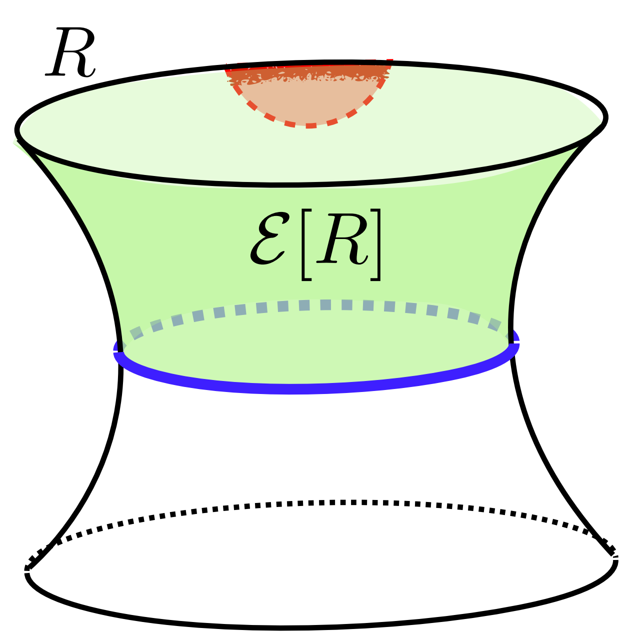
From holographic geometries to quantum code properties

Fernando Pastawski @ KITP 2016

Based on:
quant-ph/1612.tomorrow
with John Preskill
and
quant-ph/1611.07528
with Jens Eisert and Henrik Wilming

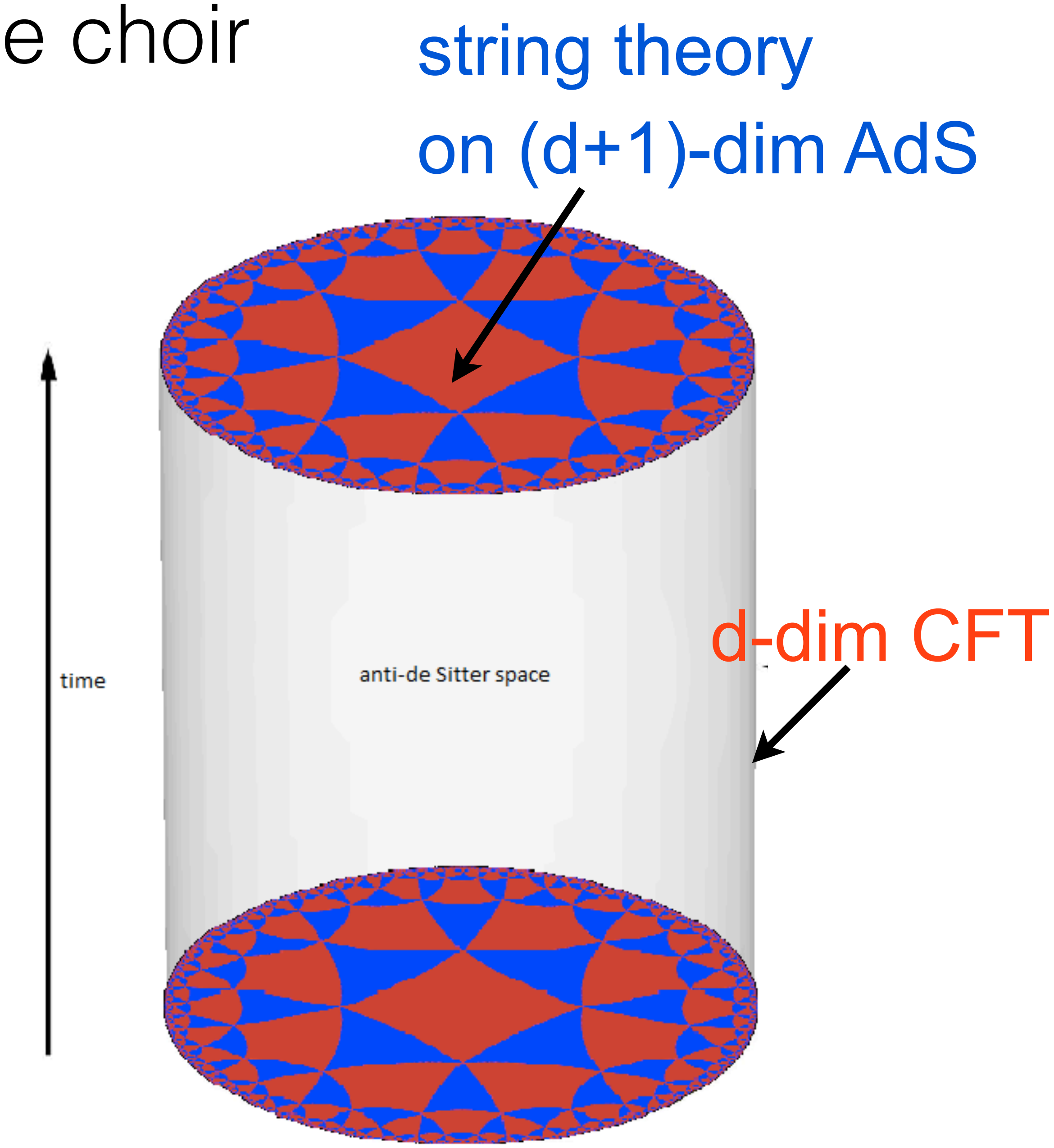


What does QEC have to do with holography?

AdS/CFT

preaching to the choir

AdS	CFT
Weakly coupled gravity	Strongly coupled
Geometric minimal surface	Entanglement entropy
Bulk operators	Boundary operators
Gravitational dynamics	Entanglement thermodynamics

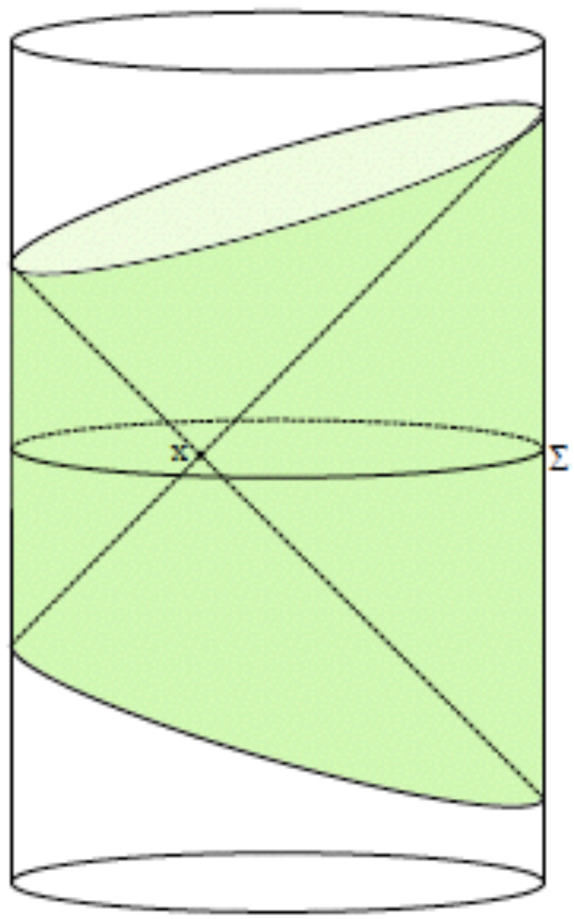


Powerful framework to study strongly-interacting systems

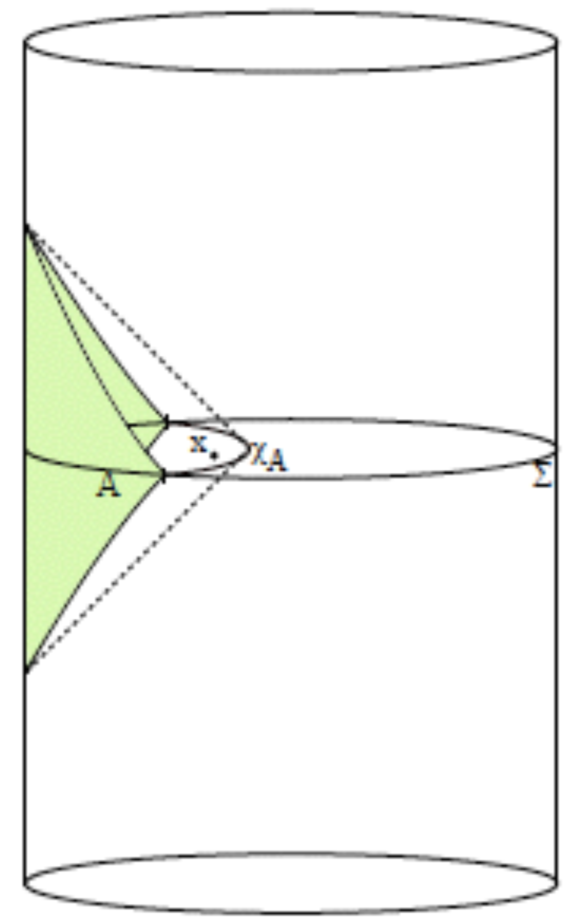
Advanced our understanding of quantum gravity

Maldacena, J. The Large-N Limit of Superconformal Field Theories and Supergravity. IJTP, 38(4), 1113–1133.

Boundary reconstruction of bulk operators



Global reconstruction

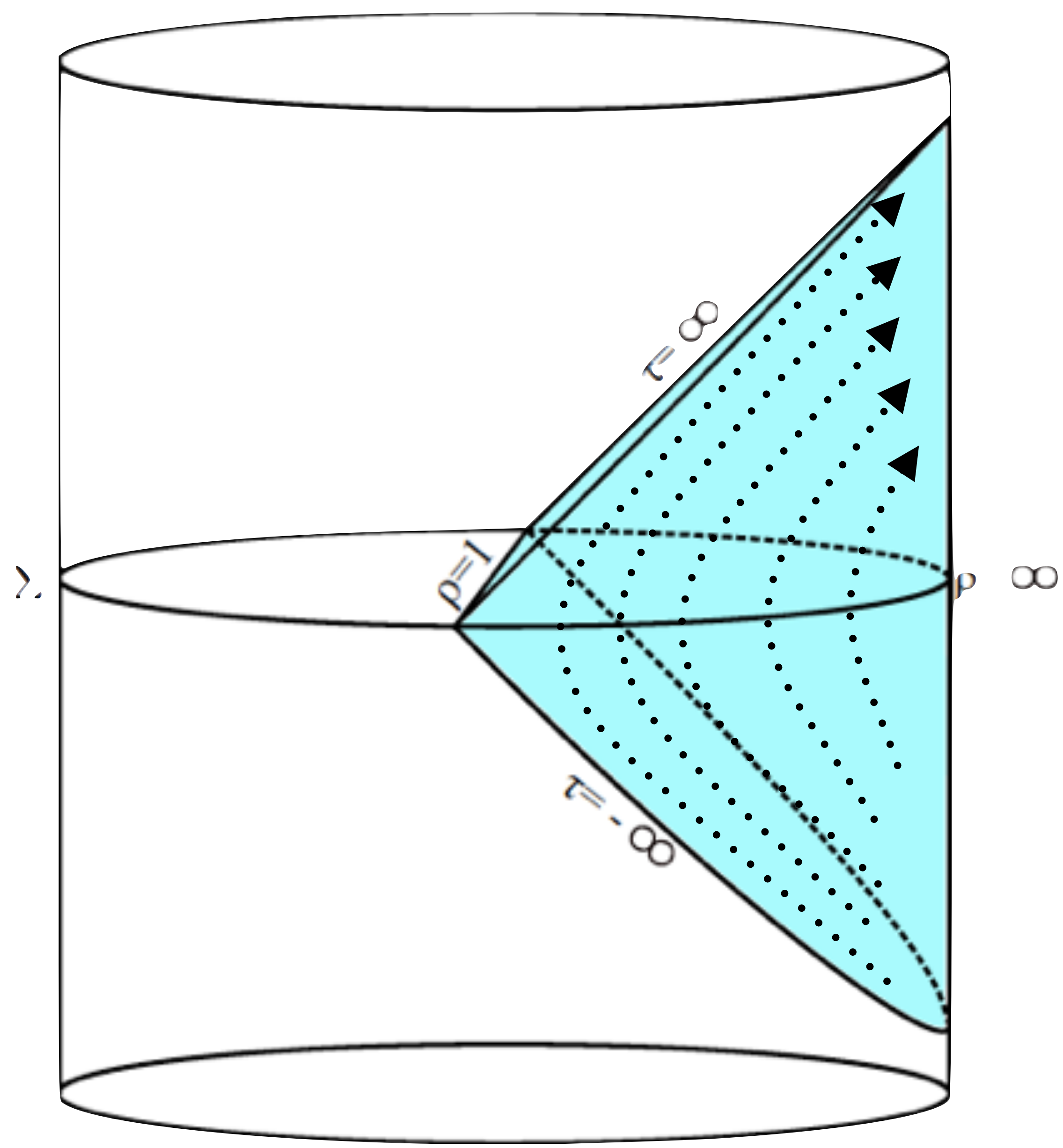


AdS-Ridler wedge reconstruction

Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. (2006). Holographic representation of local bulk operators. PRD, 74(6), 066009.

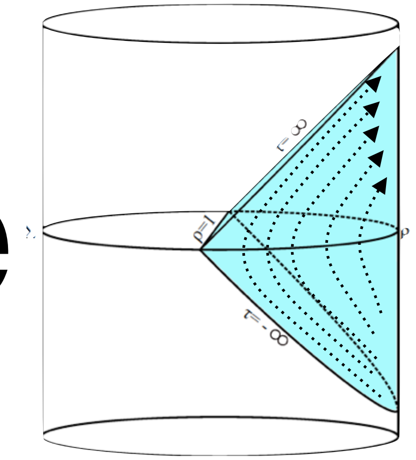
Explicit solution in metric

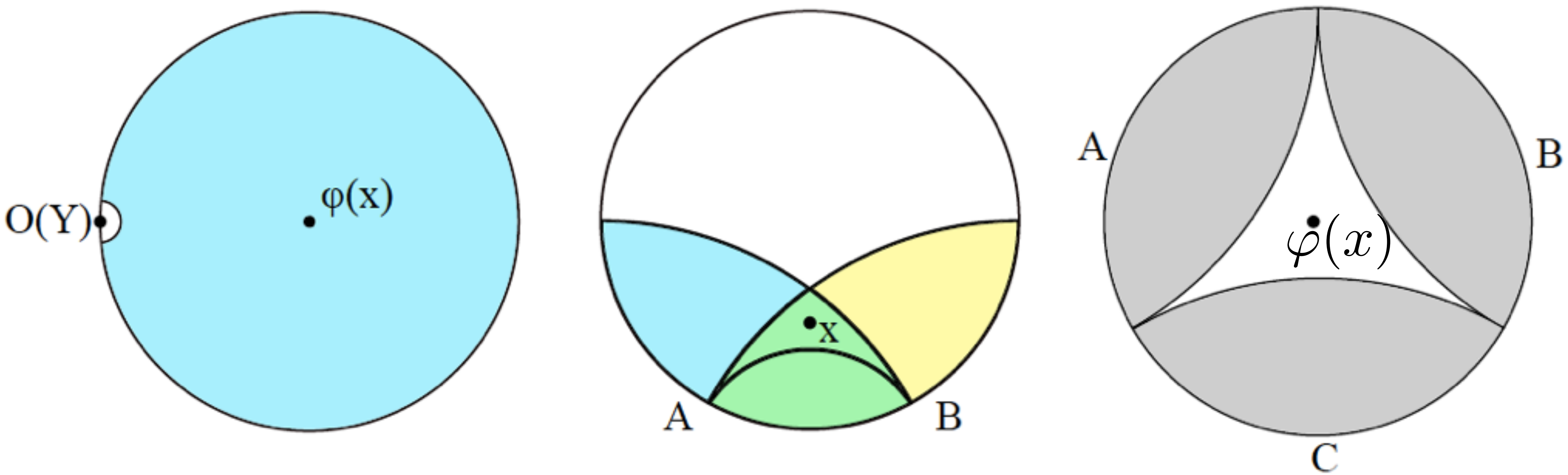
$$\mathcal{W}_C[A] \equiv \mathcal{J}^+[D[A]] \cap \mathcal{J}^-[D[A]].$$



$$\phi(x) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}} dY\, K(x;Y)\mathcal{O}(Y)$$

Reduction to spacelike slice



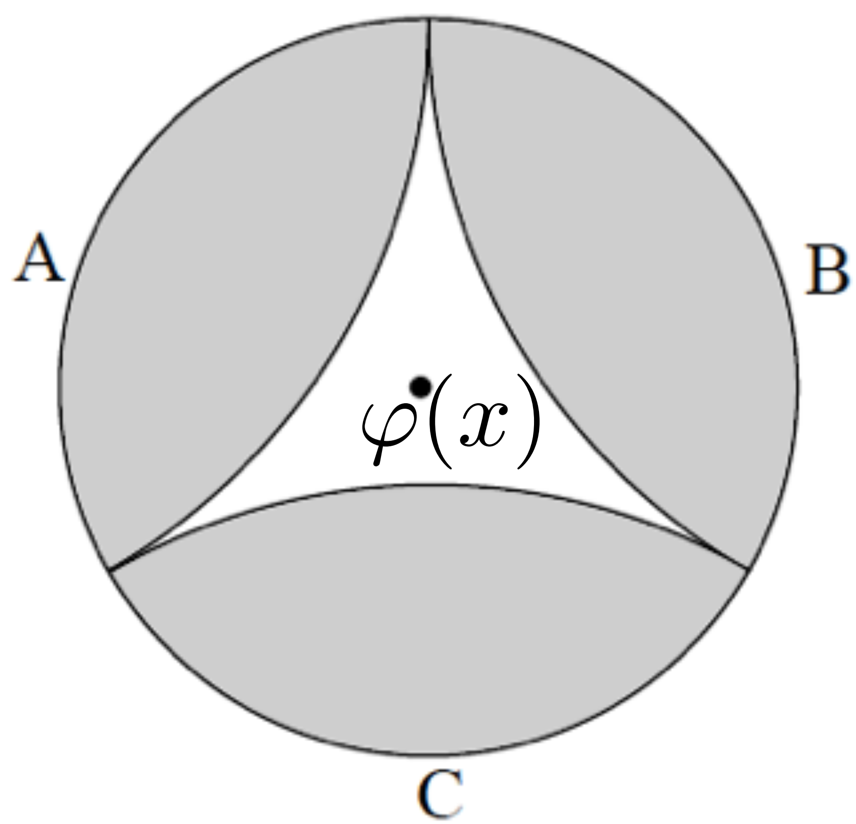


Solve boundary EOMs

$\varphi(x) \to \Phi_{AB}(x)$ $\qquad$ $\varphi(x) \to \Phi_{BC}(x)$ $\qquad$ $\varphi(x) \to \Phi_{CA}(x)$

Almheiri, A., Dong, X., & Harlow, D. (2015).
Bulk locality and quantum error correction in AdS/CFT. JHEP, 2015(4), 163.

Sharpening the paradox



$$|\Omega'\rangle = \varphi(x)|\Omega\rangle$$

$$A \cup B \cup C = Boundary$$

$$\rho'_A = \rho_A = \mathrm{tr}_{BC}[|\Omega\rangle\langle\Omega|]$$
$$\rho'_B = \rho_B = \mathrm{tr}_{CA}[|\Omega\rangle\langle\Omega|]$$
$$\rho'_C = \rho_C = \mathrm{tr}_{AB}[|\Omega\rangle\langle\Omega|]$$

$$\rho'_{AB} \neq \rho_{AB} \quad \rho'_{BC} \neq \rho_{BC} \quad \rho'_{AC} \neq \rho_{AC}$$

The effect of $\varphi(x)$ is encoded in non-local correlations.

Entanglement and Operator "teleportation"

Singlet $$|\Psi^-\rangle := \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

Stabilizer equations

$$X \otimes X|\Psi^-\rangle = Y \otimes Y|\Psi^-\rangle = Z \otimes Z|\Psi^-\rangle = -|\Psi^-\rangle$$

Operator "teleportation"

$$O_A|\Psi^-\rangle = O_B|\Psi^-\rangle$$

Resolution: Entangled ground state and low energy sector.

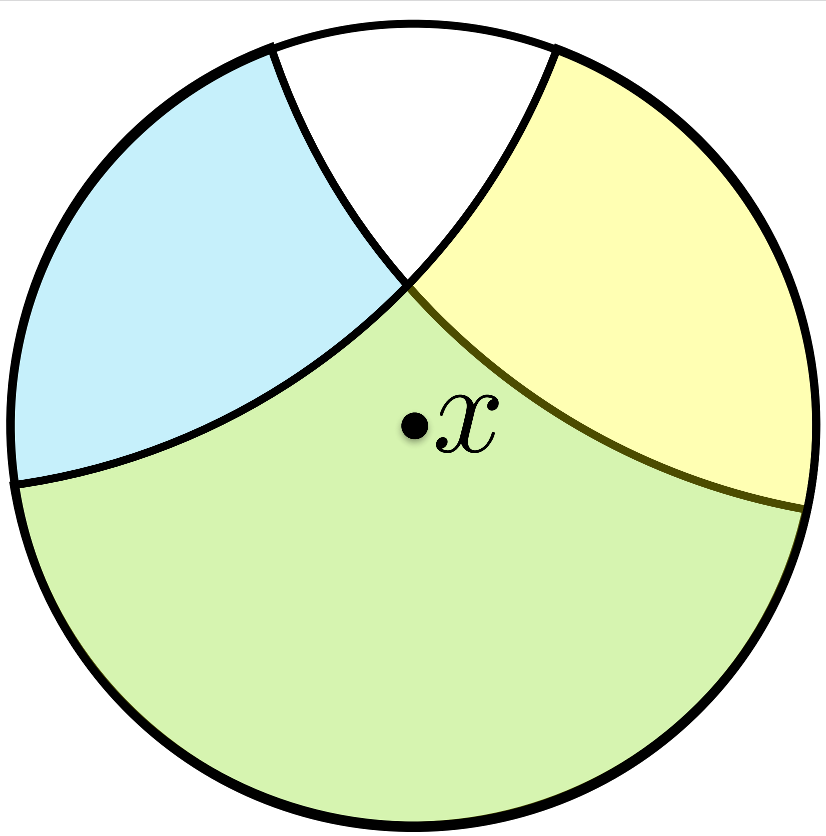
Motivation

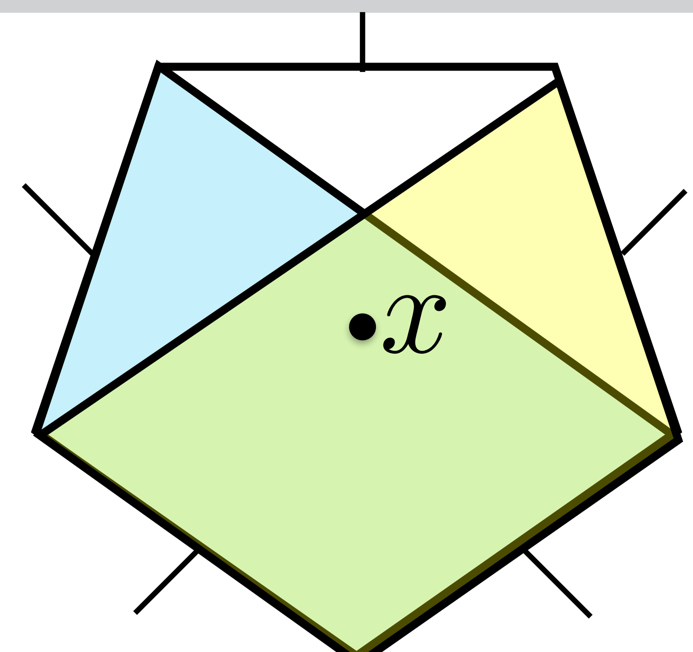
(a holography inspired code exploration)

- A connection has been established between quantum error correction and holography
- Such "holographic" codes may be fruitful for quantum information processing.
- Understanding their features could shed light on the information structure of holography and maybe even quantum gravity.

Dictionary

Holography	QECC
Bulk operators	Logical operators
Boundary operators	Physical operators
Vacuum geometry assumption	Code subspace definition
x in the entanglement wedge $\mathscr{E}[R]$	R^c correctable with respect to $\mathcal{A}_x$ Operators in $\mathcal{A}_x$ may be represented in R





$[[5,1,3]]_2$ code
Single bulk location

Operator Algebra Quantum Error Correction (OAQEC)

"If you don't eat your meat, you can't have any pudding, how can you have any pudding if you don't eat your meat!"

Bény, C., Kempf, A., & Kribs, D. (2007).
Generalization of Quantum Error Correction via the Heisenberg Picture.
PRL, 98(10), 100502.

Definition: OAQEC

Code space: $\mathcal{H}_C = P\mathcal{H}$ $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$

Noise map:

$$\mathcal{N}(\rho) = \sum_j N_j \rho N_j^\dagger$$

Noise span:

$$\mathcal{N} = \mathsf{span}\{N_a^\dagger N_b\}_{a,b}$$

$\mathcal{N}$ is correctable with respect to $\mathcal{A}$ in the code subspace $\mathcal{H}_C$ iff

i) ∃ $\mathcal{R}$ (recovery map): $\mathsf{tr}[X \mathcal{R} \circ \mathcal{N}(\rho)] = \mathsf{tr}[X\rho]$ $\rho = P\rho P$

ii) Algebraic condition $[PN_a^\dagger N_b P, X] = 0$ $\forall X \in \mathcal{A}$

distance in OAQEC

Depolarizing map: $\Delta_R(\rho) = \sigma_R \otimes \mathrm{tr}_r[\rho]$

Has all operators supported in R in its span.

Region R is correctable <=> Depolarizing R is correctable.

Logical equivalence of operators.

$\tilde{X} \sim_P X$ is $P\tilde{X}P = PXP$

Region R is correctable <=> $\mathcal{A}$ may be represented on R^c

$$\forall X \in \mathcal{A}, \exists \tilde{X}_{R^c} : \tilde{X}_{R^c} \sim_P X$$

Distance: size d of the smallest non-correctable region.

Can be relative to a sub-algebra!!

price of an algebra

Price: size p of the smallest region
where all operators can be represented.

Can be relative to a sub-algebra!!

Price: Tells us how well the information is **hidden.**
How hard it is to read.

Distance: Tells us how well the information is
protected from noise. How hard it is to modify.

Example: Repetition code

(Ferromagnetic Ising)

$$\mathcal{H}_C = \mathsf{span}\{|0\rangle^{\otimes n}, |1\rangle^{\otimes n}\}$$

$$H := -\sum_{\langle j,k \rangle} Z_j Z_k$$

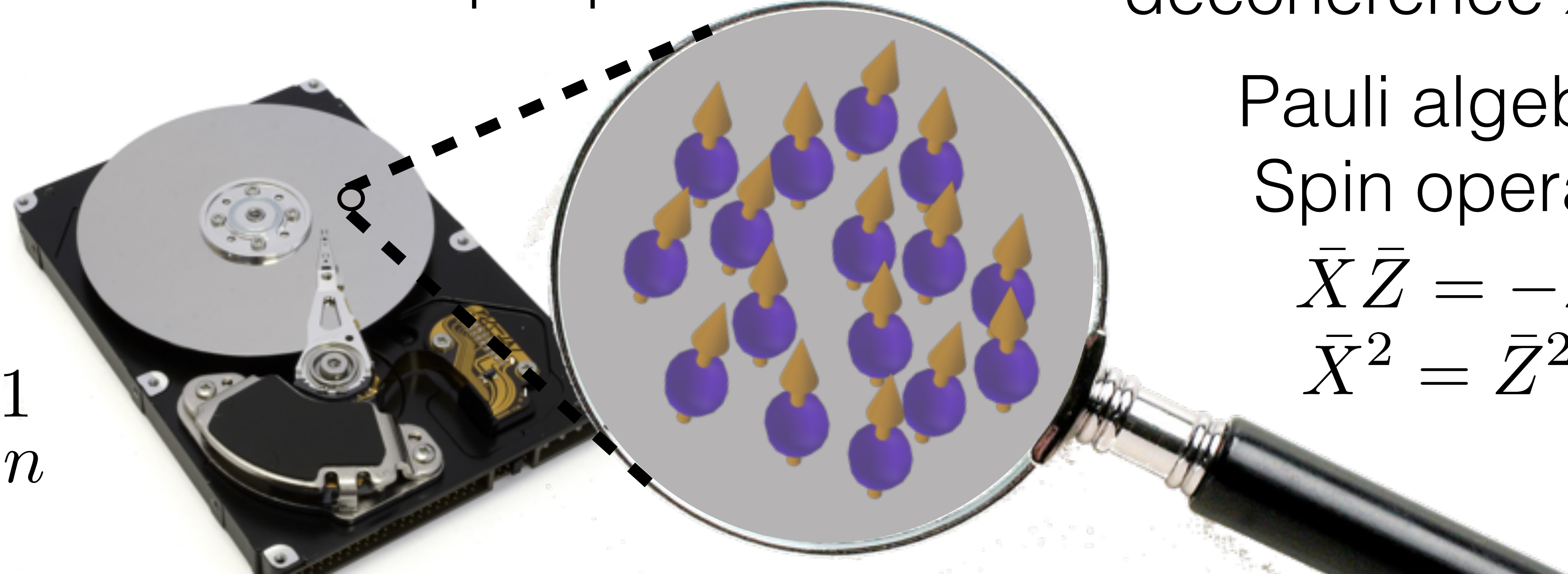
Additional conserved Quantities

$$Z_j \sim_C \bar{Z} \qquad d(\bar{Z}) = n \qquad p(\bar{Z}) = 1$$

$$\bar{X} = \bigotimes_j X_j \qquad d(\bar{X}) = 1 \qquad p(\bar{X}) = n$$

Robust macroscopic polarization.

decoherence xn !



$d = 1$
$p = n$

Pauli algebra of Spin operators

$$\bar{X}\bar{Z} = -\bar{Z}\bar{X}$$
$$\bar{X}^2 = \bar{Z}^2 = 1$$

Example:[[3,1,2]] quantum code

[[n,k,d]]   Protect non-commuting observables

$$\mathcal{H}_C = \mathsf{span}\{|\tilde{0}\rangle, |\tilde{1}\rangle, |\tilde{2}\rangle\}$$

$$|0\rangle \to |\tilde{0}\rangle = |000\rangle + |111\rangle + |222\rangle$$
$$|1\rangle \to |\tilde{1}\rangle = |012\rangle + |120\rangle + |201\rangle$$
$$|2\rangle \to |\tilde{2}\rangle = |021\rangle + |102\rangle + |210\rangle$$

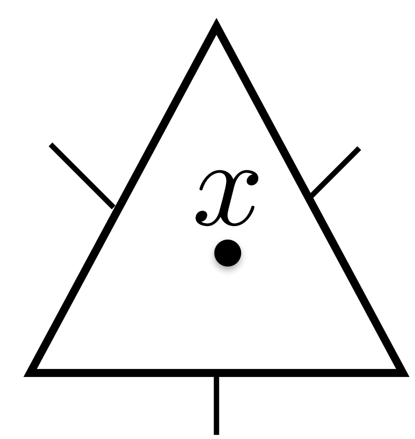
$$Z|j\rangle = \omega^j |j\rangle \qquad \omega = e^{\frac{2i\pi}{3}}$$
$$X|j\rangle = |j + 1 \bmod (3)\rangle$$

$$E = \sum_j |\tilde{j}\rangle\langle j| \qquad EE^\dagger = P_C \qquad \mathcal{E}nc(\rho) = E\rho E^\dagger$$

$$\bar{Z} \sim_C Z \otimes Z^\dagger \otimes 1 \sim_C 1 \otimes Z \otimes Z^\dagger \sim_C Z^\dagger \otimes 1 \otimes Z$$

$$\bar{X} \sim_C X \otimes X^\dagger \otimes 1 \sim_C 1 \otimes X \otimes X^\dagger \sim_C X^\dagger \otimes 1 \otimes X$$

$$d(\bar{X}) = d(\bar{Z}) = d = 2$$
$$p(\bar{X}) = p(\bar{Z}) = p = 2$$



Complementarity

Lemma 2 (reconstruction)**.** *Give code subspace* $\mathcal{H}_C = P\mathcal{H}$ *and logical subalgebra* $\mathcal{A}$*, if subsystem* R *of* $\mathcal{H}$ *is correctable with respect to* $\mathcal{A}$*, then* $\mathcal{A}$ *can be reconstructed on the complementary subsystem* R^c*. That is, for each logical operator in* $\mathcal{A}$*, there is a logically equivalent operator supported on* R^c*.*

Lemma 3 (complementarity)**.** *Given code subspace* $\mathcal{H}_C = P\mathcal{H}$ *and logical subalgebra* $\mathcal{A}$*, where* $\mathcal{H}$ *contains* n *sites, the distance and price of* $\mathcal{A}$ *obey*

$$p(\mathcal{A}) + d(\mathcal{A}) \leq n + 1. \tag{25}$$

Repetition code and qutrit code saturate complementarity.

Don't be fooled, most codes don't!!

No free-lunch

Lemma 4 (no free lunch). *Given code subspace* $\mathcal{H}_C = P\mathcal{H}$ *and non-abelian logical subalgebra* $\mathcal{A}$*, the distance and price of* $\mathcal{A}$ *obey*

$$d(\mathcal{A}) \leq p(\mathcal{A}). \tag{26}$$

Qutrit code saturates no-free lunch.

Repetition code is far from it.

Logical Z algebra in repetition violates it but is abelian.

Strong quantum singleton

Corollary 1 (strong quantum Singleton bound)**.** *Consider a code subspace $\mathcal{H}_C = P\mathcal{H}$, where $\mathcal{H}$ contains n sites, and where $k = \log\dim\mathcal{H}_c / \log\dim\mathcal{H}_0$. Then the distance d and price p of the code obey*

$$p - k \geq d - 1. \tag{38}$$

$$k \leq p - d + 1$$

We prove this for subspace codes and operator algebras in holographic codes!!

Qutrit code saturates SQSB.

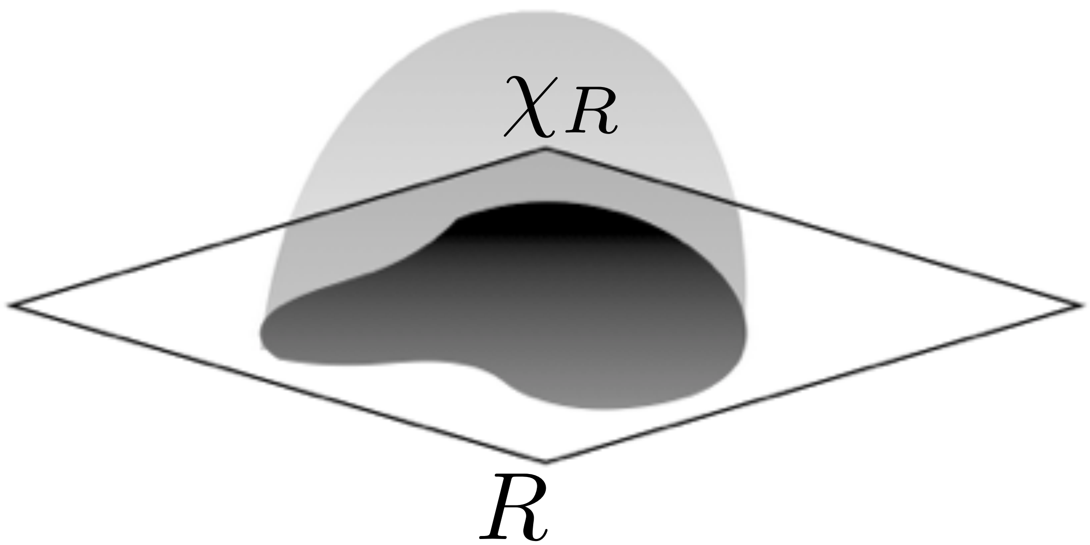
SQSB with complementarity $p + d \leq n + 1$ imply the QSB

$$k \leq n - 2(d-1)$$

Holography and QEC

Ryu-Takayanagi formula

Bulk/Boundary duality to Geometry/Entanglement duality



$$S(R) = \frac{1}{4G_N} \min_{\partial\chi_R = \partial R} \text{area}(\chi_R)$$

Entanglement ⟷ Geometry (Space-time)

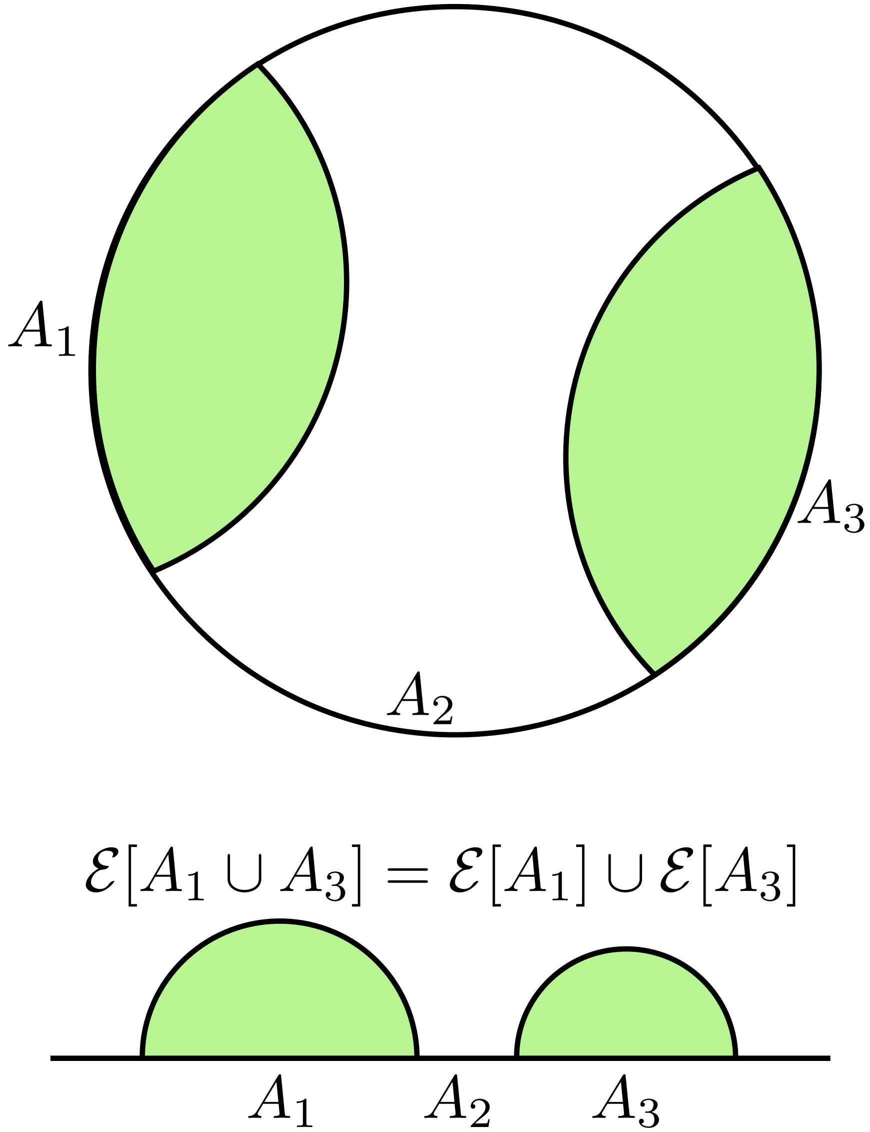
Generalization of Bekenstein-Hawking black hole entropy.

1. Static space time boundary regions
2. Fully covariant prescription.

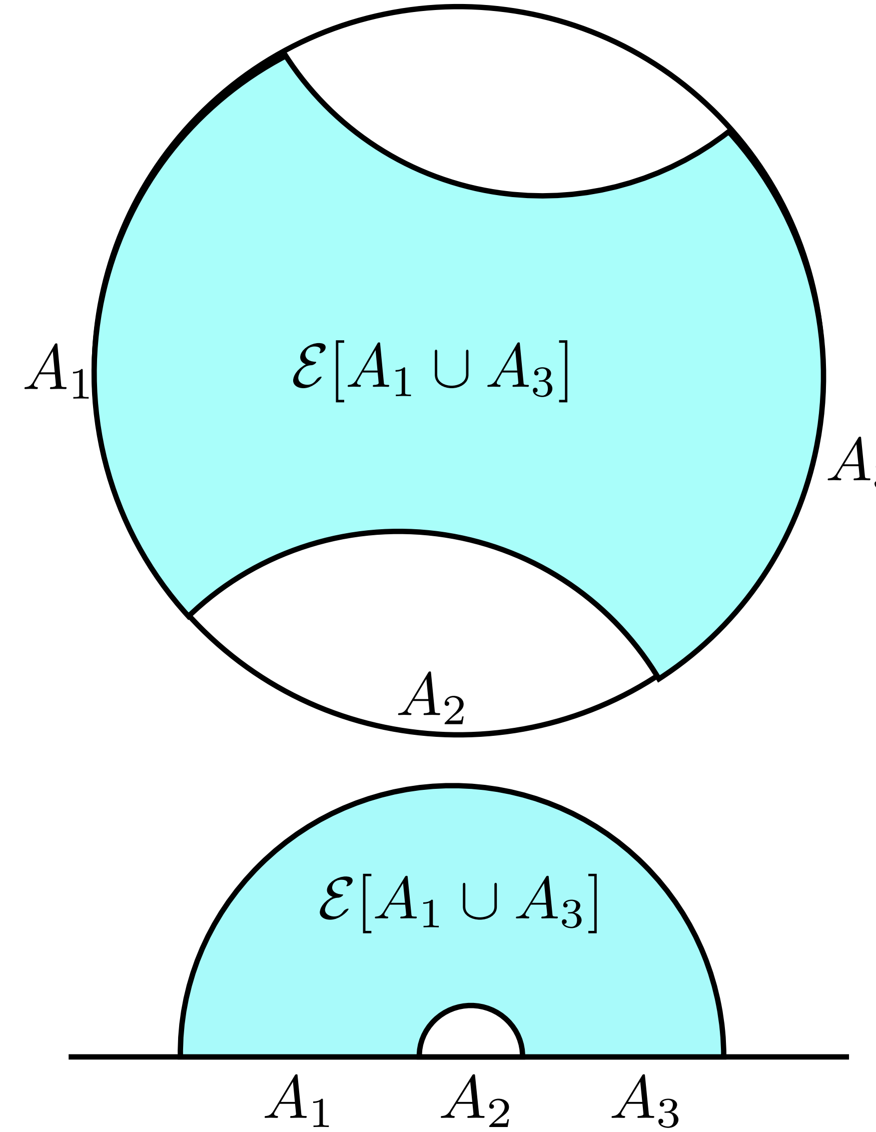
1. Ryu, S., & Takayanagi, T. (2006). Holographic Derivation of Entanglement Entropy from the anti–de Sitter Space/Conformal Field Theory Correspondence. PRL, 96(18), 181602.
2. Hubeny, V. E., Rangamani, M., & Takayanagi, T. (2007). *A covariant holographic entanglement entropy proposal.* JHEP, 2007(7), 062–062.

Entanglement wedge

(Bulk reconstruction beyond the causal wedge)



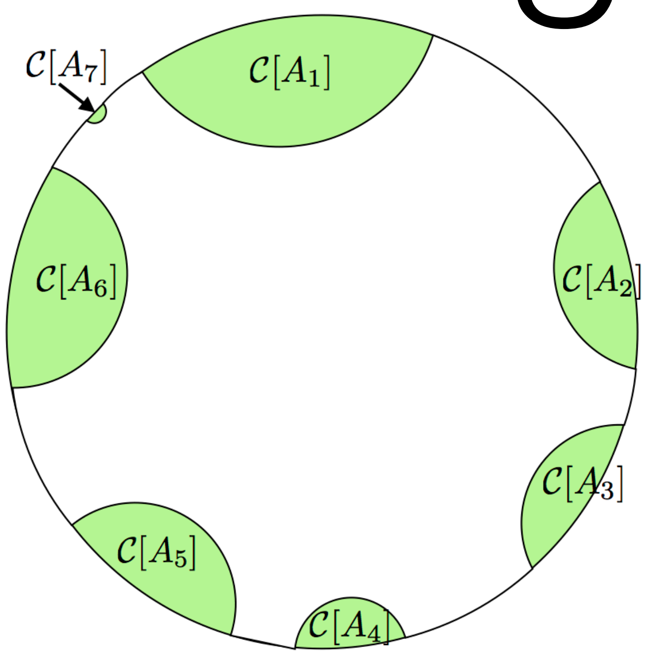
$$|A_1||A_3| \leq |A_2||A|$$

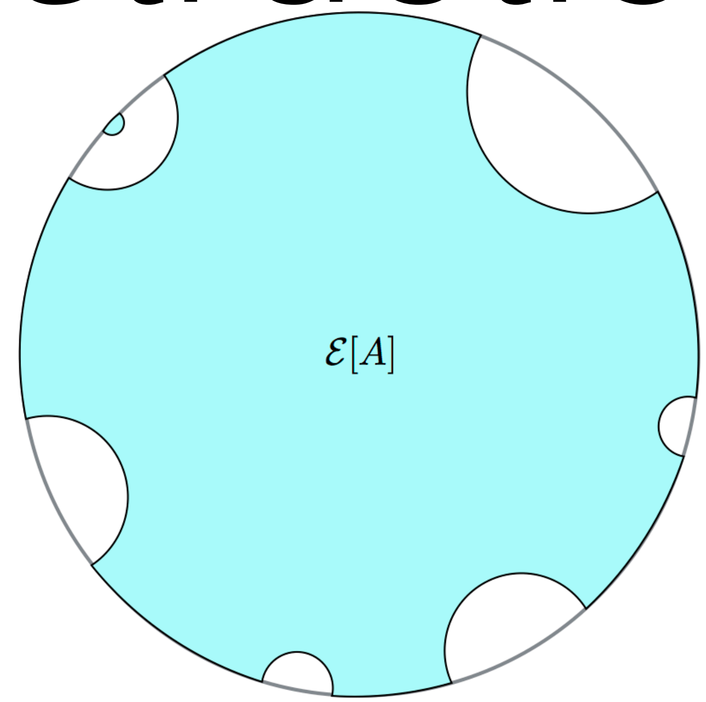


$$|A_1||A_3| \geq |A_2||A|$$

1. Bartlomiej Czech, Joanna L. Karczmarek, Fernando Nogueira, and Mark Van Raamsdonk, *The Gravity Dual of a Density Matrix.* Class. Quant. Grav. 29, 155009 (2012)
2. Matthew Headrick, Veronika E. Hubeny, Albion Lawrence, and Mukund Rangamani, *Causality & holo- graphic entanglement entropy*. JHEP 12, 162 (2014).

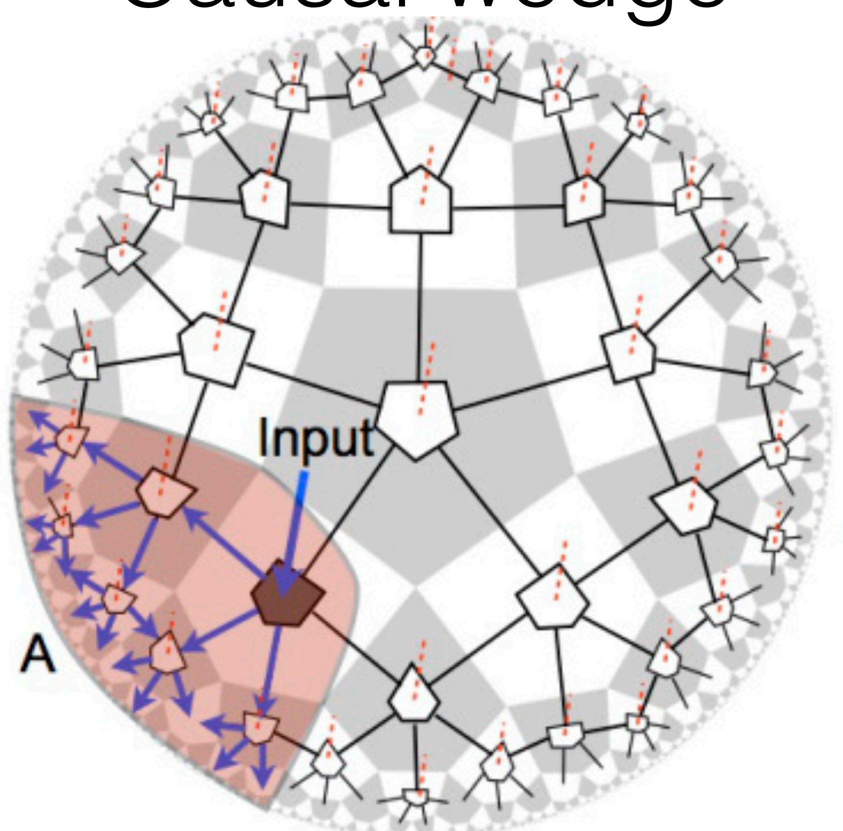
Ent. wedge reconstruction

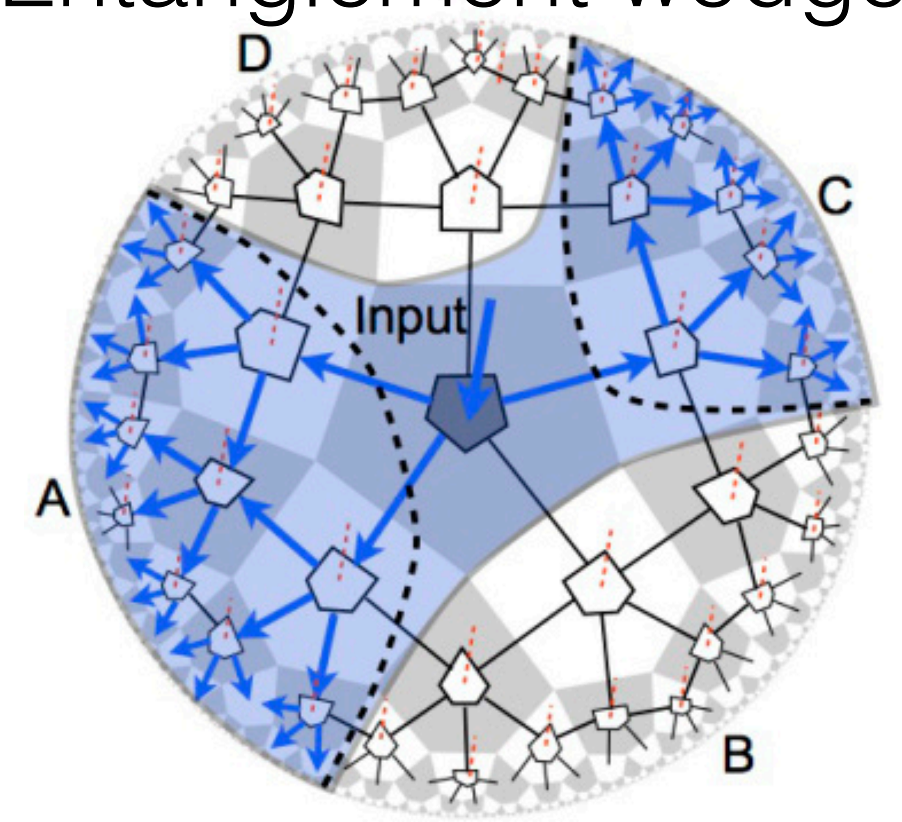




Causal wedge

Entanglement wedge





Pastawski, F., Yoshida, B., Harlow, D., & Preskill, J. (2015). Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. JHEP, 2015(6), 149.

Notation

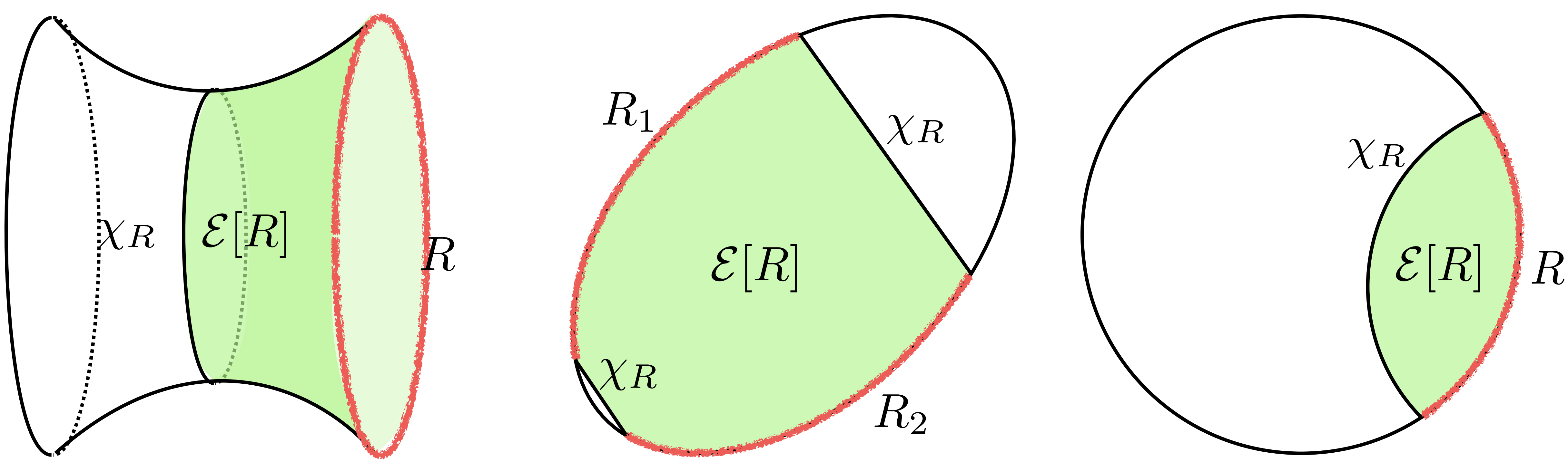
B ➡ The bulk, a Riemannian manifold (not necessarily AdS, generally finite)

∂B ➡ The boundary, (where bulk ends)

R ➡ A region of the boundary

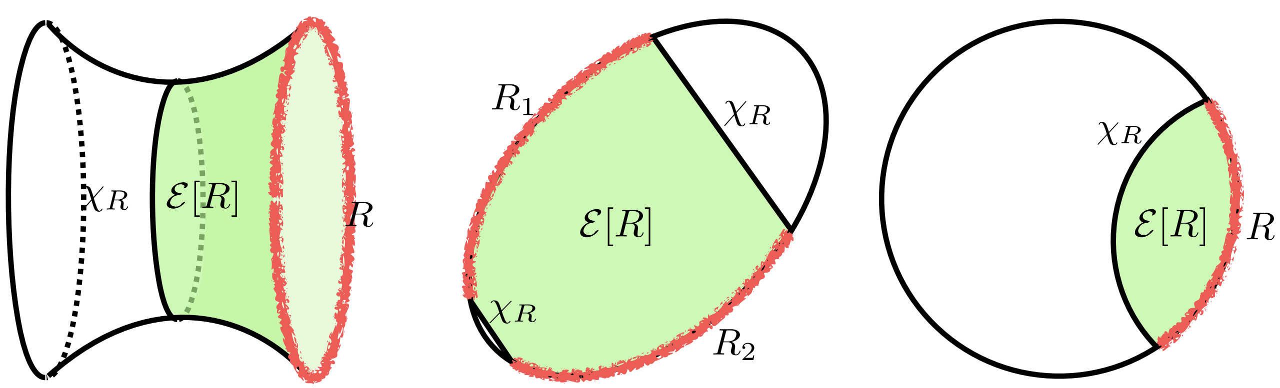
χ_R ➡ The minimal surface separating *R* from its boundary complement R^c.

$\mathcal{E}[R]$ ➡ Entanglement wedge (region between R and its minimal surface)



Riemannian entanglement wedge hypothesis

Hypothesis 2 (Entanglement wedge hypothesis)**.** *If the bulk point x is contained in the entanglement wedge $\mathcal{E}[R]$ of boundary region R, then the complementary boundary region R^c is correctable with respect to the logical bulk subalgebra $\mathcal{A}_x$. Thus for each operator in $\mathcal{A}_x$, there is a logically equivalent operator supported on R.*



$$x \in \mathcal{E}[R] \implies |\phi(x)\Omega\rangle = O_R|\Omega\rangle$$

(1) Dong, X., Harlow, D., & Wall, A. C. (2016). *Reconstruction of Bulk Operators within the Entanglement Wedge in Gauge-Gravity Duality.* PRL, 117(2), 21601

(2) Hayden, P., Nezami, S., Qi, X.-L., Thomas, N., Walter, M., & Yang, Z. (2016). *Holographic duality from random tensor networks*. JHEP, 2016(11), 9.

Dictionary

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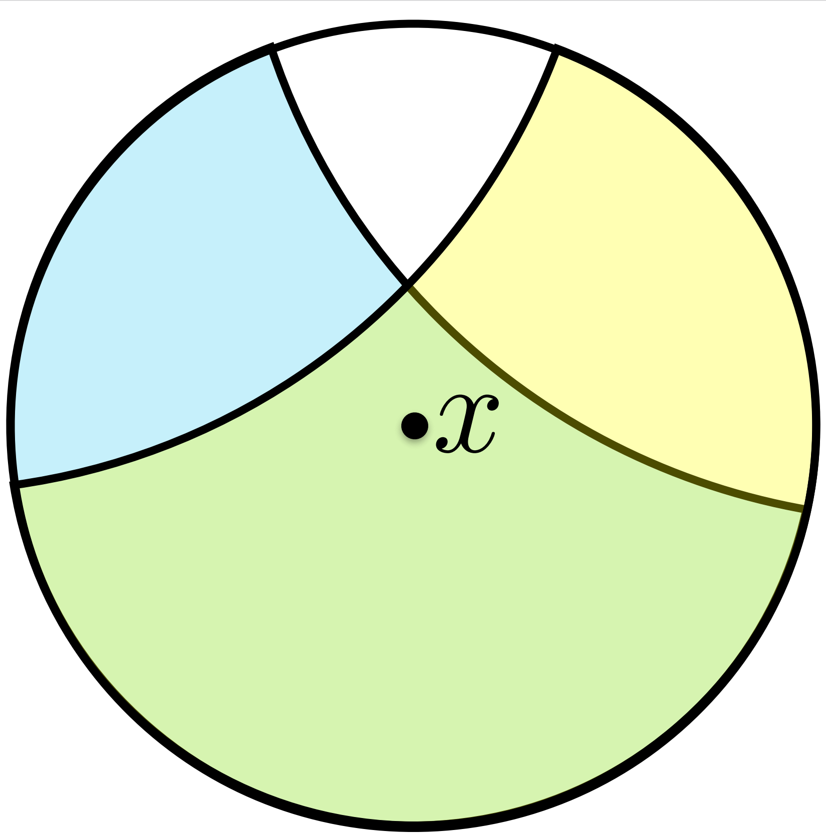
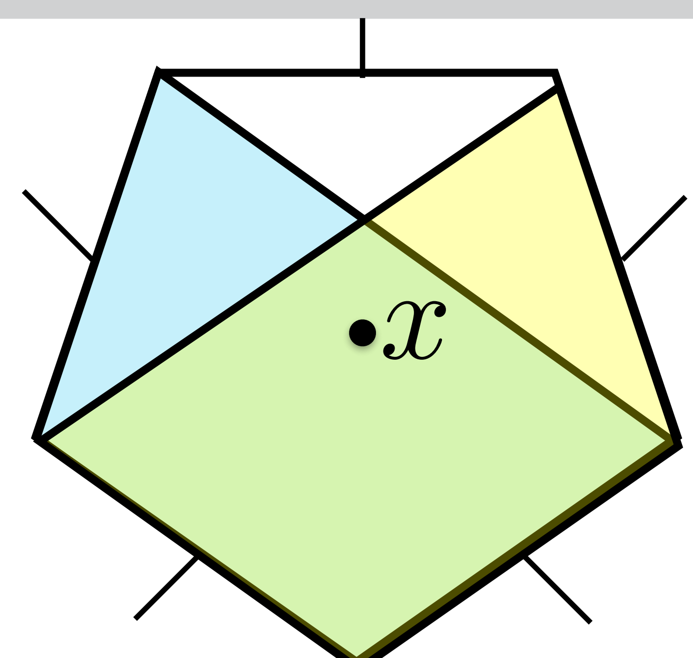


$|\partial B| = n$



$[[5,1,3]]_2$ code

Single bulk location

Geometric complementarity

Hypothesis 1 (Geometric complementarity)**.** *Given a region $R \subseteq \partial B$ and its boundary complement R^c we have that $\chi_R = \chi_{R^c} = \mathcal{E}[R] \cap \mathcal{E}[R^c]$ and $\mathcal{E}[R] \cup \mathcal{E}[R^c] = B$.*

$$d(\mathcal{A}_x) = \min_{R \subseteq \partial B : x \notin \mathcal{E}(R^c)} |R|,$$

$$p(\mathcal{A}_x) = \min_{R \subseteq \partial B : x \in \mathcal{E}(R)} |R|,$$

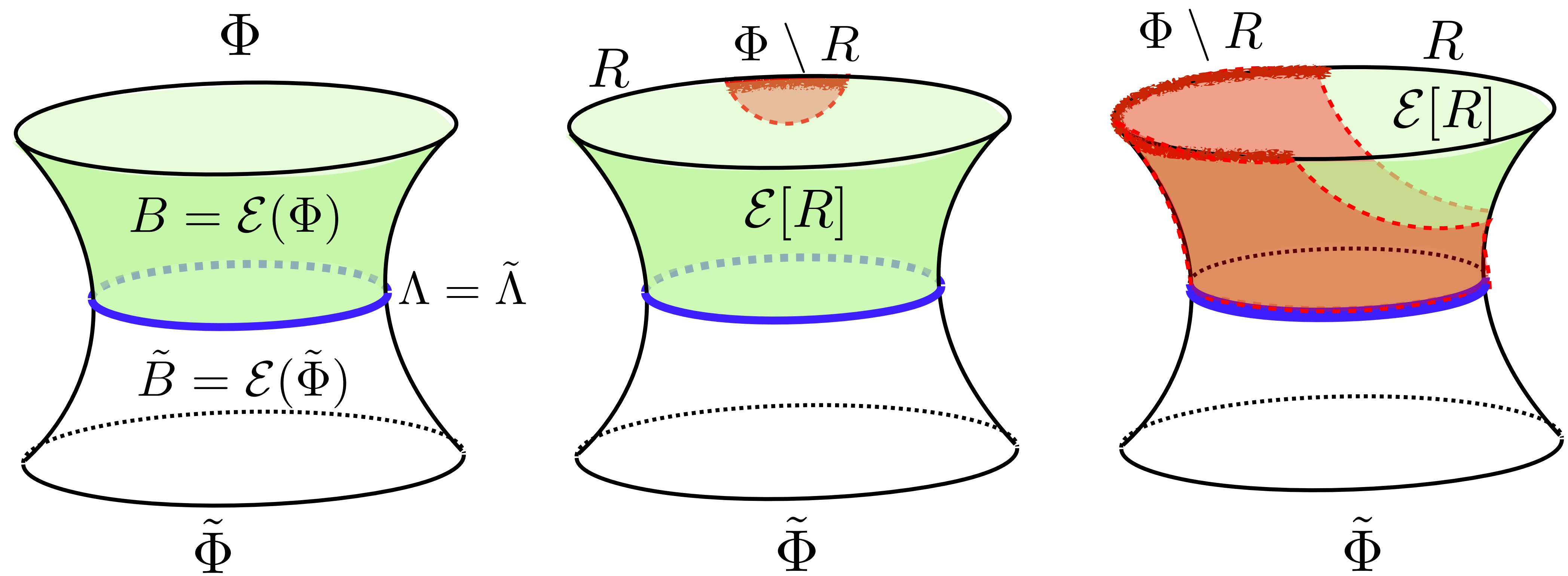
Lemma 5 (price equals distance for a point)**.** *For a holographic code, let $\mathcal{A}_x$ be the non-abelian logical algebra associated with a bulk point x. Then*

$$p(\mathcal{A}_x) = d(\mathcal{A}_x). \tag{45}$$

$$k(\mathcal{A}_x) = 0$$

Physical and logical boundaries

$$\partial B = \Phi \sqcup \Lambda$$

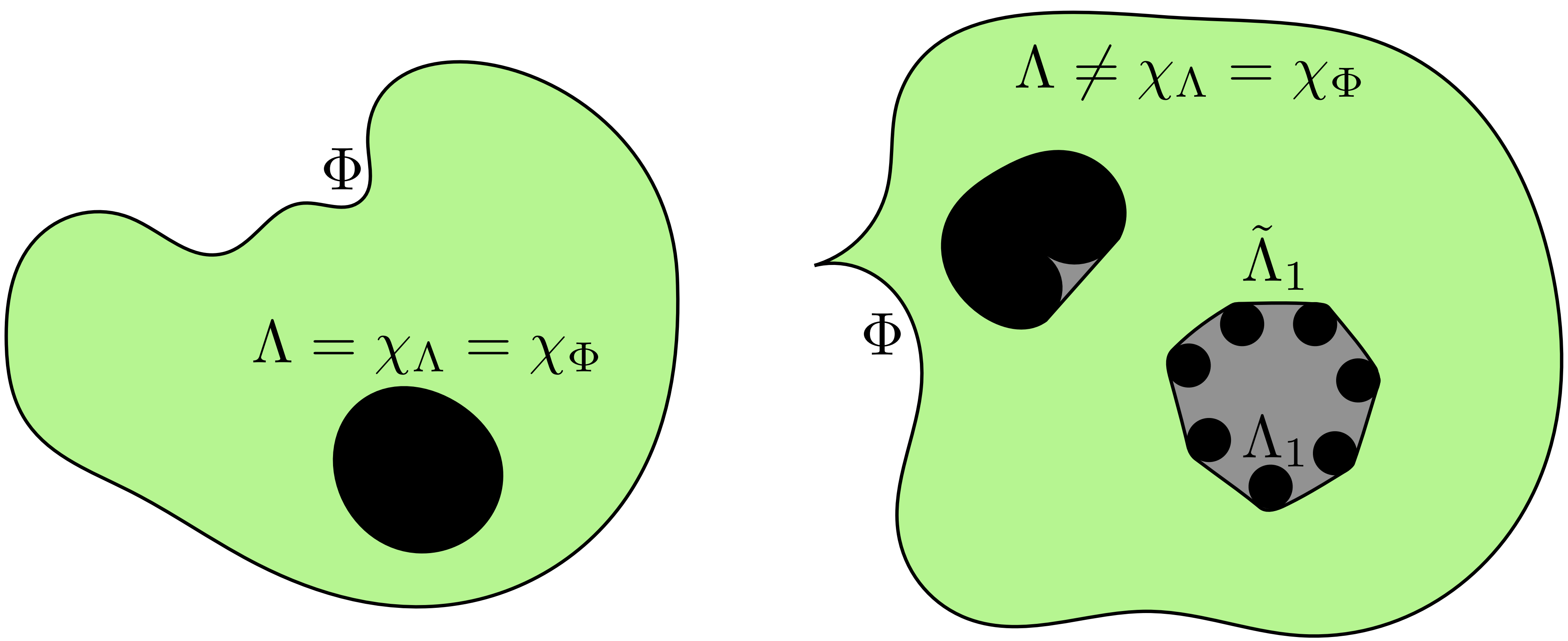


$$|\Phi| = n \qquad |\Lambda| = k$$

In 2D static geometry horizon (all or nothing).
Otherwise, minimal surface partially follow puncture.

Restrictions on Λ

$$|\Lambda| = |\chi_\Lambda|$$



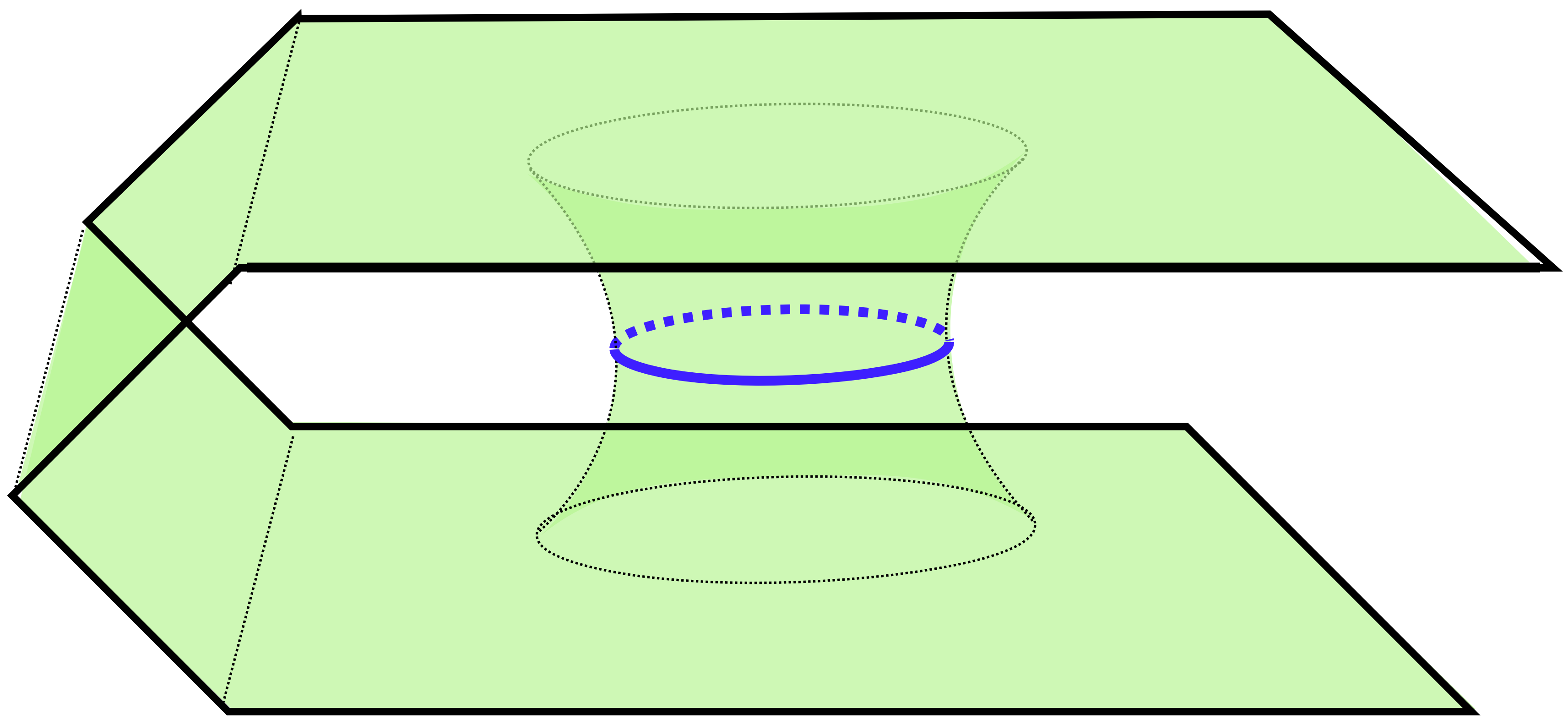
$$|\chi_\Lambda| = |\chi_\Phi| \leq |\Phi|$$

Should be thought of as Bousso bound.

Bousso, R. (1999). *A covariant entropy conjecture*. JHEP, 1999(7), 004–004.

Bulk entanglement

$$\partial B = \Phi$$



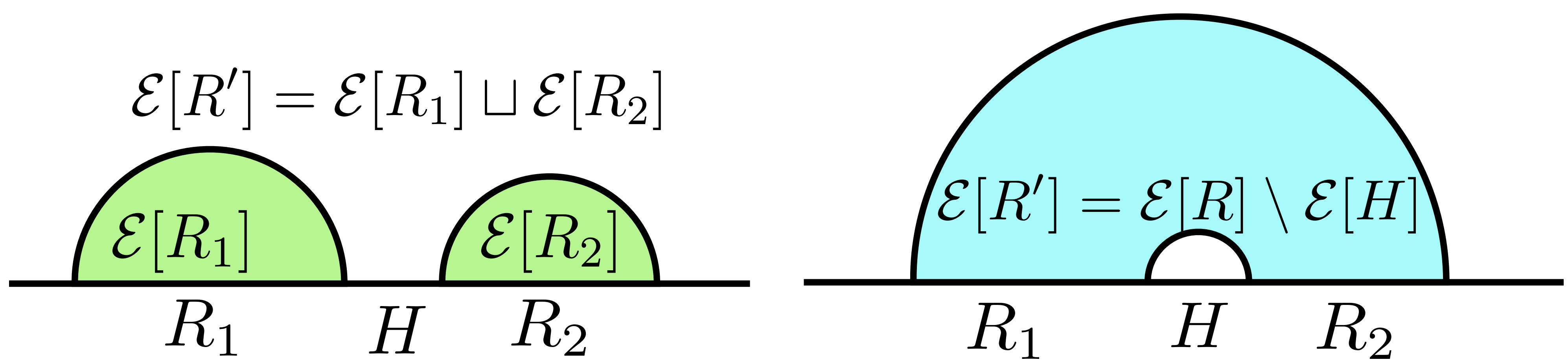
Represented through ER=EPR

Maldacena, J., & Susskind, L. (2013). *Cool horizons for entangled black holes.* Fortschritte Der Physik, 61(9), 781–811.

Uberholography

(recursive hole punching)

In negatively curved space: $|\chi_R| = 2L\log(|R|/a)$

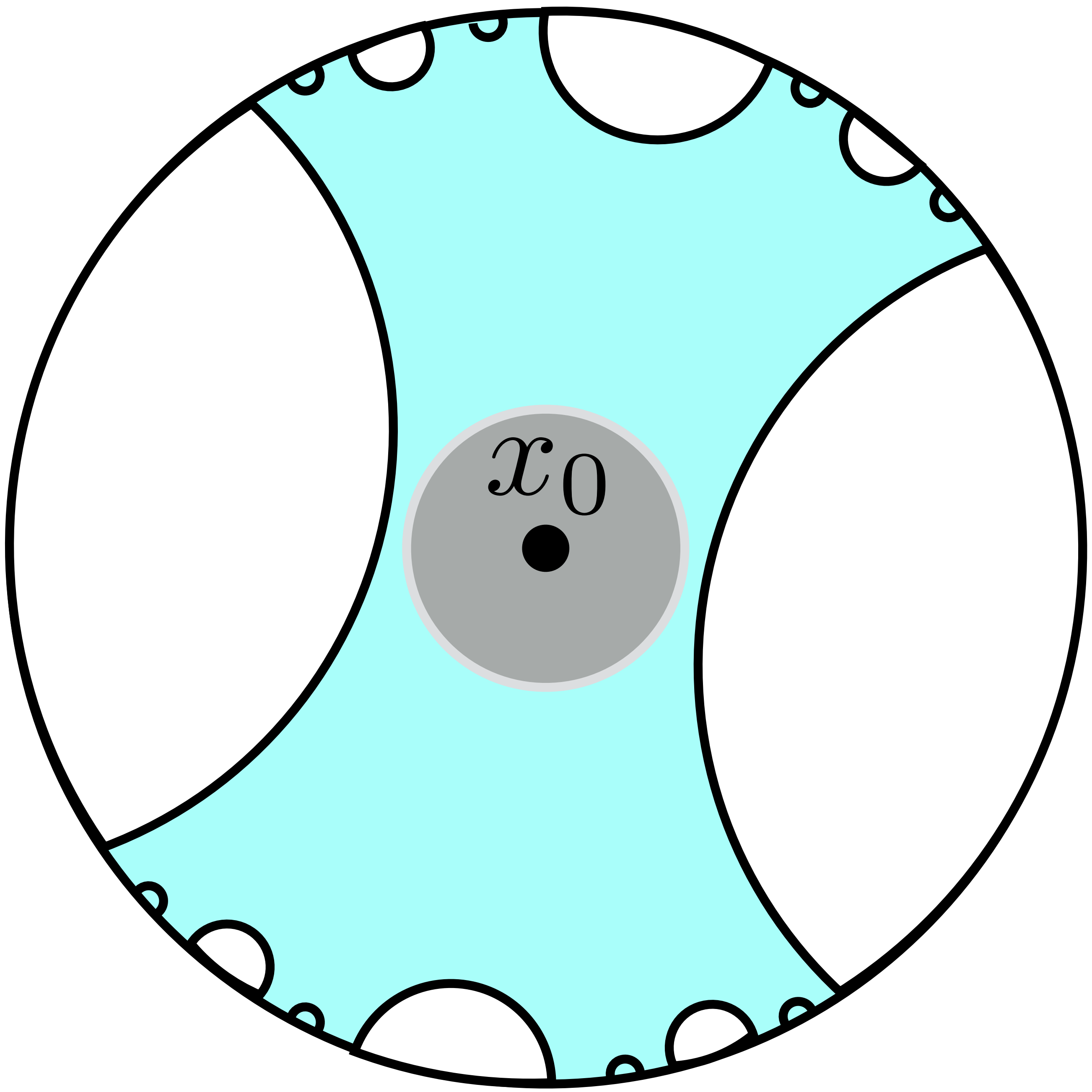


$$|R_1| = |R_2| = \left(\frac{r}{2}\right)|R|, \quad |H| = (1-r)|R|,$$

Equality at: $r = \sqrt{8} - 2 \approx 0.8284$

Uberholography

(recursive hole punching)



$$d(\mathcal{A}_{x_0}) = O(n^{0.786})$$

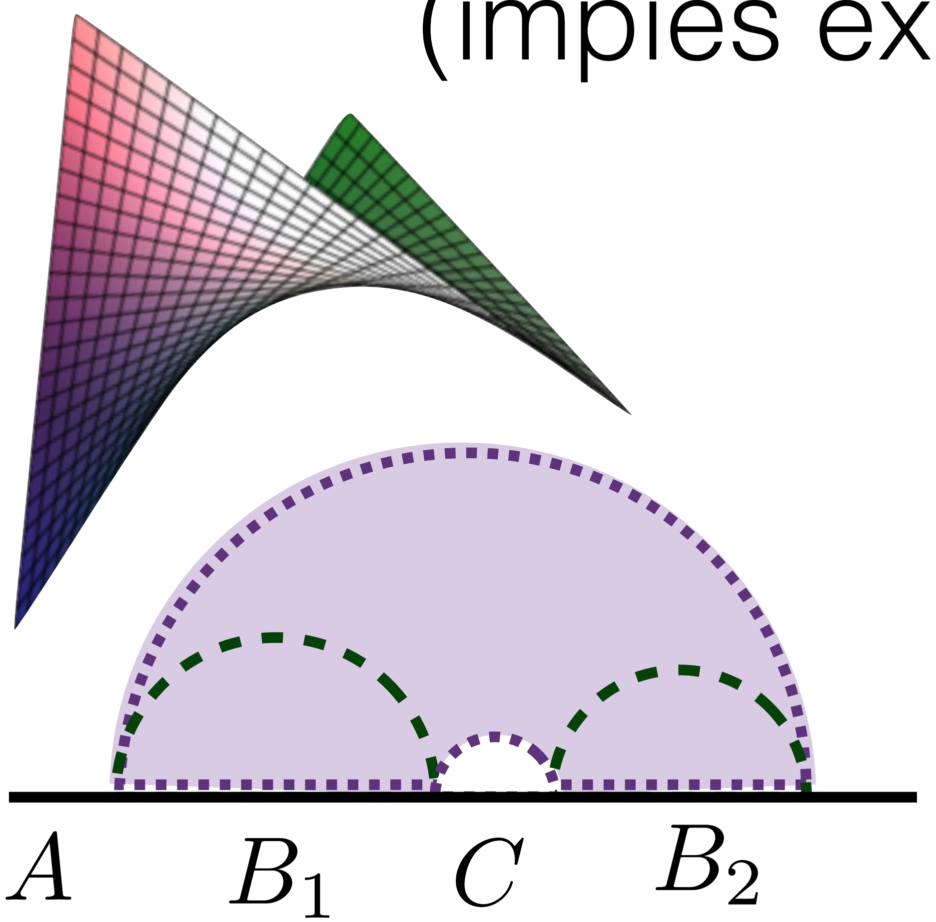
$$\alpha = \frac{\log 2}{\log(2/r)} = \frac{1}{\log_2(\sqrt{2}+1)} \approx .786.$$

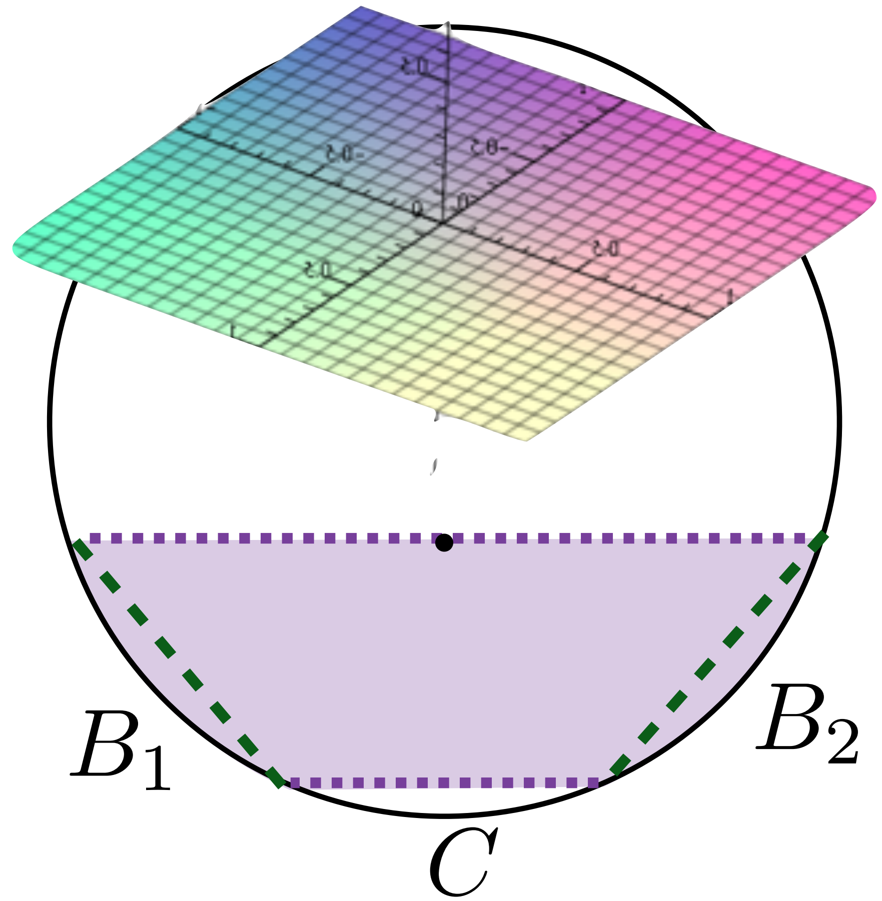
in negatively curved space

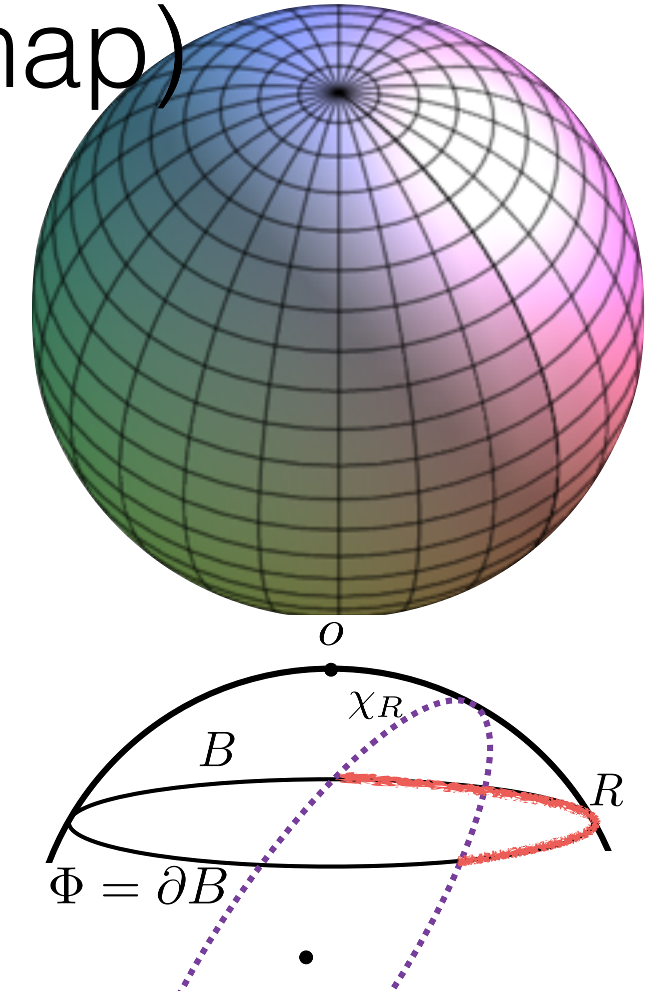
A Cantor type boundary region with fractal dimension 0.786

Quantum Markov Condition

(impies existence of recovery map)







Ryu-Takayanagi: $S(B) = S(BC) + S(C)$

Markov condition: Conditional mutual information =0

$$0 = I(A;C|B) = S(AB) + S(BC) - S(ABC) - S(B),$$

Existence of local Petz recovery map.

$$\mathcal{R}^{B\to BC} : \rho_{AB} \mapsto \rho_{ABC},$$

Petz, D. (1988). *Sufficiency of channels over Von Neumann algebras.* The Quarterly Journal of Mathematics, 39(1), 97–108.

But there are correlations!

(approximate recovery from approximate Markov)

$S(B) = S(BC) + S(C)$ if and only if

Mutual information $I(A:C) = S(A) + S(C) - S(AC) = 0$

$$1 - I(A:C|B) \leq F(\rho_{ABC}, \mathrm{id}_A \otimes \mathcal{R}_{B\to BC}\rho_{AB})^2.$$

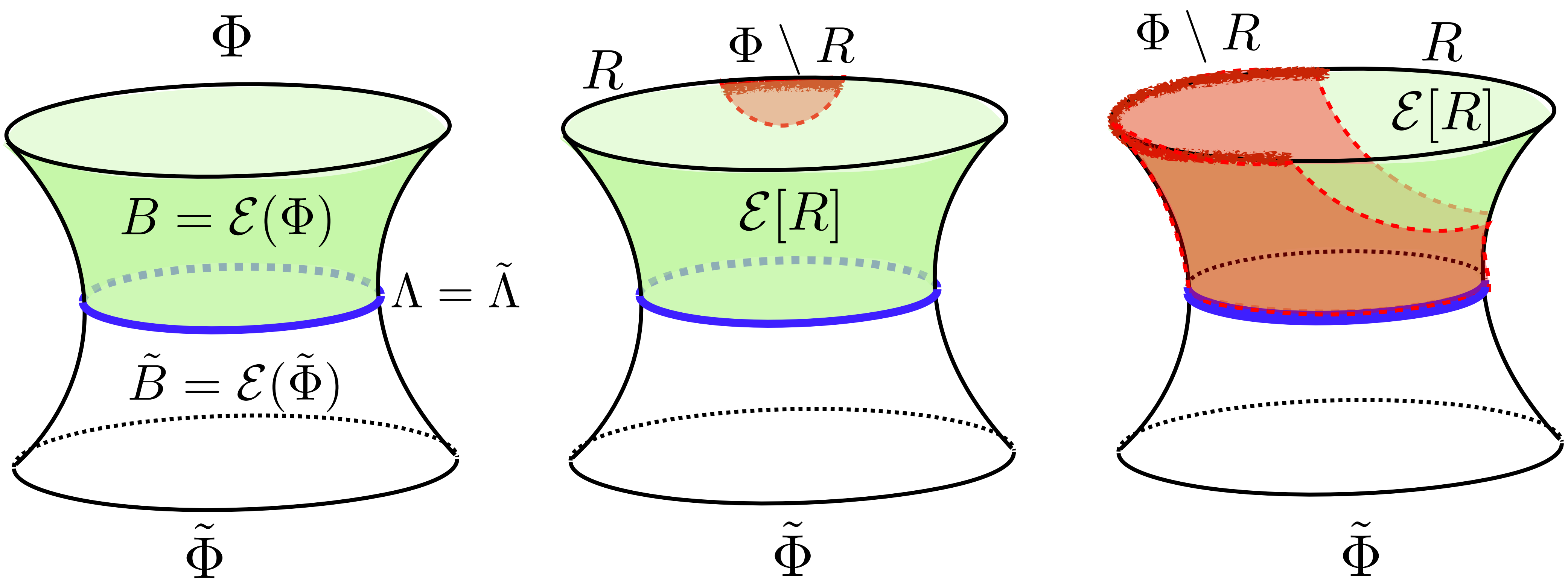
Caveat: The recovery map depends on the state on ABC

Fawzi, O., & Renner, R. (2015). *Quantum Conditional Mutual Information and Approximate Markov Chains*. CMP, 340(2), 575–611.

Quantum source-channel coding

FP, Jens Eisert and Henrik Wilming, *Quantum source-channel codes.*
quant-ph/1611.07528

Quantum source-channel codes



BTZ black hole dual to CFT thermal state.

How should I think of a CFT thermal state as a code?

For lattice models, there is no finite dimensional subspace supporting the thermal state.

Quantum source-channel codes

- Think of mixed state as **distribution** over (pure) states.

$$\rho = \int_\psi |\psi\rangle\langle\psi|\mu(\psi)d\psi.$$

- Calculate **average** fidelity the measure.

$$\bar{F}(\mu, \mathcal{E}) := \int_\psi \mu(\psi)\langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle d\psi,$$

- Bound average fidelity by **entanglement fidelity**

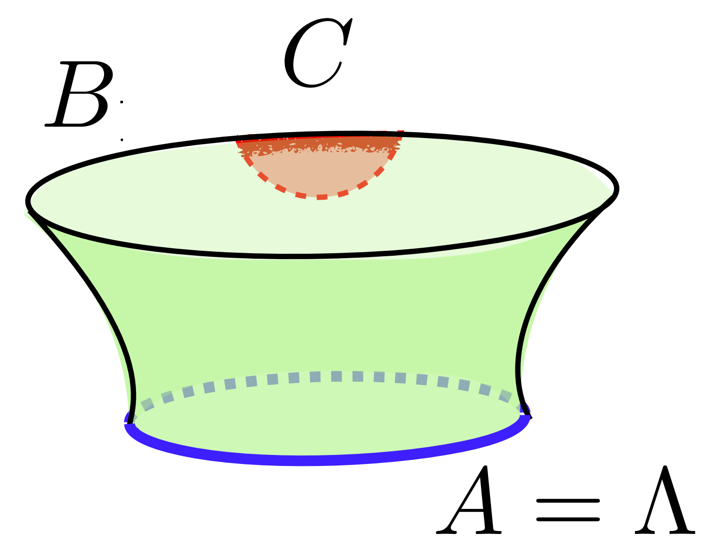
$$F_e(\rho, \mathcal{E}) := F(\mathrm{id} \otimes \mathcal{E}, |\phi\rangle), \qquad F_e(\rho, \mathcal{E}) \leq \bar{F}(\mu, \mathcal{E}),$$

- Recovery map is **independent** of purification A

$$F(\rho_{ABC}, \mathrm{id}_A \otimes \mathcal{R}_{B\to BC}\rho_{AB})^2 = F_e(\rho_{BC}, \mathcal{E}),$$

$$S_C + S_{BC} - S_B \leq \epsilon. \qquad \bar{F}(\mu, \mathcal{R}_{B\to BC} \circ \mathcal{N}_C) \geq 1 - \epsilon$$

Calculate on your favorite CFT



Study conditional mutual information as a function of system size *n.* (lattice Hamiltonian)

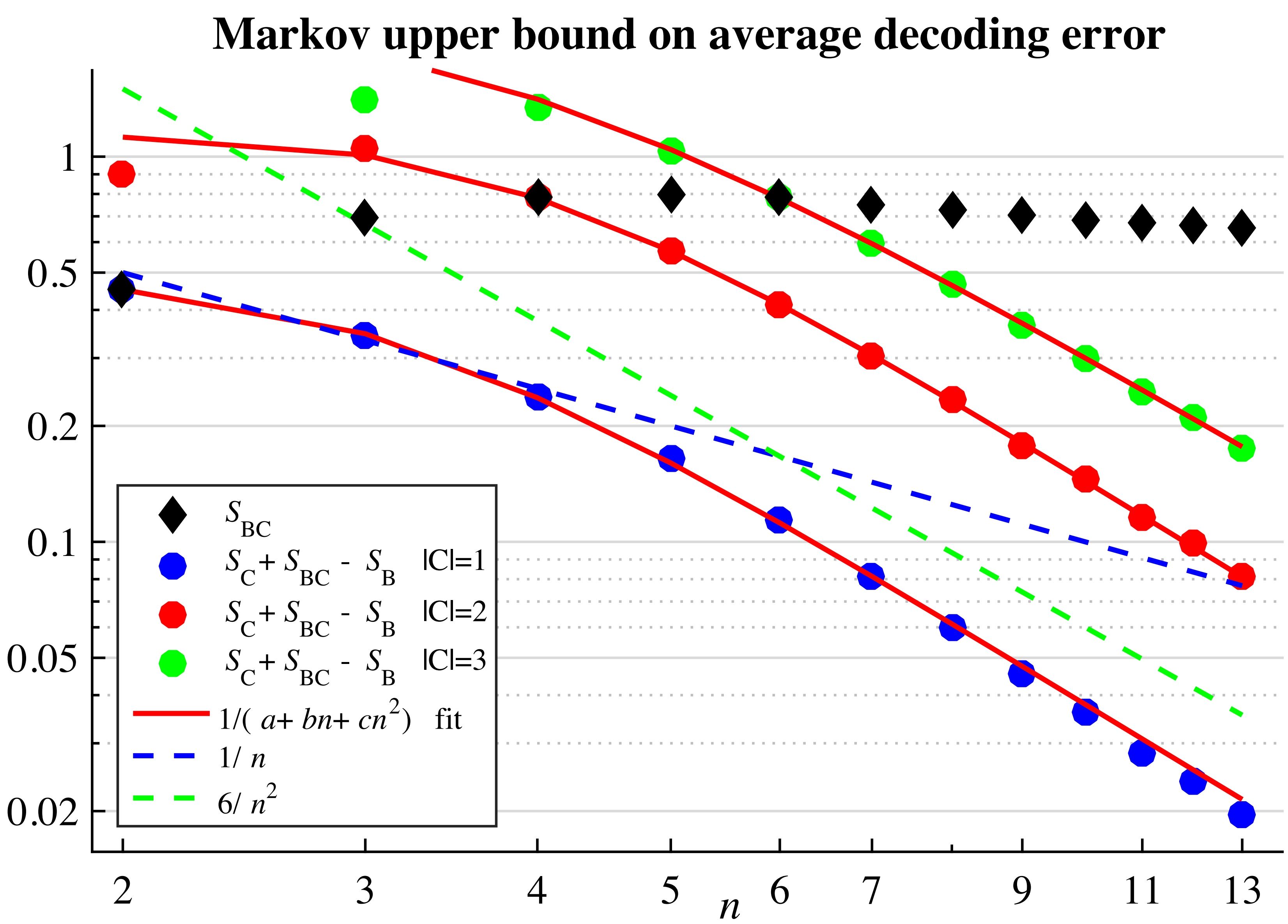
Constant temperature correspond to BH horizon a constant distance from boundary.

Scale inverse temperature with *n.* $\beta \propto n^q$ $q \in (0, 1]$

For constant $S(\rho_\beta)$ thermal entropy $\beta \propto n$

Now calculate conditional mutual information on your favorite CFT!!!

Critical transverse field Ising gapless free fermions



$\beta \propto n$

constant |C|

~constant k $S(\rho_\beta)$

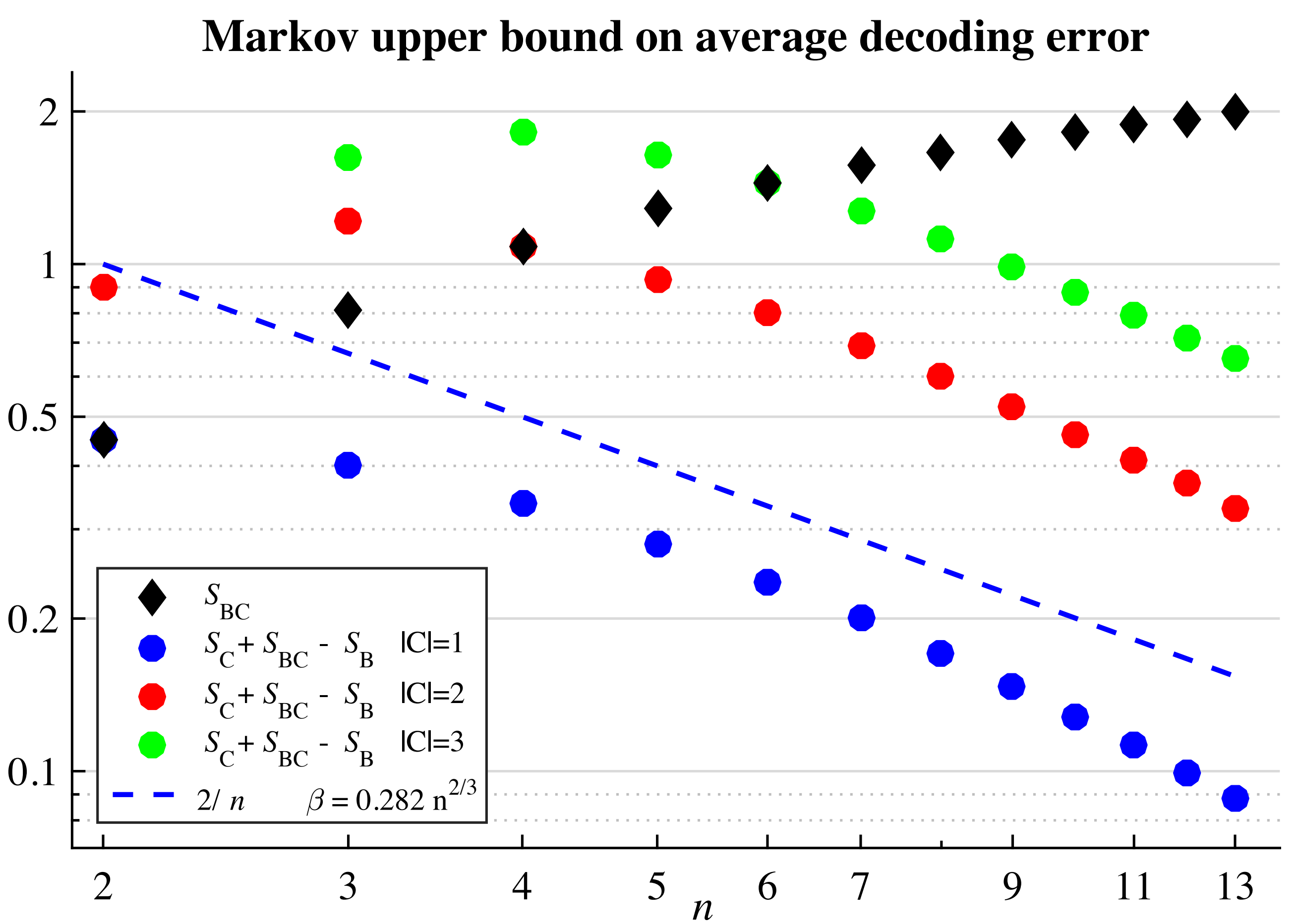
$\epsilon \propto 1/n^2$

Parity hack to forbid unphysical errors.

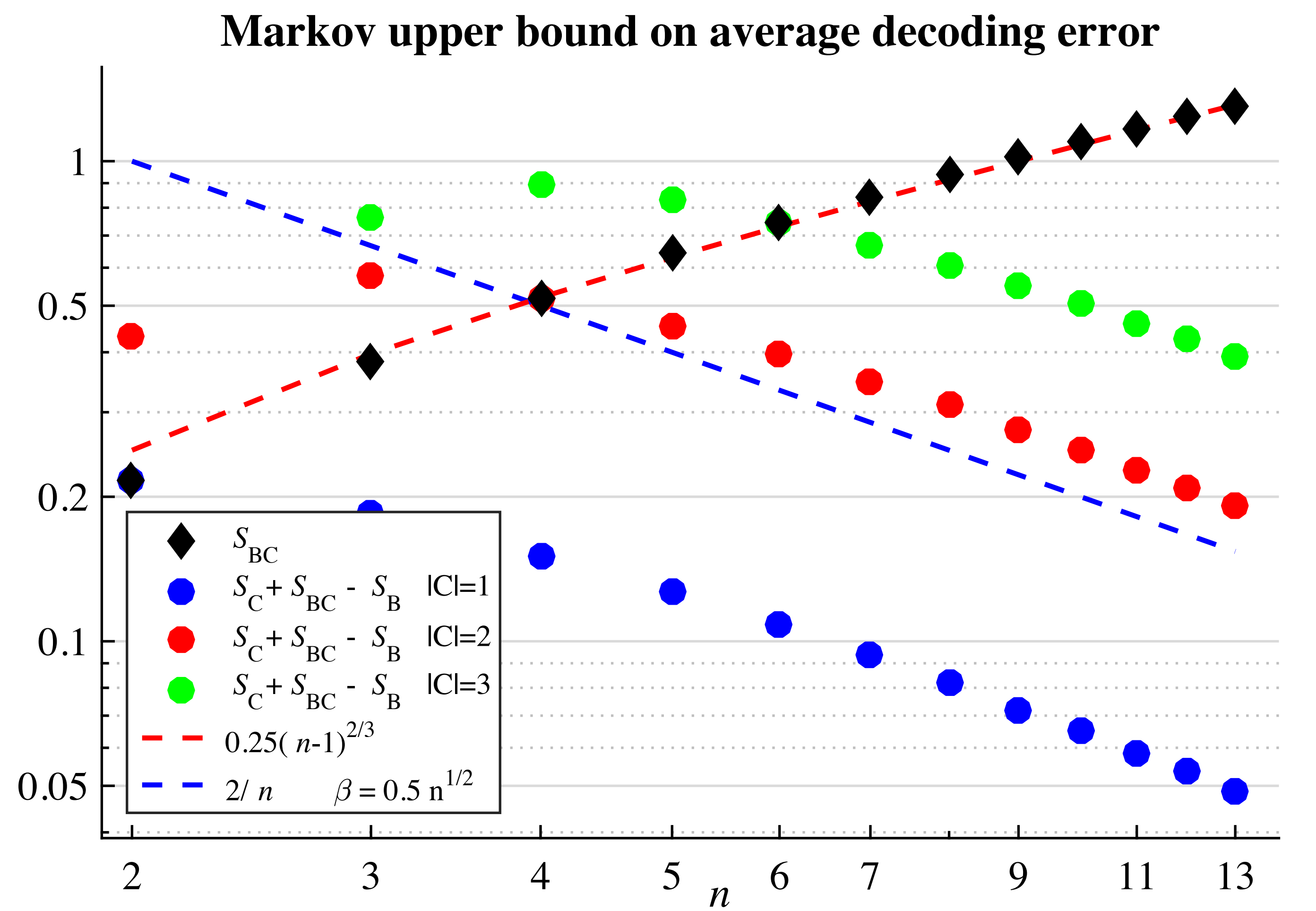
$$\rho_{BC} = \rho^{(\beta)}_{\text{even}} := \frac{P_{\text{even}} e^{-\beta H_{TF}}}{\text{tr}\left[P_{\text{even}} e^{-\beta H_{TF}}\right]}.$$

Larger BH = more logical Inf.

$\beta \propto n^{2/3}$



$\beta \propto n^{1/2}$



Conclusions
(and outlook)

- There is far more to do! Time?
- Flat/Positive curvature
    - :-)   Improves code properties. k,d
    - :-(Enhanced non-locality
- Characterizing geometry from algebraic ideas.
- Uberholography: Appearance of extra dimension > 1
- Approximate QEC extension (geometry as prior).

Petz recovery map

Original conjectured.

$$\mathcal{T}_{B\to BC} : X_B \mapsto \rho_{BC}^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}} X_B \rho_B^{-\frac{1}{2}} \otimes \mathrm{id}_C)\rho_{BC}^{\frac{1}{2}}$$

Recent versions.

$$\mathcal{R}(\cdot) := \int_{-\infty}^{\infty} \mathrm{d}t \beta_0(t) \rho_{BC}^{\frac{1+\mathrm{i}t}{2}} \rho_B^{\frac{-1+\mathrm{i}t}{2}} (\cdot) \rho_B^{\frac{-1-\mathrm{i}t}{2}} \rho_{BC}^{\frac{-1-\mathrm{i}t}{2}} .$$