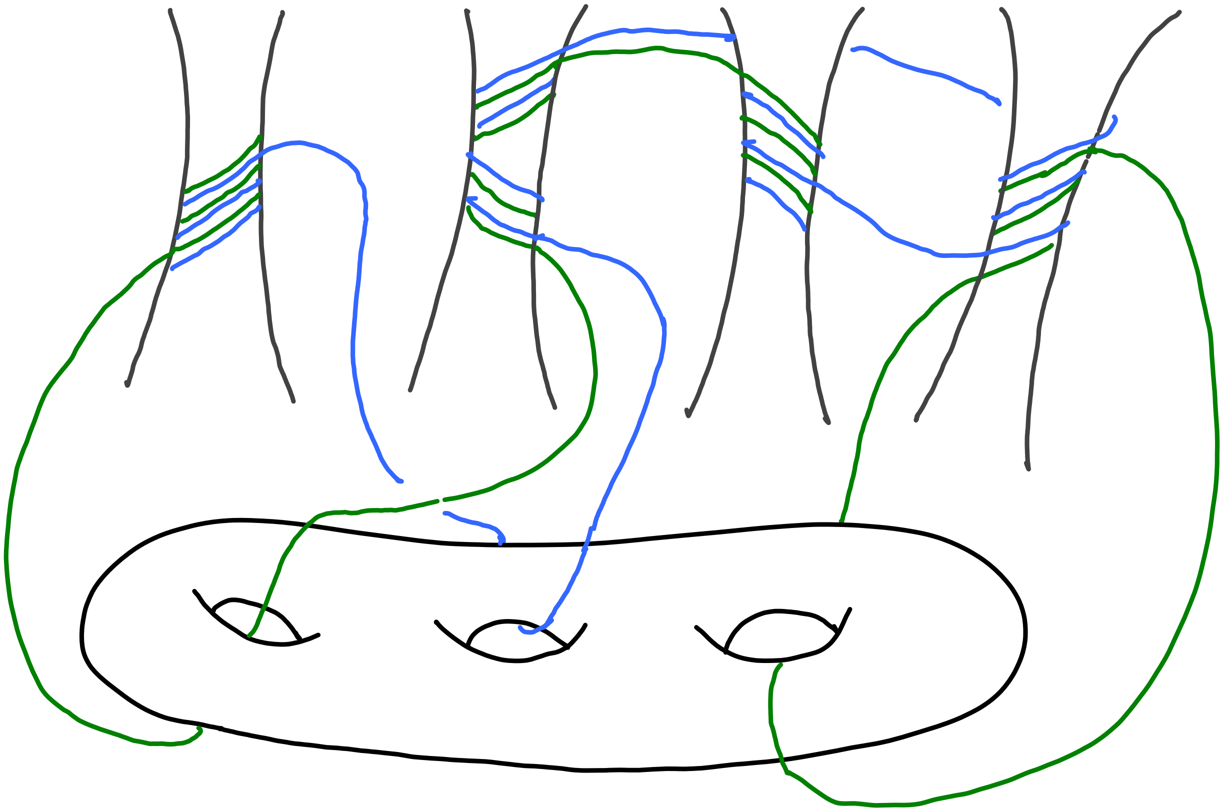
$$Z(U_1, \cdots, U_{2g-2})$$

$$= \sum_{R_1, \cdots, R_{2g-2}} Z_{R_1 \cdots R_{2g-2}} \times \mathrm{Tr}_{R_1} U_1 \cdots \mathrm{Tr}_{R_{2g-2}} U_{2g-2}$$

are topological string amplitudes for the Calabi-Yau with (2g-2) D branes wrapping Lagrangian 3-cycles.

$U_1, \cdots, U_{2g-2}$ are holonomies on the D branes.

These D branes correspond to the Omega points in the ordinary (undeformed) 2d YM.



Summary

The mixed ensemble of BPS black holes gives a non-perturbative definition of topological string theory.

This is a large N duality relating the worldvolume theory of D3 branes to topological string theory.

The topological string partition function has a mathematical interpretation as a quantization of the third cohomology of a Calabi-Yau 3-fold.

The relation to the black hole entropy shows that it has a physical interpretation as the Hartle-Hawking wave-function, including all string loop corrections.

We studied the case when the Calabi-Yau manifold is the total space of the rank 2 vector bundle.
The worldvolume theory of D4 branes wrapping one of the line bundles is related to the *q*-deformed Yang-Milles theory on the base Riemann surface.

The large N limit of the gauge theory partition function is holomorphically factorized, and the chiral blocks are interepreted in terms of the perturbative topological string theory.

Fin