Minimal String Theory

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Motivation

- Simple (minimal) and tractable string theory

- Explore D-branes, nonperturbative phenomena

- Other formulations of the theory – matrix models, holography

Descriptions

- Worldsheet: Liouville ϕ + minimal CFT (2d gravity)

- Spacetime: Dynamics in one (Euclidean) dimension ϕ

- Matrix model: Eigenvalues λ

Nonlocal relation between ϕ and λ.

Here: A Riemann surface emerges from the worldsheet description. It leads to a “derivation” of the matrix model and sheds new light on the nonlocal relation between ϕ and λ.

Roughly, ϕ and λ are conjugate (T-dual) (Moore and N.S.)

Outline

- Review of minimal CFT
- Review of Liouville theory
- Minimal string theory
- Ground ring

- Review of D-branes in Liouville theory
- D-branes in minimal string theory
- Geometric interpretation
- Conclusions

Review of minimal CFT (BPZ)

Labelled by $p < q$ relatively prime

$$c = 1 - \frac{6(p-q)^2}{p\,q}$$

Finite set of Virasoro representations

$$\Delta(\mathcal{O}_{r,s}) = \frac{(rq-sp)^2-(p-q)^2}{4p\,q}$$

$$1 \le r < p\ , \quad 1 \le s < q\ , \quad sp < rq$$

Fusion rules

$$(r_1, s_1) \times (r_2, s_2) = \sum_{r,s} (r, s)$$

with a certain range of (r, s)

Review of Liouville theory

Worldsheet Lagrangian

$$(\partial\phi)^2 + \mu\, e^{2\,b\,\phi}$$

Will set in the second term (cosmological constant) $\mu = 1$

Central charge

$$c = 1 + 6Q^2$$
$$Q = b + \frac{1}{b}$$

Virasoro primaries

$$\Delta(e^{2\alpha\phi}) = -\left(\frac{Q}{2} - \alpha\right)^2 + \frac{Q^2}{4}$$

Degenerate representations labelled by integer $r, s \geq 1$

$$2\alpha_{r,s} = \frac{1}{b}(1-r) + b(1-s)$$

have special fusion rules and allow to solve the theory (Dorn, Otto, Zamolodchikov, Zamolodchikov, Teschner)

Minimal String Theory

Combine the minimal CFT ("matter") with Liouville and ghosts.

Total $c = 26$ sets $b^2 = \frac{p}{q}$

Simplest operators in the BRST cohomology are "tachyons"

$$\mathcal{T}_{r,s} = c\,\bar{c}\,\mathcal{O}_{r,s} e^{2\,\beta_{r,s}\,\phi}$$

$$\mathcal{T}_{r,s} = \mathcal{T}_{p-r,q-s}$$

$$2\beta_{r,s} = \frac{p+q-|rq-sp|}{\sqrt{p\,q}}$$

$$1 \le r < p\ , \qquad 1 \le s < q$$

Because of degenerate matter and Liouville representations, there are additional physical operators with other ghost numbers (Lian and Zuckerman).

Ground Ring

Special operators with ghost number zero

$$\widehat{\mathcal{O}}_{r,s} = \mathcal{L}_{r,s} \cdot \mathcal{O}_{r,s} e^{2\,\alpha_{r,s}\,\phi}$$
$$2\alpha_{r,s} = \frac{p+q-rq-sp}{\sqrt{p\,q}}$$
$$1 \leq r < p \ , \quad 1 \leq s < q$$

with $\mathcal{L}_{r,s}$ a polynomial in ghosts, and Virasoro generators.

Using ghost number conservation they form a ring (modulo BRST commutators) (Witten).

Their multiplication is constrained by the fusion rules. This allows us to determine the ring relations (up to a few coefficients which are justified later)

In terms of

$$\widehat{\mathcal{O}}_{1,1} = 1$$
$$\widehat{\mathcal{O}}_{2,1} = 2X$$
$$\widehat{\mathcal{O}}_{1,2} = 2Y$$

we have

$$\widehat{\mathcal{O}}_{r,s} = U_{s-1}(X)U_{r-1}(Y)$$

$U_{s-1}(X)$ are Chebyshev polynomials

$$U_{s-1}(X = \cos\theta) = \frac{\sin s\,\theta}{\sin\theta}$$

Since $U_{s-1}(X = \cos\theta) = \frac{\sin s\,\theta}{\sin\theta}$ are $SU(2)$ characters, their products are the $SU(2)$ fusion rules (coefficients are zero or one)

The truncation to a finite number of elements is obtained by imposing the ring relations

$$U_{q-1}(X) = U_{p-1}(Y) = 0$$

(with only X present, this is familiar from the representation ring of $\widehat{SU(2)}$)

This guarantees that the ground ring multiplication is simple; i.e. all the coefficients are zero or one!

In the traditional worldsheet analysis this would arise as a surprising cancellation between complicated expressions from the minimal CFT and Liouville

This interesting structure arises because $\mu \neq 0$

The Tachyon Module

By ghost number conservation,

$$\widehat{\mathcal{O}}_{r_1,s_1}\mathcal{T}_{r_2,s_2} = \sum_{r_3,s_3} \mathcal{T}_{r_3,s_3}$$

Therefore the tachyons are a module of the ring. In particular, using

$$\mathcal{T}_{r,s} = \widehat{\mathcal{O}}_{r,s}\mathcal{T}_{1,1}$$

the coefficients above are zero or one.

$\mathcal{T}_{r,s} = \mathcal{T}_{p-r,q-s}$ leads to a new relation in the module...

$$T_p(Y) = T_q(X)$$

with $T_p(Y)$ Chebyshev polynomials

$$T_p(Y = \cos\theta) = \cos p\,\theta$$

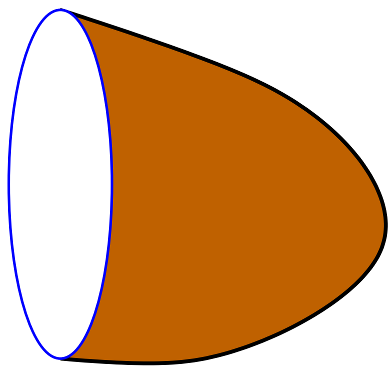
Using the ring and its module we easily derive some correlation functions; e.g.

$$\begin{aligned}
&\langle \mathcal{T}_{r_1,s_1}\mathcal{T}_{r_2,s_2}\mathcal{T}_{r_3,s_3}\rangle \\
&\qquad = \langle \widehat{\mathcal{O}}_{r_1,s_1}\widehat{\mathcal{O}}_{r_2,s_2}\widehat{\mathcal{O}}_{r_3,s_3}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\rangle \\
&\qquad = N_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}
\end{aligned}$$

This explains why the correlation functions are so simple: zero or one!

Review of Branes in Liouville

FZZT branes (Fateev, Zamolodchikov and Zamolodchikov, Teschner) – macroscopic loops in the worldsheet



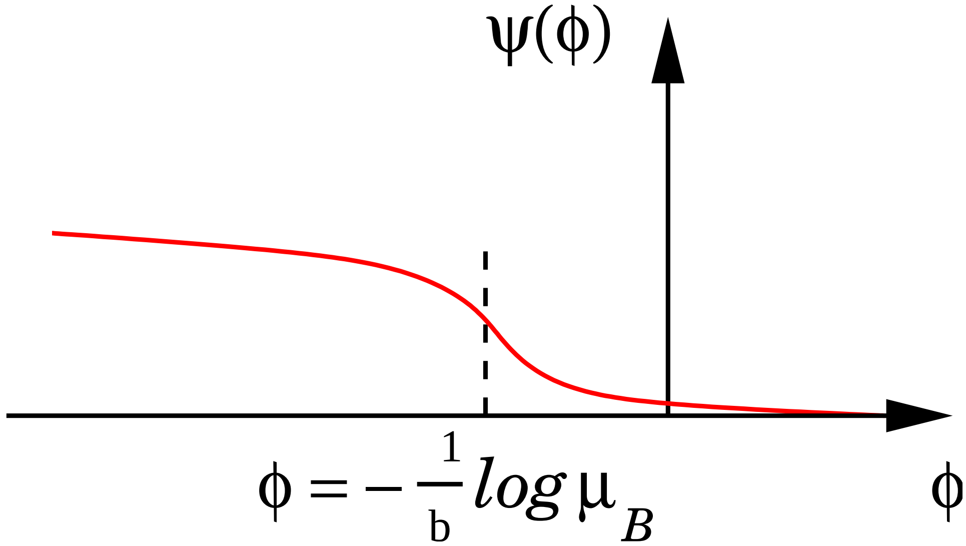
Labelled by the "boundary cosmological constant"

$$\delta S = \mu_B \oint e^{b\phi}$$

Minisuperspace wavefunction

$$\Psi(\phi) = \langle \phi | \mu_B \rangle = e^{-\mu_B e^{b\phi}}$$

The brane comes from infinity and dissolves at $\phi \approx -\frac{1}{b} \log \mu_B$.



In Cardy's formalism a brane is labelled by a representation in the open string channel

$$\mu_B = \cosh \pi \, b \, \sigma \qquad \longleftrightarrow \qquad \Delta = \frac{1}{4}\sigma^2 + \frac{Q^2}{4}$$

The boundary state is

$$|\sigma\rangle = \int_0^\infty dP \, \cos(2\pi P\sigma)\Psi(P)|P\rangle\rangle$$

$|P\rangle\rangle$ is a closed string (Ishibashi) state.

For the degenerate representations

$$\sigma = i\Big(\frac{m}{b} + nb\Big)$$

Subtracting the null vectors in the representation leads to the ZZ (Zamolodchikov and Zamolodchikov) branes

$$|m, n\rangle = |\sigma(m, n)\rangle - |\sigma(m, -n)\rangle$$

Same

$$\mu_B = (-1)^m \cos \pi\, n\, b^2$$

at $\sigma(m, \pm n)$ (Martinec).

Finite, discrete spectrum of open strings between $|m,n\rangle$ and $|m',n'\rangle$ ZZ branes: $(m,n)\otimes(m',n')$.

These branes are localized in the strong coupling region $\phi\rightarrow+\infty$.

Branes in Minimal String Theory

FZZT branes: Tensor a Liouville brane labelled by σ and a matter brane labelled by r, s

Since $b^2 = \frac{p}{q}$ is rational, and there is a limited set of operators...

- Can restrict to $r = s = 1$

- Distinct branes are labelled by $z = \cosh \frac{\pi\sigma}{\sqrt{pq}}$

ZZ branes: Tensor a Liouville brane labelled by (m, n) and the $r = s = 1$ matter brane

Simplification: the independent ZZ branes are

$$1 \leq m < p \ , \qquad 1 \leq n < q \ , \qquad np < mq$$

Eigenstates of the ring elements

$$X|m,n\rangle = (-1)^m \cos\frac{\pi p\, n}{q}|m,n\rangle$$
$$Y|m,n\rangle = (-1)^n \cos\frac{\pi q\, m}{p}|m,n\rangle$$

$\Rightarrow$ a simple derivation of the ring relations.

Geometric Interpretation

The disk amplitude $Z(\mu_B)$ is not a single valued function of

$$x \equiv \mu_B = \cosh \pi b\, \sigma \ , \qquad b^2 = \frac{p}{q}$$

Instead, x and

$$y \equiv \partial_{\mu_B} Z(\mu_B) = \cosh \frac{\pi \sigma}{b}$$

satisfy

$$T_p(y) = T_q(x)$$

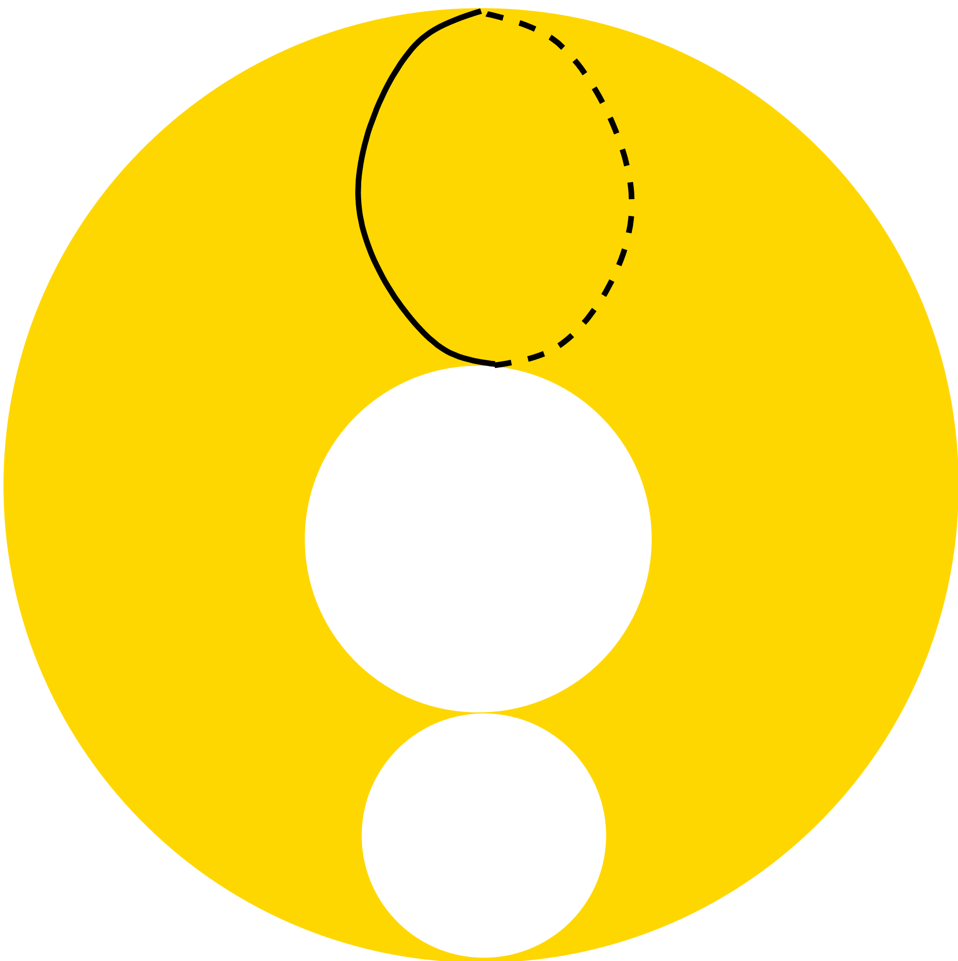
$$T_p(y = \cos\theta) = \cos p\,\theta$$

Another perspective on the relation in the tachyon module.

This is a genus $\frac{(p-1)(q-1)}{2}$ Riemann surface with $\frac{(p-1)(q-1)}{2}$ pinched A-cycles

σ gives an infinite cover.

$z = \cosh \frac{\pi\sigma}{\sqrt{pq}}$ gives a single cover



Line integrals of $y\,dx$ lead to branes:

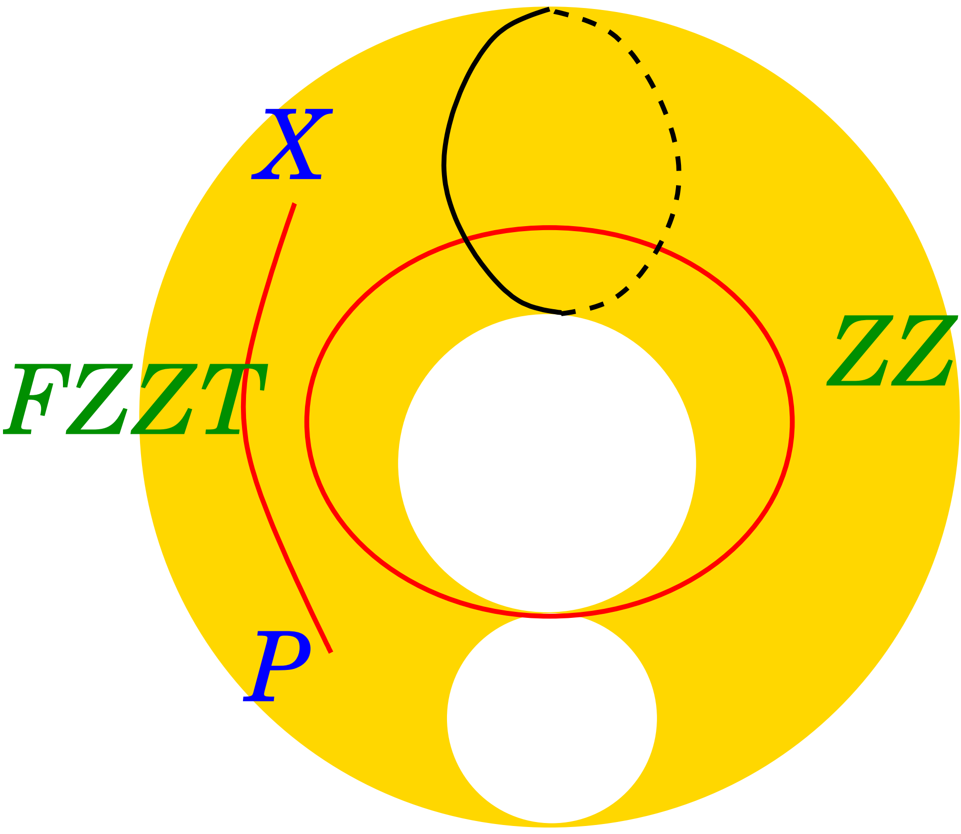
An FZZT brane is an open line integral

$$Z(x) = \int_P^x y\,dx'$$

A ZZ brane is a difference between two FZZT branes, and hence it is an integral along a closed contour. We will show that it passes through a singularity; it is an integral along a B-cycle

$$Z(m,n) = \oint_{B_{m,n}} y\,dx$$

FZZT and ZZ branes on the Riemann surface:



$(x_{m,n}, y_{m,n})$ at the singularities are the eigenvalues of the ring generators X and Y. Recall, the ZZ branes are eigenstates.

More explicitly, at the singularities we have

$$T_p(y) = T_q(x)$$
$$T'_p(y) = p\,U_{p-1}(y) = 0$$
$$T'_q(x) = q\,U_{q-1}(x) = 0$$

These are the ring relations and the relation in the tachyon module!

With $N \sim \mathcal{O}(1/g_s)$ ZZ branes of type (m,n) the corresponding pinched cycle opens up:

$$\oint_{A_{m,n}} y dx = g_s N$$

This is conjugate to $\oint_{B_{m,n}} y\, dx$ which creates the ZZ brane

Matrix Model

Consider $(p = 2, q = 2l + 1)$, which corresponds to the one matrix model

Our surface is

$$2y^2 - 1 = T_q(x)$$

It has two copies of the complex x plane which are connected along a cut $(-\infty, -1)$ and l singularities (pinched cycles)

$$\left(x_n = \cos\frac{2\pi n}{q}\ ,\ y_n = 0\right)\ , \qquad n = 1, ..., l$$

Interpretation:

y is the singular part of the resolvent

Discontinuity along the cut is the eigenvalue density $\rho(x) = \mathrm{Im}\sqrt{2 + 2\,T_q(x)}$

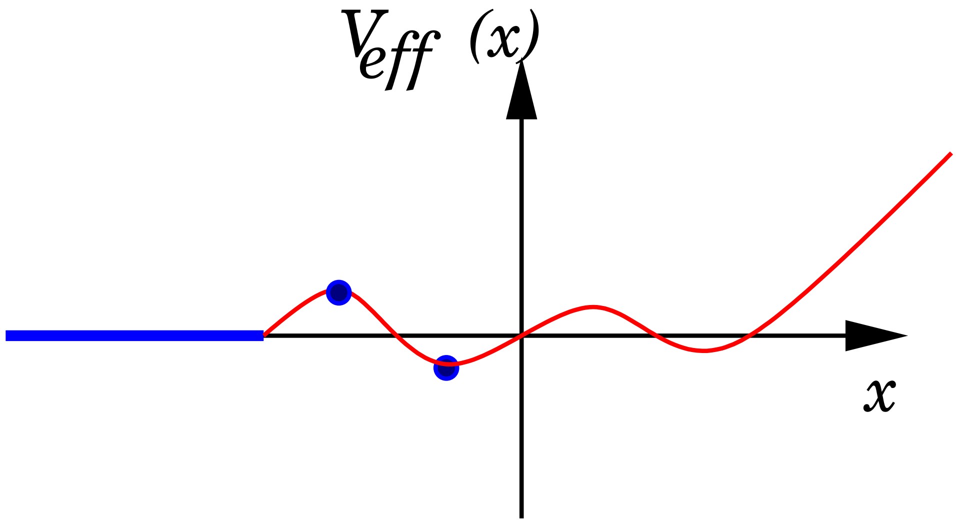
y is the force on an eigenvalue. $y = 0$ at the singularities.

The disk amplitude of FZZT brane

$$Z(x) = \int^{x} y\, dx' = V_{eff}(x)$$

is the effective potential of a probe eigenvalue.

ZZ brane: Eigenvalue at a stationary point of $V_{eff}(x)$ (where $y = 0$).



Nonperturbative instability

Conclusions

- We translated the features of the minimal string theory to simple properties of an underlying Riemann surface.

- In the simplest situation most of the A-cycles are pinched.

- Observables are deformations of the Riemann surface (did not do here).

- The ring relations control
  - the correlation functions
  - the defining equation of the surface
  - its singularities

- D-branes are contour integrals of a certain one form:
  - FZZT branes are open contours
  - ZZ branes are associated with the B-cycles.

This gives a worldsheet “derivation” of the matrix model, and adds a new perspective to the understanding that

the eigenvalues are associated with D-branes (Polchinski, McGreevy, Verlinde, Klebanov, Maldacena, N.S., Martinec...)

Extensions/Generalizations

- Minimal type 0 theory

- $c = 1$

- $\hat{c} = 1$

- Quantization

- ???