DEFECTS AND TOPOLOGY IN LIQUID CRYSTALS: *A PERSPECTIVE USING WHITEHEAD PRODUCTS*

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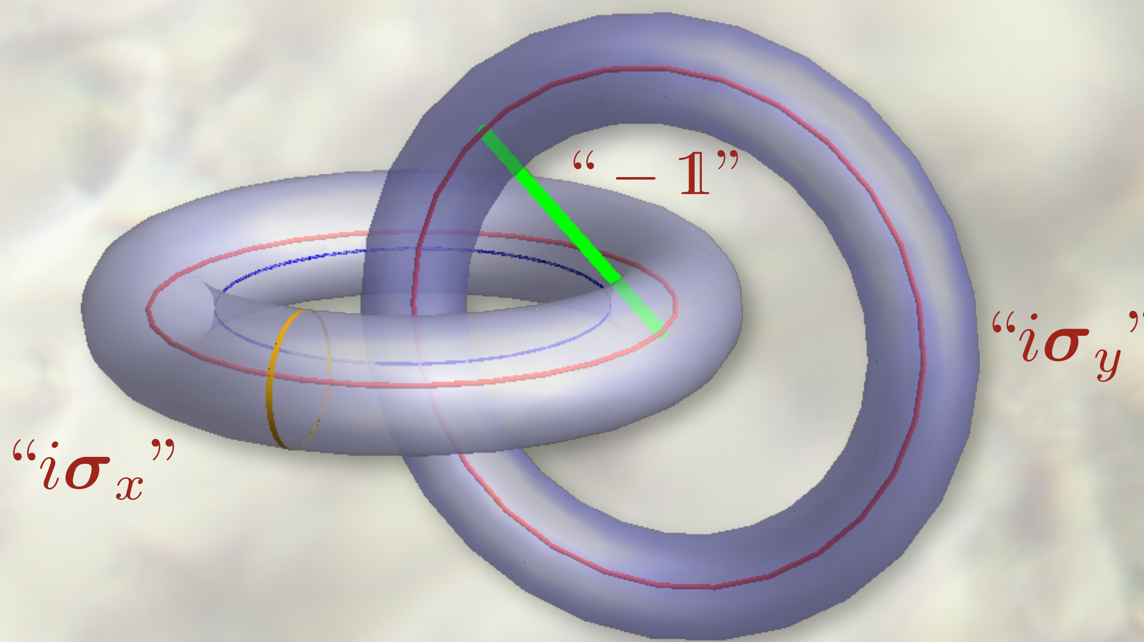

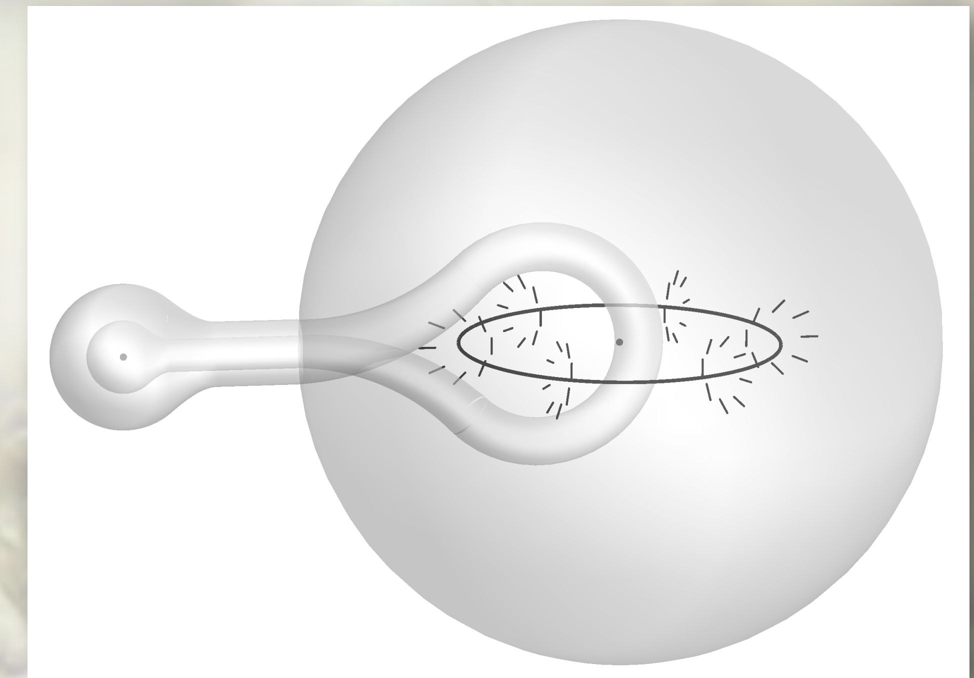

Knotted Fields, KITP, Santa Barbara
9th July 2012

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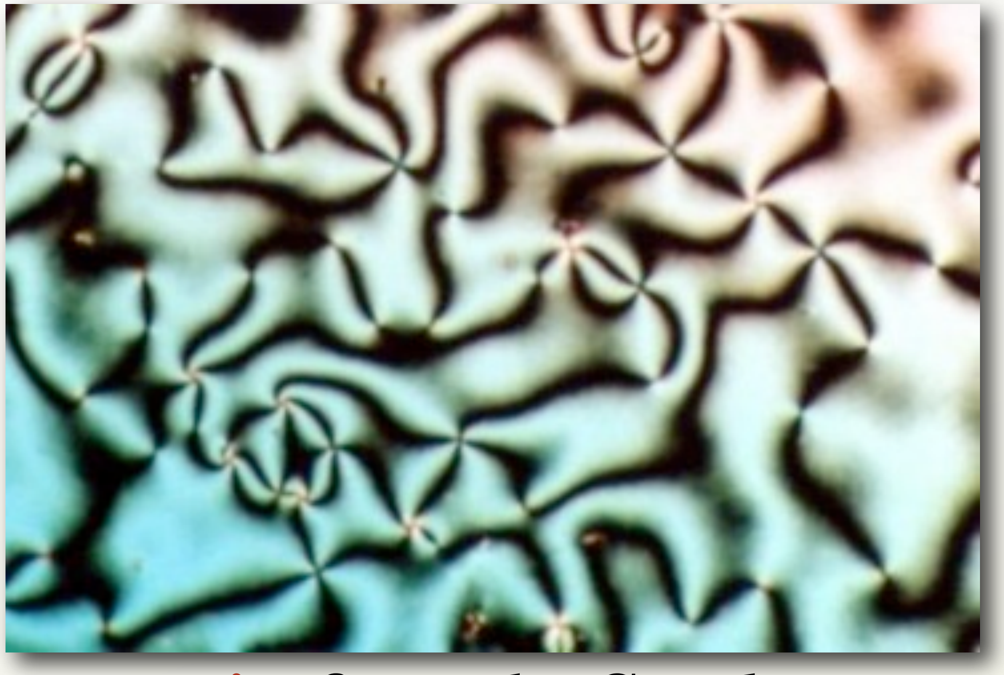
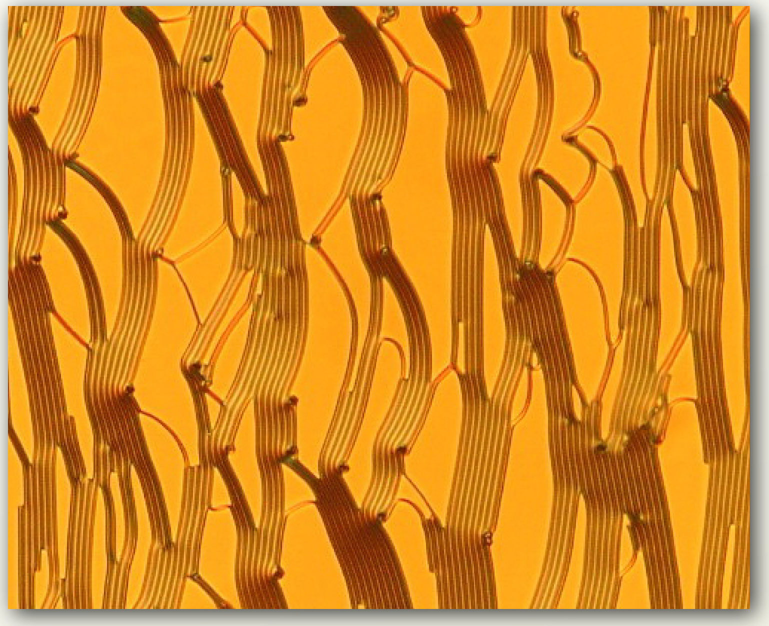
TOPOLOGY AND LIQUID CRYSTALS



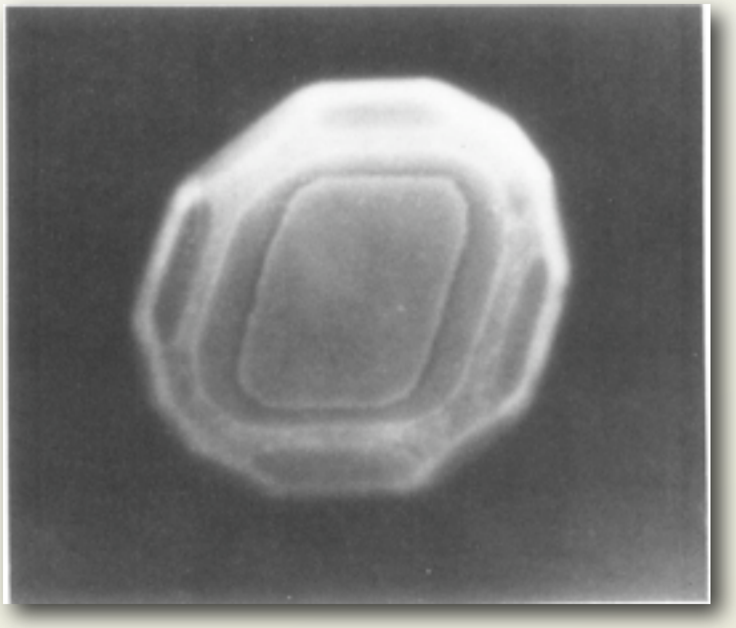
nematic: from the Greek νημα meaning 'thread-like'

Photo by Michi Nakata



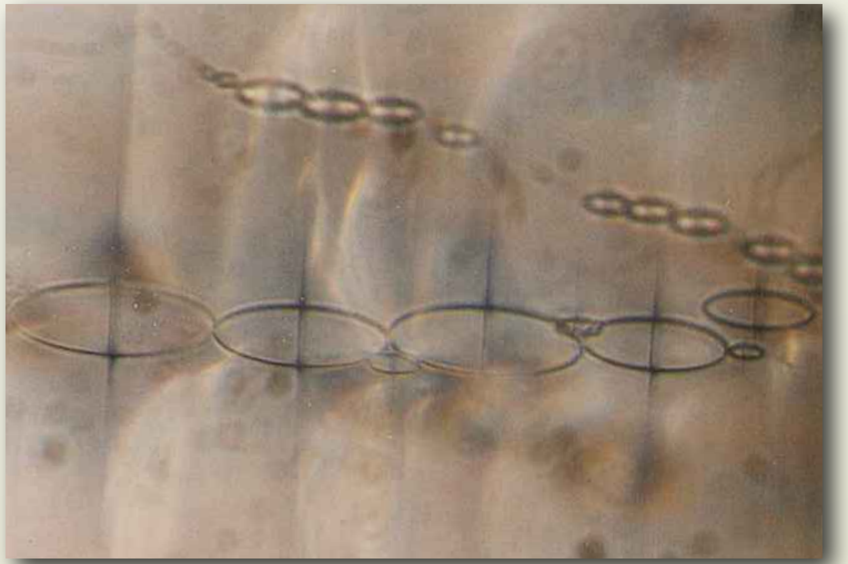
oily streak: in a cholesteric

Pieranski et al, *Phys. Rev. A* **31**, 3912 (1985)



blue phase: faceted monodomain

courtesy of Noel Clark



focal conic: characteristic texture in smectics

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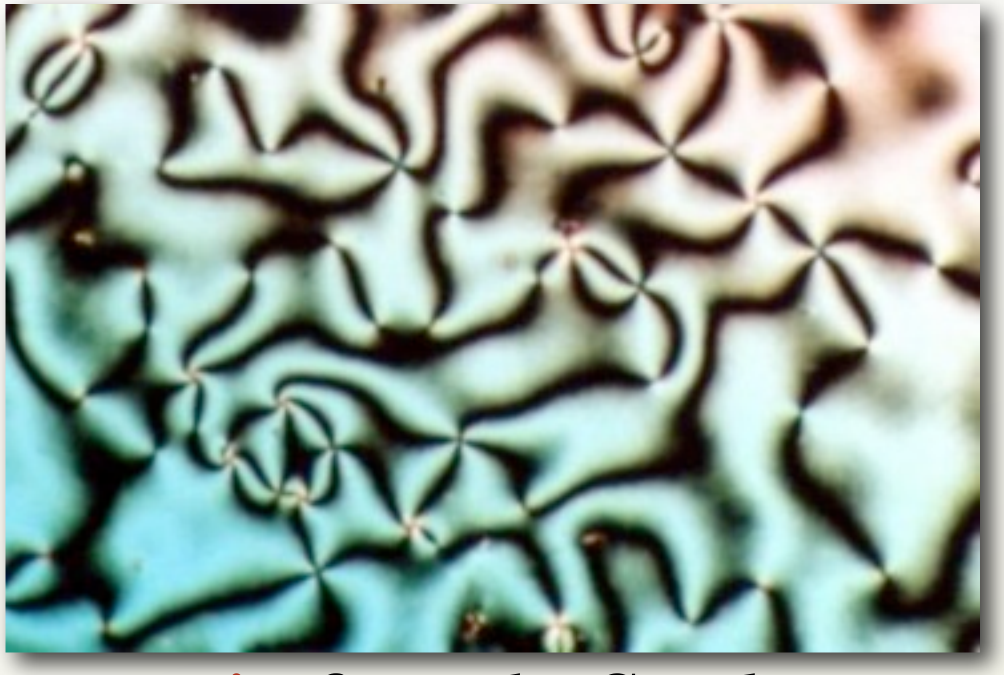

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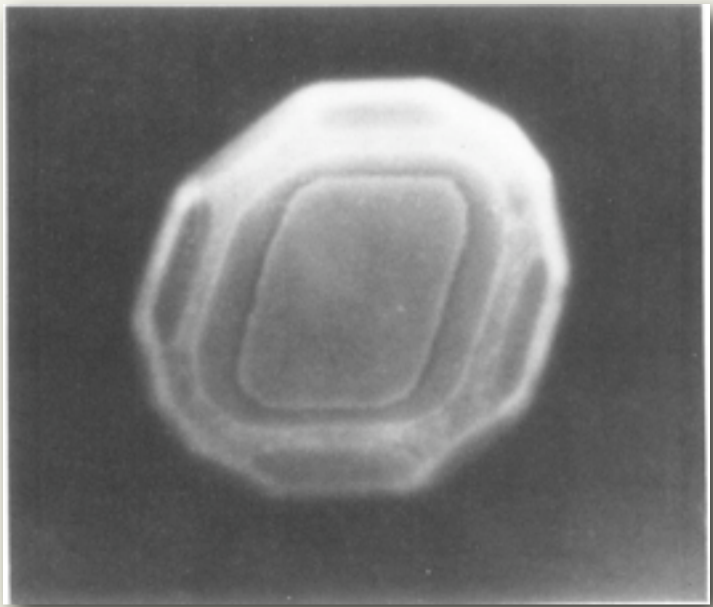

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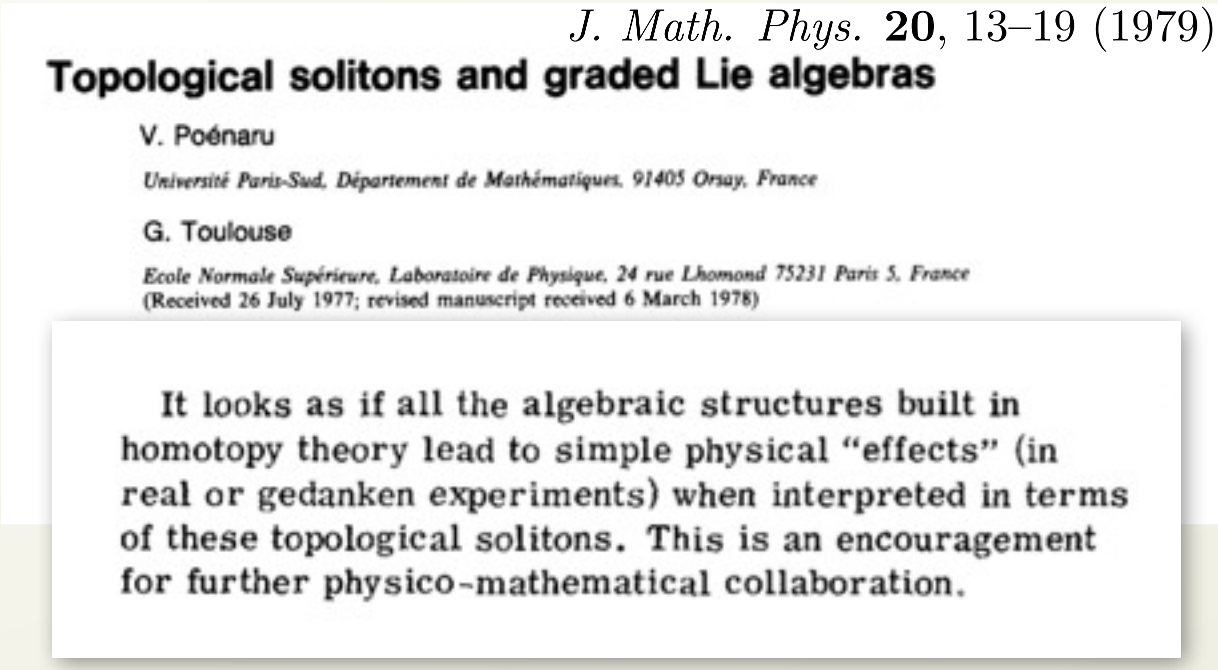

J. Math. Phys. **20**, 13–19 (1979)

Topological solitons and graded Lie algebras

V. Poénaru

Université Paris-Sud, Département de Mathématiques, 91405 Orsay, France

G. Toulouse

Ecole Normale Supérieure, Laboratoire de Physique, 24 rue Lhomond 75231 Paris 5, France

(Received 26 July 1977; revised manuscript received 6 March 1978)

It looks as if all the algebraic structures built in homotopy theory lead to simple physical "effects" (in real or gedanken experiments) when interpreted in terms of these topological solitons. This is an encouragement for further physico-mathematical collaboration.

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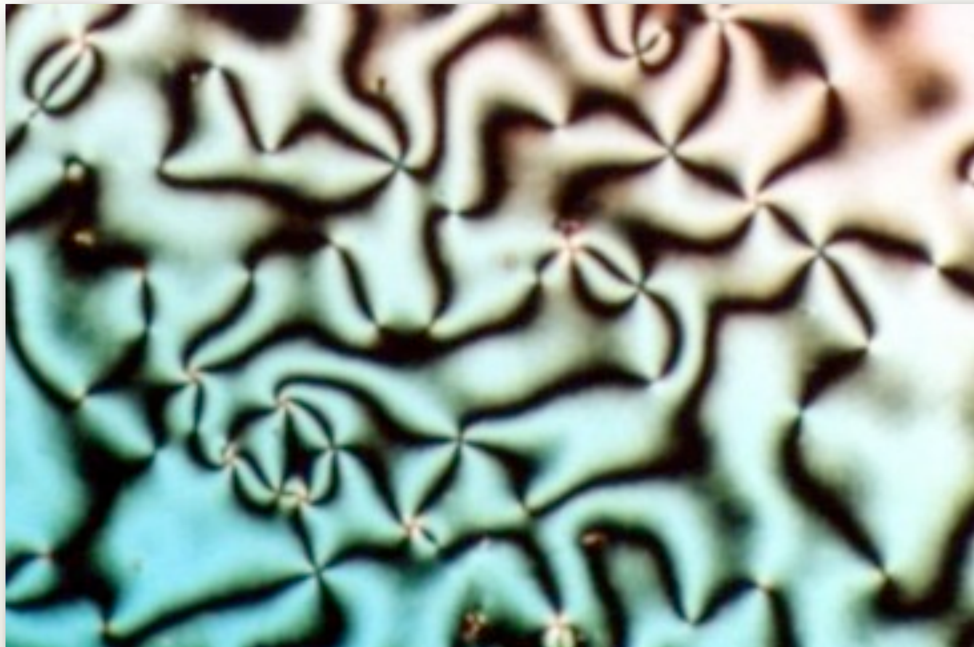
TOPOLOGY AND LIQUID CRYSTALS



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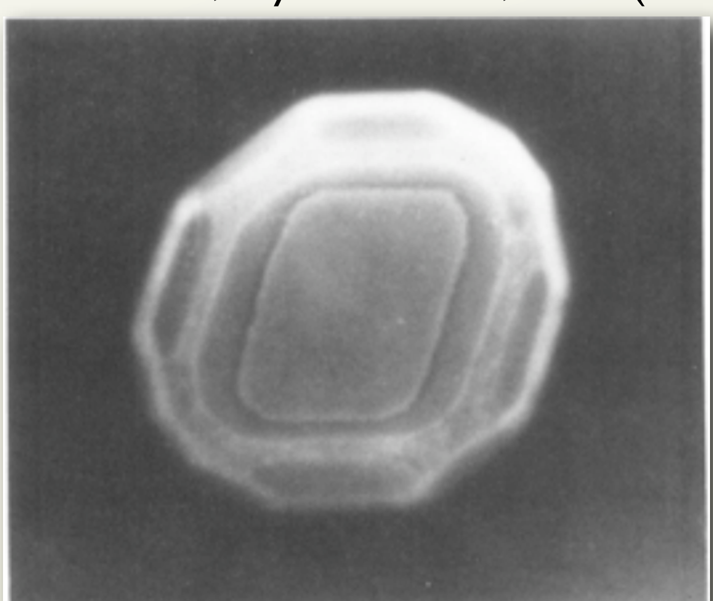
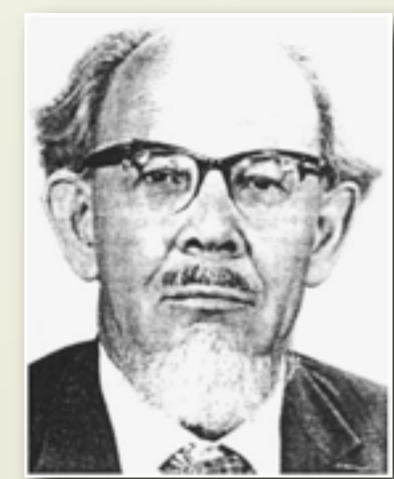


blue phase: **faceted monodomain**

"They are totally useless, I think, except for one important intellectual use, that of providing tangible examples of topological oddities, and so helping to bring topology into the public domain of science, from being the private preserve of a few abstract mathematicians and particle theorists."



Sir Charles Frank, 1983

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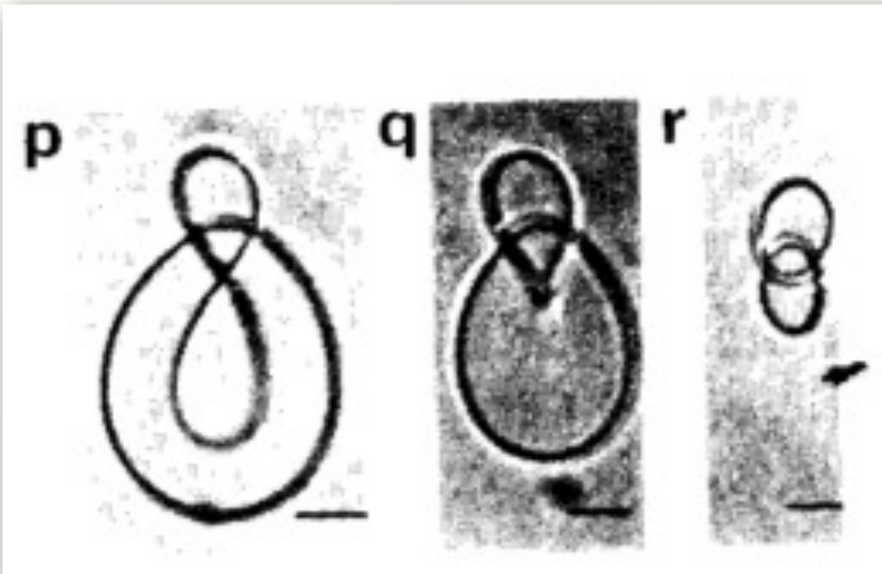
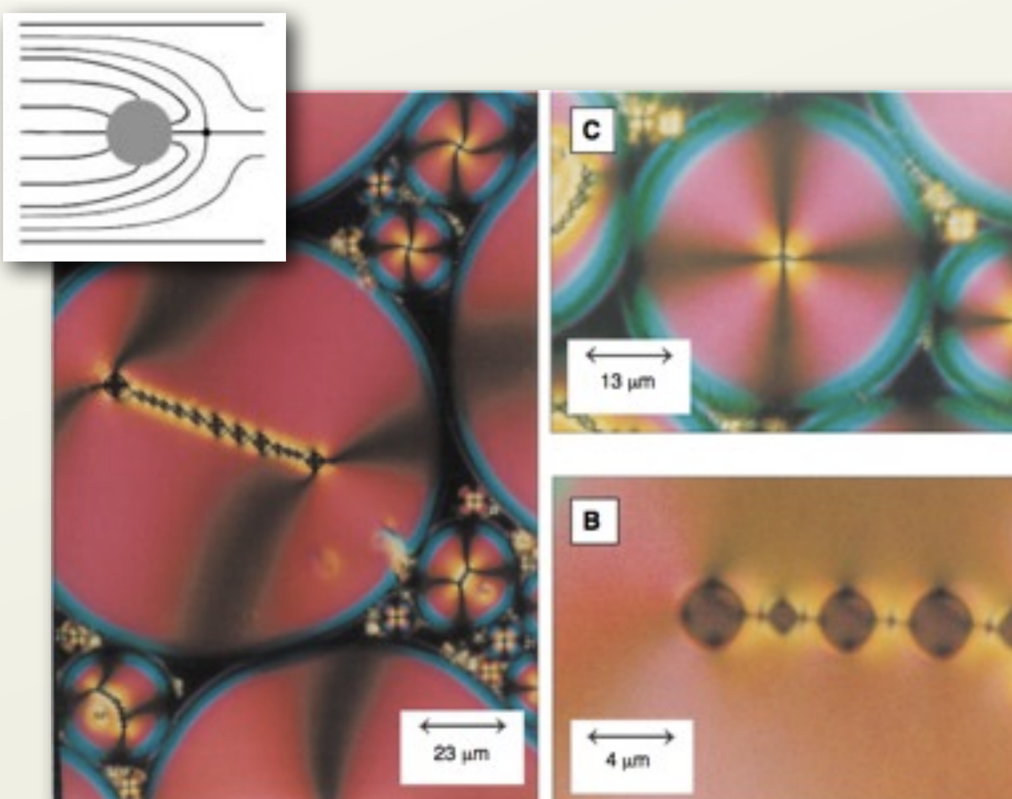
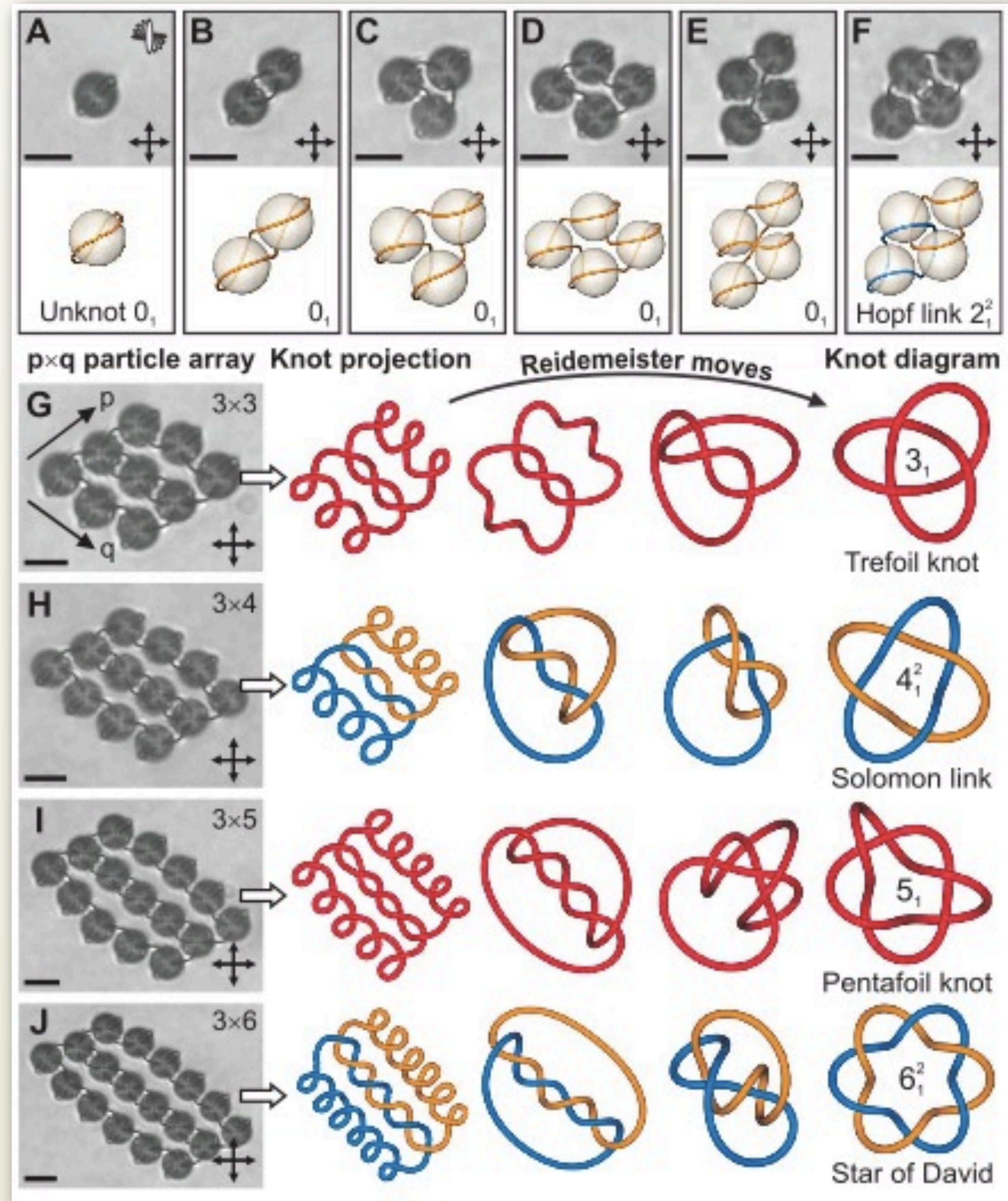
COLLOIDS IN LIQUID CRYSTALS



BOULIGAND *J. Phys. France* **35**, 959–981 (1974)

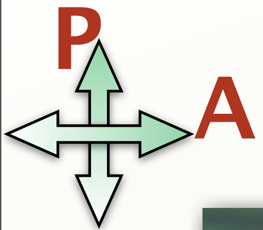


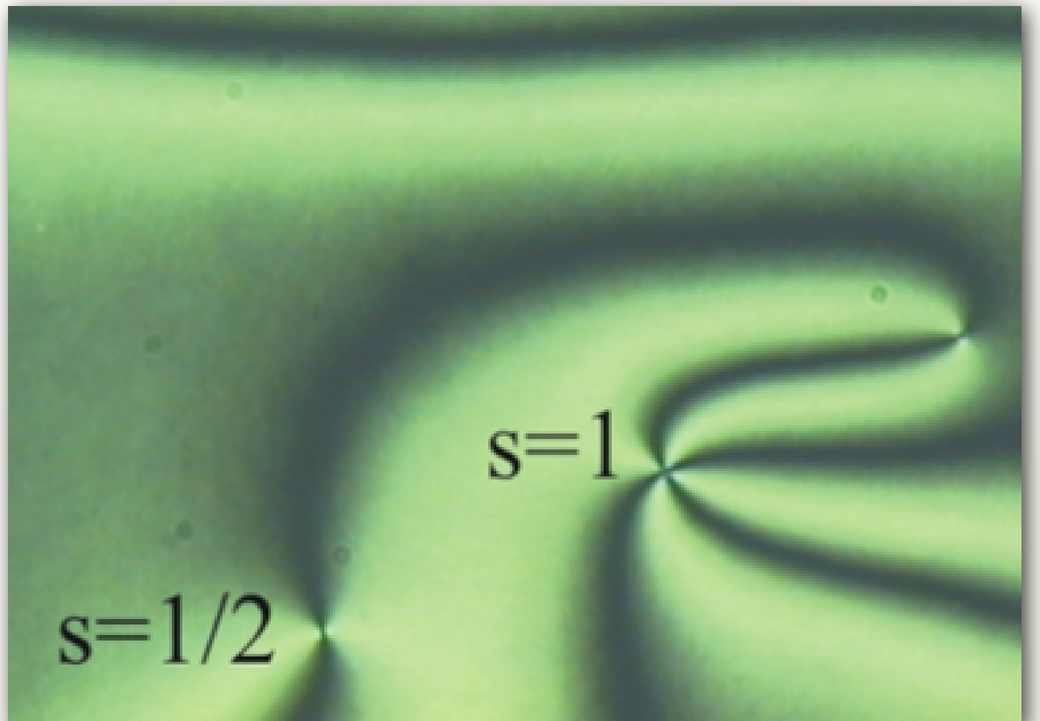
POULIN ET AL *Science* **275**, 1770–1773 (1997)



TKALEC ET AL *Science* **333**, 62–65 (2011)

ORDER AND SYMMETRY BREAKING





courtesy of Ingo Dierking

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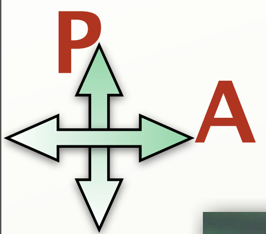

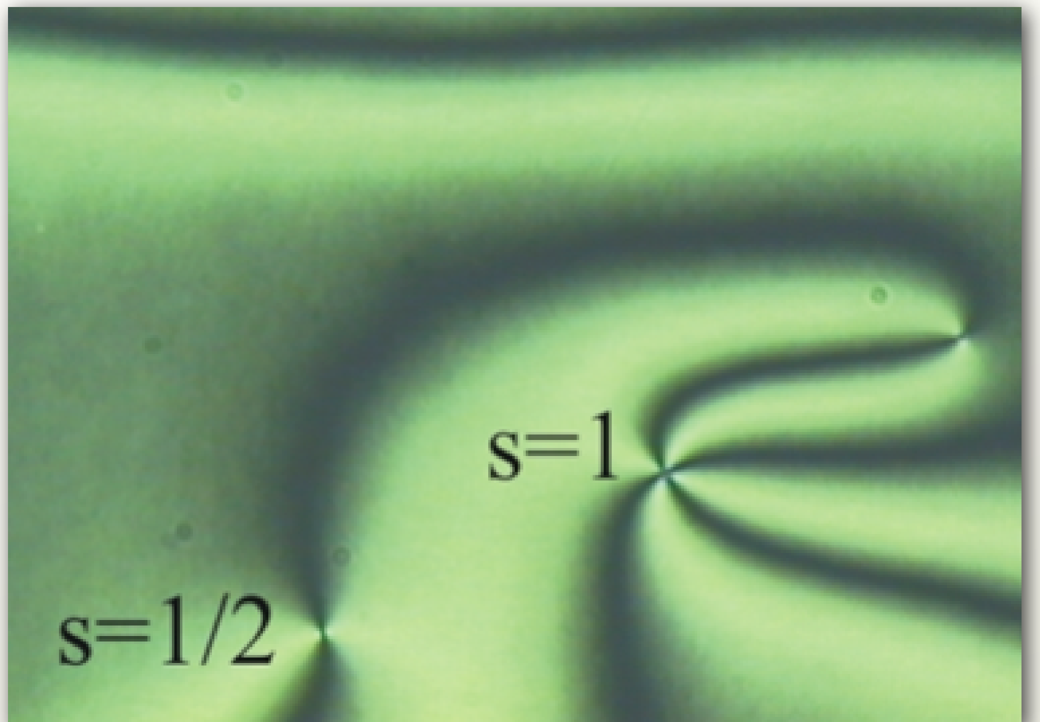

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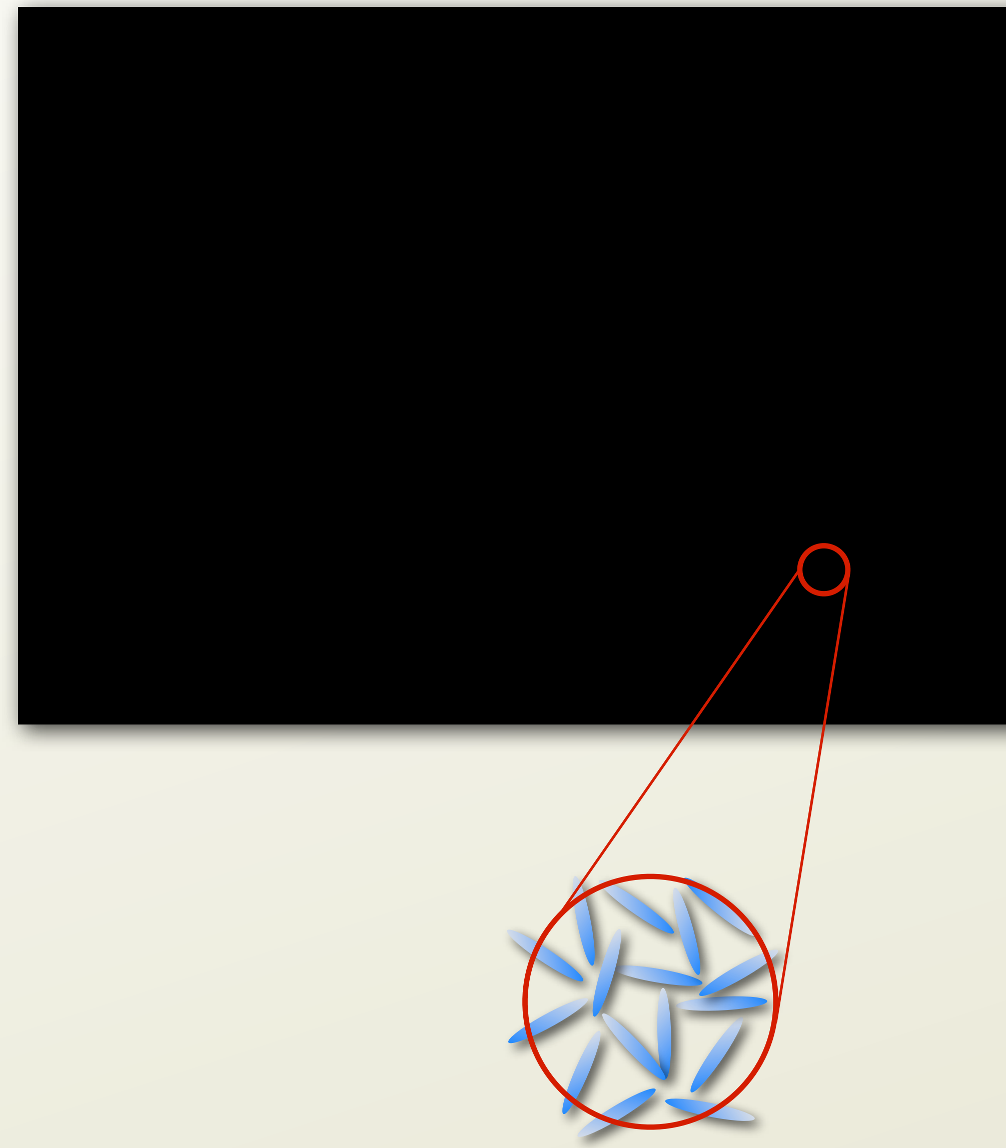

symmetry $SO(3)$

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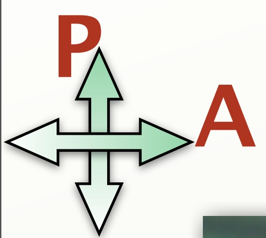

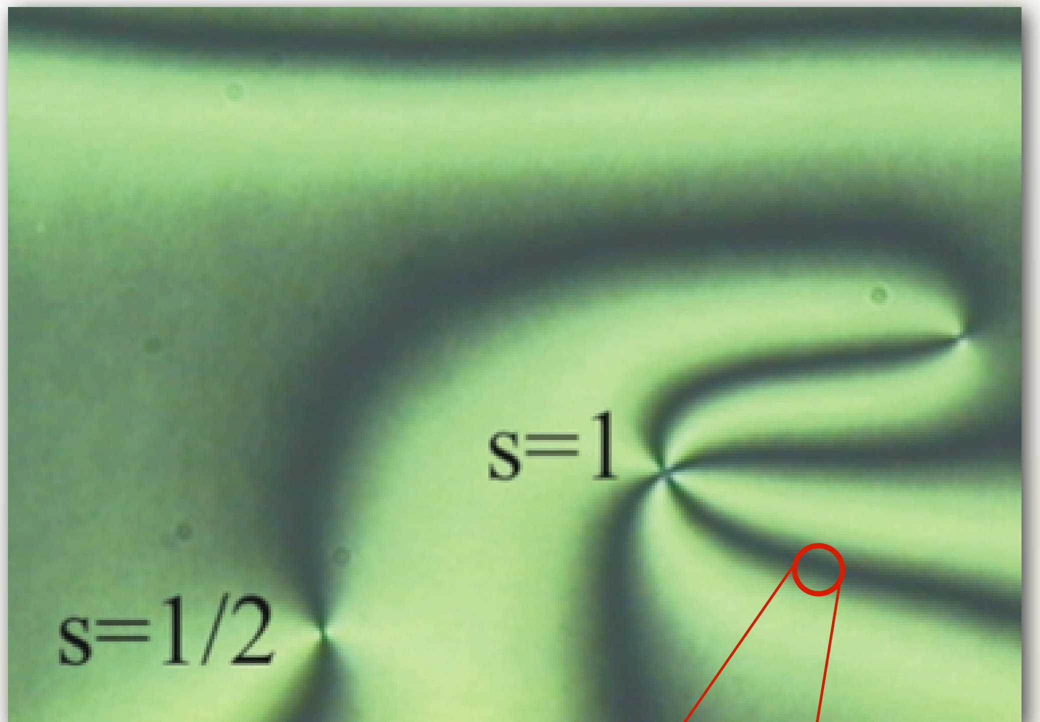

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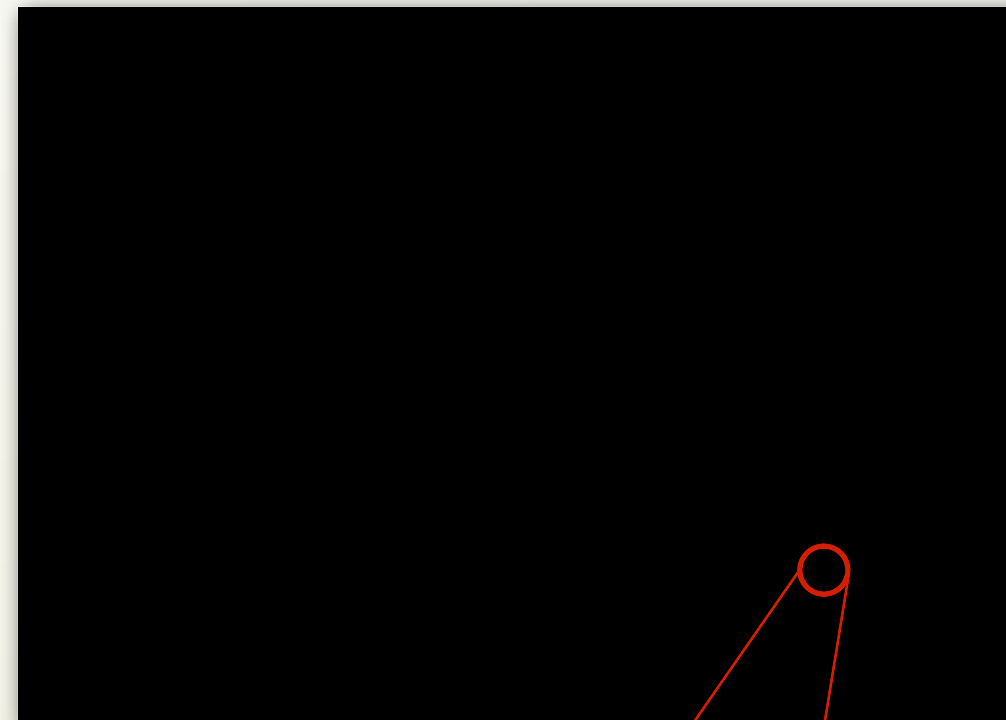
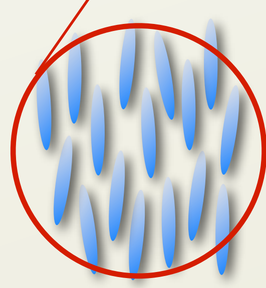
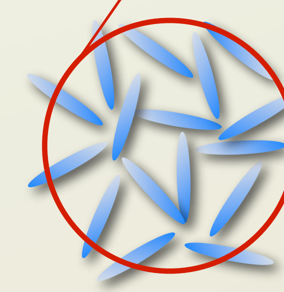




symmetry D_∞



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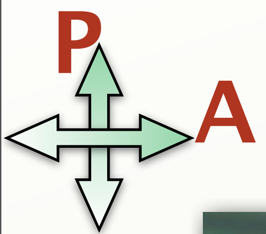

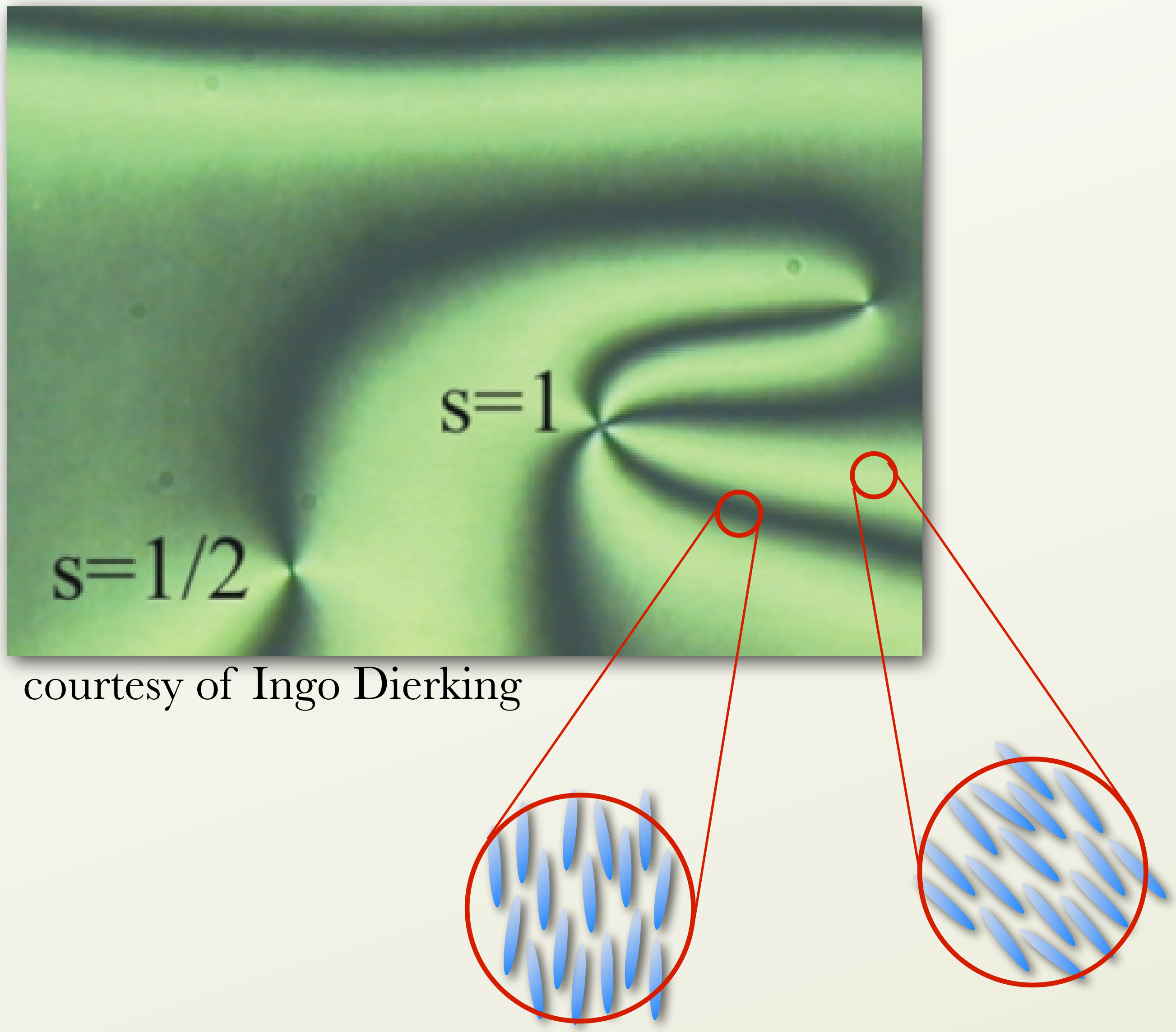

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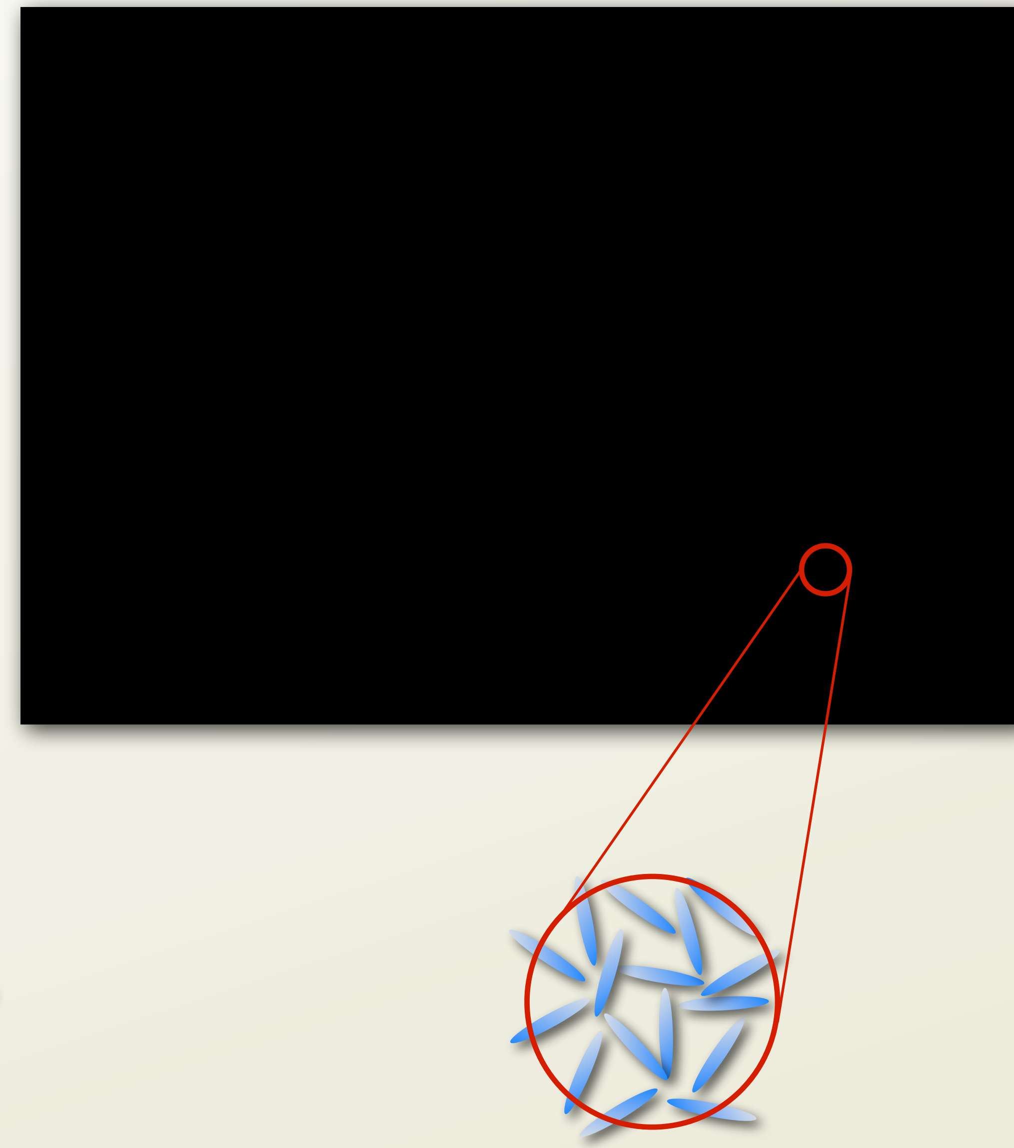

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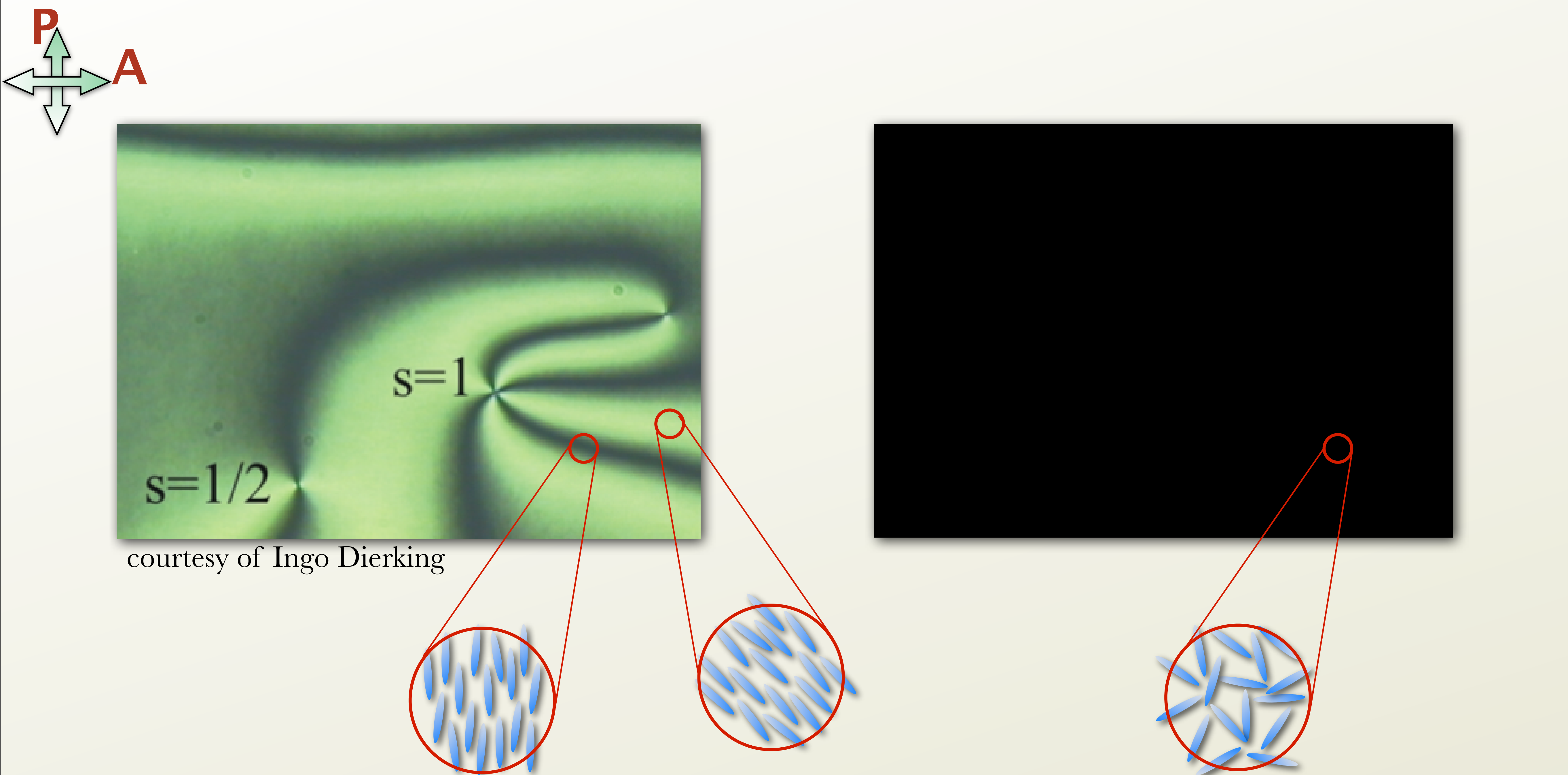

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symmetry D_∞

symmetry $SO(3)$

ground state manifold $SO(3)/D_\infty = \mathbb{RP}^2$

uniaxial nematic

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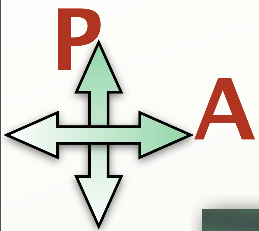

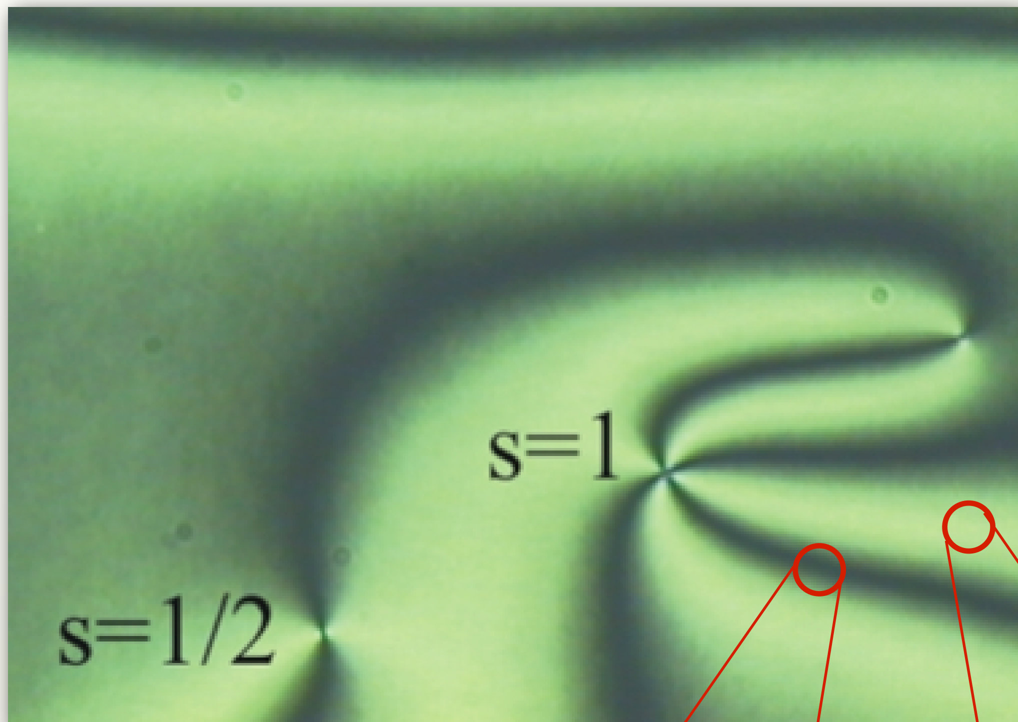

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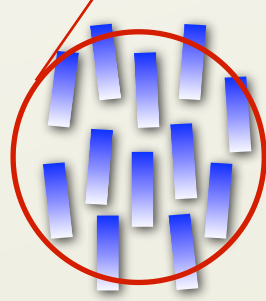

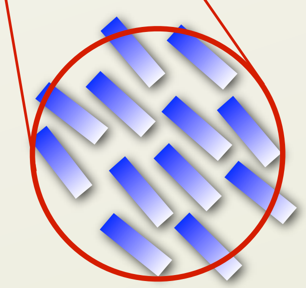

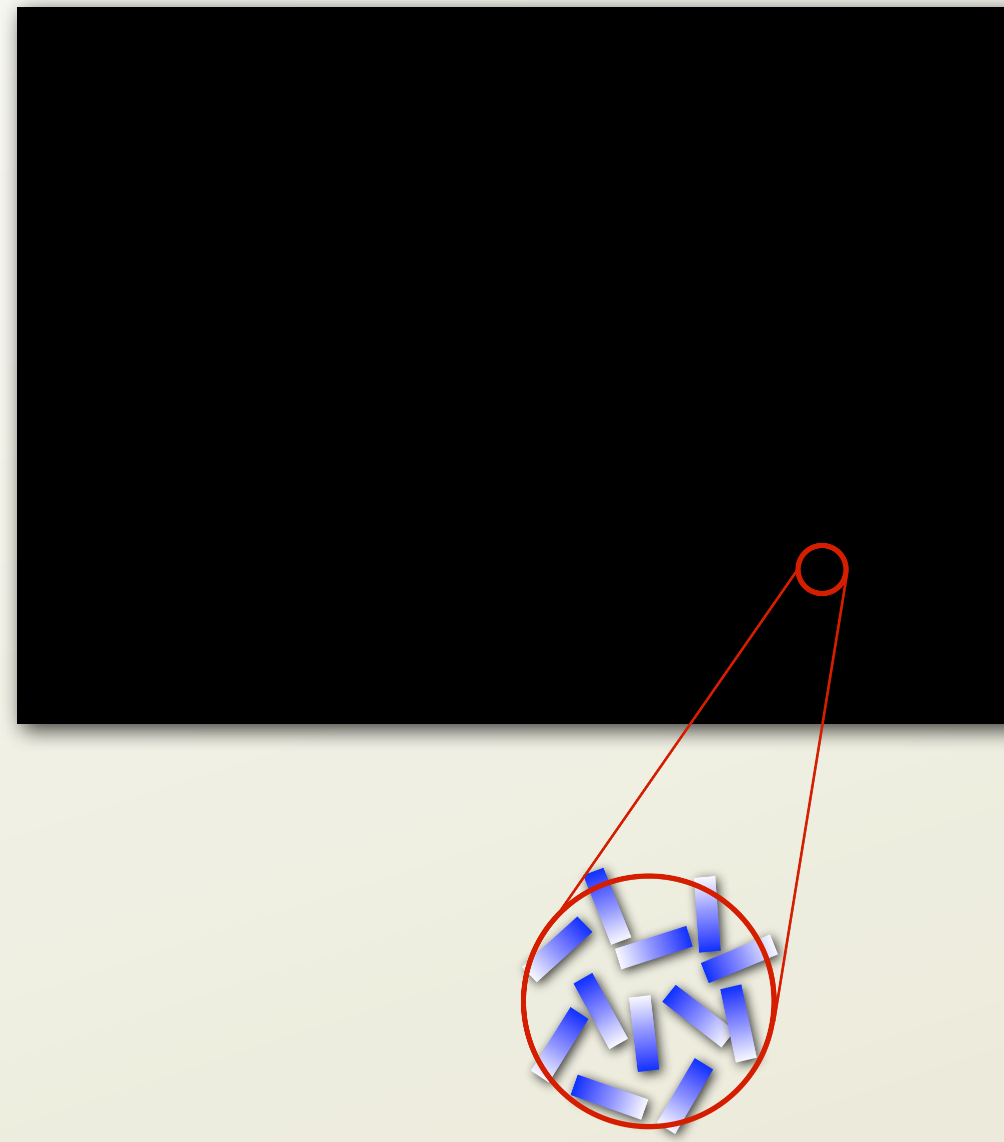

symmetry D_2

symmetry $SO(3)$

ground state manifold $SO(3)/D_2$

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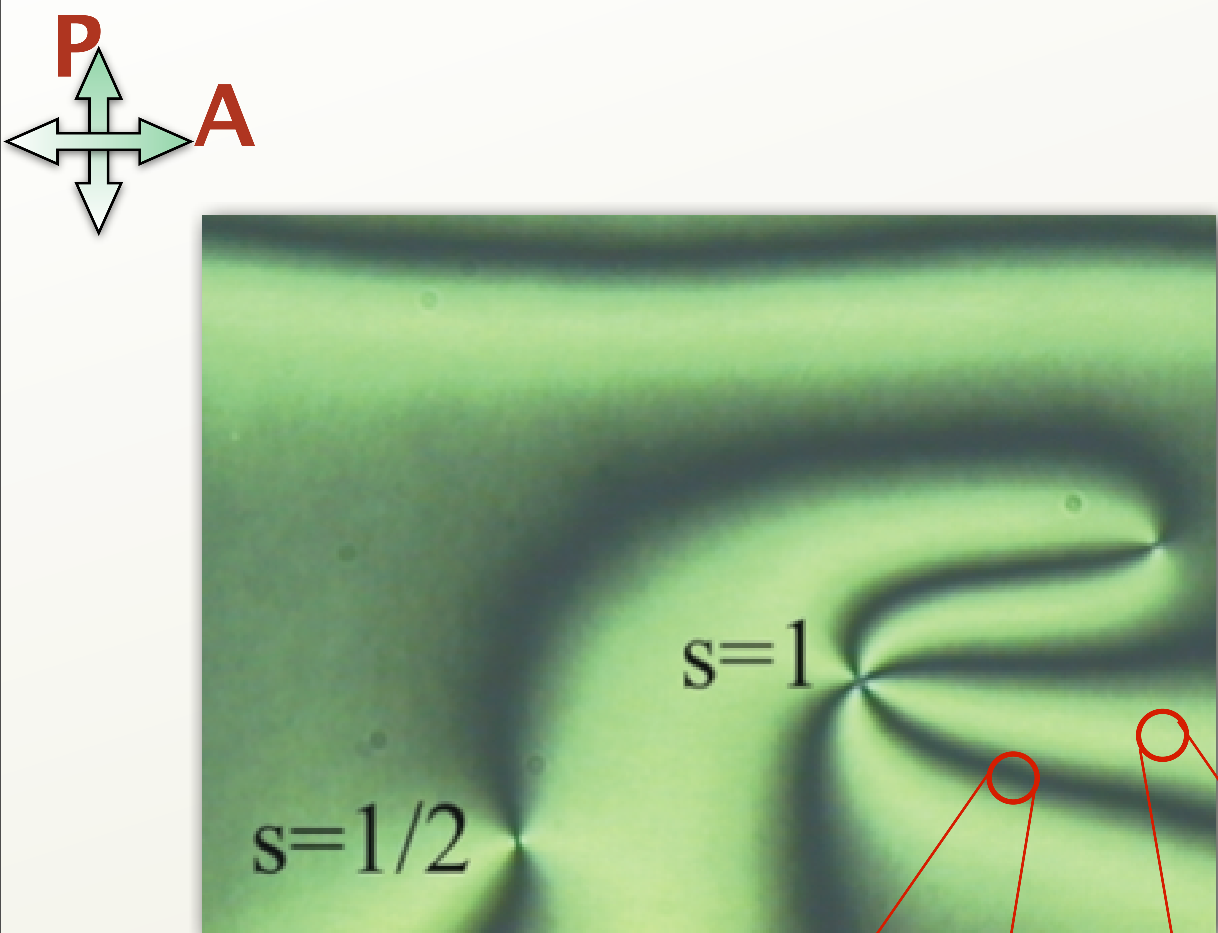

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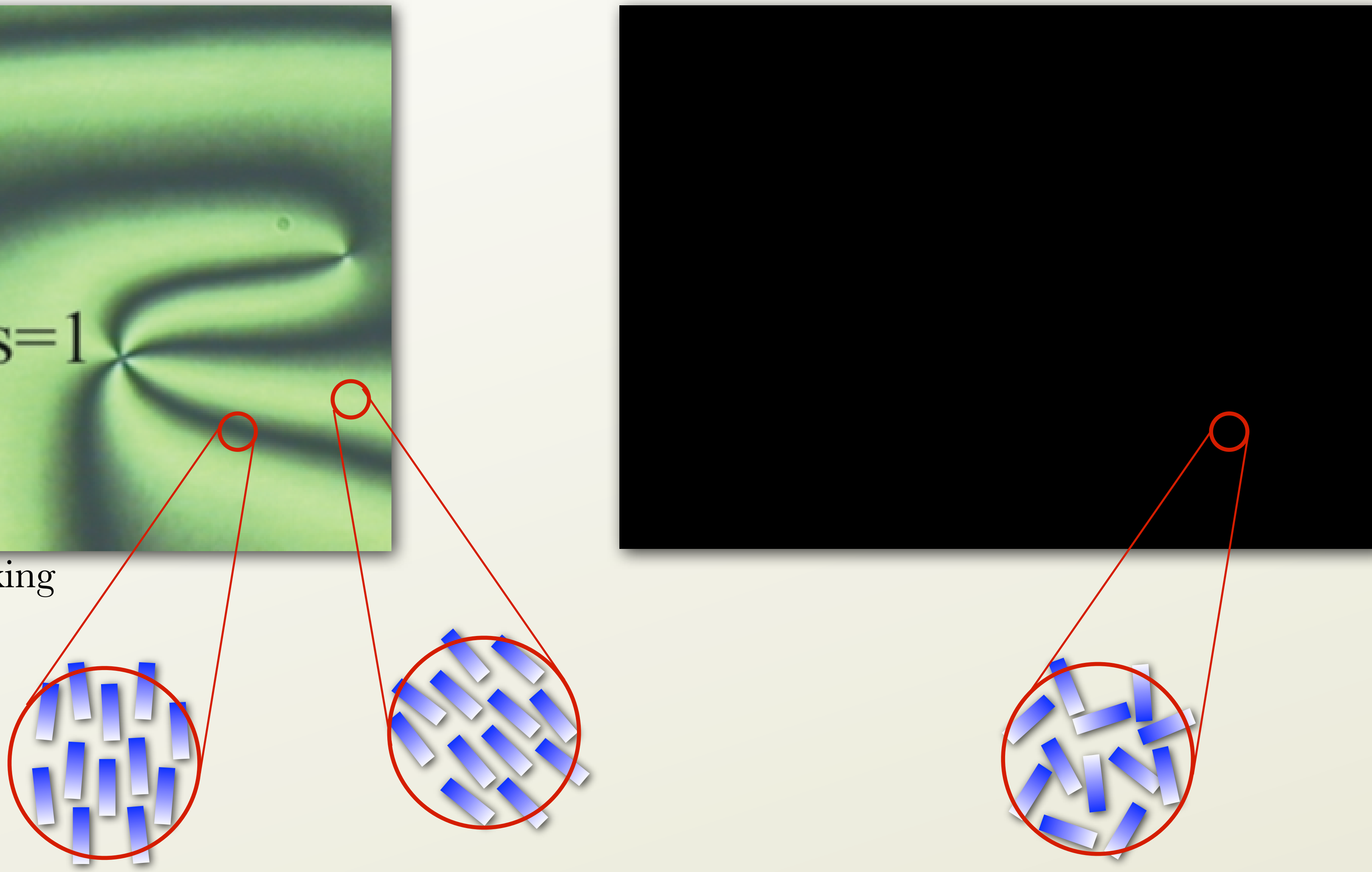

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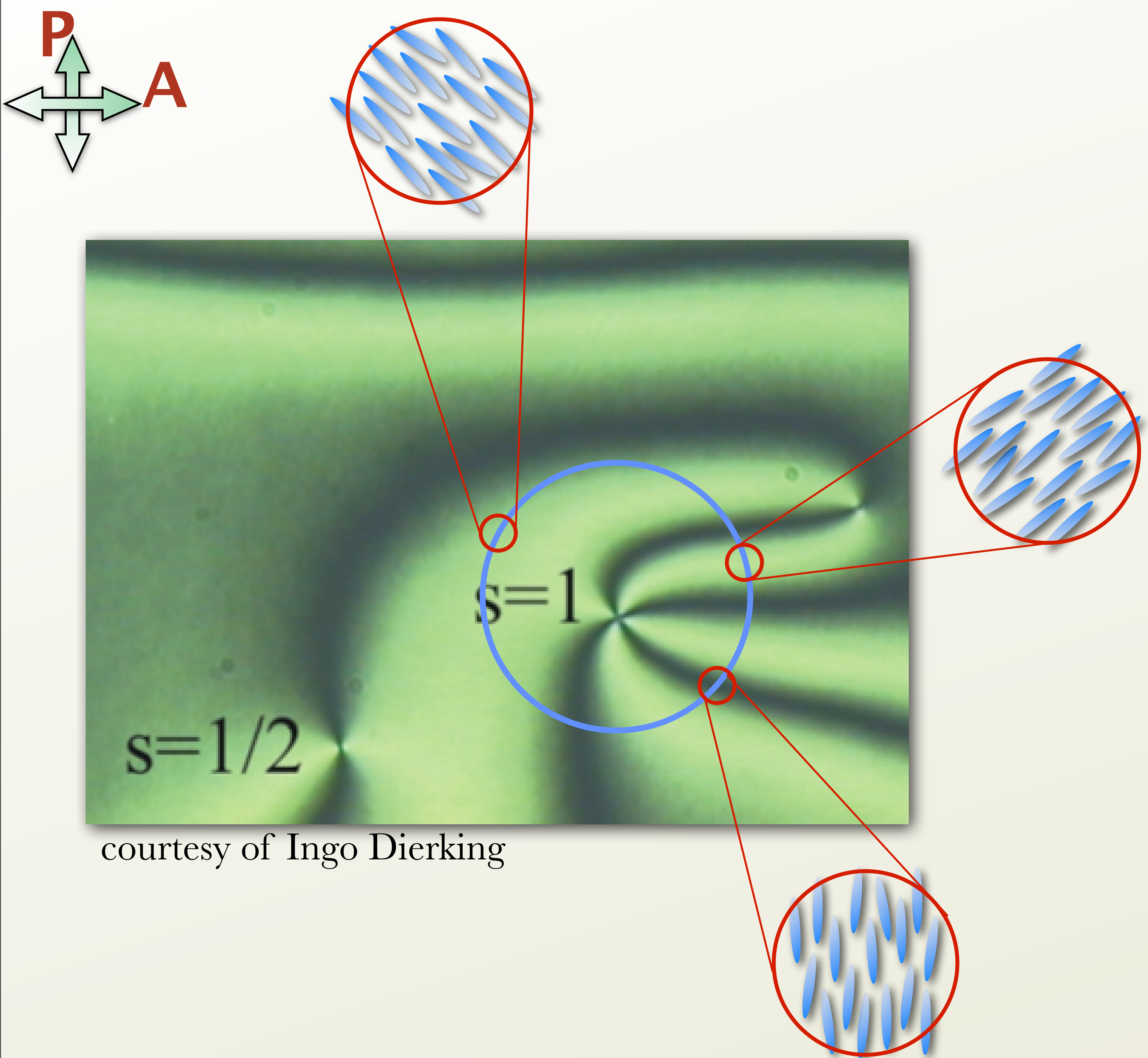
ground state manifold $SO(3)/D_2$

biaxial nematic

DISCLINATIONS: *POINT SINGULARITIES IN TWO DIMENSIONS*



courtesy of Ingo Dierking

measure the texture on some loop encircling the defect

$$\text{map}: \quad S^1 \to X$$

classify defects using homotopy groups

$\pi_1(X)$ **based**

$[S^1, X]$ **free**

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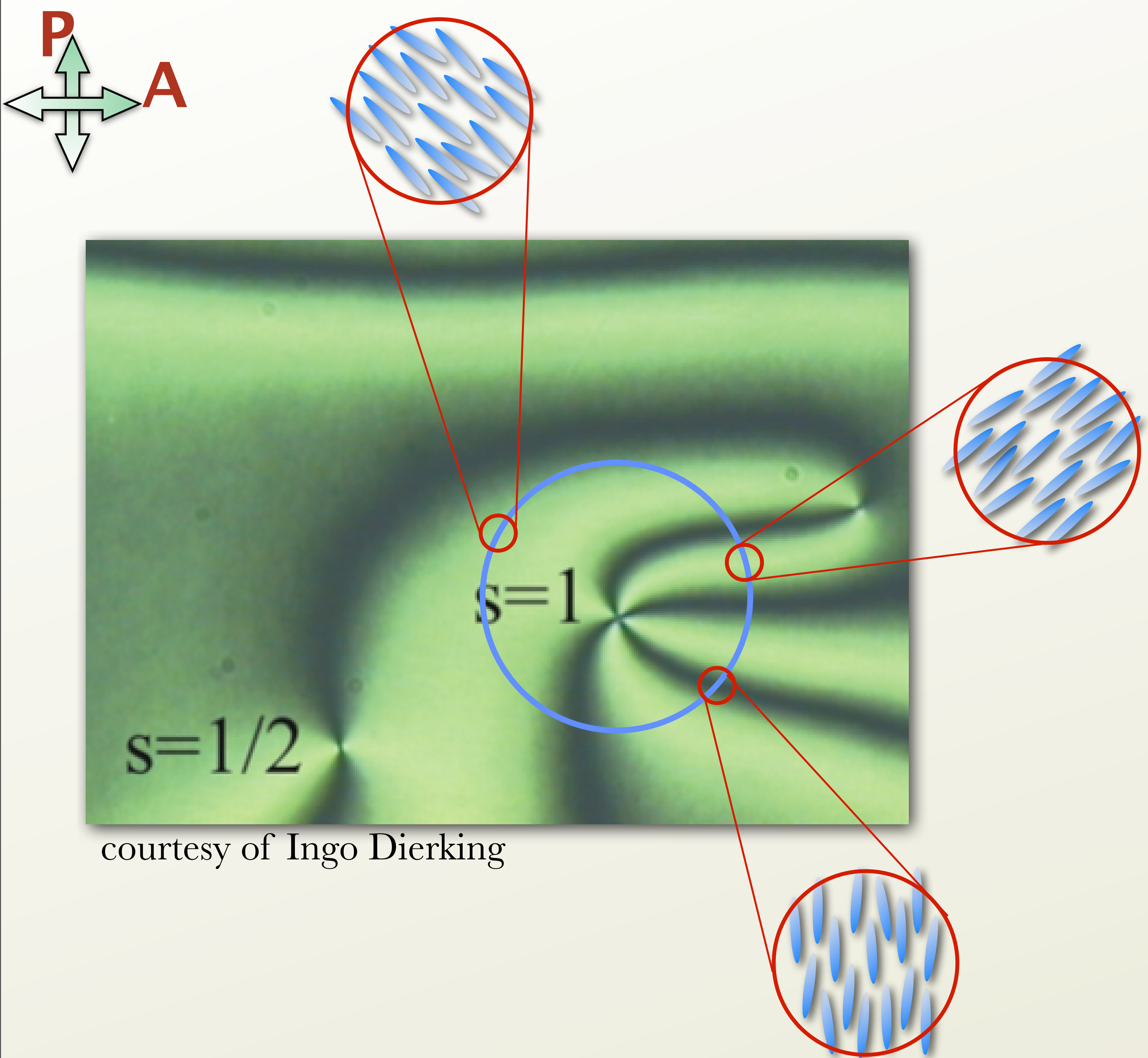

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planar nematic $\pi_1(\mathbb{RP}^1) = \frac{1}{2}\mathbb{Z}$

nematic $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

biaxial $\pi_1(SO(3)/D_2) = Q_8$

BASED AND FREE: 1/2 DISCLINATIONS

- $\pm\frac{1}{2}$ are homotopic in uniaxial nematics
- but not in biaxials

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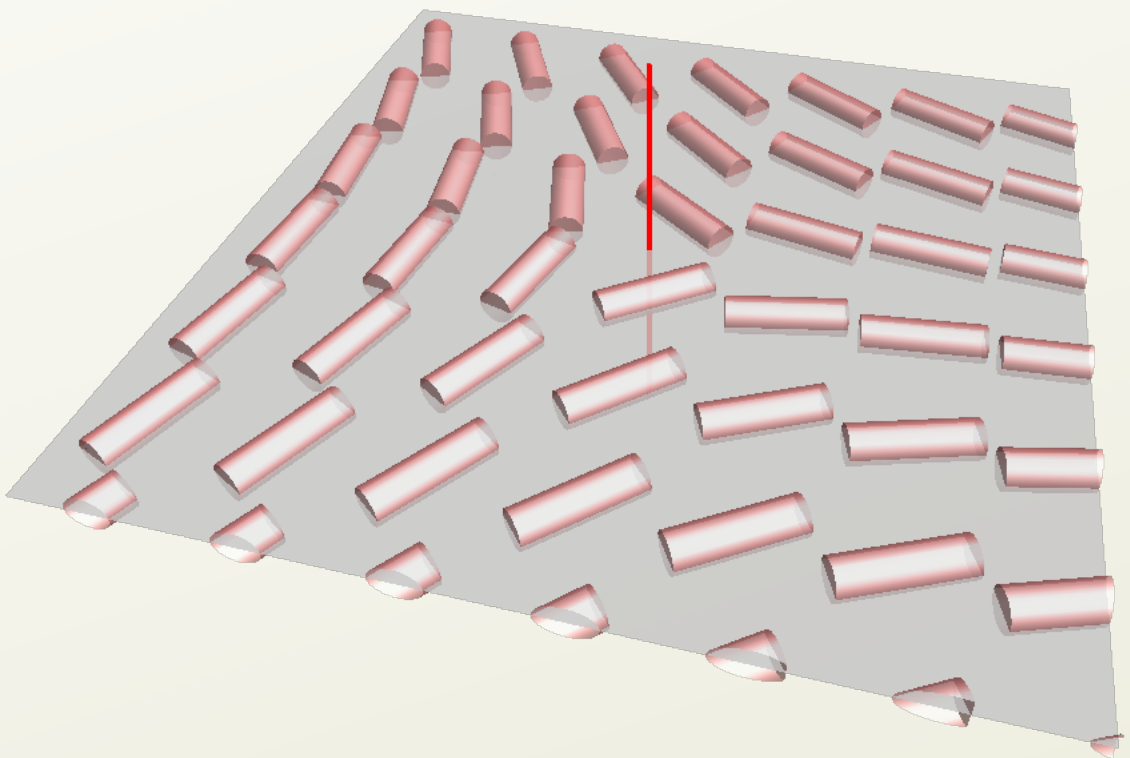

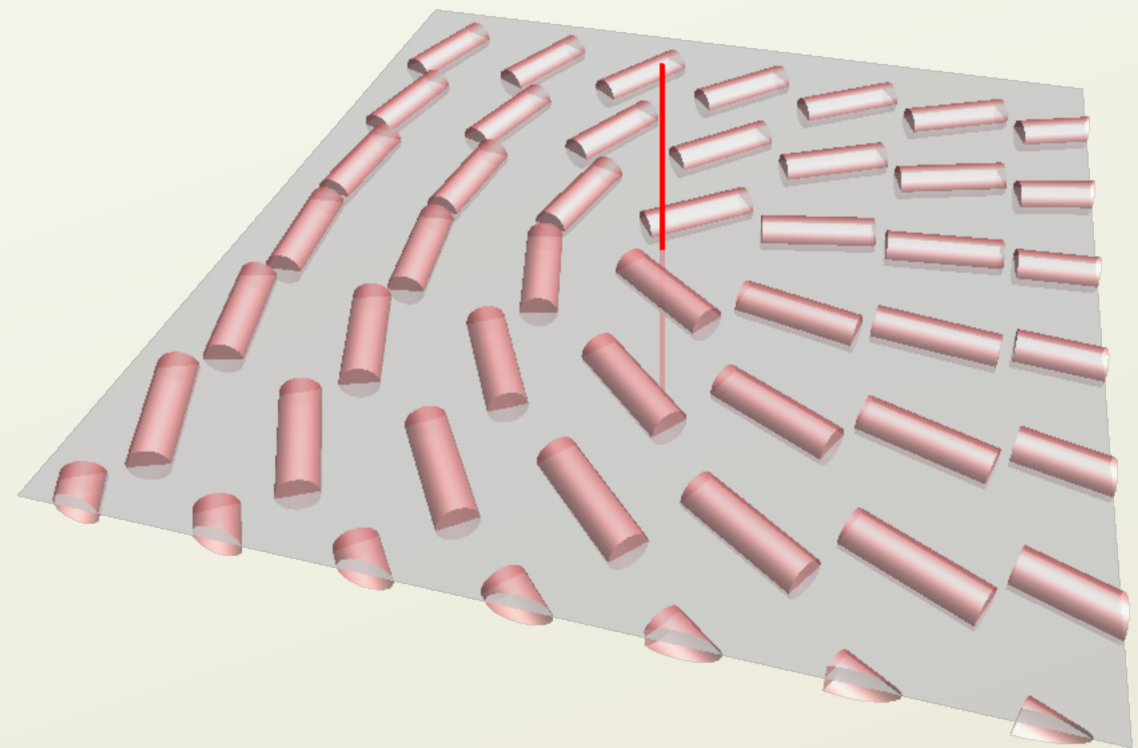

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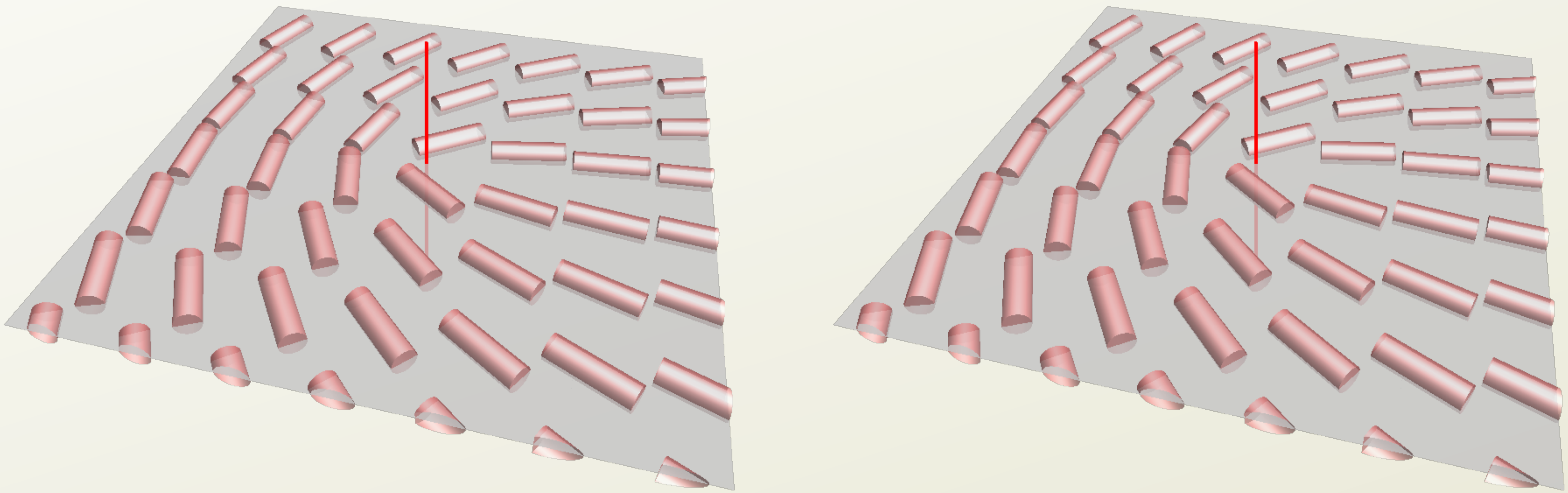

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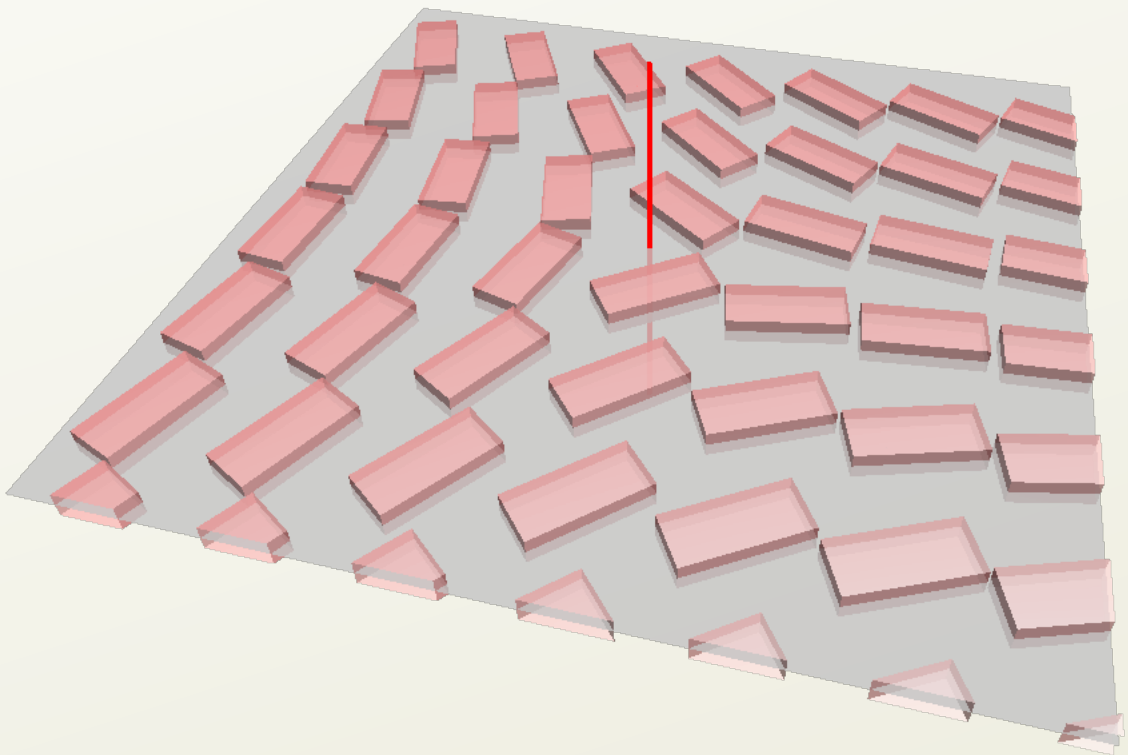

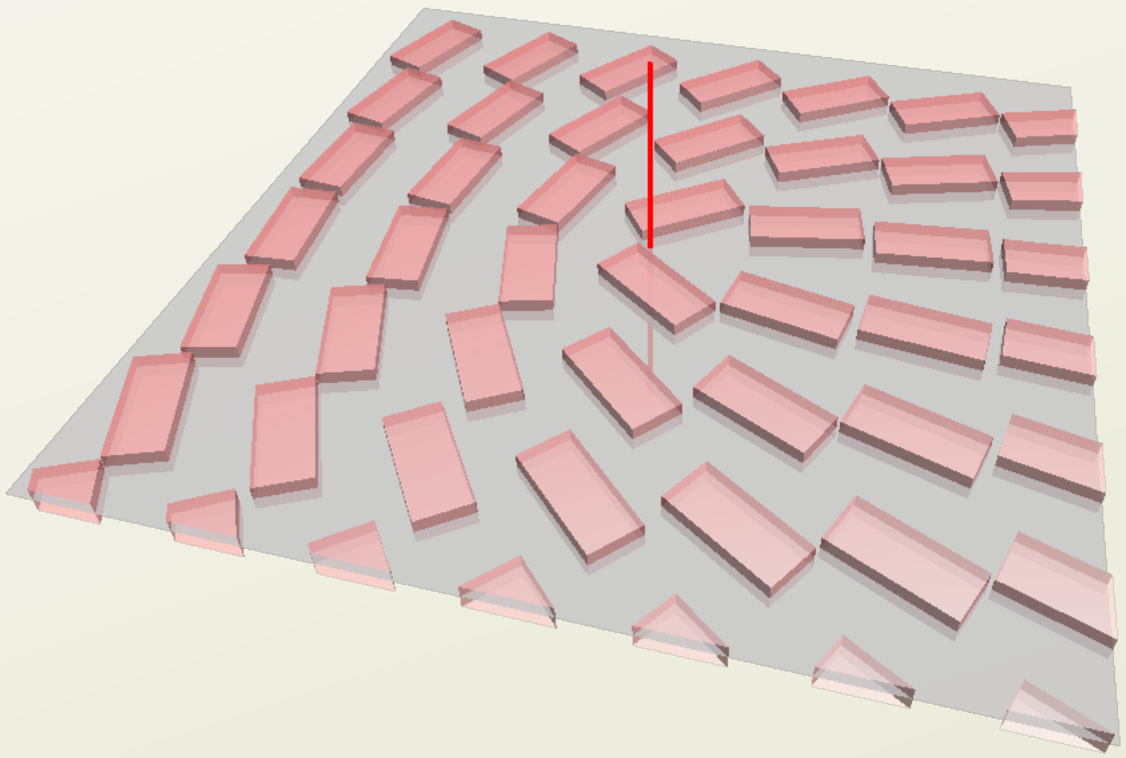

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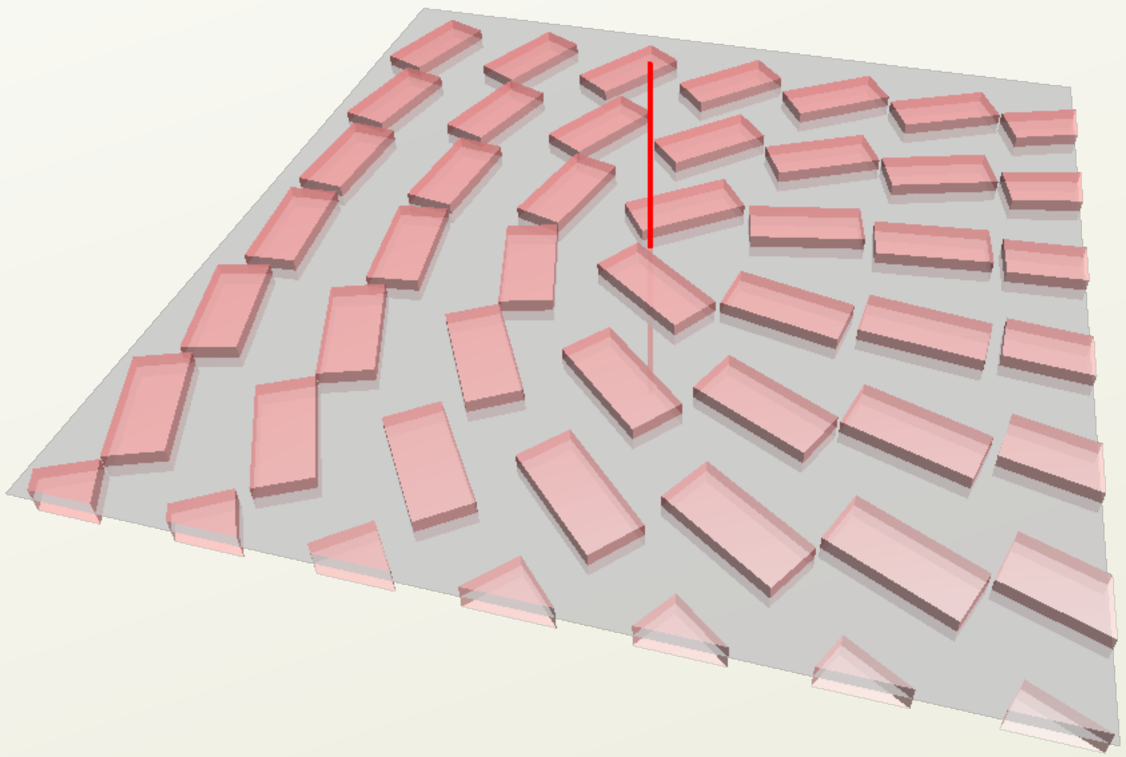

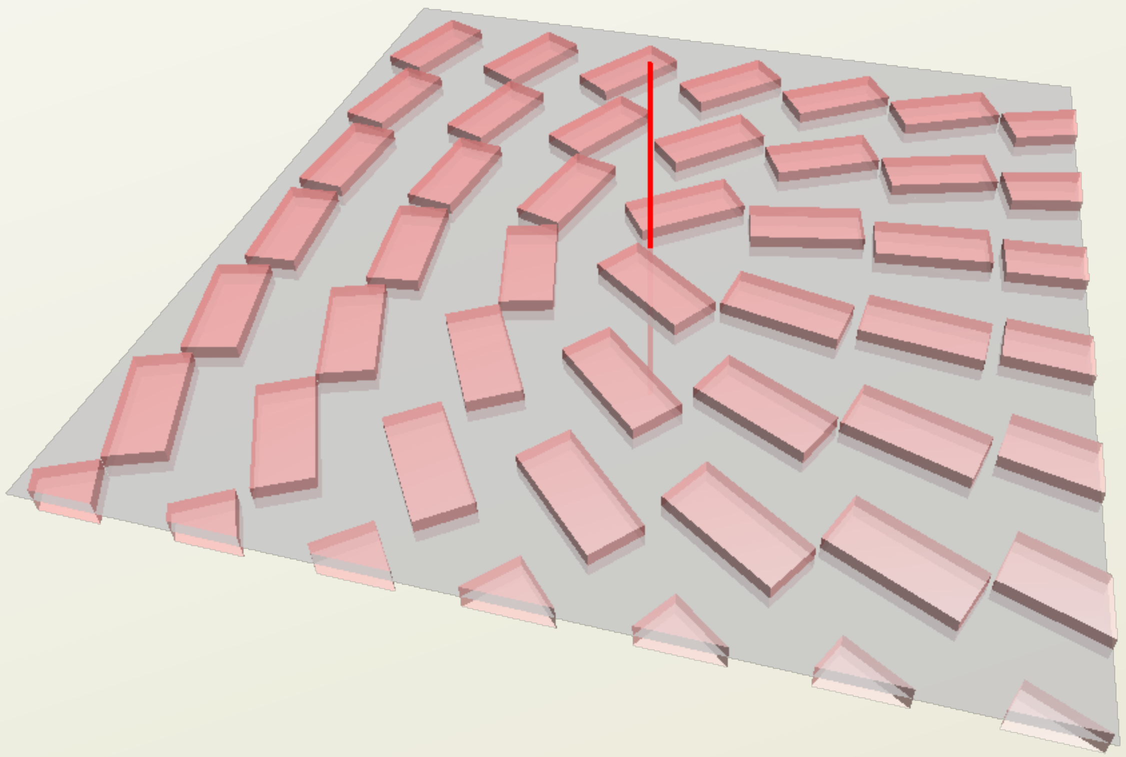

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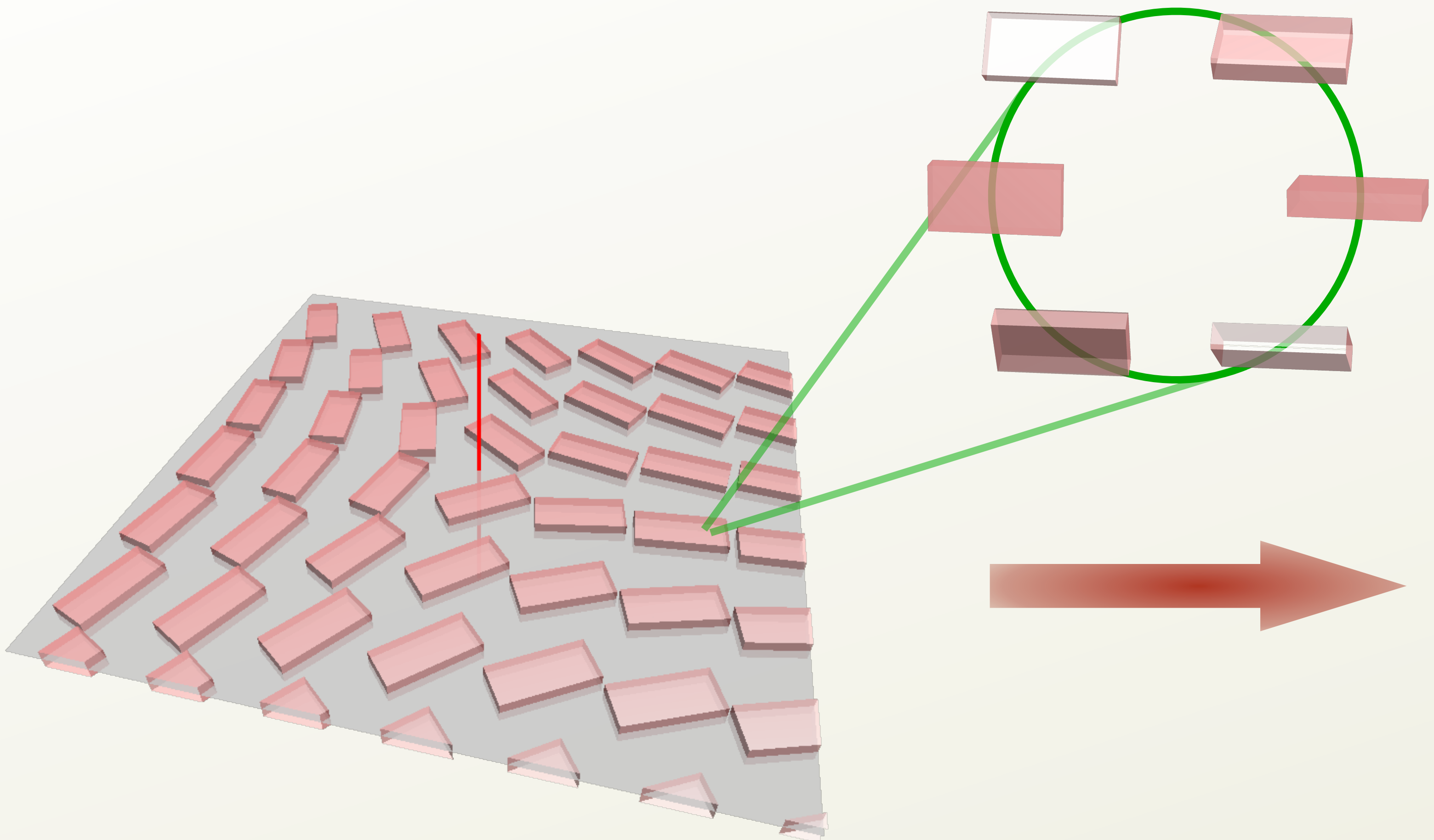
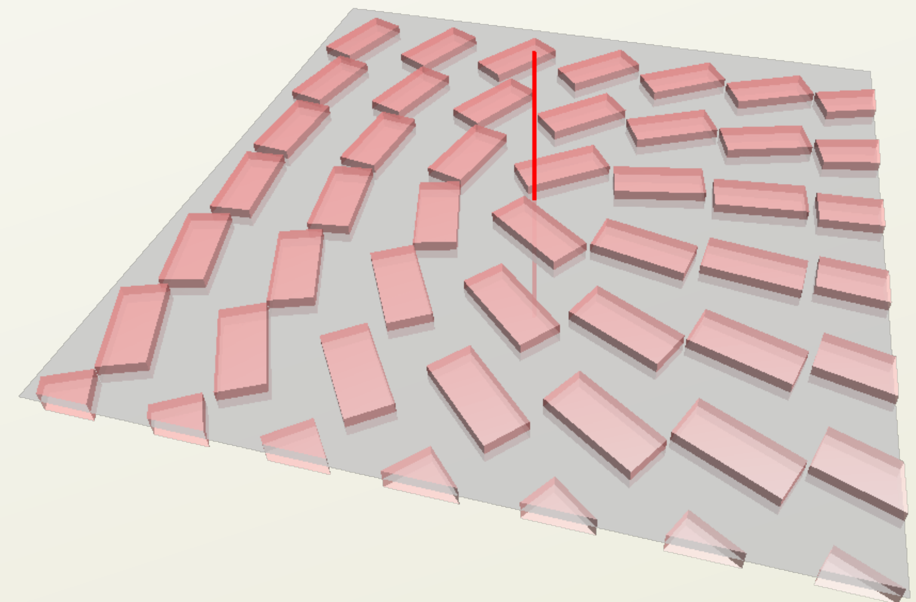
ACTION OF π_1 ON ITSELF



base point traverses a loop in the GSM $\beta \in \pi_1(X, x_0)$



initial defect in class

$\alpha \in \pi_1(X, x_0)$

$-i\boldsymbol{\sigma}_y$

final defect in class

$\alpha^\beta \in \pi_1(X, x_0)$

$(i\boldsymbol{\sigma}_x)(-i\boldsymbol{\sigma}_y)(i\boldsymbol{\sigma}_x)^{-1} = (i\boldsymbol{\sigma}_y)$

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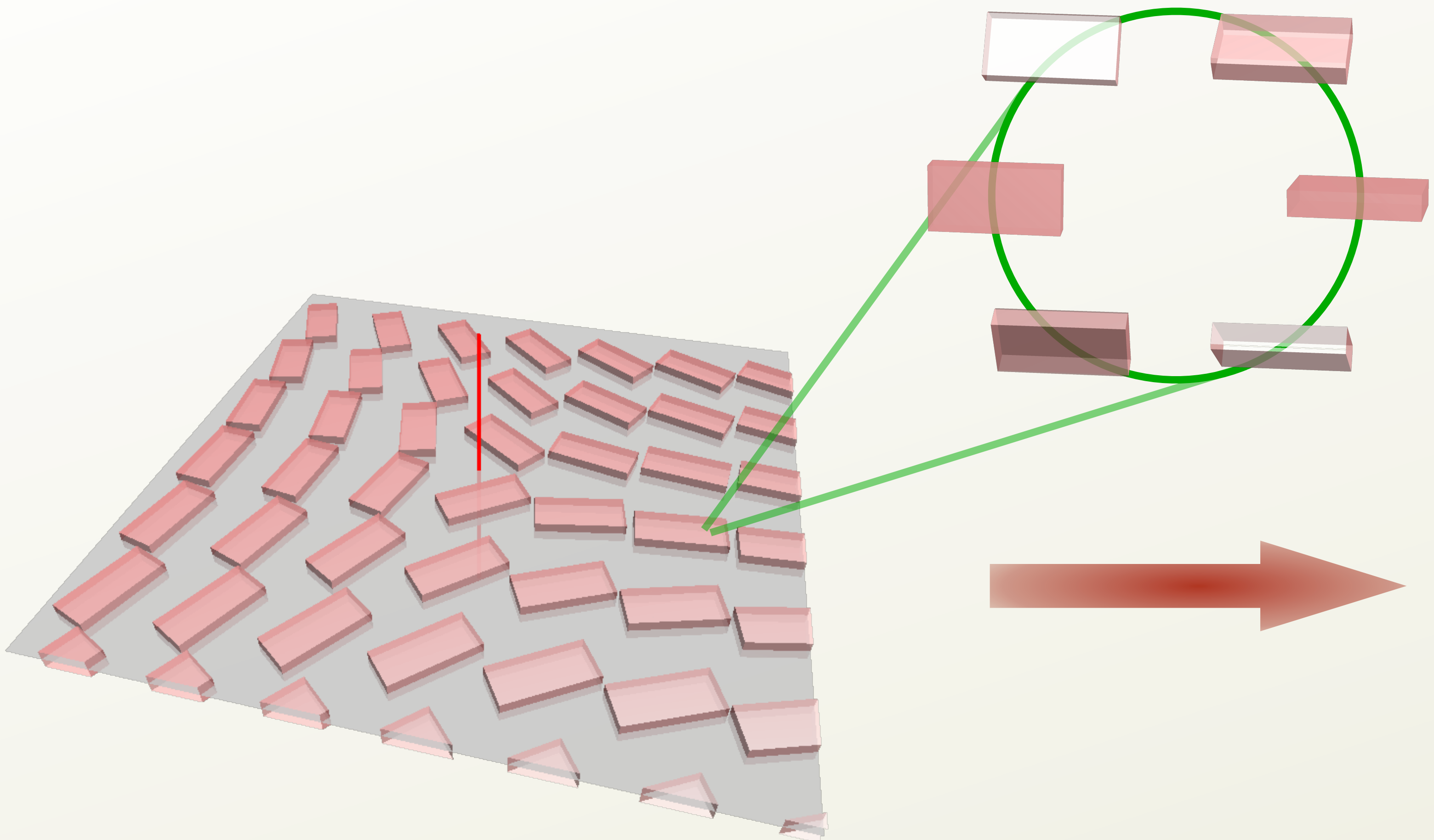

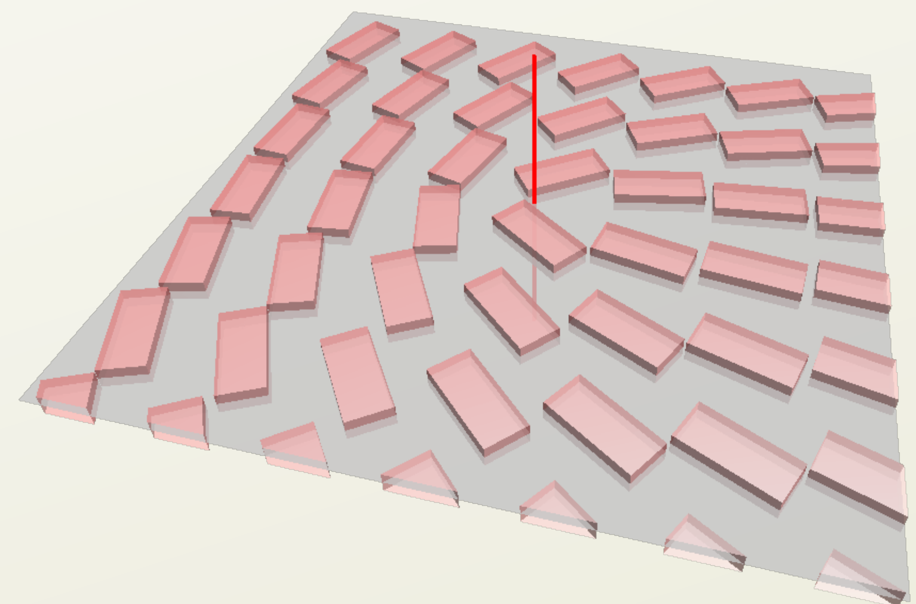

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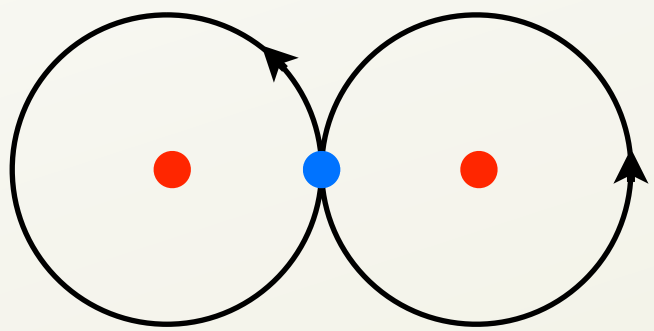
where is the defect β *?*

FREE HOMOTOPY BY MOVING DEFECTS

"drag one defect around another"

$\beta \in \pi_1(X, x_0)$



initial measurement

$\alpha \in \pi_1(X, x_0)$

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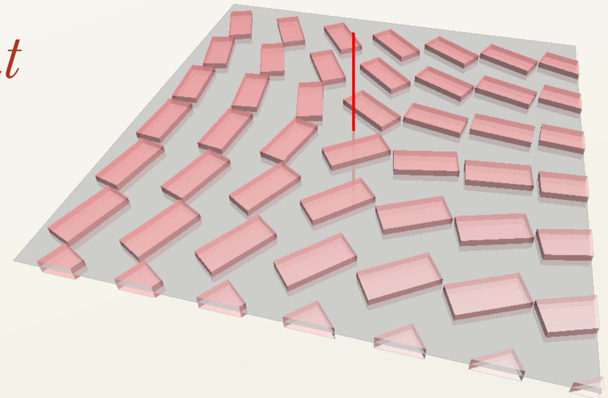
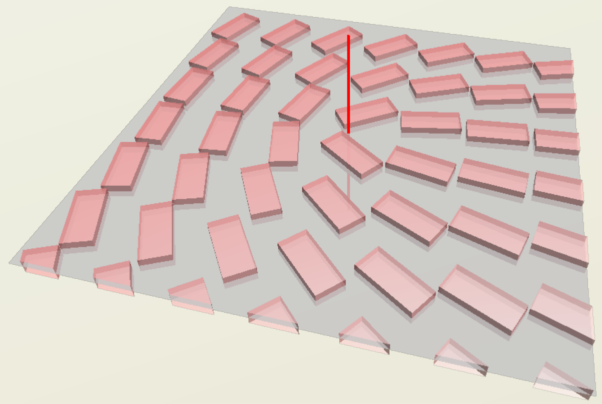


new measurement

$\alpha^\beta \in \pi_1(X, x_0)$

$+i\boldsymbol{\sigma}_y$



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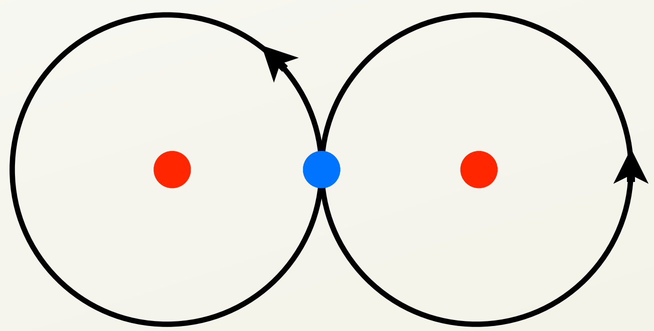

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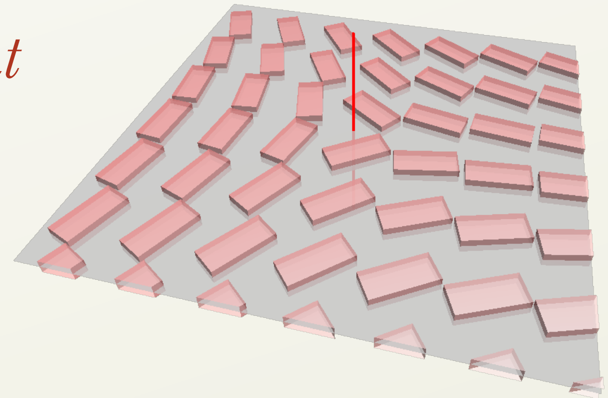

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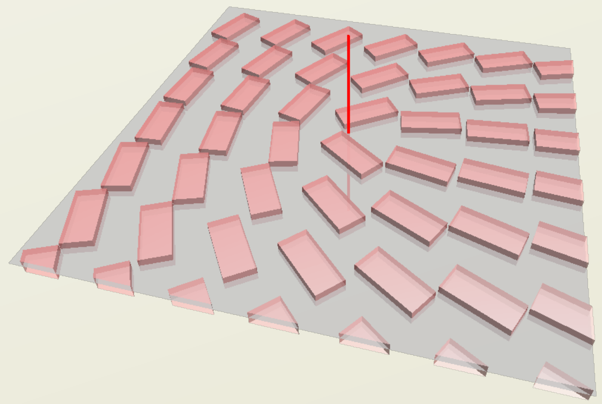

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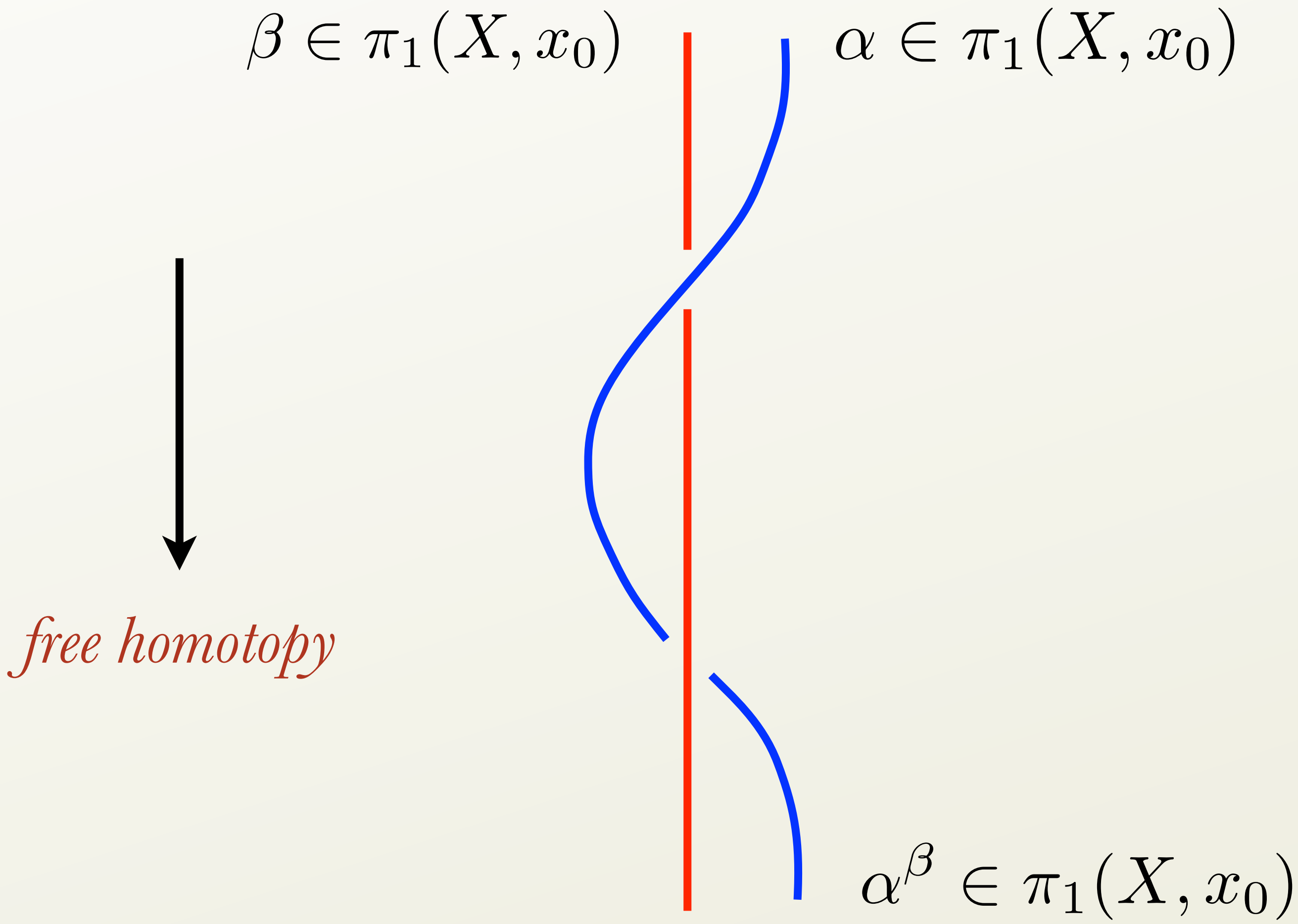
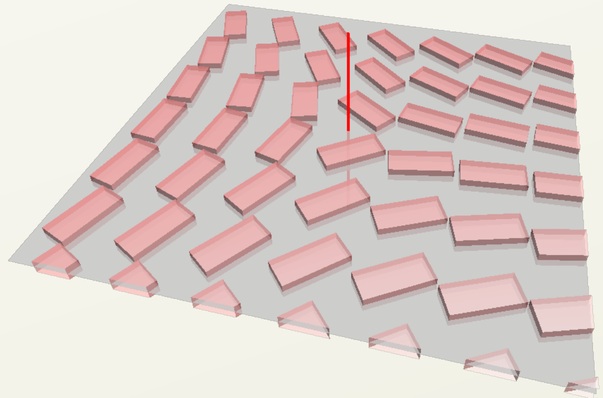
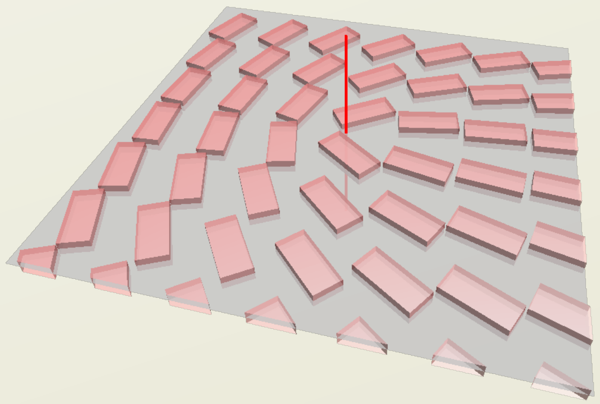


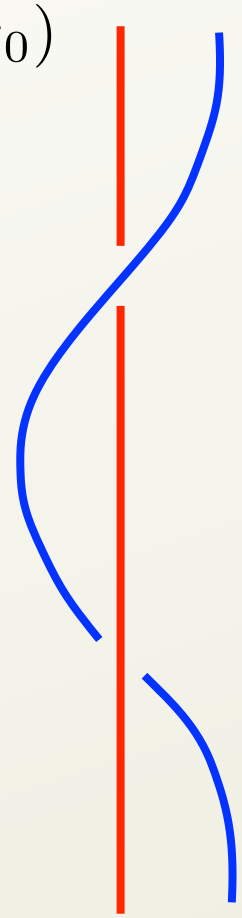




FREE HOMOTOPY BY MOVING DEFECTS

"drag one defect around another"

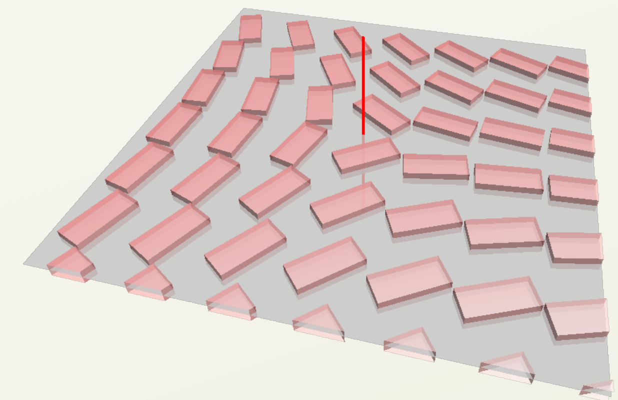
$\beta \in \pi_1(X, x_0)$

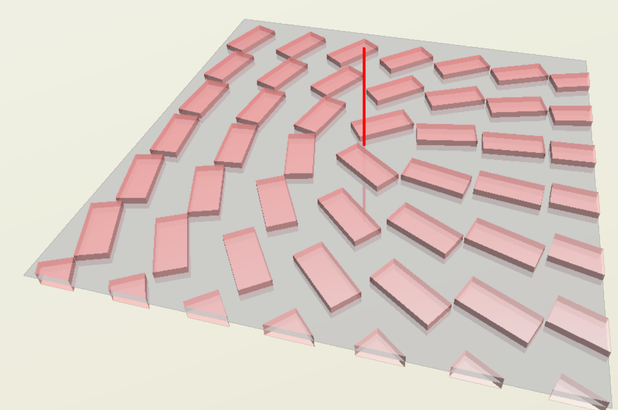


$\alpha \in \pi_1(X, x_0)$

free homotopy

$\alpha^\beta \in \pi_1(X, x_0)$

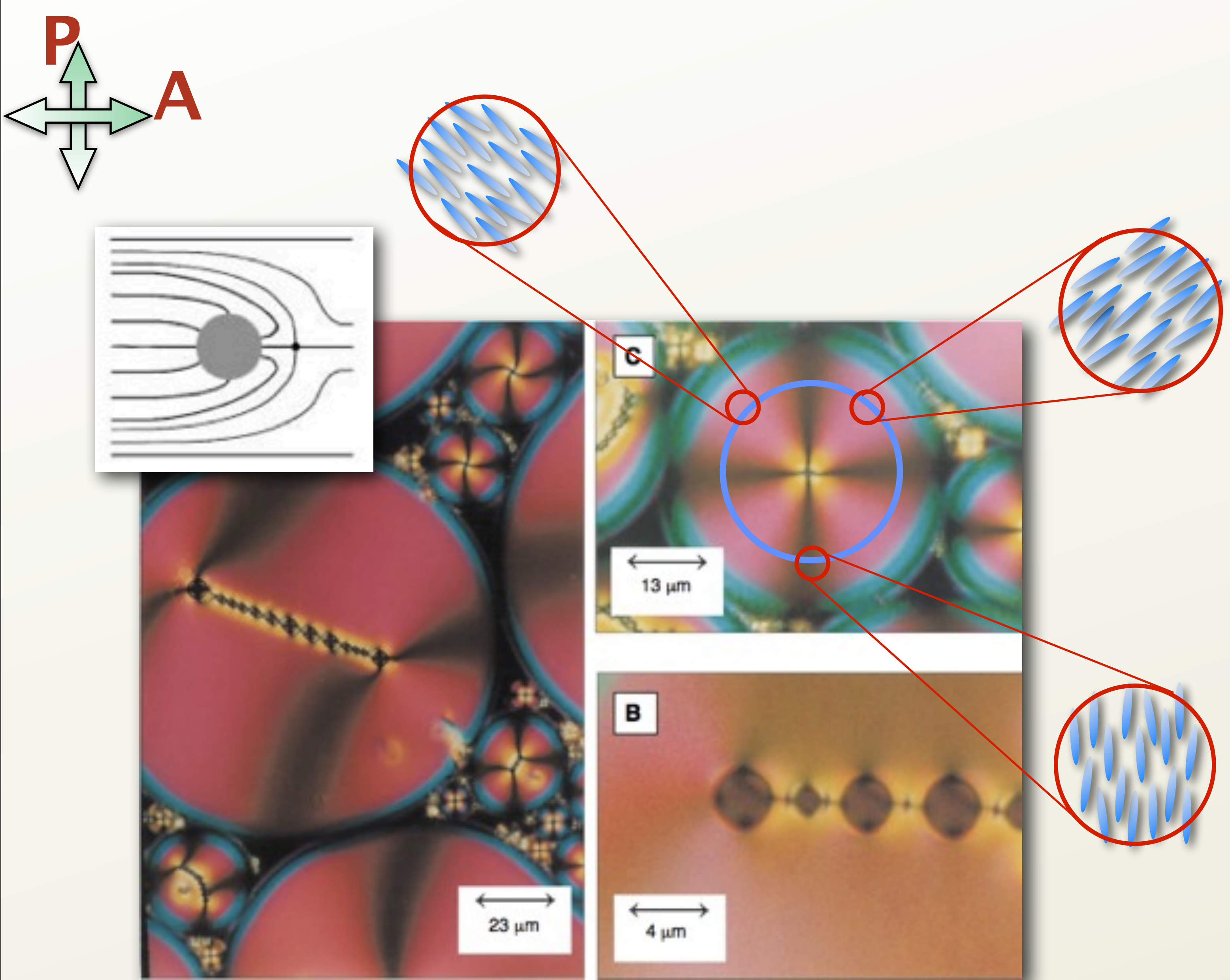




the change in homotopy class $\alpha^\beta \circ \alpha^{-1} = [\beta, \alpha]$
is a Whitehead product

WARWICK

HEDGEHOGS: *POINT SINGULARITIES IN THREE DIMENSIONS*



POULIN ET AL *Science* **275**, 1770–1773 (1997)

measure the texture on some sphere enclosing the defect

$$\text{map}: \quad S^2 \rightarrow X$$

classify defects using homotopy groups

$\pi_2(X)$ **based**

$[S^2, X]$ **free**

nematic $\pi_2(\mathbb{RP}^2) = \mathbb{Z}$

biaxial $\pi_2(SO(3)/D_2) = 0$

WARWICK

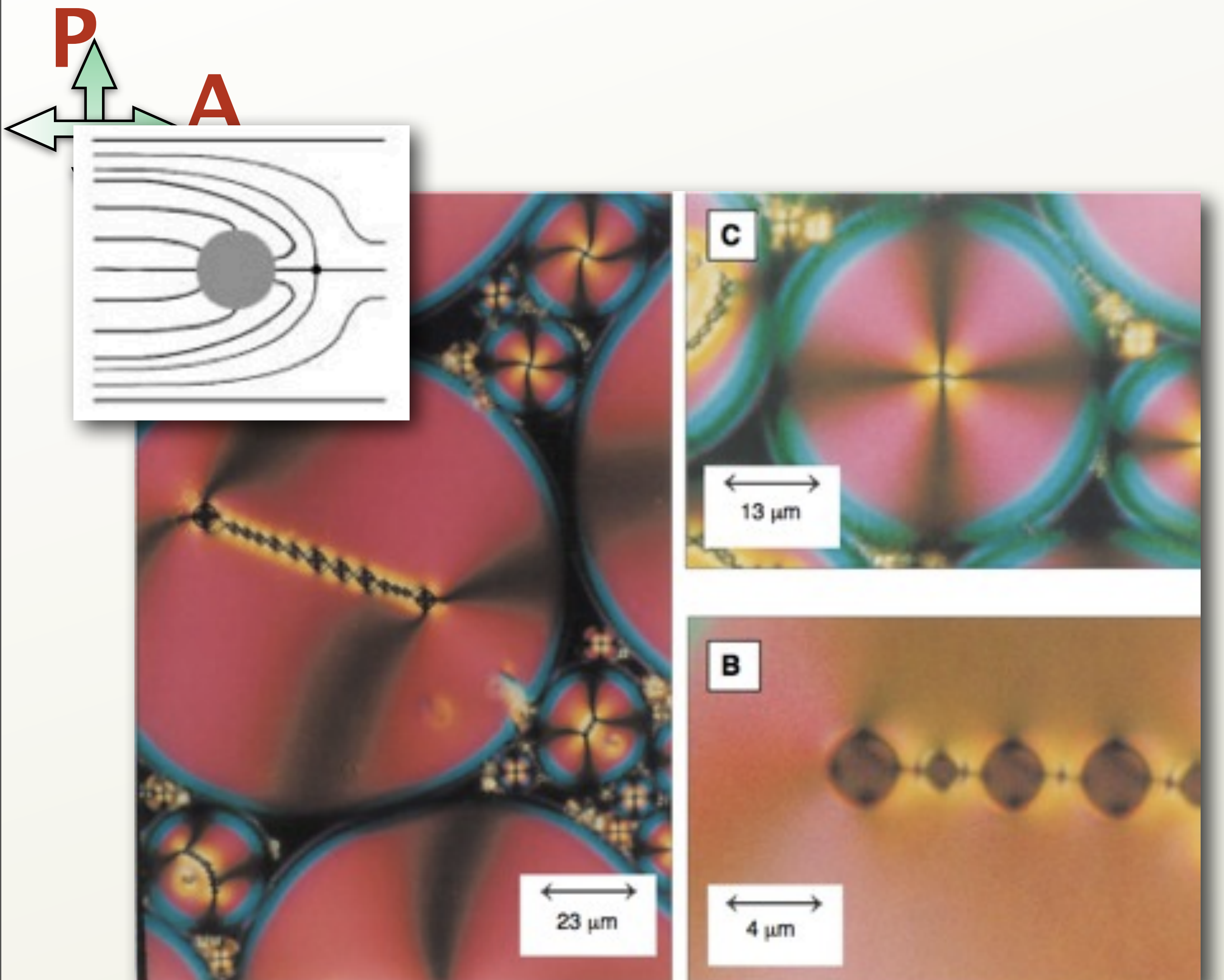
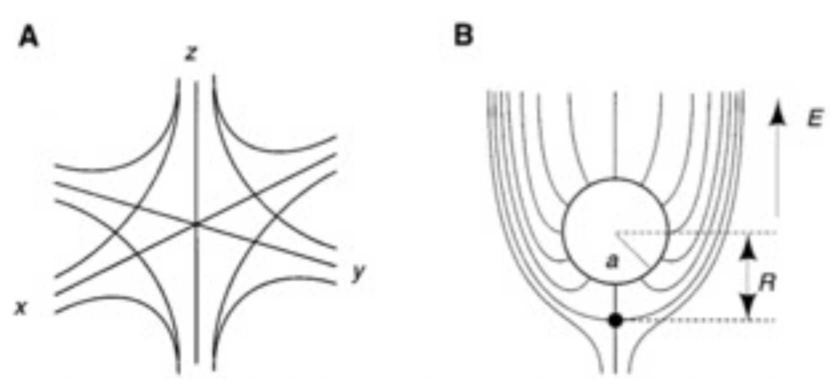


Fig. 2. (**A**) Director configuration for a hyperbolic defect. The director is tangent to the lines shown. (**B**) A droplet-defect dipole. The director configuration around the spherical droplet is that of a radial hedgehog. The point defect is a hyperbolic hedgehog. Rotation of this figure about the vertical axis produces the three-dimensional director configuration, which is uniform and parallel to the vertical axis from the dipole. In the electrostatic analog, the droplet becomes a conducting sphere with charge *Q* in an external electric field *E*, which produces the field lines determining the orientation of the director.



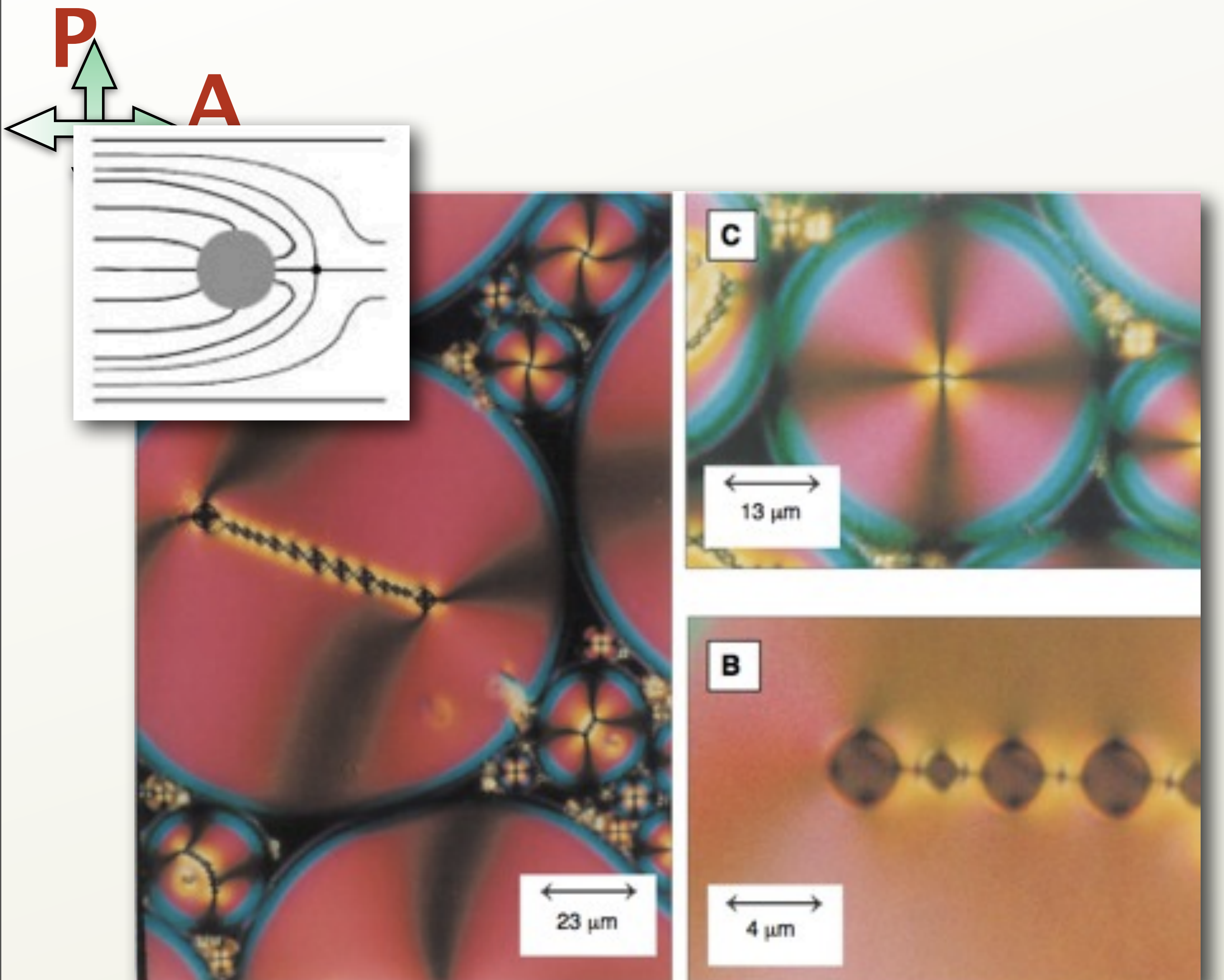
POULIN ET AL *Science* **275**, 1770–1773 (1997)

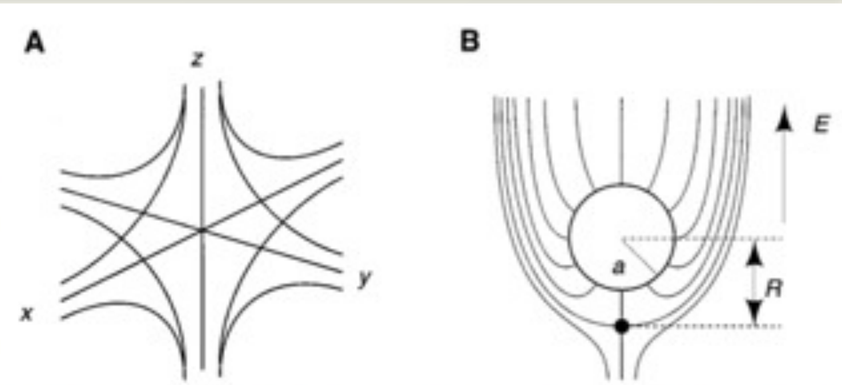
what's the sign of the hedgehog?

$$\pi_2(\mathbb{RP}^2) = \mathbb{Z} \qquad \textbf{based}$$

$$[S^2, \mathbb{RP}^2] = \mathbb{N} \qquad \textbf{free}$$

WARWICK





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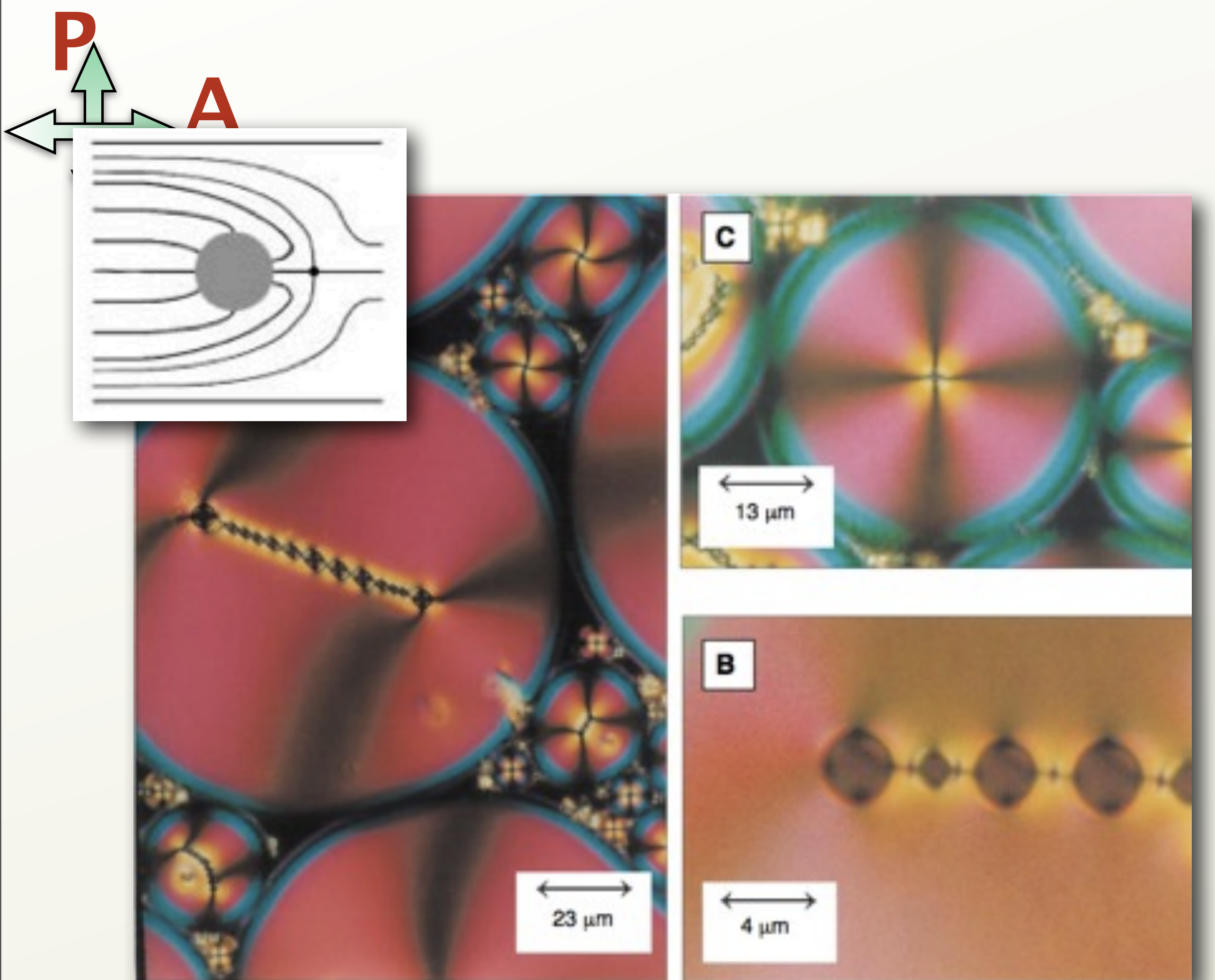
the hedgehog charge can be computed by a degree integral

$$\deg(\mathbf{n}) = \frac{1}{4\pi} \int_{S^2} \mathbf{n} \cdot \partial_\theta \mathbf{n} \times \partial_\phi \mathbf{n}$$

WARWICK

BASED VERSUS FREE



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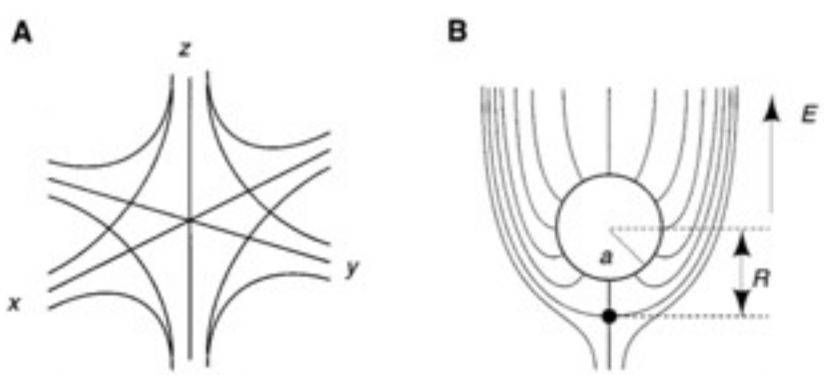

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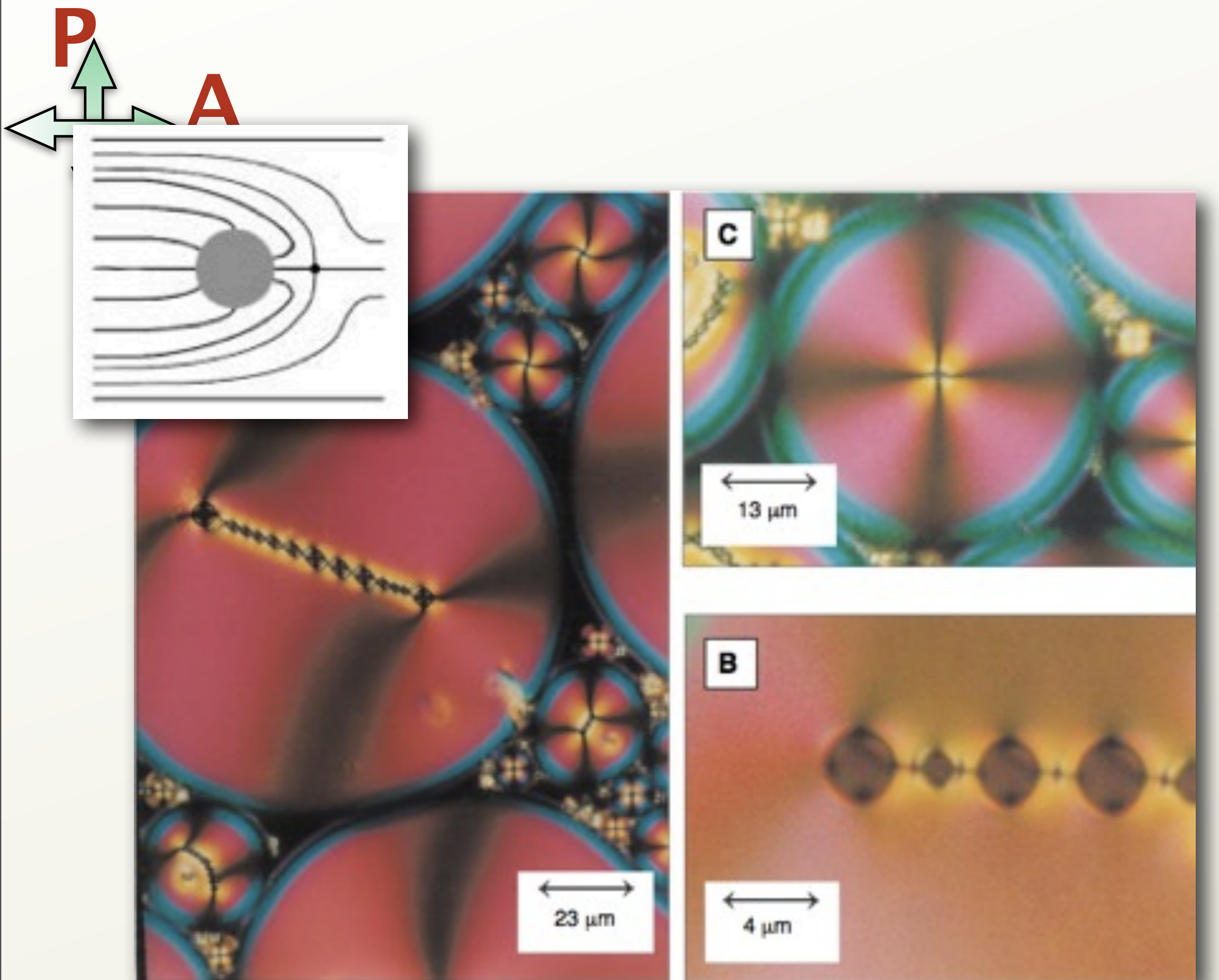
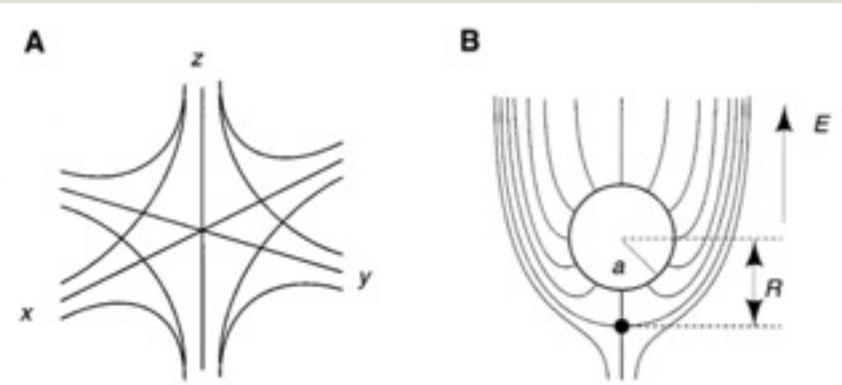
odd in $\mathbf{n}$

$$\deg(\mathbf{n}) \sim -\deg(\mathbf{n})$$





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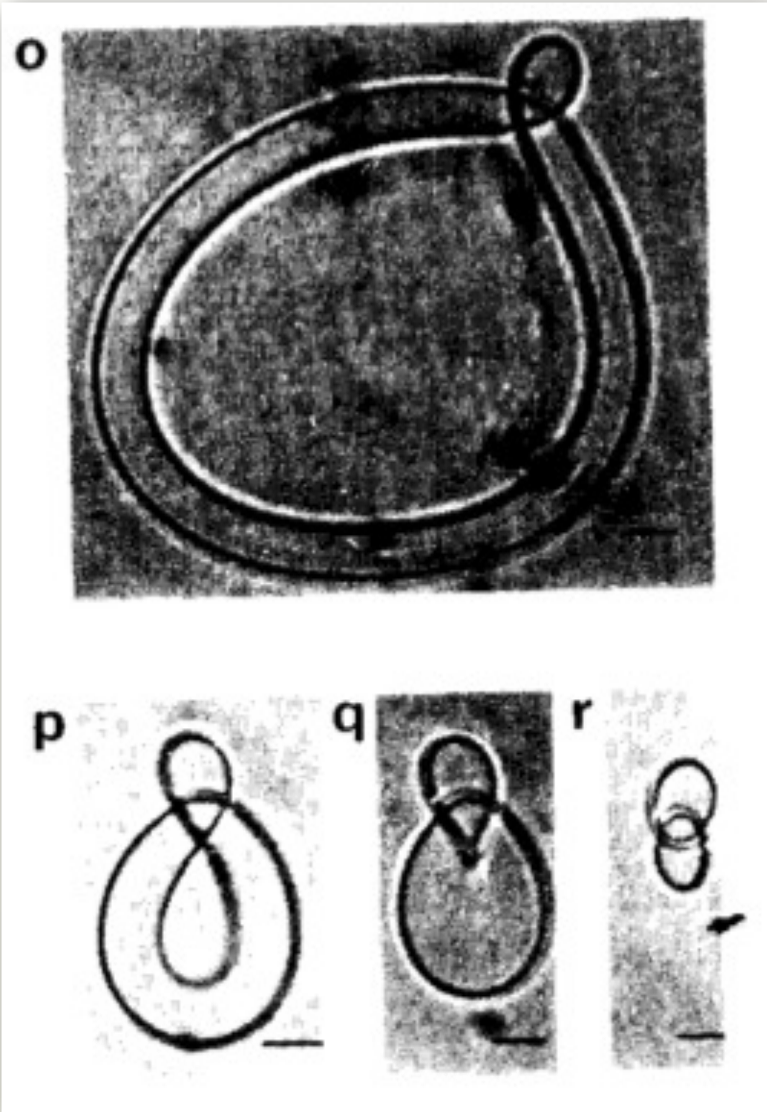
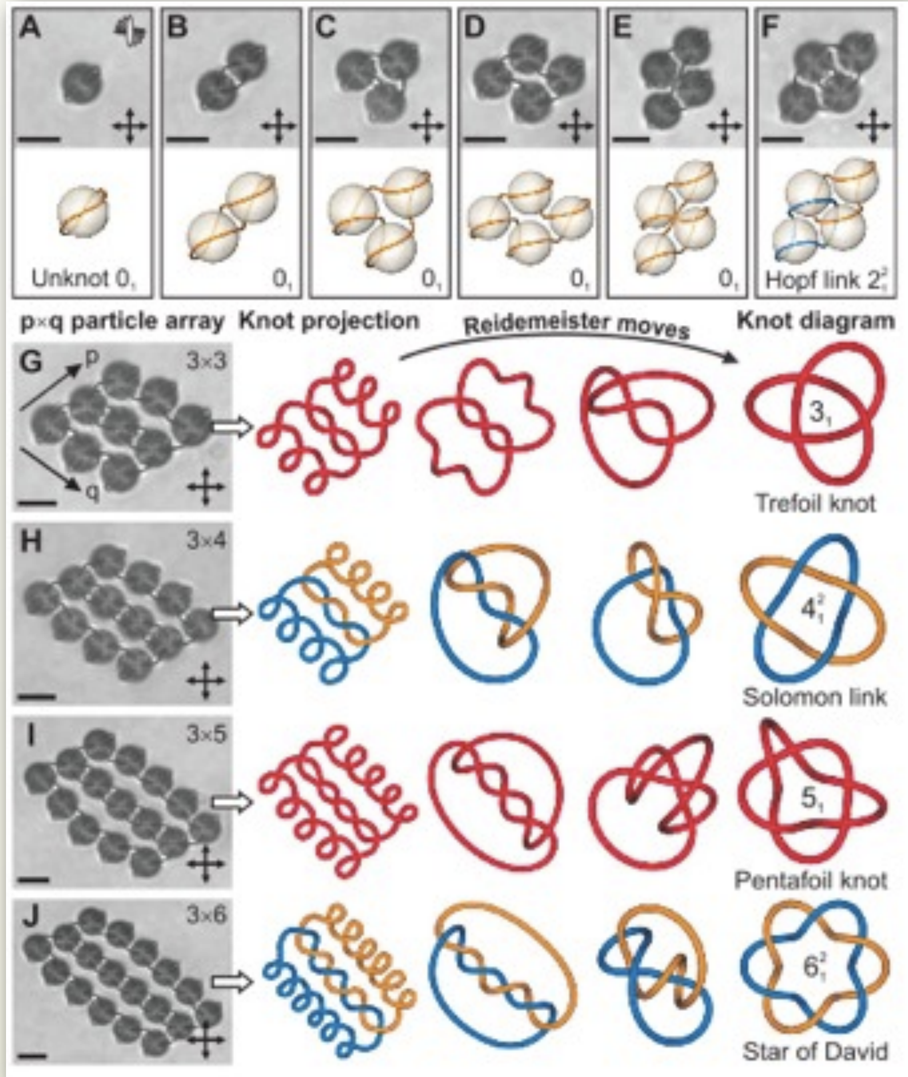
can you construct a free homotopy?

WARWICK

DISCLINATION LOOPS



BOULIGAND *J. Phys. France* **35**, 959–981 (1974)

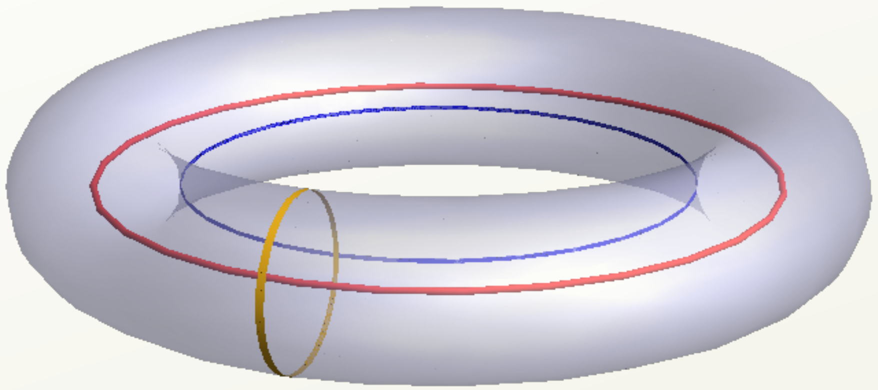


TKALEC ET AL *Science* **333**, 62–65 (2011)

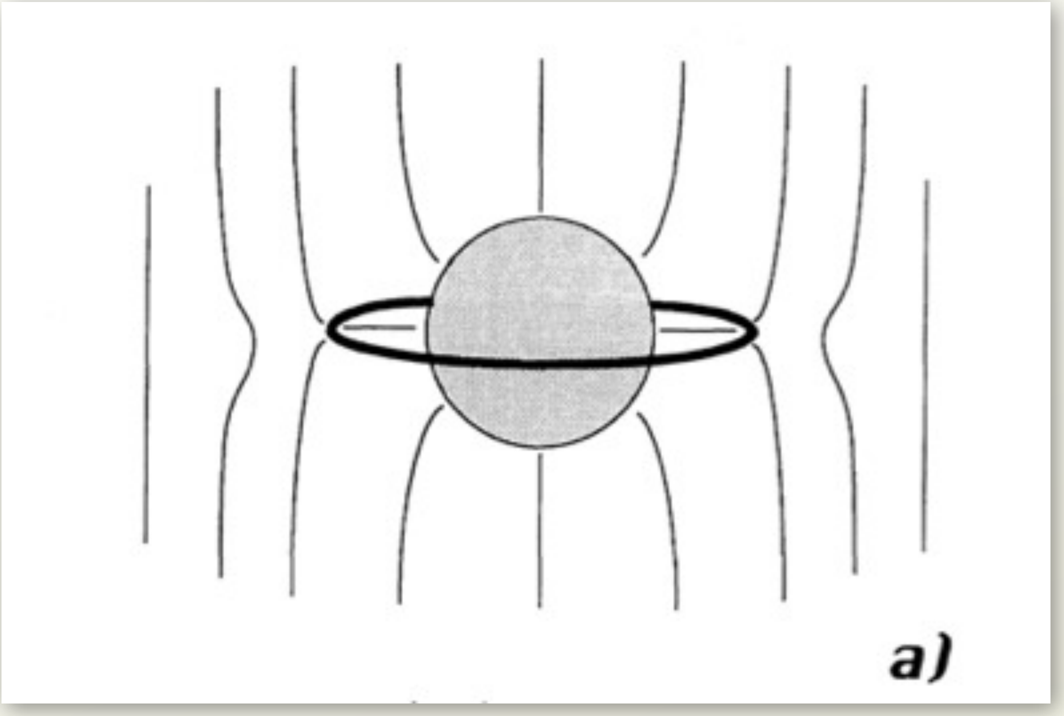
disclination loops are extended objects

WARWICK

DISCLINATION LOOPS: *SIMPLE REMARKS*



contour length of the disclination provides another natural measuring loop



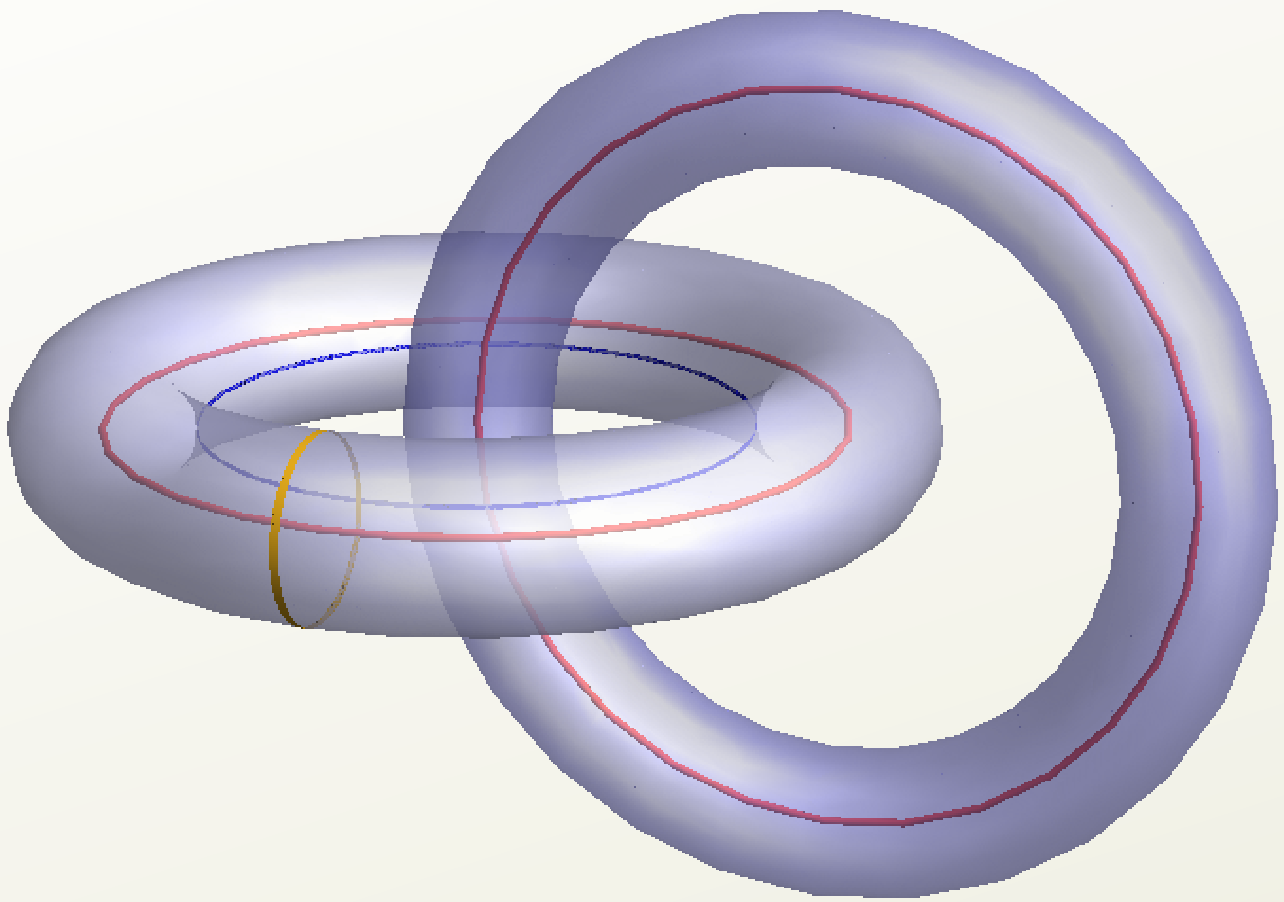
TERENTJEV *Phys. Rev. E* **51**, 1330–1337 (1995)

orange circle measures $\alpha \in \pi_1(X, x_0)$

blue circle measures $\beta \in \pi_1(X, x_0)$

WARWICK

LINKED LOOPS



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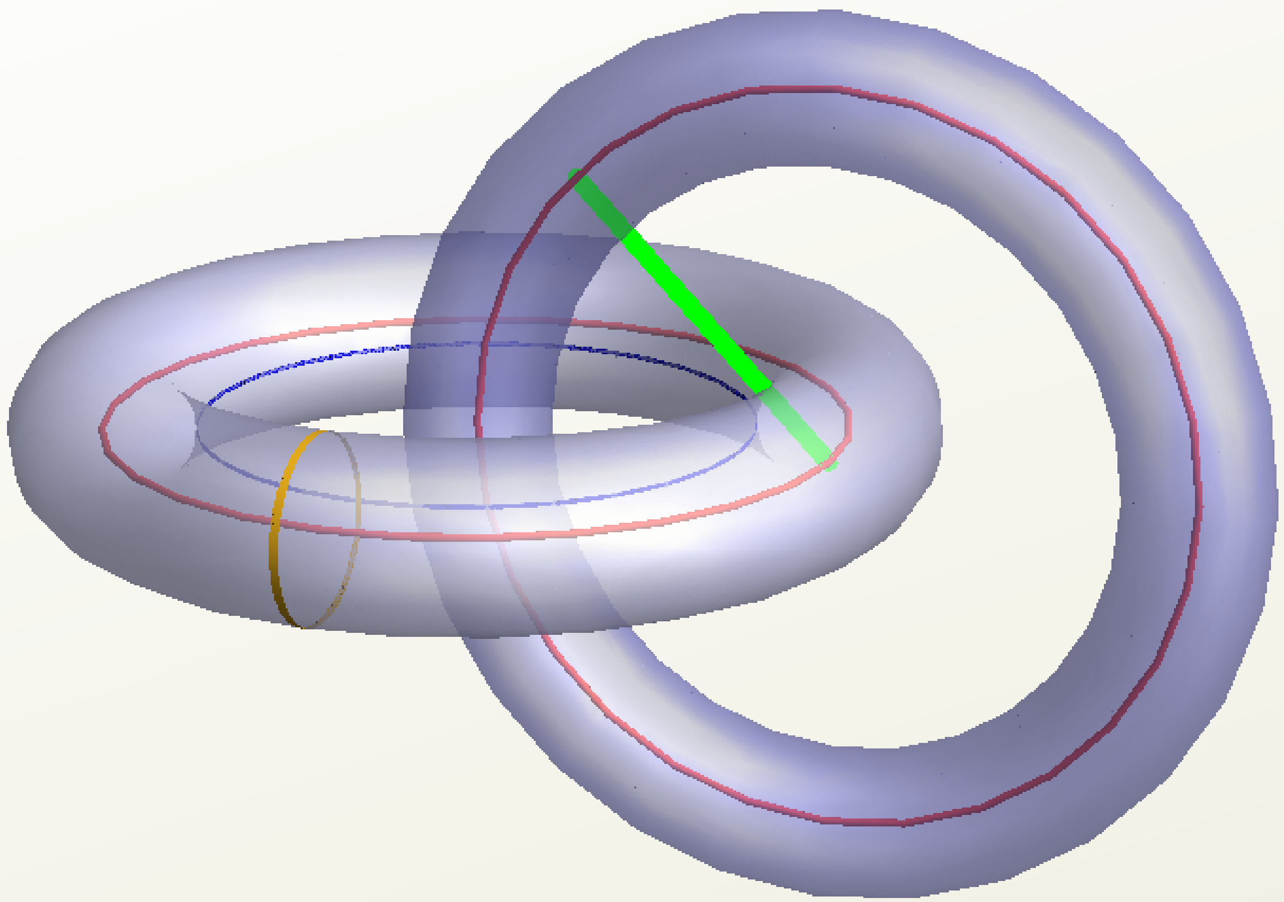
TOPOLOGICAL ENTANGLEMENT



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two defects collectively define a third

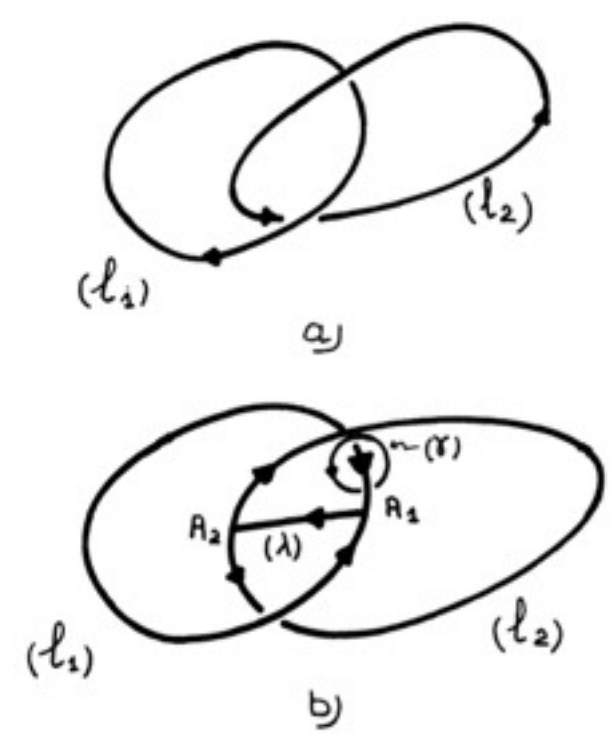
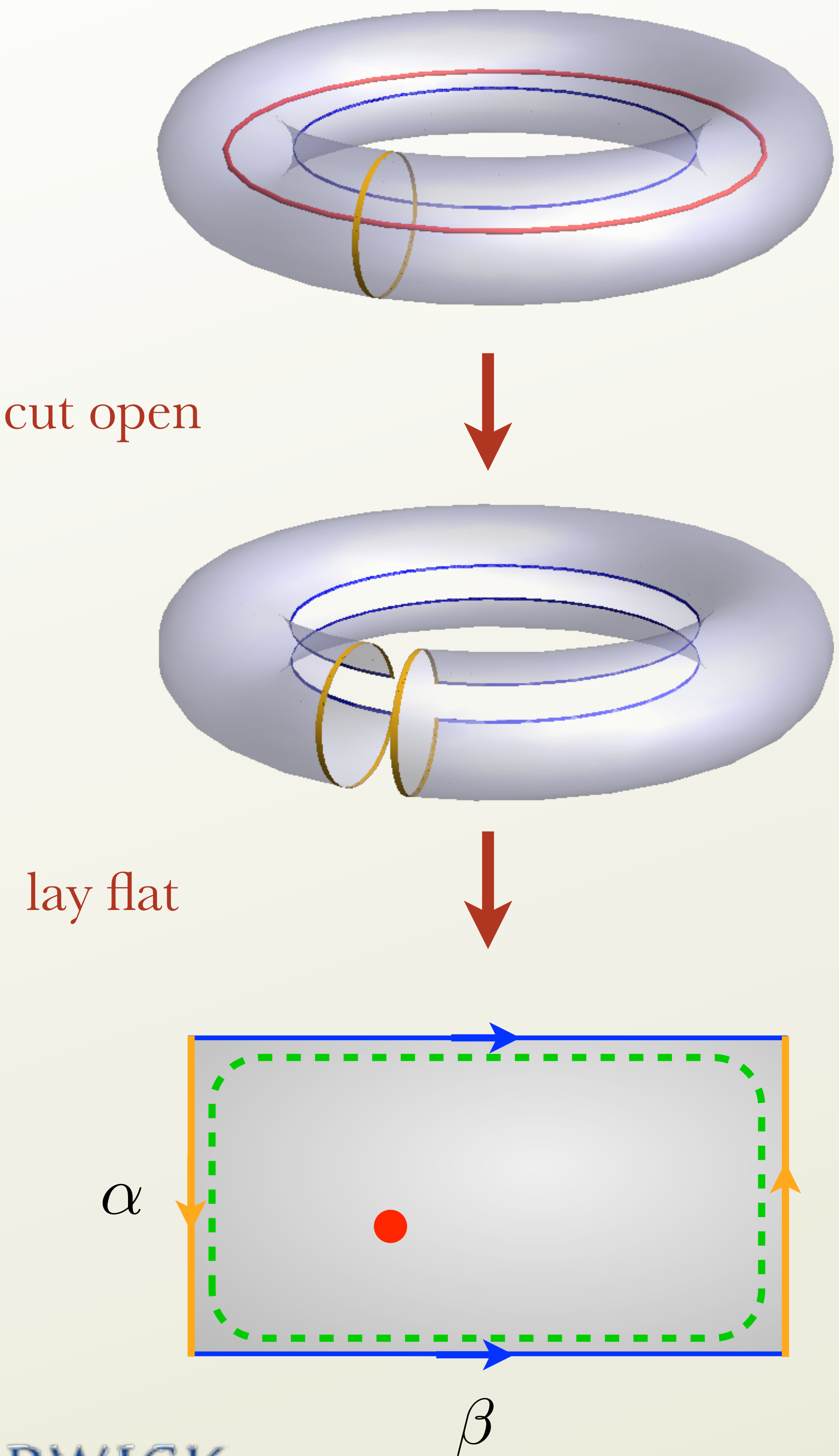


FIG. 2. — The homotopy classes of two linked lines must commute (*a*). If they do not, the two lines must be joined by a line whose homotopy class is a commutator (*b*).

KLÉMAN *J. Phys. France Lett.* **38**, 199–202 (1977)

TOPOLOGICAL ENTANGLEMENT



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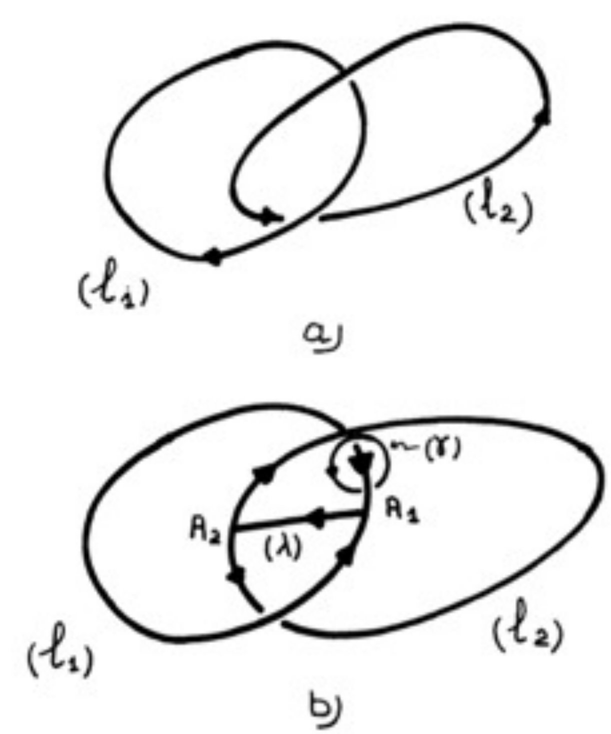
connecting tether $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$

Whitehead product



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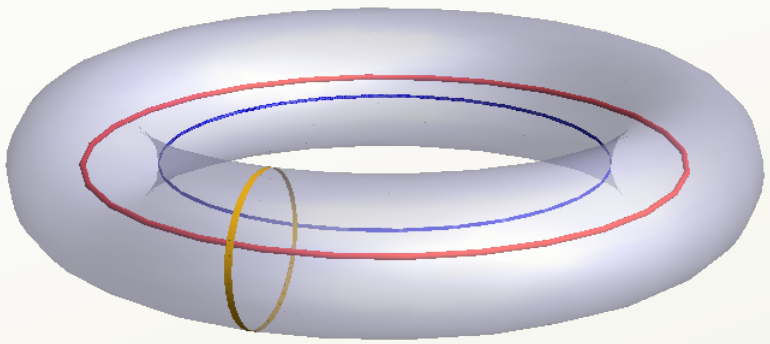
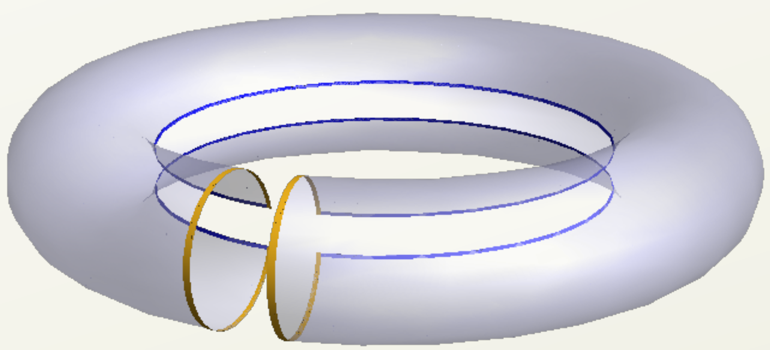
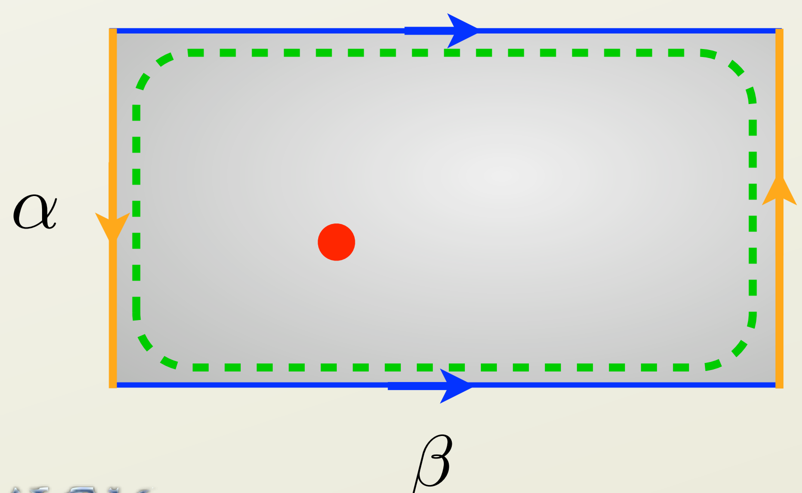


cut open



lay flat



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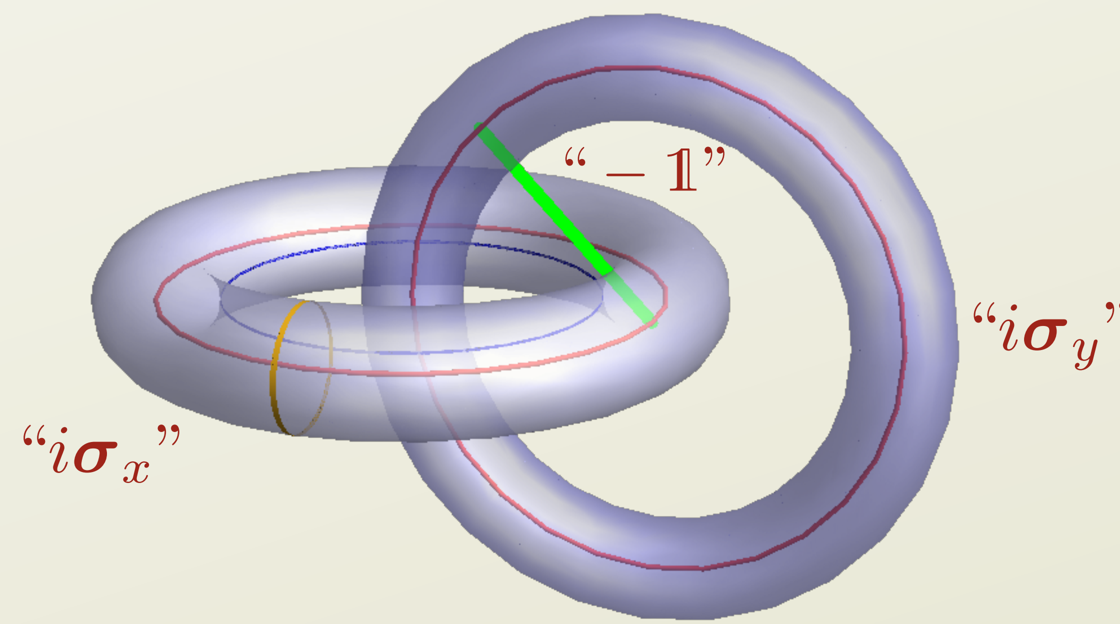
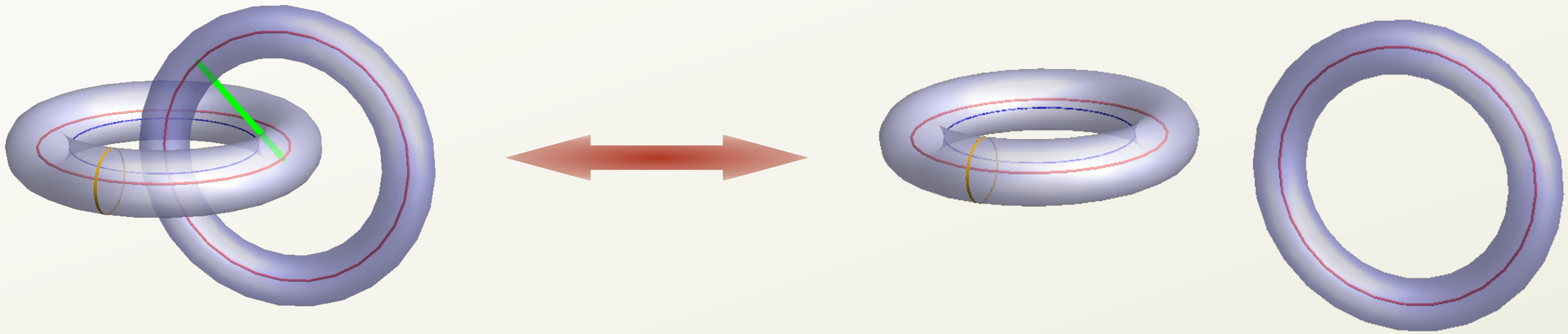


DEFECT CROSSING

when defects cross,
sometimes there's a tether



WARWICK

Poénaru & Toulouse *J. Phys. France* **38**, 887–895 (1977)

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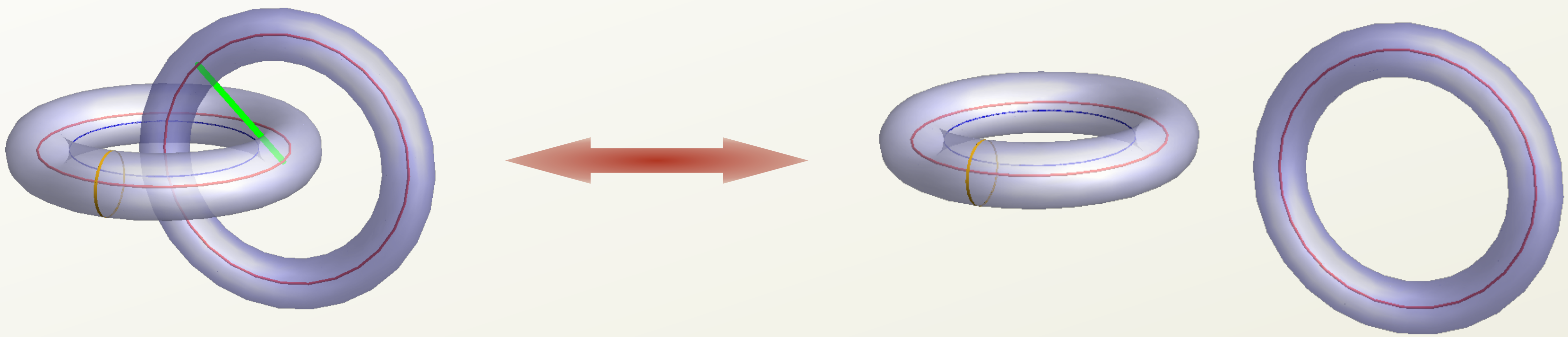

Tome 38 | N° 8 | AOÛT 1977

LE JOURNAL DE PHYSIQUE

Classification
Physics Abstracts
1.110 — 7.160 — 7.222

THE CROSSING OF DEFECTS IN ORDERED MEDIA AND THE TOPOLOGY OF 3-MANIFOLDS

V. POENARU

Université Paris-Sud, Département de Mathématiques, 91405 Orsay, France

and

G. TOULOUSE

Ecole Normale Supérieure, Laboratoire de Physique, 24, rue Lhomond, 75231 Paris, France

(*Reçu le 15 février 1977, accepté le 26 avril 1977*)

Résumé. — Ce travail est un premier pas dans l'étude des obstructions topologiques pour déformer les défauts des milieux ordonnés. Il s'inscrit dans la ligne des théories physiques récentes qui classifient les défauts en termes de groupes d'homotopie d'une certaine variété V, caractéristique pour l'ordre en question.

On montre ici que les seules obstructions pour le croisement des lignes de défaut, dans un échantillon 3-dimensionnel, sont les commutateurs dans le groupe fondamental de V. Ceci est un phénomène qualitativement nouveau pour une certaine classe de matériaux, à π_1 V *non commutatif*, qu'on espère voir synthétisés bientôt. Il en résulte aussi la nécessité d'une révision de certains concepts traditionnels dans la physique de la matière condensée.

Le présent travail contient un cadre mathématique rigoureux pour la description des défauts non commutatifs, une discussion des applications physiques, ainsi que quelques problèmes ouverts.

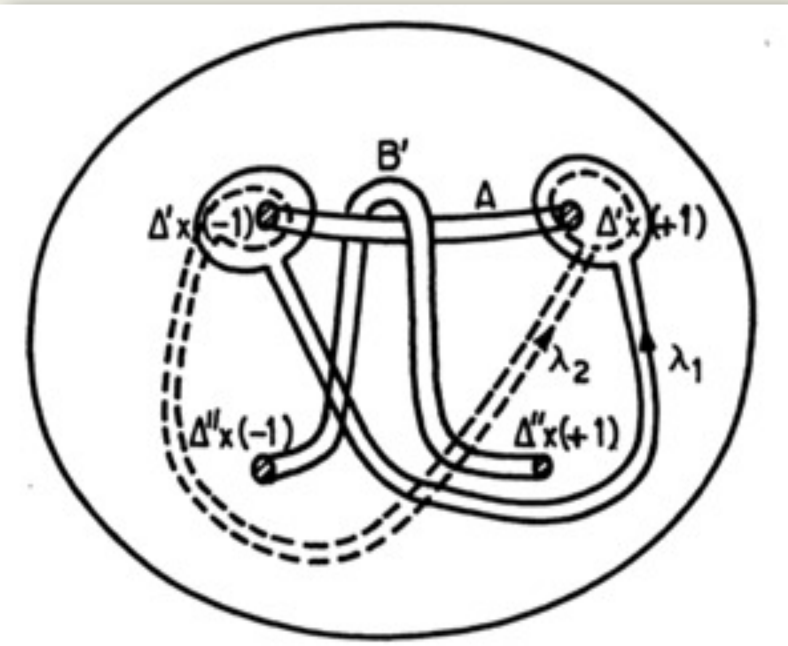

Fig. 7.

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Poénaru & Toulouse *J. Phys. France* **38**, 887–895 (1977)

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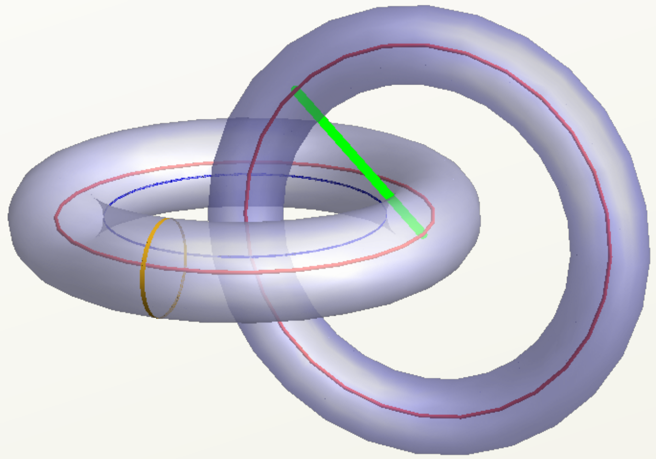

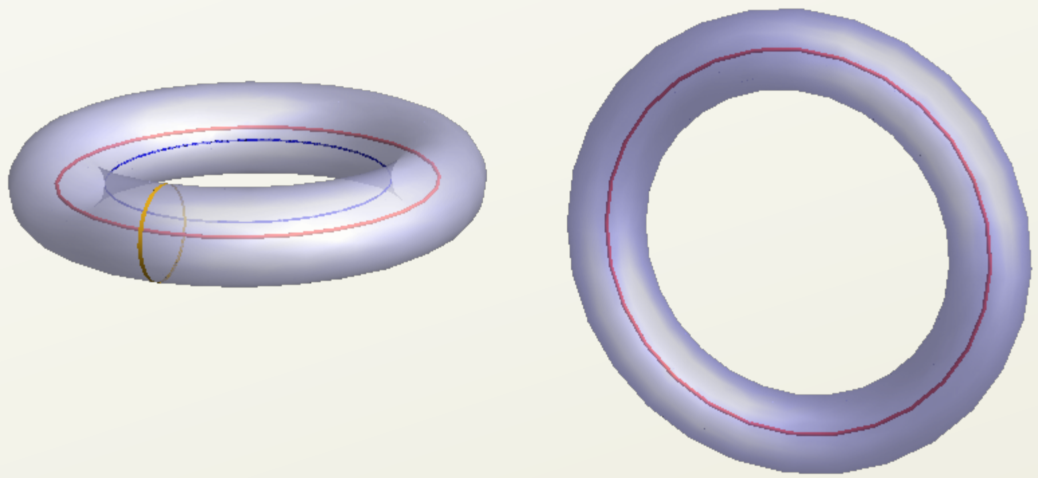

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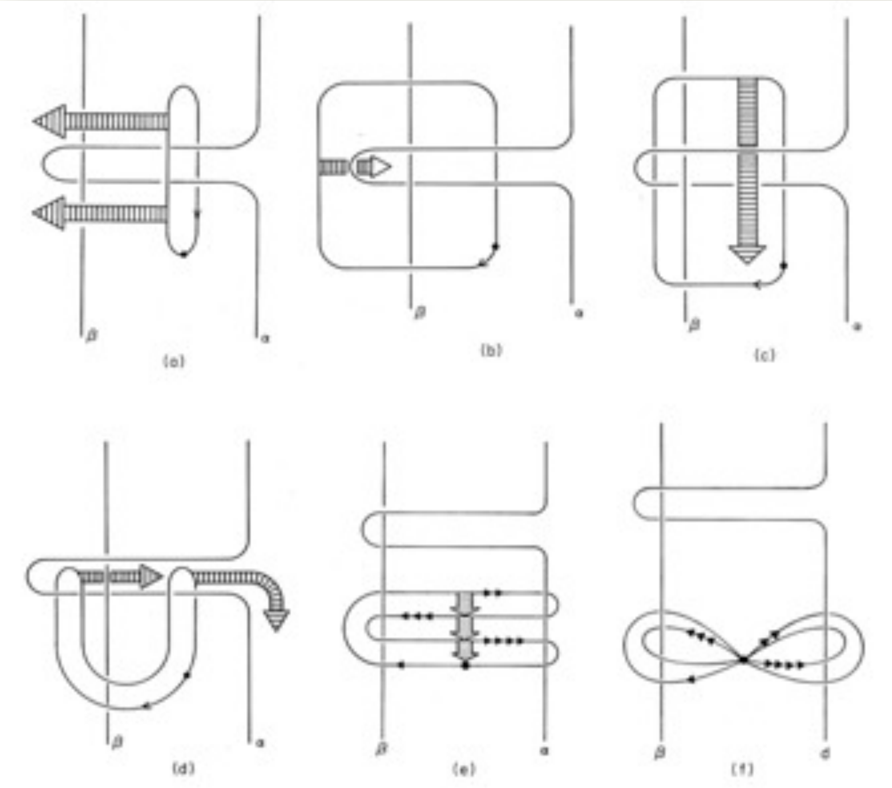

POÉNARU & TOULOUSE *J. Phys. France* **38**, 887–895 (1977)

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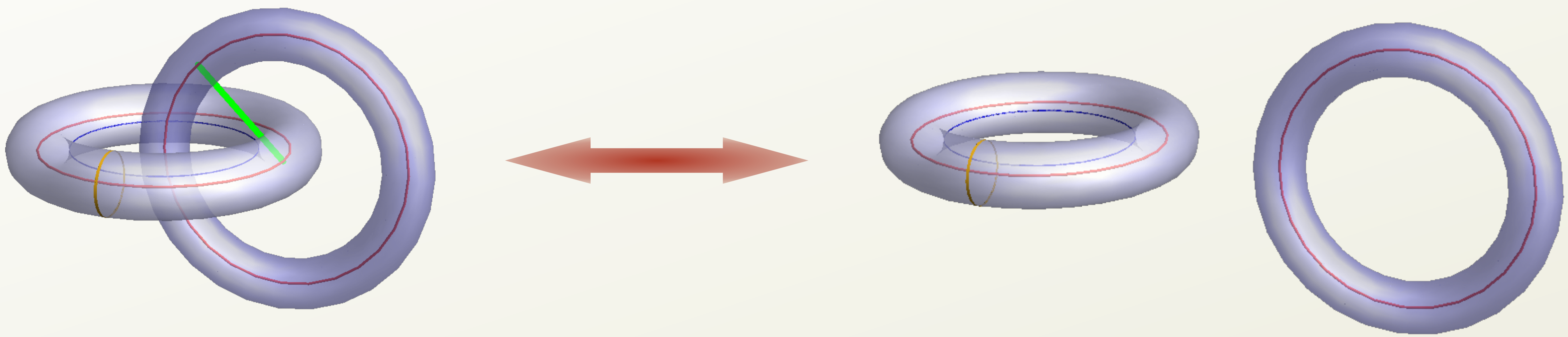
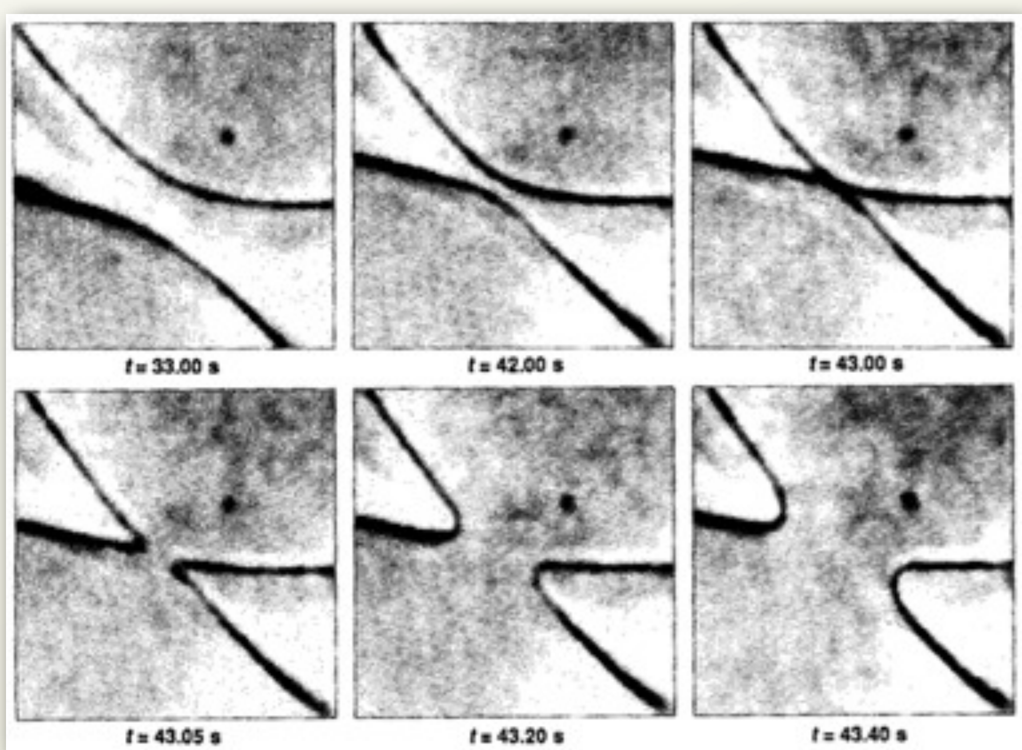




CHUANG, DURRER, TUROK & YURKE *Science* **251**, 1336–1342 (1991)

no experiments yet …

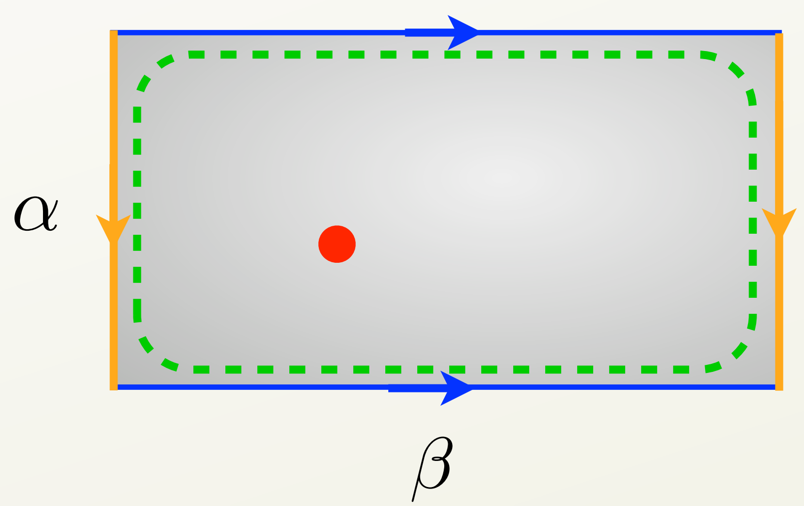
POÉNARU & TOULOUSE *J. Phys. France* **38**, 887–895 (1977)

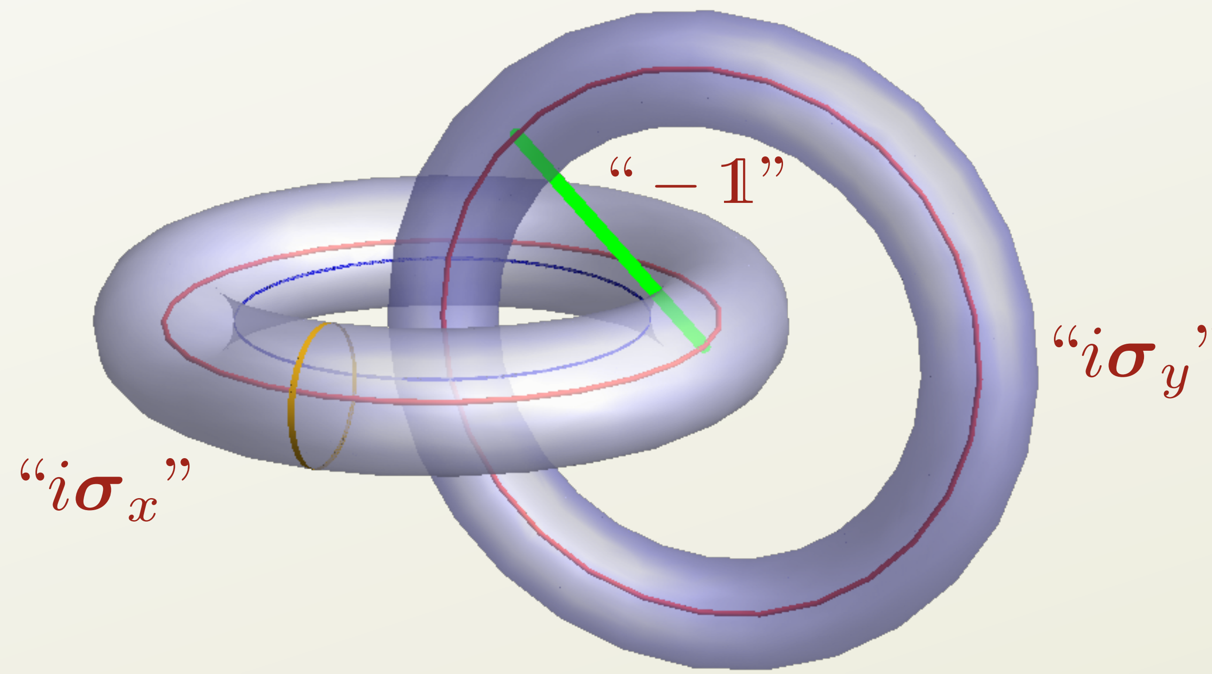
Whitehead products

two defects collectively define a third

orange circle measures $\alpha \in \pi_1(X, x_0)$

blue circle measures $\beta \in \pi_1(X, x_0)$





Given two defects α, β in the form of linked loops they collectively define a third

$$\alpha, \beta \quad \longrightarrow \quad [\alpha, \beta]$$

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \quad \longrightarrow \quad \pi_1(X, x_0)$$

Whitehead product

Whitehead *Ann. of Math.* **42**, 409–428 (1941)
Kléman *J. Phys. France Lett.* **38**, 199–202 (1977)
Poénaru & Toulouse *J. Phys. France* **38**, 887–895 (1977)

Whitehead products

two defects collectively define a third

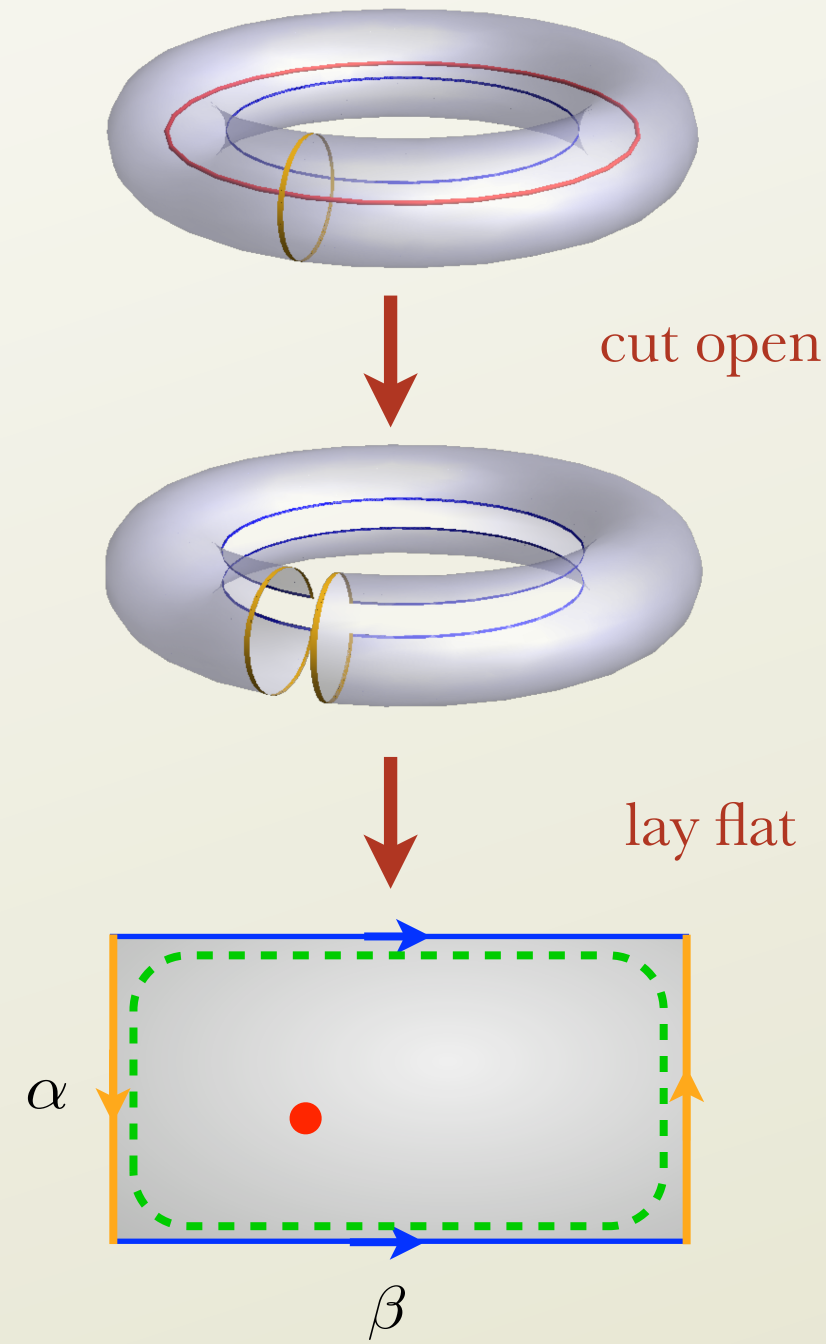
$$\pi_p(X,x_0)\times\pi_q(X,x_0) \quad\to\quad \pi_{p+q-1}(X,x_0)$$

think of a "p-defect" linking a "q-defect" in $\mathbb{R}^{p+q+1}$

surround the "p-defect" with a $S^p\times S^q$

cut this open along a $S^p\vee S^q$ to give a D^{p+q}

the map on the boundary $\partial D^{p+q}=S^{p+q-1}$
is the Whitehead product



Whitehead *Ann. of Math.* **42**, 409–428 (1941)

WHITEHEAD PRODUCTS

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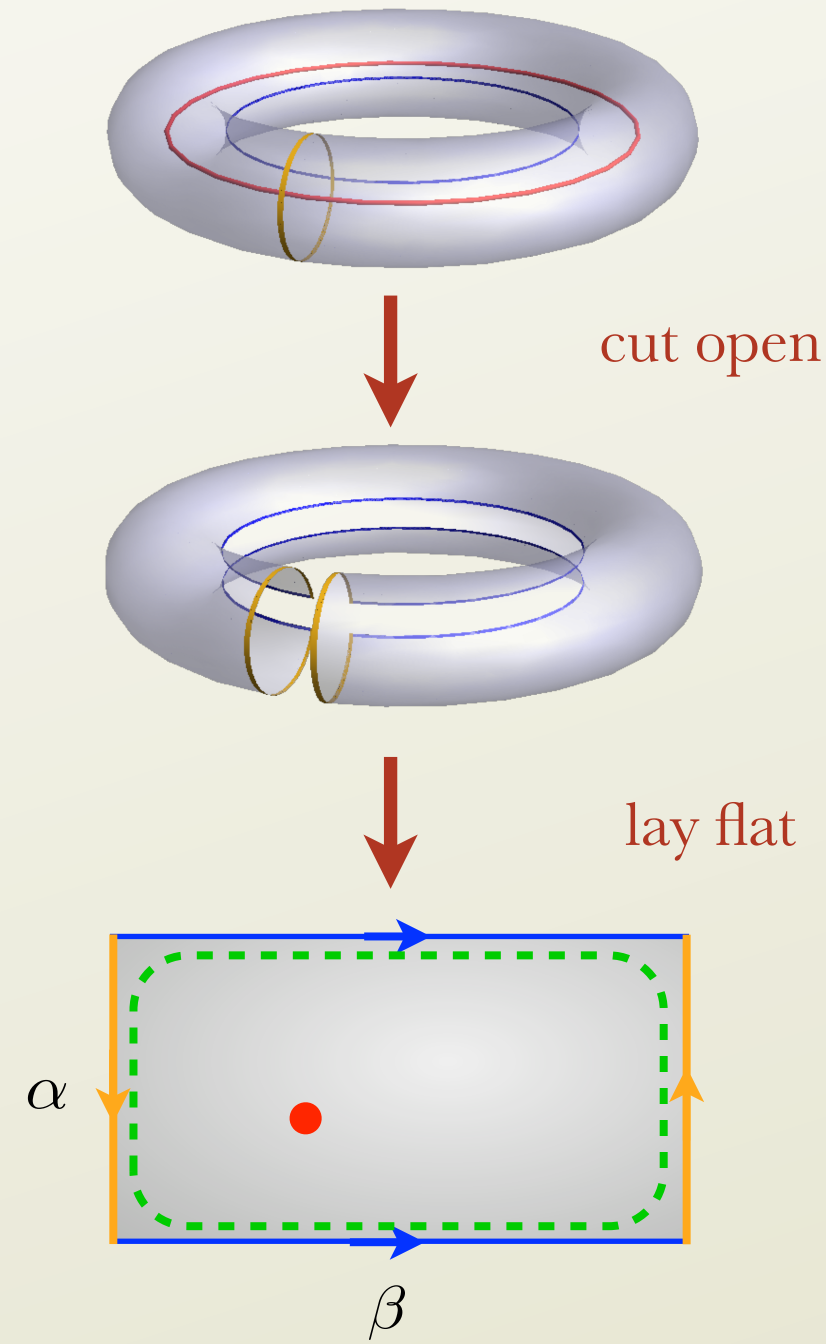
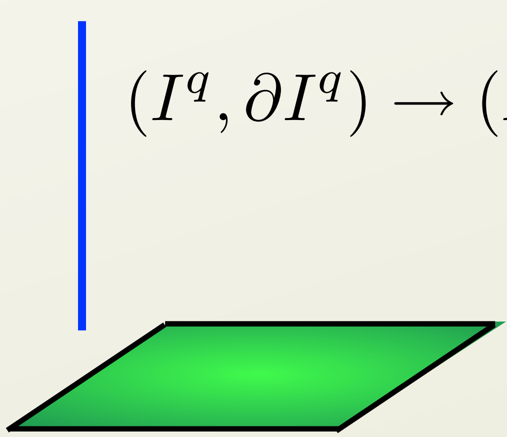
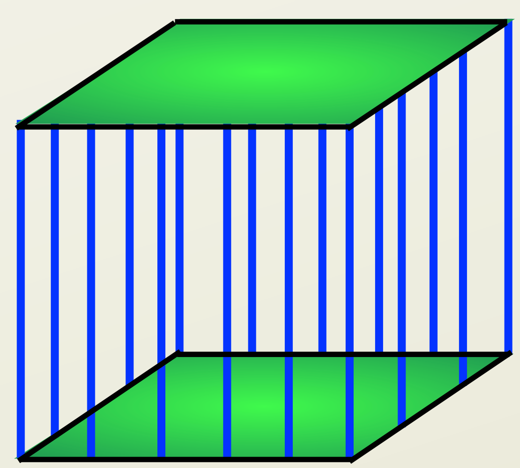




$(I^q, \partial I^q) \to (X, x_0)$

$(I^p, \partial I^p) \to (X, x_0)$

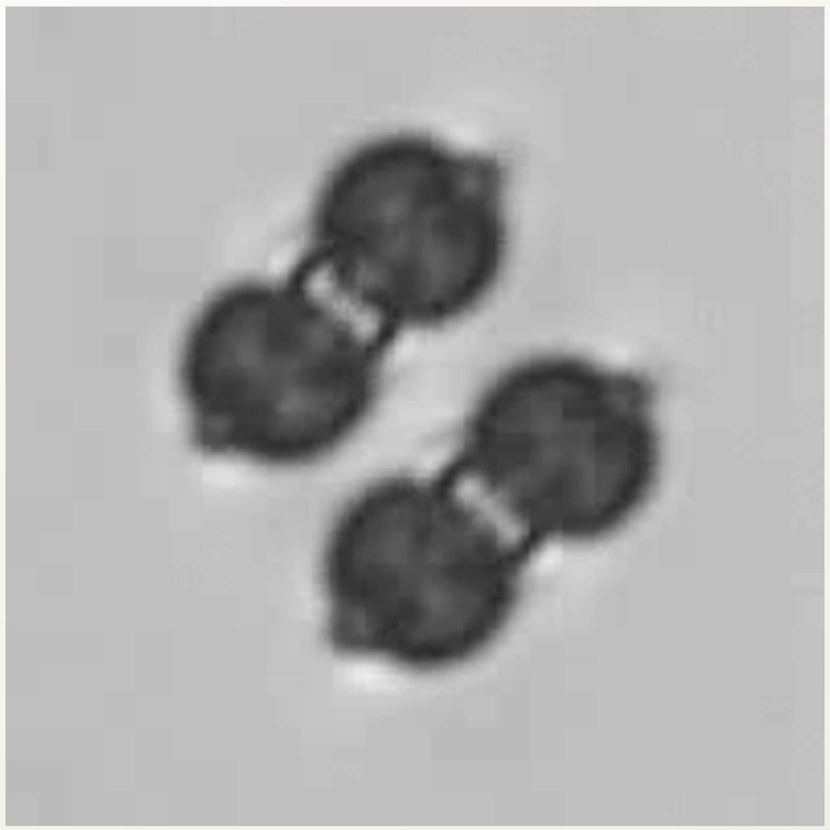


$\partial I^{p+q} \to (X, x_0)$

WHITEHEAD *Ann. of Math.* **42**, 409–428 (1941)

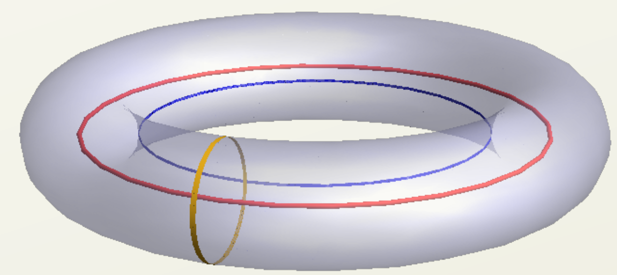
DISCLINATION LOOPS AND HEDGEHOG CHARGE

line and point defects coexist



TKALEC ET AL *Science* **333**, 62–65 (2011)

what is the hedgehog charge of a disclination loop?



classify using both $\pi_1(X)$ and $\pi_2(X)$

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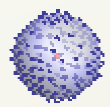
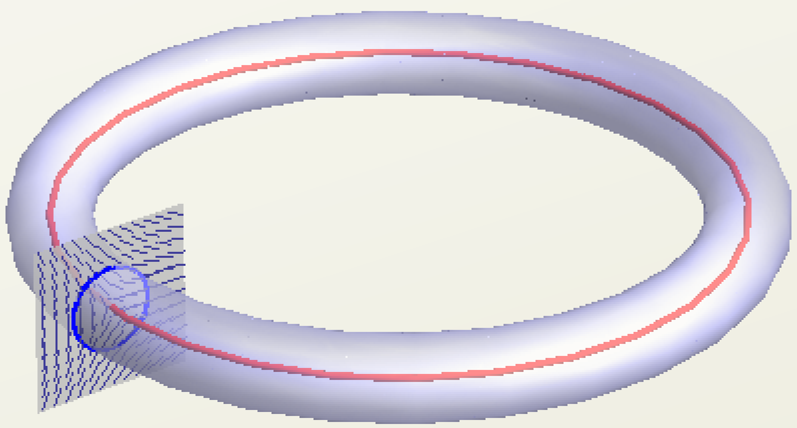
MOVING A HEDGEHOG AROUND A DISCLINATION

the wicked ways of an evil experimentalist

hedgehog

$p \in \pi_2(\mathbb{RP}^2, x_0)$





$-1 \in \pi_1(\mathbb{RP}^2, x_0)$

disclination loop

what's its charge?

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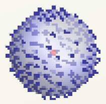

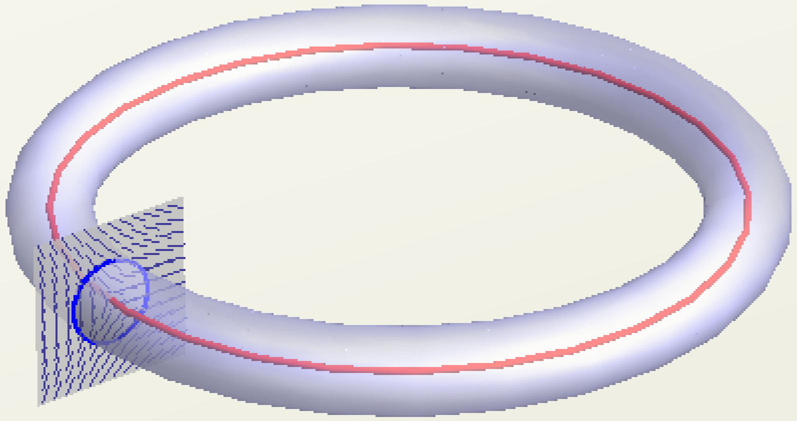

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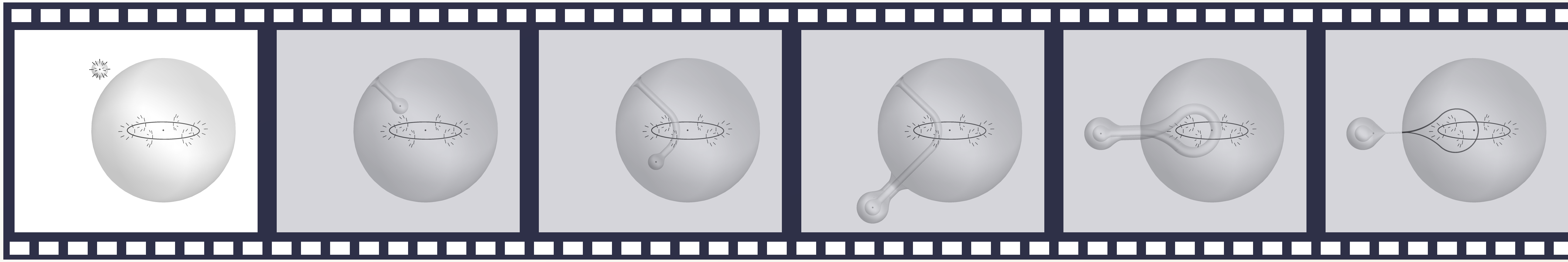

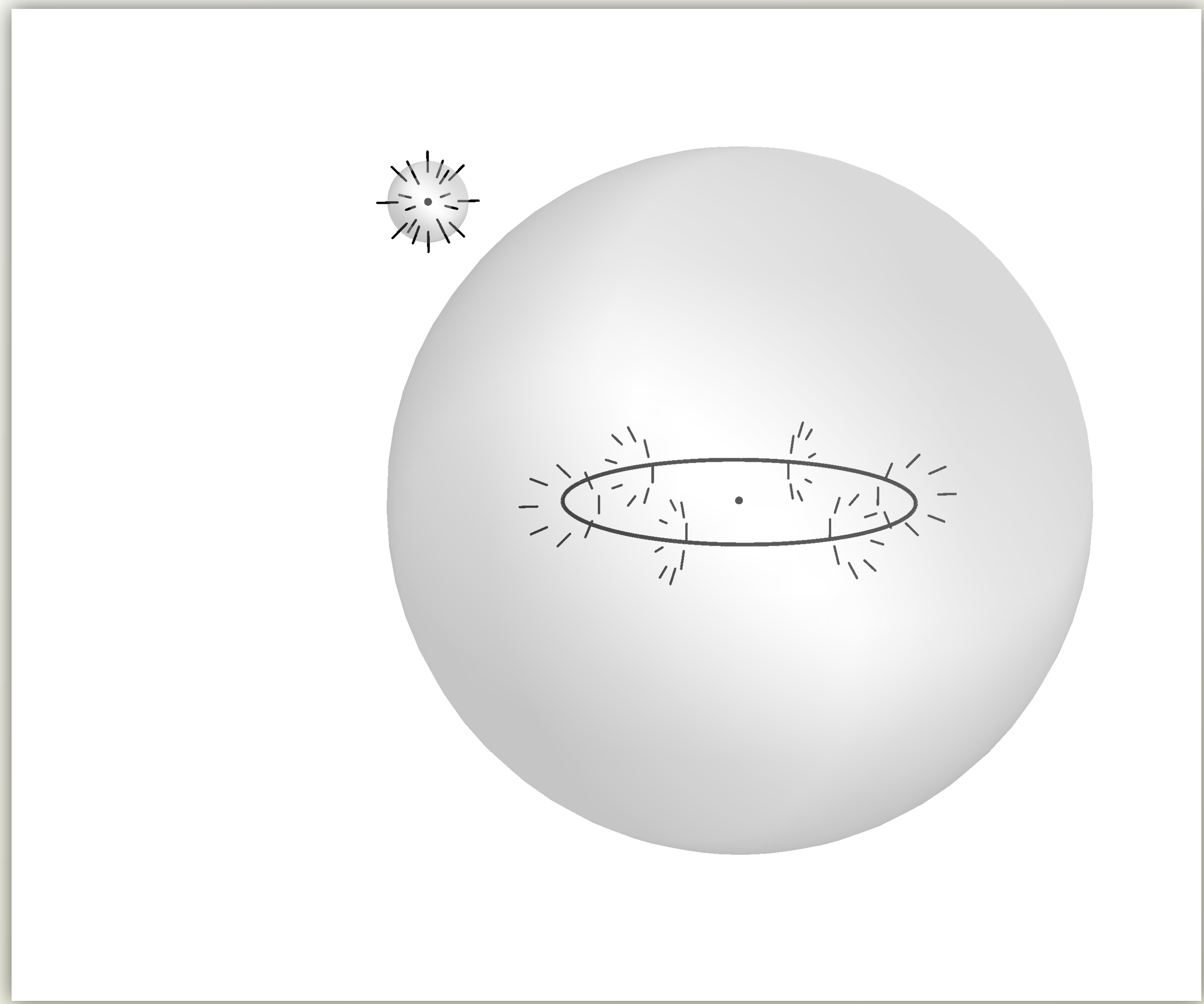

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images: Sabetta Matsumoto

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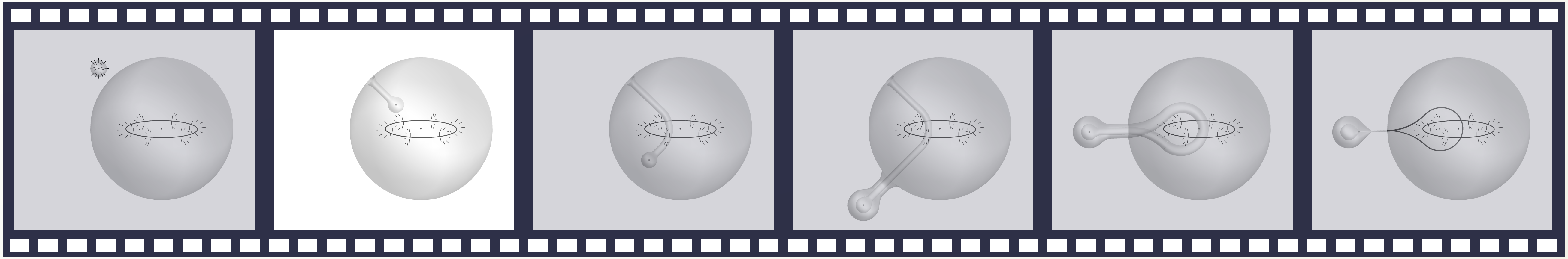

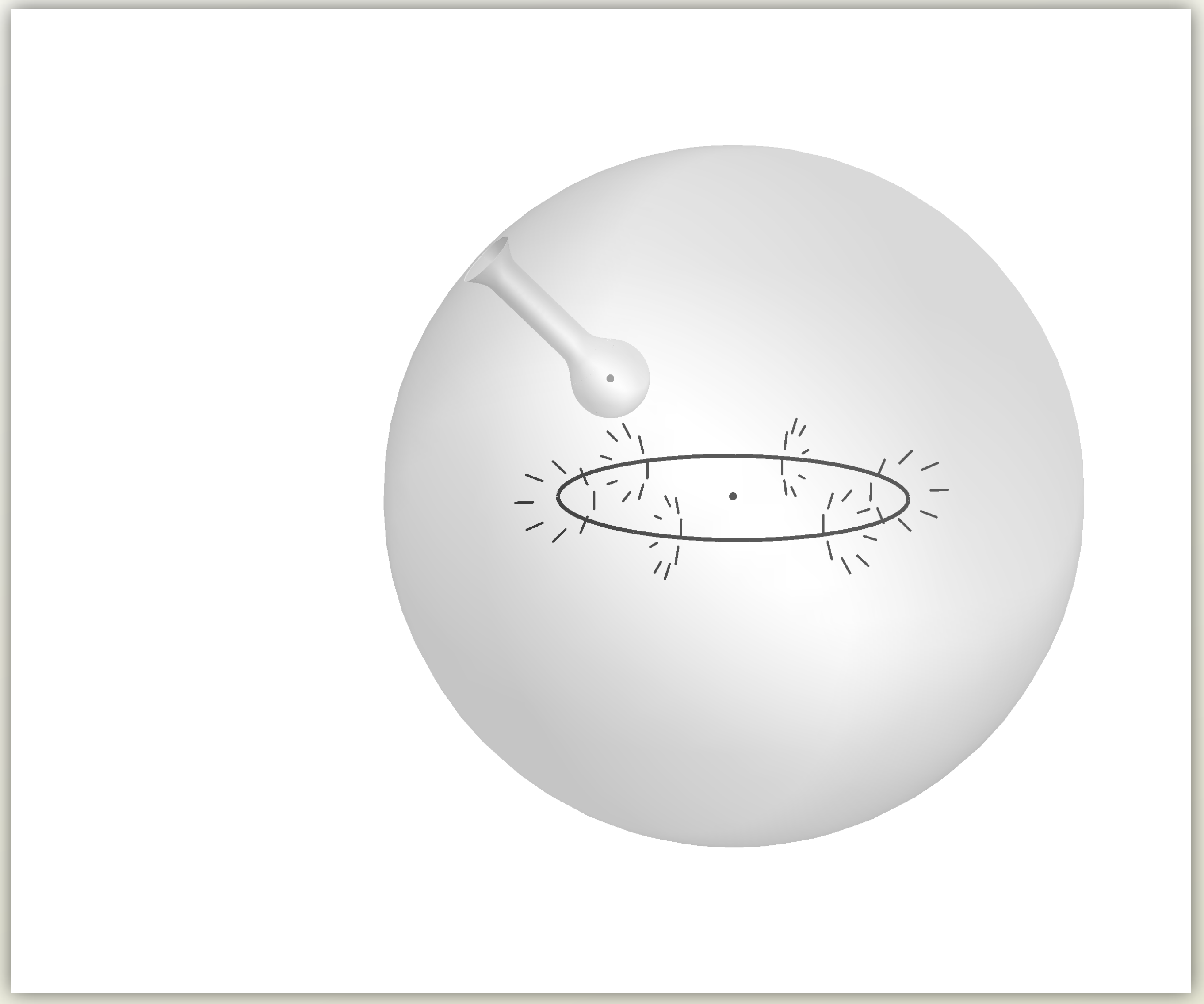

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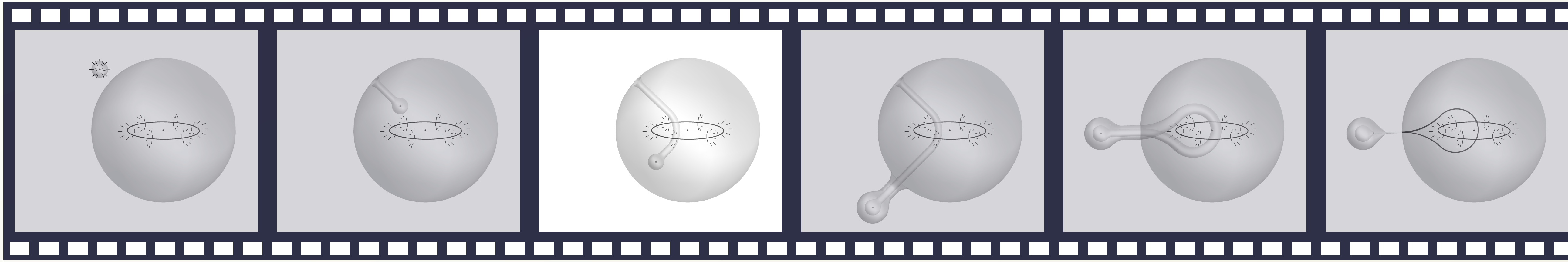

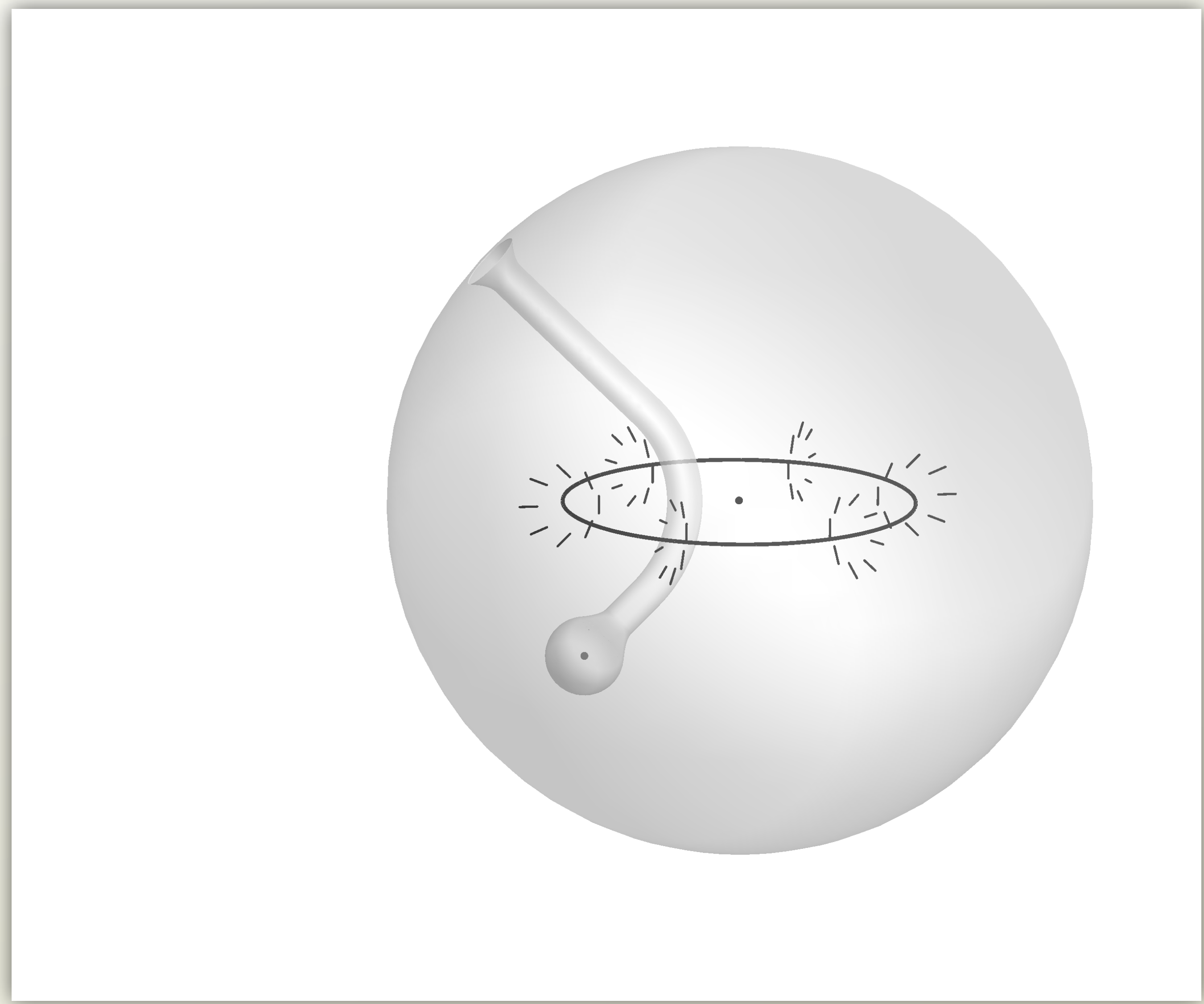

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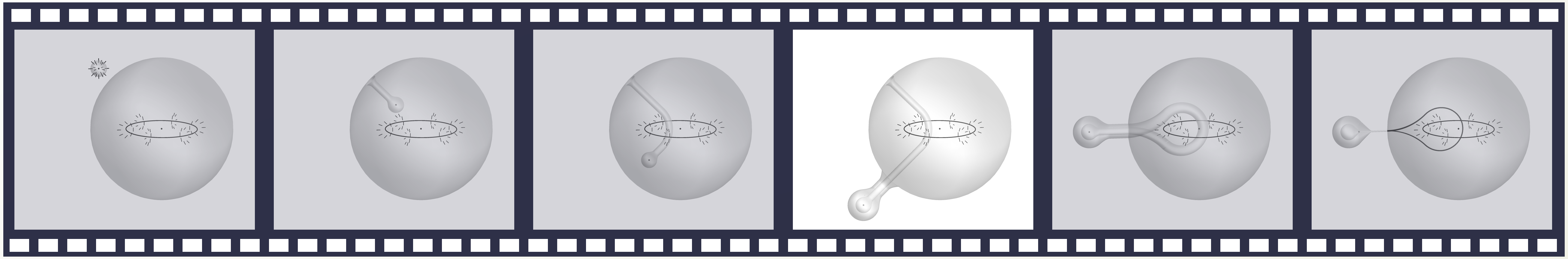

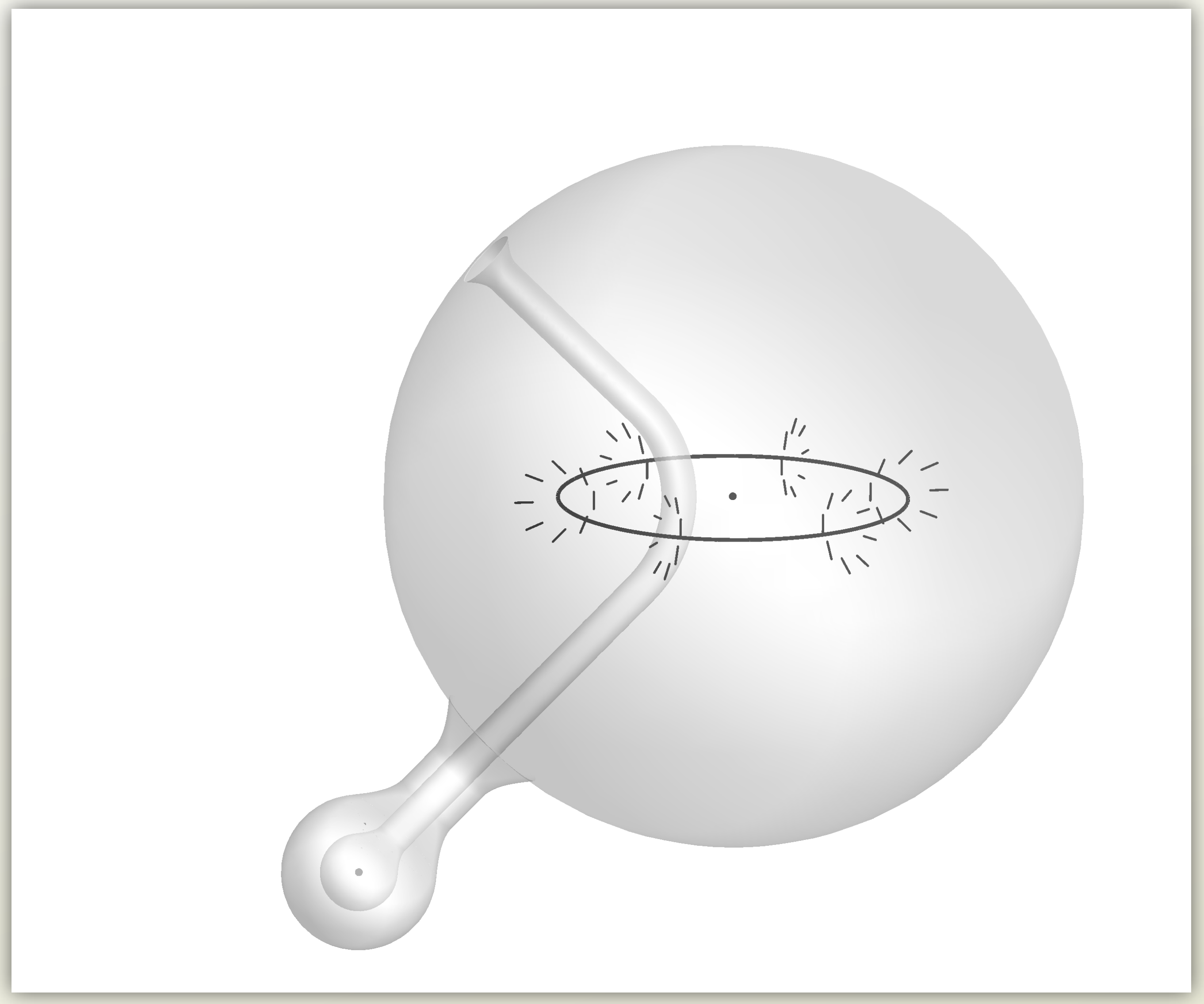

images: *Sabetta Matsumoto*

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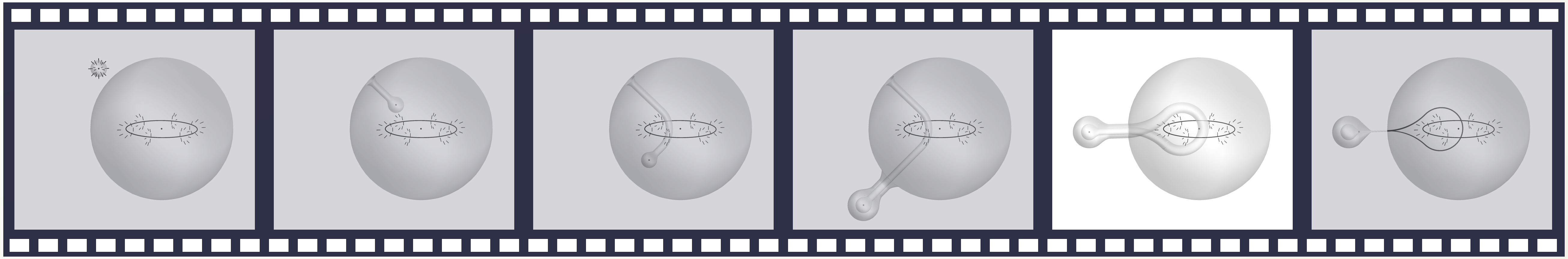

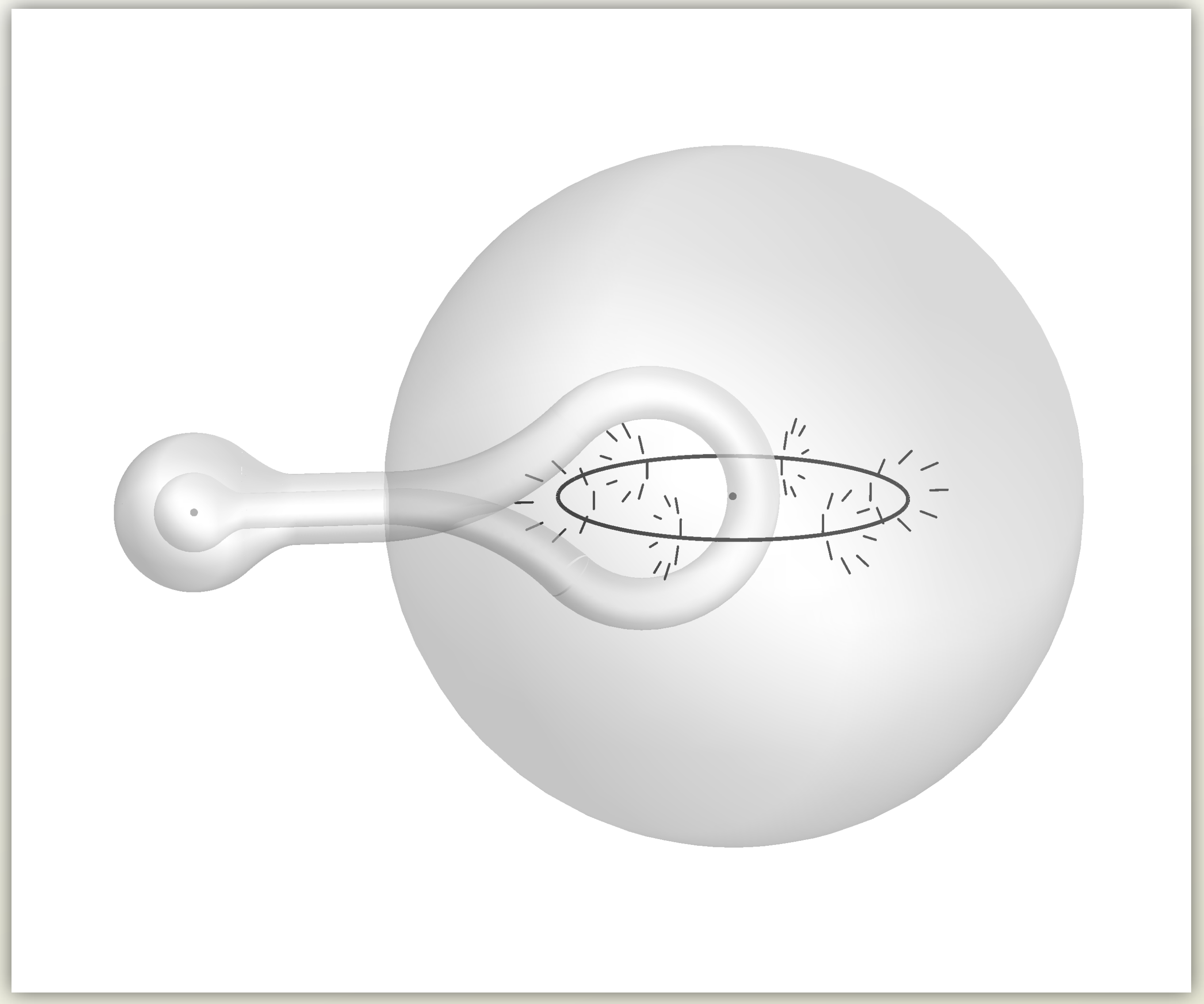

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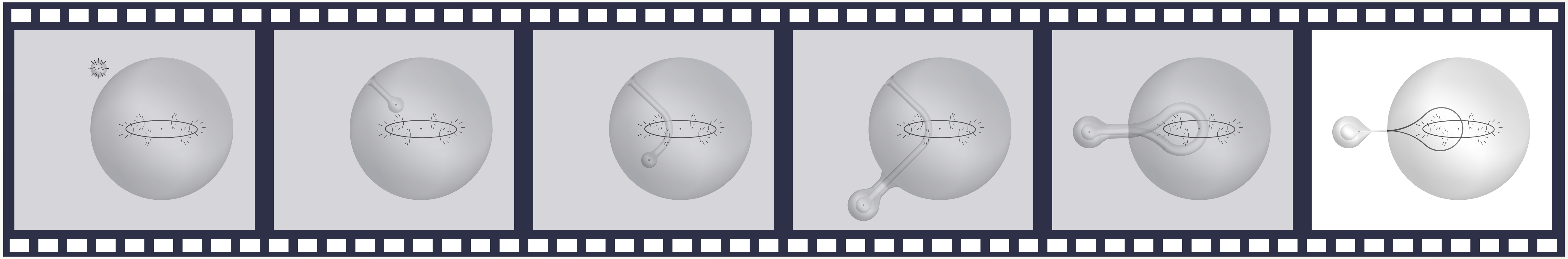

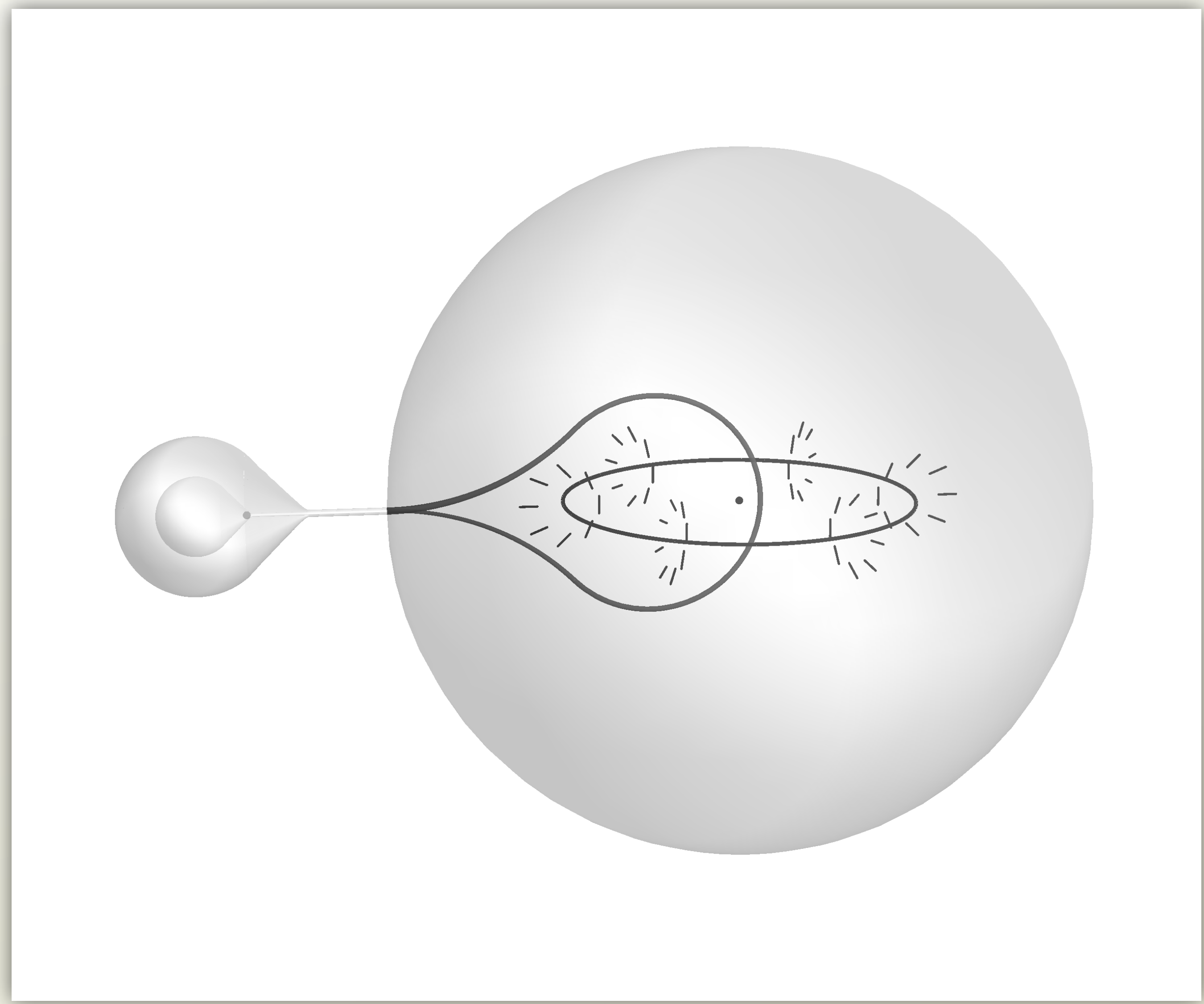

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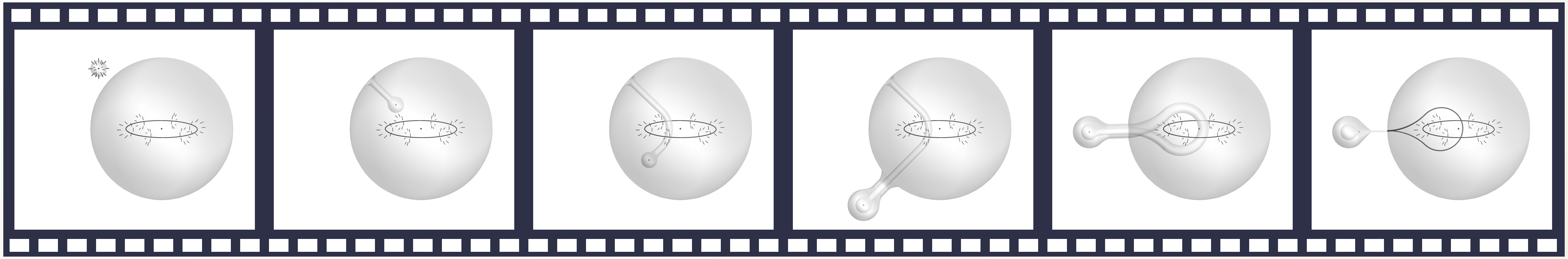

motion of the hedgehog removes charge from the disclination loop $[-1, p] = 2p$

Whitehead product

disclination loops only carry a hedgehog charge $\mathbb{Z}/2\mathbb{Z}$

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Jänich *Acta Appl. Math.* **8**, 65–74 (1987)
GPA, Chen, Matsumoto, Kamien *Rev. Mod. Phys.* **84**, 497–514 (2012)

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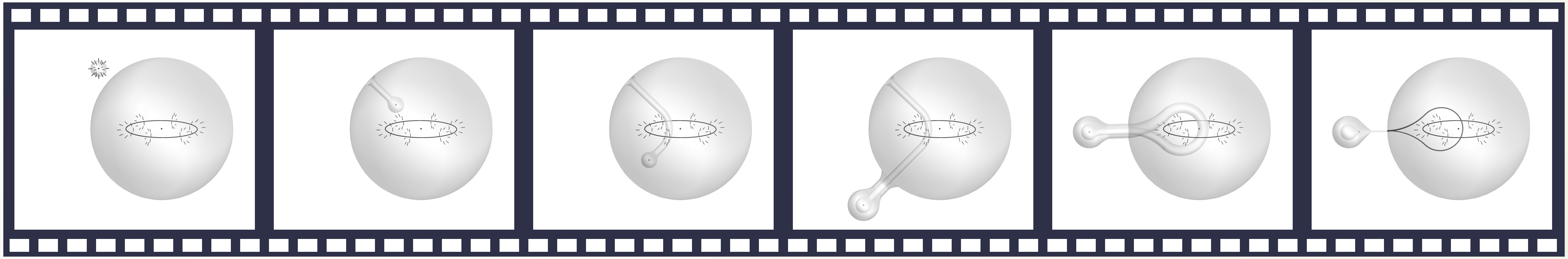

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Whitehead product

disclination loops only carry a $\mathbb{Z}/2\mathbb{Z}$ **hedgehog charge**

there are four types of disclination loops: ***linked or unlinked, and even or odd hedgehog charge***

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TOPOLOGY OF DISCLINATION LOOPS

- Neighbourhood of a disclination loop is a ***torus*** $S^1 \times S^1$
- Each meridian is a non-trivial map $S^1 \to \mathbb{RP}^2$
- Disclination loops are characterised by how this local texture changes around the torus

$$S^1 \to \mathrm{map}^{(-1)}(S^1, \mathbb{RP}^2)$$

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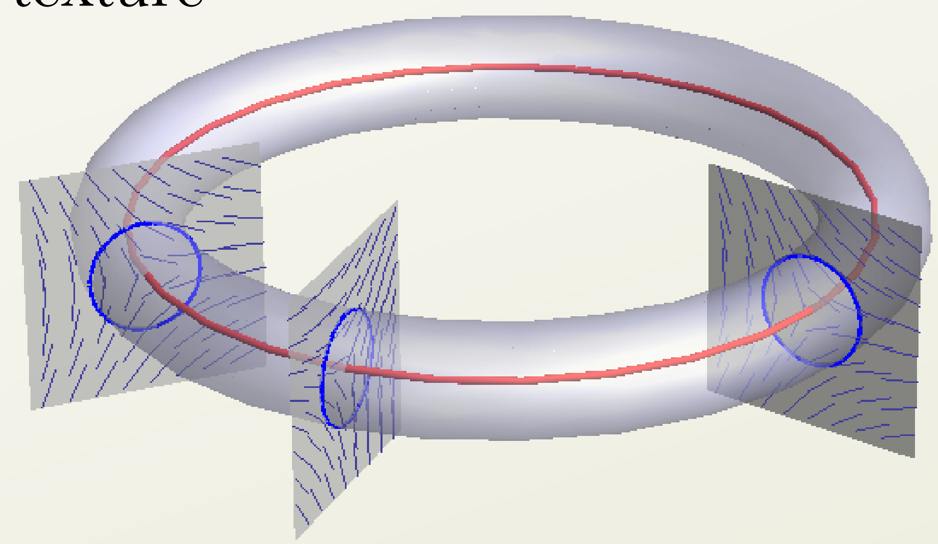

Whitehead *Ann. Math.* **47**, 460–475 (1946)
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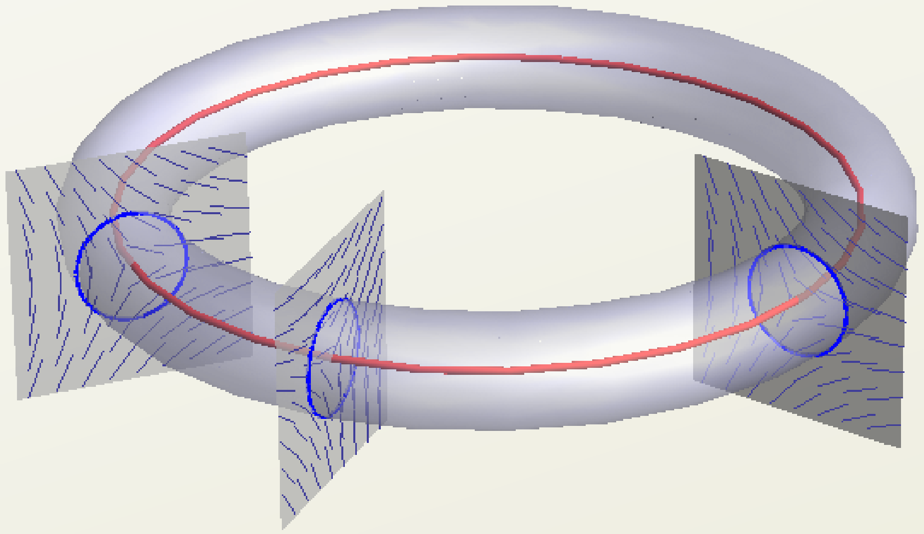

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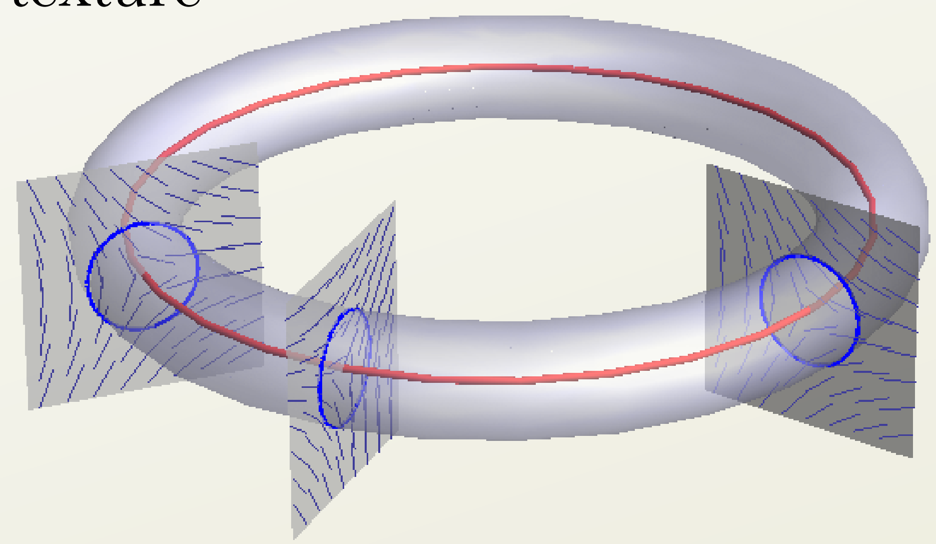

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$$\cdots \to \pi_2(\mathbb{RP}^2) \to \pi_1(\mathrm{map}_*^{(-1)}(S^1, \mathbb{RP}^2)) \to \pi_1(\mathrm{map}^{(-1)}(S^1, \mathbb{RP}^2)) \to \pi_1(\mathbb{RP}^2) \to \cdots$$

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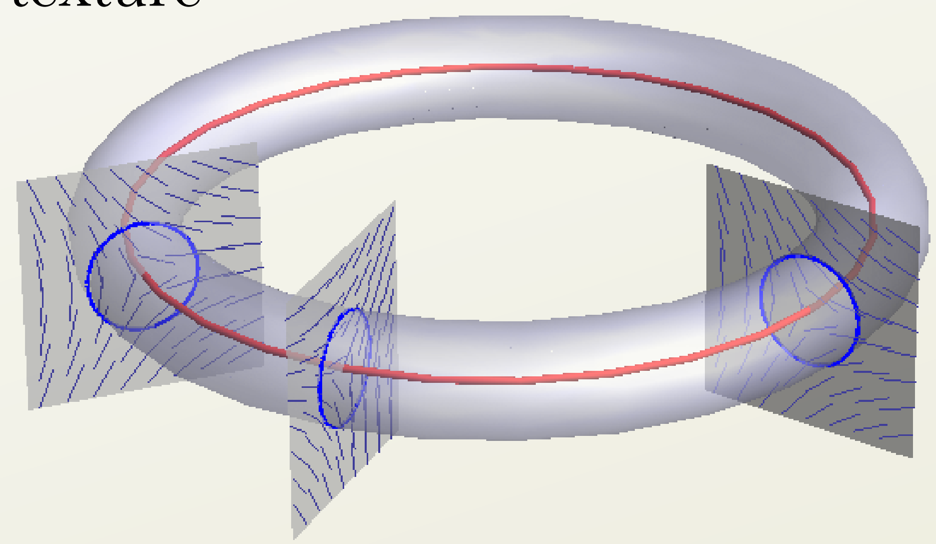

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$$\pi_2(\mathbb{RP}^2)$$

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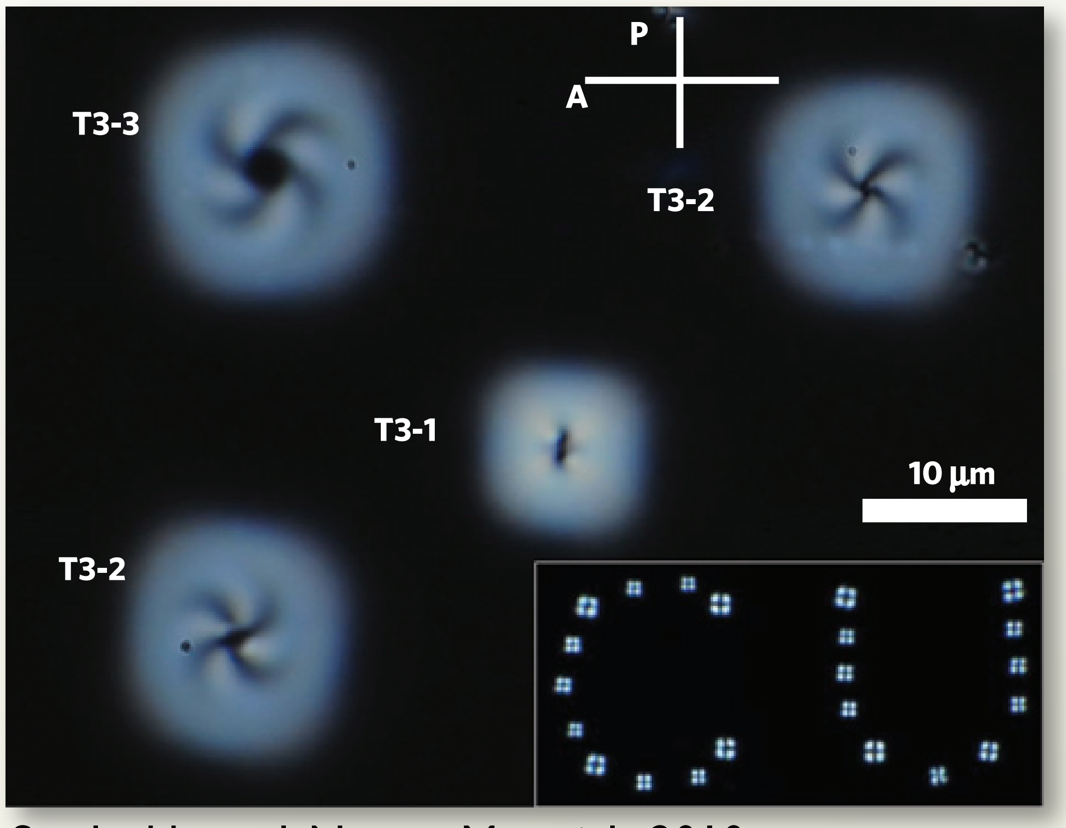
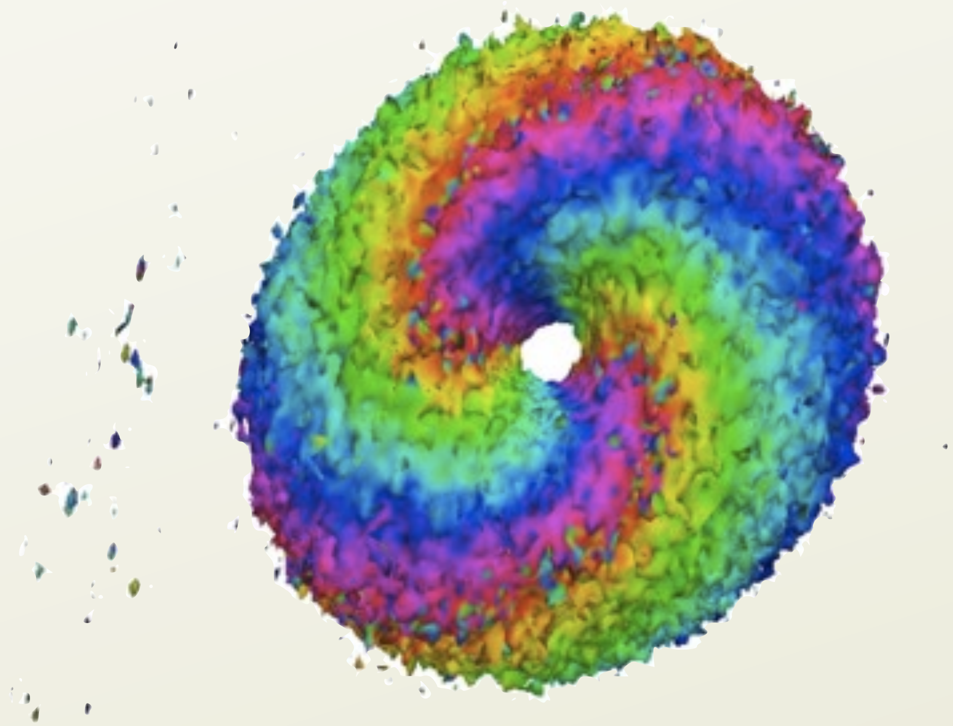
TEXTURES

textures are non-trivial maps $S^3 \to \mathbb{RP}^2$

characterised by a Hopf charge $h \in \pi_3(\mathbb{RP}^2) = \mathbb{Z}$



Smalyukh et al, Nature Materials 2010



Experimental data: Ackerman and Smalyukh

think of a texture as a point defect in $\mathbb{R}^4$ (S^4)

what about line defects?

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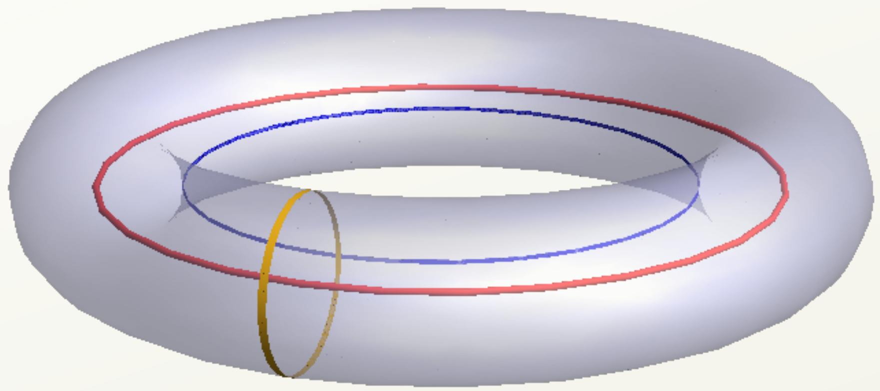
GEDANKEN EXPERIMENT: *HEDGEHOG LOOPS*

think of a line defect in $\mathbb{R}^4$ (S^4) **a *hedgehog loop***



orange sphere measures $p \in \pi_2(\mathbb{RP}^2)$

blue circle measures $\alpha \in \pi_1(\mathbb{RP}^2)$

enclose in a 3-sphere and measure a Hopf index $h \in \pi_3(\mathbb{RP}^2)$

Whitehead product $\pi_2(\mathbb{RP}^2) \times \pi_2(\mathbb{RP}^2) \to \pi_3(\mathbb{RP}^2)$

$p \qquad q \qquad [p, q] = 2pq$

hedgehog loop only carries a Hopf charge $\mod 2p$

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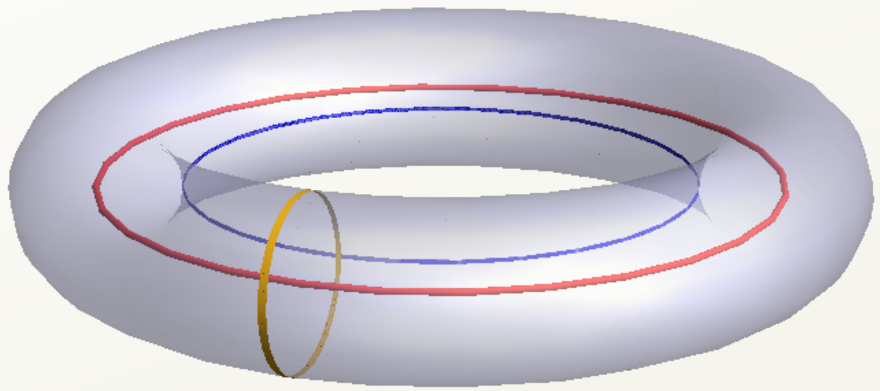

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periodic textures in $\mathbb{R}^3$ ***??***

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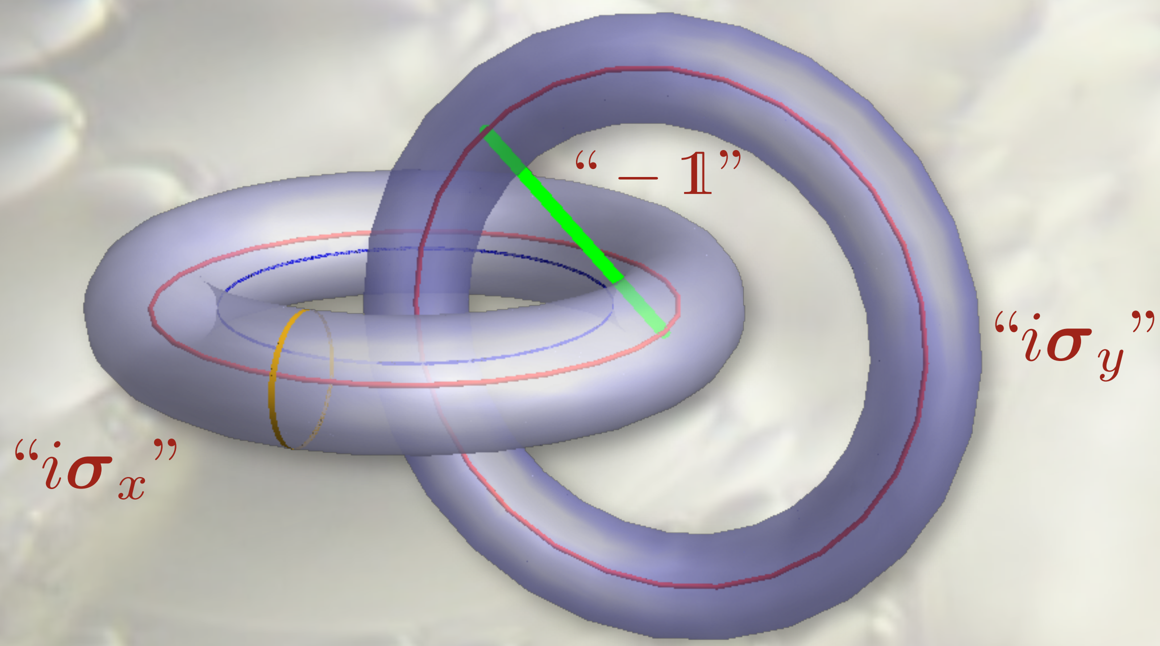
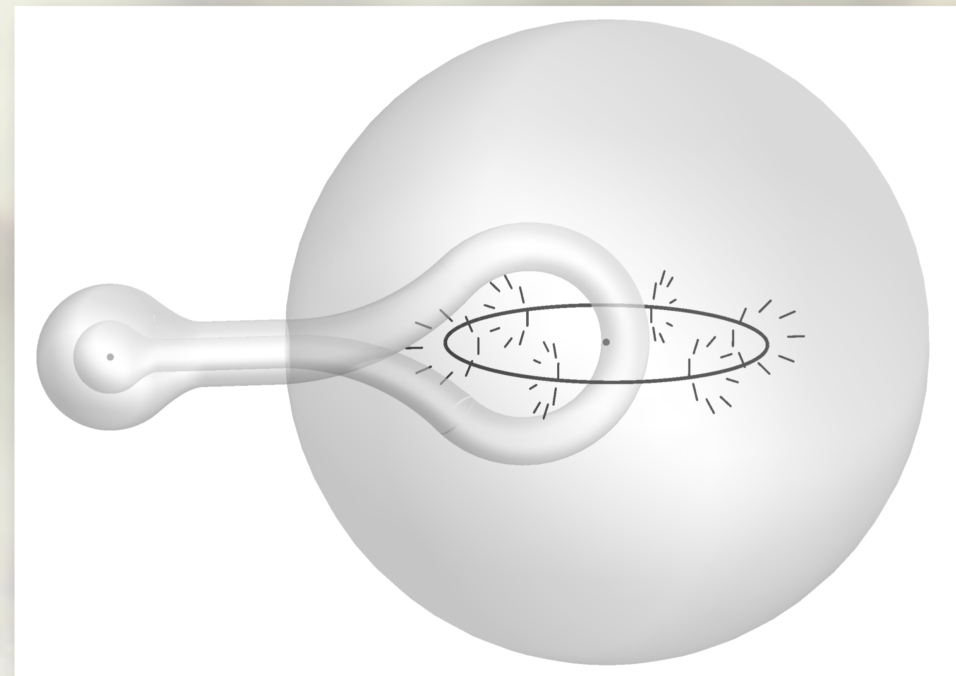
THANKS!

Bryan Gin-ge Chen, Elisabetta Matsumoto, Randall Kamien





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FUNDING

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NSF DMR05-47230, DMR05-20020 (Penn MRSEC), EPSRC