Braid groups, their applications and connections

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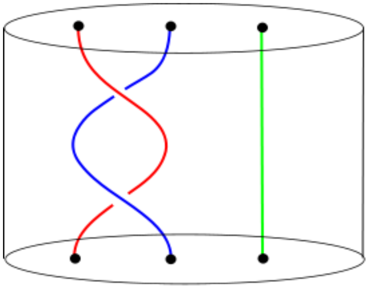
Introduction:

- Artin's braid groups are at the confluence of several basic mathematical structures.
- The purpose of this lecture is to illustrate one such confluence by comparing "crude" motions of points in the plane to analogous motions of points in the two-dimensional sphere, and to give topological features of these motions, features to be made precise below.
- The focus here is on properties of spaces of braids and their fundamental groups rather than the geometry of individual braids.

A (crude) picture of a braid obtained from 3 particles moving through time in a plane:



- Three particles mill about, but do not collide.

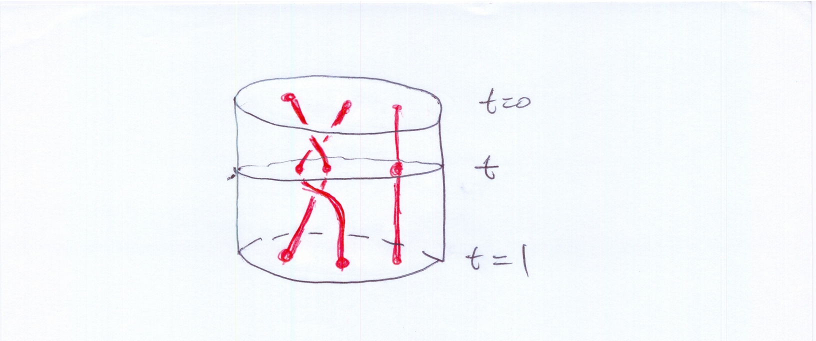
Motivation:

- The main point of this lecture is that particles “milling about” in certain prescribed ways correspond to several different, related subjects.
- Classical “discrete” structures given by the homotopy groups of spheres as well as modular forms arise naturally arise by keeping track of these motions.

- The purpose of this lecture is to illustrate these two connections.

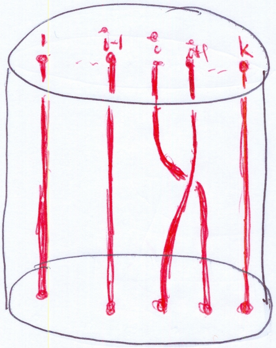
Artin's braid groups and paths of distinct points in the plane:

- Artin's braid group can be though of as paths of particles of points in the plane as the particles move from time $t = 0$ to time $t = 1$ for which the particles are not allowed to collide, that is the space of paths of particles in the configuration space.
- A picture returns:



A picture of certain natural elements in Artin's braid group:

- Consider the braids σ_i for $1 \leq i \leq k-1$ pictured next.



Artin's presentation of the braid group:

- The braid group on k strands denoted

$$Br_k = Br_k(\mathbb{R}^2)$$

is generated by elements

$$\sigma_i$$

for $1 \leq i \leq k-1$,

- with a complete set of relations given by

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $1 \leq i \leq k-2$, and

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $|i-j| \geq 2$.

Configuration spaces:

- The braid group is the fundamental group of certain 'unordered' configuration spaces to be defined shortly.

- The configuration space of **ordered k-tuples of distinct points in a space M** is

$$\mathrm{Conf}(M,k) = \{(m_1,\ldots,m_k) \in M^k \mid m_i \neq m_j \text{ if } i \neq j\}.$$

Configuration spaces continued:

- The symmetric group on k-letters

$$\Sigma_k$$

acts naturally on $\mathrm{Conf}(M, k)$ by permutation of coordinates.

- The orbit space

$$\mathrm{Conf}(M, k)/\Sigma_k$$

is the "unordered" configuration space sometimes called 'unlabeled'.

- In case $M = \mathbb{C}$, this last space would have been familiar to you in high school as we are about to see !

A basic example:

- Consider the space of all monic, complex polynomials of degree k

  $$f(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0 = (z - r_1)\cdots(z - r_k)$$

  denoted

  $$\mathsf{Polynomials}_{\mathbf{k}}(z) = \{f(z)\}.$$

A basic example continued:

- By identifying a polynomial $f(z)$ with its coefficients, the space of polynomials

  $$\text{Polynomials}_{\mathbf{k}}(z)$$

  is homeomorphic to complex k-dimensional space

  $$\mathbb{C}^k.$$

- By identifying a polynomial with its' roots, the space $\text{Polynomials}_{\mathbf{k}}(z)$ is homeomorphic to the k-fold symmetric product

  $$\mathrm{Sym}^k(\mathbb{C}) = (\mathbb{C}^k)/\Sigma_k$$

  the space of *unordered* roots of the polynomials.

A basic example continued:

- Thus the space $\mathrm{Polynomials}_{\mathbf{k}}(z)$ is homeomorphic to both $\mathbb{C}^k$ and to $(\mathbb{C}^k)/\Sigma_k$ via the homeomorphism

$$\mathrm{roots} : (\mathbb{C}^k)/\Sigma_k \to \mathbb{C}^k$$

which sends the roots of a polynomial to the coefficients, the elementary symmetric functions of the roots (up to sign).

A basic example continued:

- A homework problem of Galois was to see how the homeomorphism

$$\text{roots} : (\mathbb{C}^k)/\Sigma_k \to \mathbb{C}^k$$

can be inverted !

A basic example continued:

- Restrict to

$$\mathsf{Conf}(\mathbb{C}, k)/\Sigma_k \subset (\mathbb{C}^k)/\Sigma_k$$

to see that this configuration space

$$\mathsf{Conf}(\mathbb{C}, k)/\Sigma_k$$

is *the space of monic, complex polynomials of degree* k with k distinct roots.

Braid groups for other surfaces:

- A braid can be thought of as the graph of a loop in the configuration space

$$\text{Conf}(\mathbf{S}, k)/\Sigma_k$$

for any surface **S**.

- The k-stranded braid group of a surface $\mathbf{S}$

$$Br_k(\mathbf{S})$$

is the fundamental group of the 'unordered' configuration space $\text{Conf}(\mathbf{S}, k)/\Sigma_k$.

Braids on the Riemann sphere:

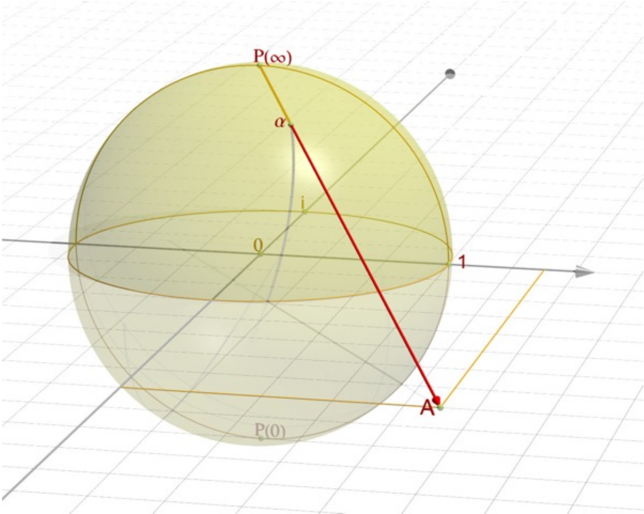
- The next few sections compare the braid groups for the complex numbers $\mathbb{C}$ or the Riemann sphere

$$\mathbb{C} \cup \{\infty\} = S^2.$$

- The Riemann sphere is also identified as the space of complex lines through the origin in $\mathbb{C}^2$

$$\mathbb{CP}^1.$$

The Riemann sphere:

▶ 

Configurations in the Riemann sphere:

- The configuration space

$$\mathrm{Conf}(S^2, 3)$$

is a disguise for

$$PSL(2, \mathbb{C})$$

dating back to classical work of Poincaré.

Configurations in the Riemann sphere:

- That is, consider the configuration given by

$$(0, 1, \infty) \in \mathsf{Conf}(S^2, 3).$$

- The map which sends a point

$$\rho \in PSL(2, \mathbb{C})$$

to

$$(\rho(0), \rho(1), \rho(\infty))$$

gives a homeomorphism

$$PSL(2, \mathbb{C}) \to \mathsf{Conf}(S^2, 3).$$

Configurations in the Riemann sphere:

Thus there are homeomorphisms

- $$PSL(2, \mathbb{C})/SO(3) \to \mathrm{Conf}(S^2, 3)/SO(3)$$

  where $SO(3)$ is regarded as the maximal compact subgroup,

- $$PSL(2, \mathbb{C})/SO(3) \to \mathbb{H}^3$$

  one sheet of hyperbolic three space.

Configurations in the Riemann sphere:

The notation

$$\mathbb{H}^3$$

means one sheet of hyperbolic three space given by

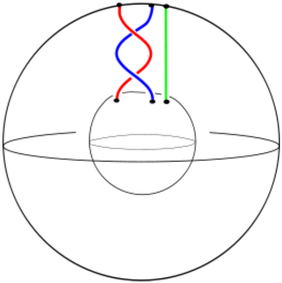
$$\{(t, x, y, z) \in \mathbb{R}^4 \mid t > 0,\ t^2 - (x^2 + y^2 + z^2) = 1\}.$$

Configurations in the Riemann sphere:

- Furthermore, if $k \geq 3$, there are homeomorphisms

$$\mathsf{Conf}(S^2, k)/SO(3) \to \mathbb{H} \times \mathsf{Conf}(\mathbb{C} - \{0, 1\}, k - 3).$$

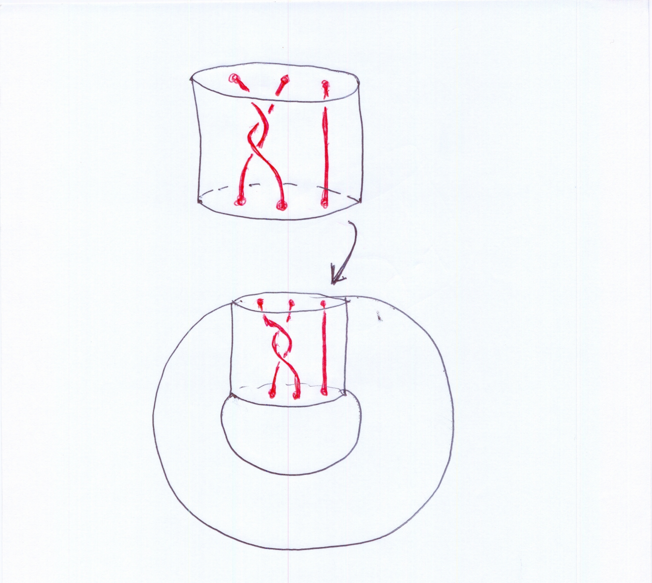
A three-stranded braid for the Riemann sphere:



The map from the braid group for the plane to the braid group for the Riemann sphere:

One goal of this lecture is to compare braids for the plane $\mathbb{C}$ to those for the Riemann sphere S^2.

A picture of the map from the braid group for the plane to the braid group for the Riemann sphere:



The map from the braid group for the plane to the braid group for the Riemann sphere again:

- The homomorphism

$$Br_k(\mathbb{R}^2) \to Br_k(S^2)$$

  has a kernel in case $k \geq 3$.

- How and why is this interesting ?

- The next slide is the starting point of how these features 'fit' together.

Borromean “creatures” (usually ‘Borromean rings’:)

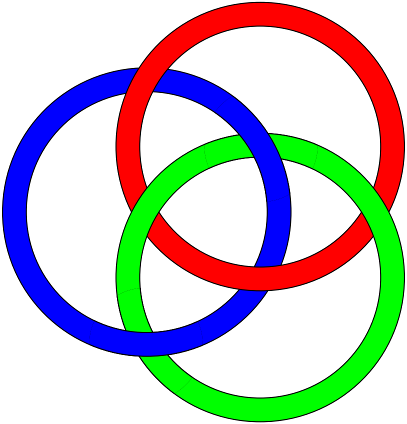


Figure: The Borromean rings

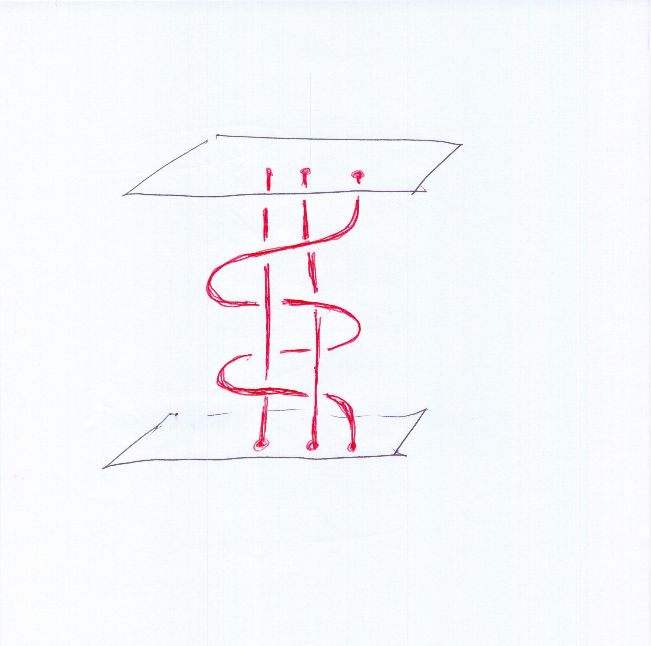
Borromean braids for the plane and for the Riemann sphere:

- One property of the Borromean rings is that if any circle is removed, the remaining pair of circles is unlinked.
- Define the Borromean k-stranded braids for a surface S as those k-stranded braids which "become trival" if any strand is deleted.
- The group of Borromean k-stranded braids is

$$\mathsf{Borr}_k(S).$$

,

An example of a Borromean braid for the plane:



Borromean braids for the plane and for the sphere continued:

- The group of Borromean braids

$$\mathrm{Borr}_k(\mathbf{S})$$

is usually a countably infinitely generated free group.

- One "small example" is

$$\mathrm{Borr}_4(S^2)$$

which is also the principal congruence subgroup of level 4 in $SL(2,\mathbb{Z})$.

Borromean braids for the plane and for the sphere continued:

- The first homework problem is to compare

$$\mathrm{Borr}_k(\mathbb{C}) \to \mathrm{Borr}_k(S^2).$$

induced by the natural inclusion

$$\mathbb{C} \to S^2 = \mathbb{C} \cup \infty,$$

and

$$\mathrm{Conf}(\mathbb{C}, k) \to \mathrm{Conf}(S^2, k).$$

- These maps are the subject of joint work with J. Berrick, Y. Wong, J. Wu and the speaker which are discussed next.

A theorem about Borromean braids in the plane and in the sphere:

Assume that $k \geq 5$.

- The image of

$$\text{Borr}_k(\mathbb{C}) \to \text{Borr}_k(S^2).$$

induced by the natural inclusion

$$\mathbb{C} \to S^2 = \mathbb{C} \cup \infty.$$

is a normal subgroup with quotient group given by

$$\pi_{k-1} S^2$$

the $(k-1)$-st homotopy group of the two sphere.

Some background:

Assume that $k \geq 5$.

- The homotopy groups of spheres are currently not well understood, and have been the subject of much work by many people.
- One theorem of Serre (in his thesis) gives that

  $$\pi_{k-1}S^2$$

  is a finite abelian group for all $(k-1) > 3$. Thus the image of $\mathrm{Borr}_k(\mathbb{C})$ in $\mathrm{Borr}_k(S^2)$ is a finite index subgroup.
- **Homework:** Identify a more direct way to see this fact or to identify the size of $\pi_{k-1}S^2$ arising from geometric properties of braids.

Some background continued:

Assume that $k \geq 5$.

- Write
$$\mathrm{Borr}_k(S^2)/\mathrm{Borr}_k(\mathbb{C})$$
for the quotient of $\mathrm{Borr}_k(S^2)$ modulo the image of $\mathrm{Borr}_k(\mathbb{C})$.

- Thus the groups $\mathrm{Borr}_k(S^2)/\mathrm{Borr}_k(\mathbb{C})$ are finite abelian.

- In addition, these finite abelian groups have no 2-torsion elements of order ≥ 8 by work of I. M. James or p-torsion of order p^2 for p an odd prime by results of either P. Selick, or Moore, Neisendorfer, and the speaker.

A theorem about Borromean braids in the plane and in the sphere continued:

Assume that $k \geq 5$.

- There is an exact sequence

$$1 \to \mathrm{Borr}_{k+1}(S^2) \to \mathrm{Borr}_k(\mathbb{C}) \to \mathrm{Borr}_k(S^2) \to \pi_{k-1}(S^2) \to 1.$$

Remarks, homework, as well as **wild speculation**:

- Explicit computations of $\pi_{k-1}S^2$ are similar in flavor to calculating values of the classical partition function (the number of ordered partitions of a fixed integer).

- This remark suggests forming the generating function

$$\Theta(t) = 1 + \sum_{t \geq 4} (1/k!)|\mathsf{Borr}_k(S^2)/\mathsf{Borr}_k(\mathbb{C})|t^k$$

  where $|\mathsf{Borr}_k(S^2)/\mathsf{Borr}_k(\mathbb{C})|$ denotes the order of the (finite abelian) group.

Remarks, homework, as well as **wild speculation**:

- Is the function $\Theta(t)$ analytic ?
- Do the actions of $PSL(2, \mathbb{C})$ reflect non-trivial symmetry properties of the generating function $\Theta(t)$?
- Can the coefficients of $\Theta(t)$ be estimated via analytic methods as in the case of the classical partition function ?

Other spaces:

- A mild variation of these methods gives an analogous picture for all spheres.
- **However, these methods have not been useful for concrete computations.**

- **Homework:** Identify methods to keep track of the size of these groups. For example, automorphisms of surfaces may give invariance properties as exemplified next.

A final comment/accident:

- The braid groups have natural symplectic representations

$$Br_{2g+2}(\mathbb{R}^2) \to Sp(2g, \mathbb{Z})$$

which arise from diffeomorphisms of Riemann surfaces.

- A natural question is to ask about the cohomology of the braid group with coefficients in symmetric powers of this representation.

Calculations:

- The first case arises with a quotient of B_3 given by $SL(2,\mathbb{Z})$.
- There is a direct sum decomposition

  $$H^*(SL(2,\mathbb{Z});\mathbb{Z}[x,y]) = \mathcal{M} \oplus Tors$$

  where

  $$\mathcal{M}$$

  is a free abelian group while

  $$Tors$$

  is a torsion group.
- This case in cohomology was first considered in the 1950's by G. Shimura, or at least **the torsion free summand** of the cohomology groups

Calculations:

- The purpose of this section is to give the torsion in the cohomology groups

$$H^*(SL(2,\mathbb{Z});\mathbb{Z}[x,y])$$

obtained in joint work with M. Salvetti, and F. Callegaro.

- To do so, first recall that there exist p-local fibrations for each odd prime $p > 3$ given by

$$S^{2n-1} \to T_p(2n+1) \to \Omega(S^{2n+1})$$

in work of D. Anick as well as Gray-Theriault.

- In particular, there are spaces $T_p(2n+1)$ which are total spaces of p-local fibrations, $p > 3$.

Calculations:

- **Shimura's Theorem**: The graded abelian group

$$\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{C}$$

is given by the classical ring of modular forms based on the $SL(2, \mathbb{Z})$ action on the upper $\frac{1}{2}$-plane by fractional linear transformations.

Calculations:

Theorem 11

- If p is a prime with $p > 3$, then the p-torsion in

$$H^*(SL(2,\mathbb{Z});\mathbb{Z}[x,y])$$

is given by the reduced cohomology of

$$T(2p+3) \times T(2p^2 - 2p + 1).$$

- In case $p = 2, 3$, the answers are not quite as nice.

Remarks:

- The spaces $T_p(2n+1)$ have long been conjectured to deloop the fibre of the classical double suspension map

$$S^{2n-1} \to \Omega^2(S^{2n+1}).$$

- **Homework:** Explain this accident.

Thank you very much.

- Please remember to hand in the homework !