Topological invariants of framed knots in nematics

Simon Čopar

University of Ljubljana, Slovenia

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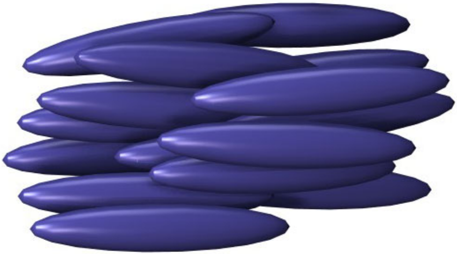
Supervisor: *prof. Slobodan Žumer*
Coworkers: *T. Porenta, D. Seč, M. Ravnik*
Collaborations: *U. Tkalec, V. S. R. Jampani, prof. I. Muševič*

Nematics and disclination lines

- Nematic liquid crystals: director member of $\mathbb{R}P^2$.
- π_1 differentiates only between *defect* and *nondefect*.
- In $2D$, defects are $\mathbb{Z}$: it seems to work in $3D$.
- In the company of colloidal particles, mostly closed $-1/2$ defect loops.
- Is the topological classification of a restricted system different?



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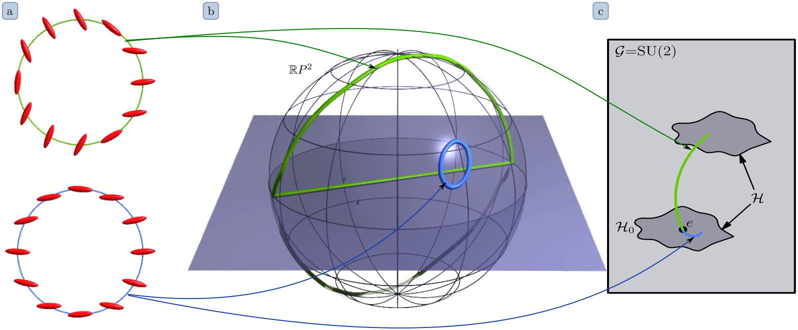

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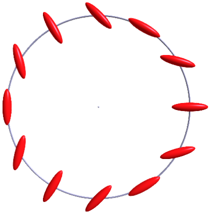
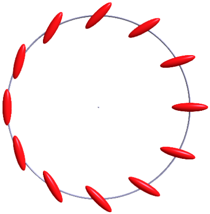
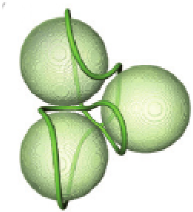
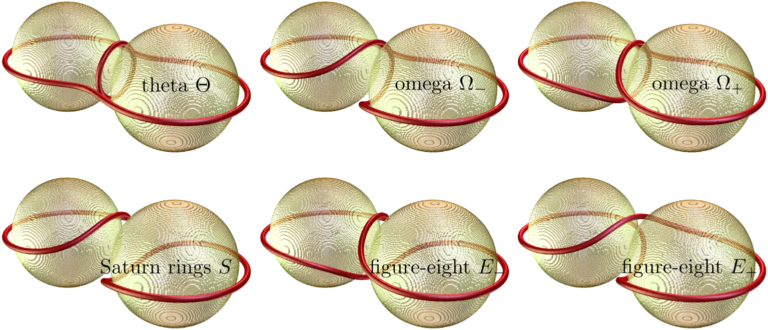




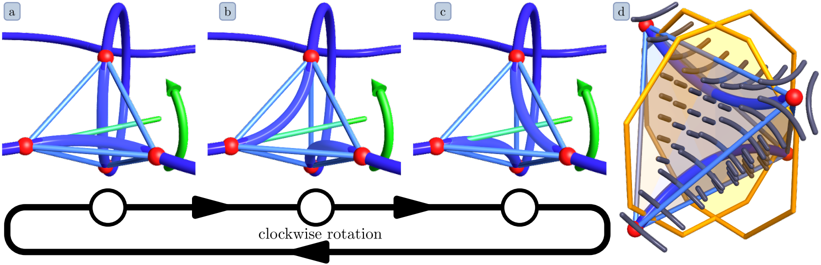


M. Ravnik and S. Žumer, Soft Matter **5**, 4520 (2009)


theta Θ
omega Ω_-
omega Ω_+
Saturn rings S
figure-eight E_-
figure-eight E_+

Tetrahedral rotations

- Surface texture fits in all orientations
- The mismatch of symmetry (T/D_{2d}) gives 3 orientations
- Only topological match required: real structures can be deformed



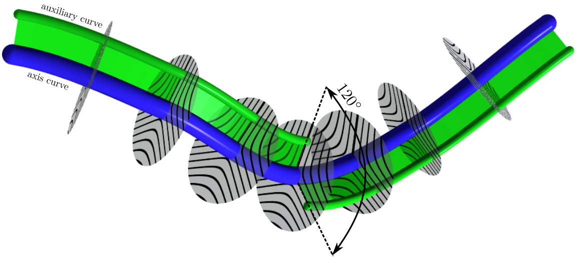
S. Čopar and S. Žumer, Phys. Rev. Lett. **106**, 177801 (2011)

Self-linking number

- Călugăreanu theorem: $Sl = Tw + Wr$
- Twist does not change under rewiring!
- Disclination symmetry $\mapsto$ 1/3 quantization of Sl
- Inversion symmetry $\mapsto$ everything is zero
- Why zero twist? Colloidal confinement!

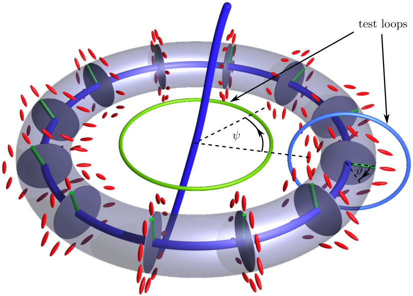
$$Lk, Sl, Wr = \oint \mathrm{d}\vec{r}_1 \times \mathrm{d}\vec{r}_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$Tw = \frac{1}{2\pi} \oint \vec{t}(s) \cdot (\vec{u}(s) \times \partial_s \vec{u}(s))\,\mathrm{d}s$$



Single-loop topology

- Torus homotopy
- π_1 of small circle: $-1/2$ cross section
- π_1 of large circle: depends on the Sl
- No linking: $Sl = \cdots, -\frac{2}{3}, 0, \frac{2}{3}, \frac{4}{3}, \cdots$
- Linking: $Sl = \cdots, -1, -\frac{1}{3}, \frac{1}{3}, 1, \cdots$
- More precisely: Sl encapsulates cyclic $\mathbb{Z}_4$ topological index



K. Janich, Acta Appl. Math. **8**, 65-74 (1987)
G. Alexander, B. Chen, E. Matsumoto, R. Kamien, Rev. Mod. Phys. **84**, 497 (2012)

Rewiring and charge conservation

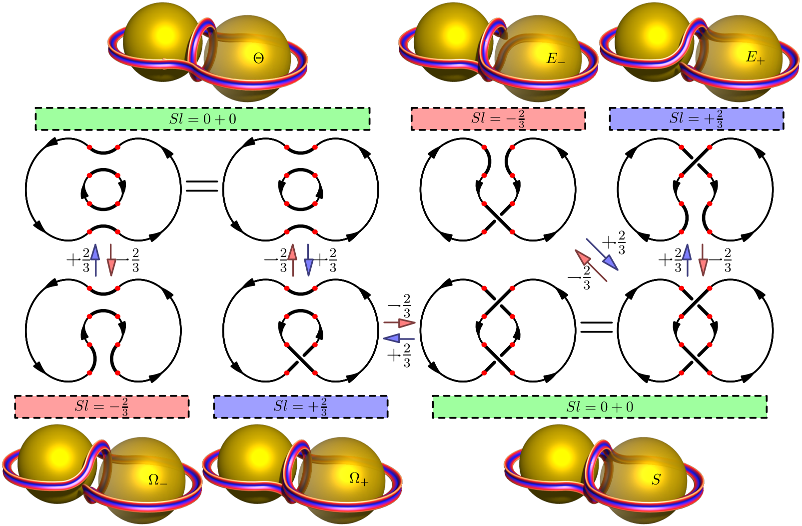
- **Oriented** rewiring changes self-linking number for $\pm\frac{2}{3}$.
- Rewiring changes the number of loops by ± 1.
- Rewiring is local: π_2 charge is conserved.
- Many loops: linking numbers also count!

$$\boxed{\frac{3}{2}\left(\sum_i Sl(A_i) + 2\sum_{i>j} Lk(A_i, A_j)\right) + n = q \mod 2}$$

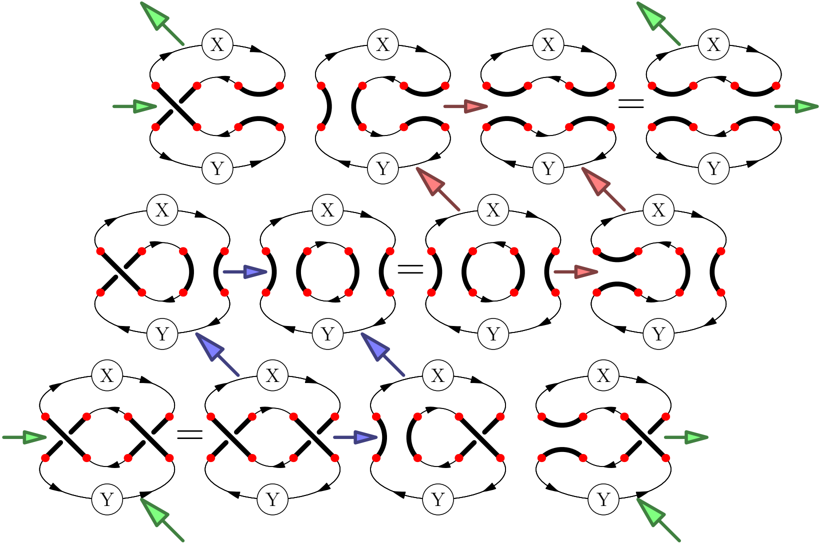
Compare with:

$$\boxed{\sum q_i = \frac{1}{2}\left(\nu_i - \sum_{i\neq j} Lk(A_i, A_j)\right) = q \mod 2}$$

K. Janich, Acta Appl. Math. **8**, 65-74 (1987)



S. Čopar and S. Žumer, Phys. Rev. Lett. **106**, 177801 (2011)



X
Y
X
Y
X
Y
X
Y
X
Y
X
Y
X
Y
X
Y
X
Y
X
Y
X
Y
X
Y

Linking matrix

Gauss integral is bilinear $\mapsto$ linking invariants of an union of loops are components of a matrix.

$$L_{ij} = \begin{cases} Sl(A_i) & i = j \\ Lk(A_i, A_j) & \text{true} \end{cases}$$

Conservation law states the topological charge of a set of loops is simply a matrix element:

$$u_i(\frac{3}{2}L_{ij} + \delta_{ij})u_j = q \mod 2$$

$u_i \in (-1, 0, 1)$ encodes orientations of loops (± 1) or lack of interest for a particular loop (0).
Linking number is orientation-sensitive, trace $Sl = L_{ii}$ is a good link invariant.

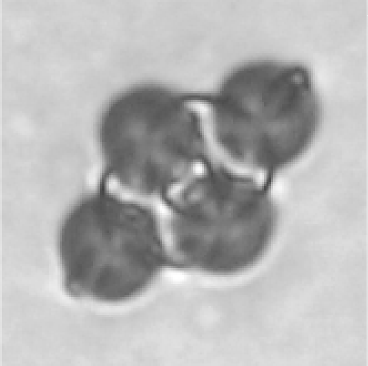
$-\frac{1}{2}$ disclinations vs. general disclinations

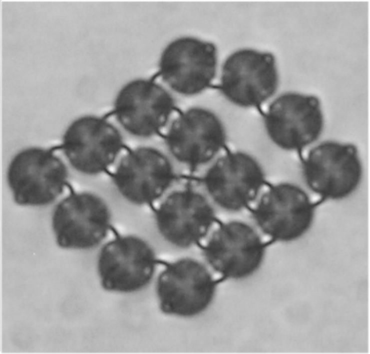
$-1/2$ disclinations

- Fixed 3-fold profile
- Ribbons with well defined Sl
- Full integer classification
- Tetrahedral rewiring
- Most nematic and chiral nematic colloids

general nematic disclinations

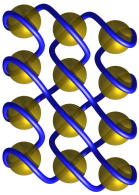
- Any profile
- No reference for the framing
- Only $\mathbb{Z}_4$: linked/unlinked, even/odd charge
- No simple geometric rewiring formalism
- Special frustration, highly chiral phases



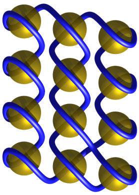


Obtaining knots

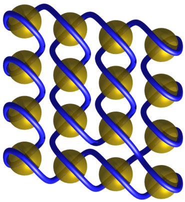
- Every rewiring is allowed $\mapsto$ every knot is possible
- Topological constraints are satisfied by the self-linking number
- Full topological information of a framed knot: **knot or link type + linking matrix**
- A particular colloidal grid: still all knots possible or not?



trefoil (+)



trefoil (−)



7_4

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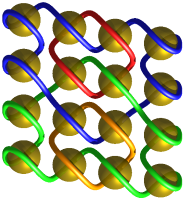
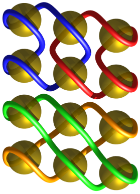
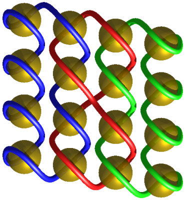


$L_{10}n_{104}$



hopf + 2 loops



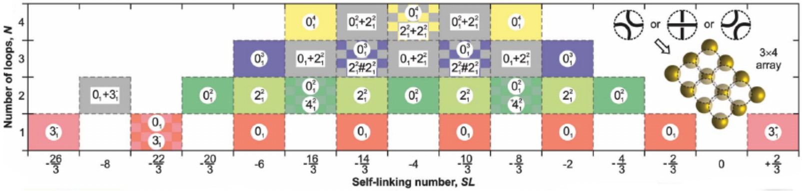
Borromean rings

Classification scheme

Conservation law:

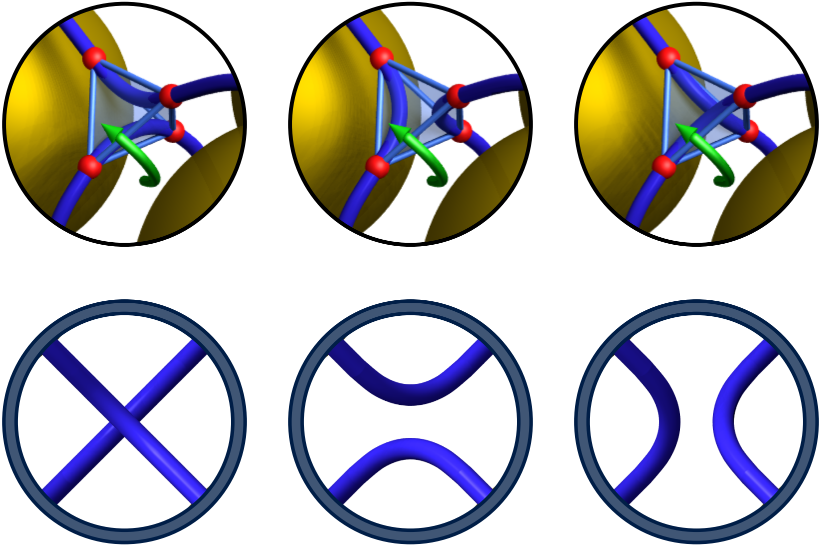
$$\underbrace{\frac{3}{2}\sum_i Sl(A_i)}_{Sl} + \underbrace{\frac{3}{2}\sum_{i>j} Lk(A_i, A_j)}_{\text{even/odd link}} + n = q \mod 2$$

Plot knots into (Sl, n) diagram.



U. Tkalec, M. Ravnik, S. Čopar, S. Žumer, I. Muševič, Science **333**, 62 (2011)

Kauffman (unoriented) tangles:

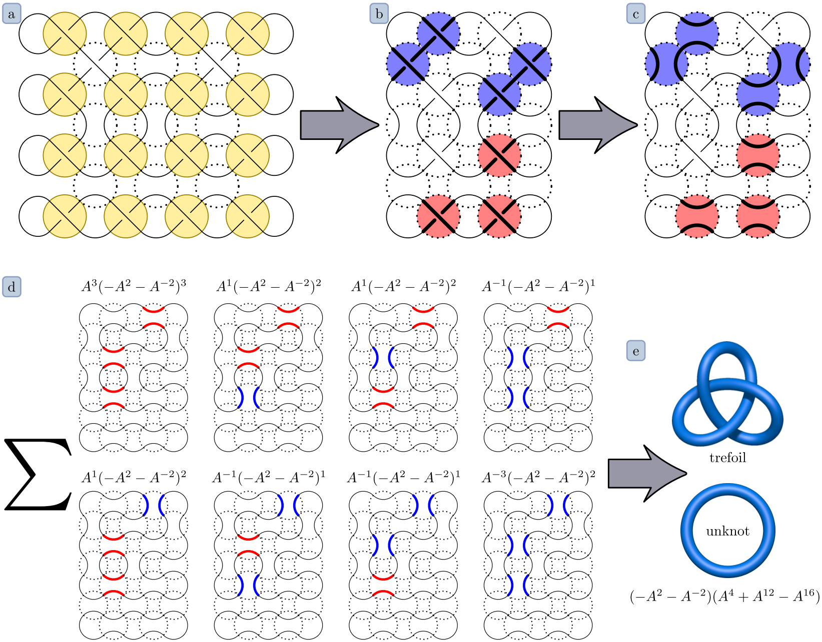


The Jones polynomial

- Writhe calculation: numerical integration ✓
- Lk: counting crossings **or** numerical integration ✓
- Knot classification?

Orient links $\Rightarrow$ obtain integer writhe $\Rightarrow$ summation formula for Kauffman bracket $\Rightarrow$ recover Jones polynomial $\Rightarrow$ table lookup $\Rightarrow$ hope the Jones polynomial gives the right answer

$$X(K) = (-A)^{-3w(K)} \sum A^{a-b}(-A^2 - A^{-2})^{n-1}$$



a
b
c
d
$A^3(-A^2 - A^{-2})^3$
$A^1(-A^2 - A^{-2})^2$
$A^1(-A^2 - A^{-2})^2$
$A^{-1}(-A^2 - A^{-2})^1$
$A^1(-A^2 - A^{-2})^2$
$A^{-1}(-A^2 - A^{-2})^1$
$A^{-1}(-A^2 - A^{-2})^1$
$A^{-3}(-A^2 - A^{-2})^2$
e
trefoil
unknot
$(-A^2 - A^{-2})(A^4 + A^{12} - A^{16})$

Extending the theory

- Four-point junction as an object
- $+1/2$ rewiring formalism
- Generalization to other (non-three fold) director field profiles
- Higher order nontetrahedral building blocks
- Any suggestions?

Thank you!